GRAPH INVARIANTS IN KNOT THEORY

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ABSTRACT

GRAPH INVARIANTS IN KNOT THEORY

This thesis reviews the establishment of the link between knot theory and graph theory. The Chromatic polynomial, the dichromatic polynomial and the Tutte polynomial are examined in detail as graph invariants related to the vertex coloring of a graph. Signed planar graphs are one-to-one correspondence with links and knots via medial construction. This correspondence reveals the relation between the Tutte polynomial and Kaufmann bracket polynomial, hence a Jones polynomial. Furthermore, we explore Virtual Knot Theory, introduced by Kauffman, which generalizes classical knot theory. The Bollobás-Riordan polynomial is presented as a generalization of the Tutte polynomial for ribbon graphs. We show the relationship between the Kauffman bracket polynomials of virtual links and the Bollobás-Riordan polynomials of ribbon graphs.

ÖZET

Düğüm Teorisinde Graf Değişmezleri

Bu tez, düğüm teorisi ile graf teorisi arasındaki bağın kurulmasını incelemektedir. Graf değişmezleri olarak incelenen Chromatic polinomu, Dichromatic polinomu ve Tutte polinomu, bir grafın köşe boyamaları ile ilişkilidir. Düzlemsel işaretli grafın medial yapısı, linkler ve düğümler ile birebir bir ilişkiye sahiptir. Bu ilişki, Tutte polinomu ile Kauffman bracket polinomu arasındaki bağı, dolayısıyla Jones polinomu ile olan ilişkiyi ortaya koyar. Ayrıca, klasik düğüm teorisini genelleyen Virtual Düğüm Teorisi'ni, Kauffman'ın tanıttığı şekliyle inceliyoruz. Bollobás-Riordan polinomu, ribbon grafikler için Tutte polinomunun bir genellemesi olarak sunulmaktadır. Son olarak, sanal linklerin Kauffman bracket polinomları ile ribbon grafilerin Bollobás-Riordan polinomları arasındaki ilişkiyi gösteriyoruz.

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CHAPTER 1

INTRODUCTION

Graph theory plays a crucial role in various fields such as computer science, biology, and social network analysis. It provides powerful tools for solving problems related to network connectivity, optimization, and data structure. Knots are embeddings of a circle in three-dimensional space. Knot theory is fundamental for understanding the properties of physical knots in DNA, polymers, and other molecular structures. Knot theory also has applications in physics, particularly in the study of quantum field theory and statistical mechanics. Both fields rely on invariants to classify and analyze these structures. Invariants are quantities that remain unchanged under specific transformations, making them crucial for distinguishing different graphs and knots.

The chromatic polynomial is first introduced by the English mathematician George David Birkhoff in 1912 (Birkhoff, 1912). Birkhoff developed this polynomial to solve problems related to graph colorings. Tutte generalized the chromatic polynomial by introducing the dichromatic polynomial, which is now known as the Tutte polynomial (Tutte, 1954). Tutte defined his polynomial based on the activity states arising from the edge labels of a graph. Kauffman further generalized Tutte's definition for signed graphs (Kauffman, 2006). In this context, there is a one-to-one correspondence between planar graphs and link and knot diagrams via a medial construction. This correspondence establishes a relationship between the Tutte polynomial and Kauffman, and consequently with the Jones polynomial.

The Bollobás-Riordan polynomial was introduced by Béla Bollobás and Oliver Riordan for ribbon graphs (Bollobás and Riordan (1999), Bollobás and Riordan (2001), Bollobás and Riordan (2002)). Ribbon graphs represent graphs embedded on surfaces. The Bollobás-Riordan polynomial is a generalization of the Tutte polynomial. Additionally, Chmutov and Pak showed that the Kauffman bracket polynomials of checkerboard colorable virtual links is related to the Bollobás-Riordan polynomial of the corresponding ribbon graph of those links (Chmutov and Pak, 2006). This result generalizes the relationship between the Tutte and Bracket polynomials.

This thesis is structured as follows;
Chapter 2 introduces the basic definitions and properties of graphs and knots, laying the groundwork for the subsequent chapters.

Chapter 3 explores polynomial invariants in detail, including the Chromatic polynomial, the Dichromatic polynomial, and the Tutte polynomial in graph theory, as well as the Bracket polynomial and the Jones polynomial in knot theory. The relationship between these polynomials is reviewed.

Chapter 4 discusses the fundamental definitions and theorems of the generalization of classical knots and graphs.

Chapter 5 focuses on the Bollobás-Riordan polynomial, a generalization of the Tutte polynomial, and the relationship between the Bollobás-Riordan polynomial and the Jones polynomial for virtual knots.

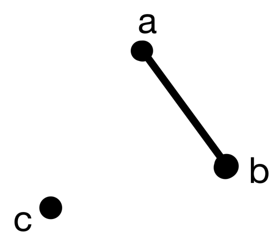
CHAPTER 2

PRELIMINARIES

This chapter introduces some basic terminology related to graphs and knots that are used later. Section 2.1 provides a review of graph theory while 2.2 introduces the basics of knot theory.

2.1. On Graph Theory

Graph theory is a foundational branch of discrete mathematics. The origins of graph theory can be traced back to the 18th century, with Leonhard Euler's The Seven Bridges of Königsberg Problem, where he introduced the concept of a graph to analyze the problem's underlying structure. Graph theory is utilized in different fields, including computer science, operations research, social sciences, and biology. In this section, we shall present the basic concepts of graph theory and their properties. The following sources are referenced: Bondy and Murty (2008), Bollobás (2013), Diestel (2005).



We shall now give the definition of a graph:

Definition 2.1 A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges, disjoint from $V(G)$, together with an *incidence function* $\psi_{(G)}$ that associates with each edge of G an ordered pair of vertices of G. The incidence function $\psi_{(G)}$

$$\psi_{(G)} : E(G) \to V(G) \times V(G)$$

is defined as

$$\psi_G(e) = (u, v)$$

where $e \in E(G)$ and $u, v \in V(G)$. In this case, we say e is *incident* to (or *join*) u and v which are called the *ends* of e. Additionally, the vertices u and v are termed adjacent (or neighbors), and the set of all neighbors of a vertex v is denoted by $N_G(v)$.

Different types of edges give rise to specific classifications of graphs.

Definition 2.2 A vertex v of a graph G is *adjacent* to itself, i.e., there exists an edge $e = vv$ of G. Such edges are called *loop*. If two or more edges of a graph G has same ends, then these edges are called *multiple edges*. A graph is called *multigraph* if it has multiple edge. A *simple* graph is a graph that does not contain loops and multiple edges.

Example 2.1 Suppose $G = (V(G), E(G))$ where

$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$
$$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, \}$$

and

$$\psi_G(e_1) = v_1v_2 \quad \psi_G(e_2) = v_2v_3 \quad \psi_G(e_3) = v_3v_2 \quad \psi_G(e_4) = v_4v_6$$

$$\psi_G(e_5) = v_5v_4 \quad \psi_G(e_6) = v_6v_6 \quad \psi_G(e_7) = v_4v_1$$

and the diagram of the graph G is shown in the figure 2.1.

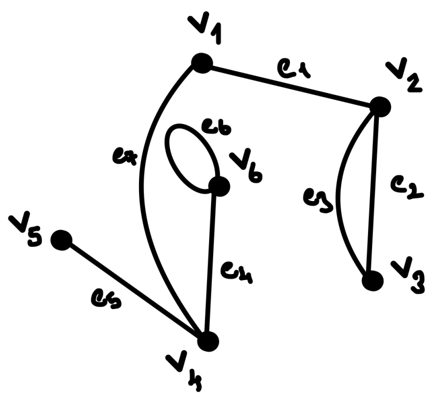


Figure 2.1. The graph G

Definition 2.3 The *order* of the graph G is the number of vertices of G and the *size* of the graph G is the number of edges in G, denoted by $|G|$ (or $v(G)$) and $e(G)$, respectively. If the order of a graph G is finite (or infinite), then G is called a finite (or infinite) graph.

The focus of this thesis is on finite graphs, and the term 'graph' refers to a finite graph.

Definition 2.4 A graph G is called empty if $V(G)$ and $E(G)$ are empty sets. If $|G| = 1$, then G is called a trivial graph. If $|G| = 0$, then G is called a null graph.

Definition 2.5 A graph H is called a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning subgraph* H of a graph G is defined as a subgraph where $V(H) = V(G)$.

Example 2.2 Suppose $G = \{V(G), E(G)\}$ is defined in Example 2.1. For a subgraph H of G,

$$V(H) = \{v_1, v_2, v_4, v_6\}$$

$$E(H) = \{e_1, e_6, e_7,\}$$

and

$$\psi_G(e_1) = v_1v_2 \quad \psi_G(e_6) = v_6v_6 \quad \psi_G(e_7) = v_4v_1.$$

A spanning subgraph S of G is defined as $E(S) = \{e_1\}$ and $V(S) = V(G)$.

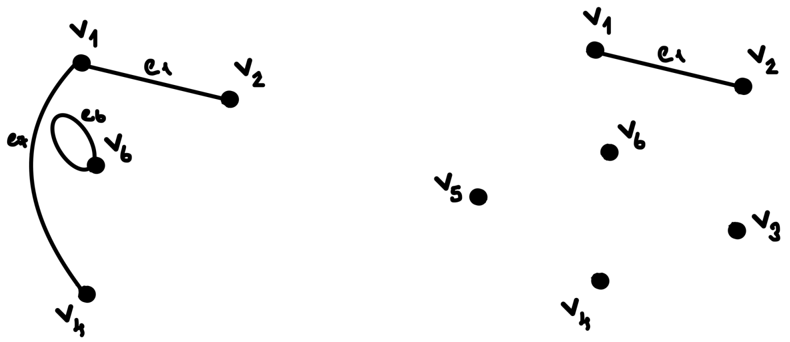
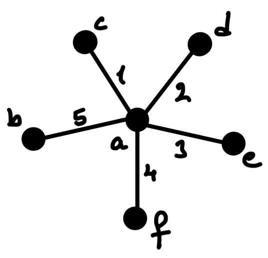


Figure 2.2. The diagrams of subgraph H and spanning subgraph S.

Definition 2.6 A *degree* $d(v)$ of the vertex v is the number of edges that are incident to the vertex v. An *isolated* vertex is a vertex with degree 0. A graph G is called *r-regular* if every vertex in G has degree r.

Let G be a graph with $V(G) = \{a, b, c, d, e, f\}$ and $E(G) = \{1, 2, 3, 4, 5\}$ as given



then $d(a) = 5$ and for the other vertices $d(i) = 1$, $i \in V(G) - a$.

Each edge contributes to the degree at both of its ends. For a graph G, this gives the following relation:

$$2e(G) = \sum_{i \in V(G)} d(i).$$

As a result, there are an even numbers of vertices with odd degrees, since the number of edges in G is half the sum of the degrees of its vertices.

Definition 2.7 A *path* $P = (V, E)$ is a graph with

$$V(P) = \{v_0, v_1, ..., v_j\} \qquad E(P) = \{v_0v_1, v_1v_2, ..., v_{j-1}v_j\},$$

denoted by $v_0v_1...v_j$. The vertices v_0 and v_j are called the *ends* of P and the *length* of P is the number of edges $e(P)$. If a path P is of the form $v_i...v_i$, i.e., its ends are the same vertex v_i, then P is called *cycle*.

Let us see an example of the path of a graph:

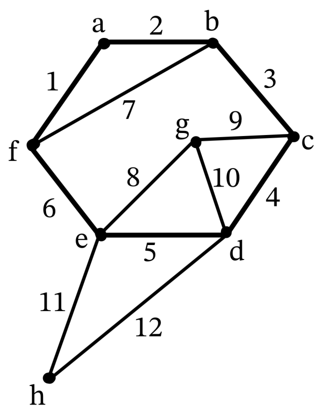
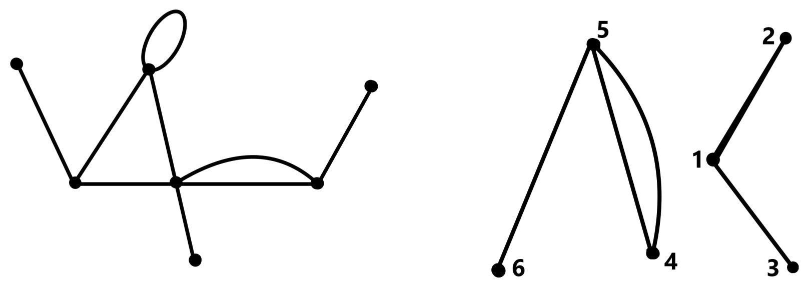


Figure 2.3. A graph G

Example 2.3 Let G be a graph as shown the Figure 2.3. Some paths of G are:

$$P_1 = a2b3c4d6e11h$$
$$P_2 = f1a2b7f$$
$$P_3 = e6f7b3c9g8e$$
$$P_4 = e8g10d4c3b2a1f$$
$$P_5 = h12d10g8e6f1a2b$$

Definition 2.8 A graph G is called *connected* if, for every pair of vertices $v_i, v_j \in V(G)$, $i \neq j$, there exists a path from v_i to v_j.



(a) (b)

Figure 2.4. (a) connected graph; (b) disconnected graph

In the graph shown in Figure 2.4b, we cannot find any path between the vertices 1, 2, 3 and the vertices 4, 5, 6. Therefore, it is a disconnected graph.

Definition 2.9 A *tree* T is a connected graph with no cycles. A *spanning tree* H of a graph G is a tree with $V(H) = V(G)$.

2.1.1. Spanning Trees

We shall explore some properties of the spanning trees of a graph G. In this section, we assume all graphs are connected. We use the notion of the spanning trees in Chapter 3.

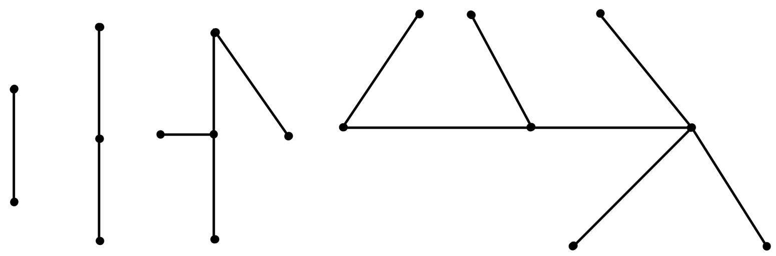


Figure 2.5. Some tree examples

Theorem 2.1 (Bondy and Murty, 2008) Let G be a connected graph. Then G has a spanning tree.

Proof We prove by induction on the number of vertices and the number of edges. Let G be a connected graph with 1 vertex and no edge. Therefore, the spanning tree of G is itself. Let G be a connected graph with n vertices and m edges. If G has no cycle, then G itself is a spanning tree. If G has cycle, then for an edge e in the cycle. Suppose $G-e$ is a subgraph of G with no cycle and $G-e$ is connected. Since $G-e$ has $m-1$ edges, then it must have a spanning tree from the hypothesis. This spanning tree is also a spanning tree of G. Hence G has a spanning tree. □

This theorem confirms the existence of the spanning tree of a connected graph G. Let $S(G)$ be the set of spanning trees of G. For an edge $e \in G$, let $G \backslash e$ be a graph with contracting edge e, i.e, collapsing e and its ends to a new vertex v', whose neighbors are all the neighbors of the ends of e. Also $G-e$ is graph obtained by deleting edge e from G. Both operations are illustrated in Figure 2.7. We have the following result:

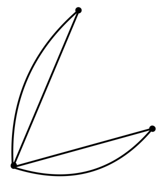
Proposition 2.1 (Bollobás, 2013) Let G be a connected graph and $e \in G$. For the set $S(G)$ of spanning trees of G, we have

$$S(G) = S(G-e) \sqcup S(G \backslash e).$$

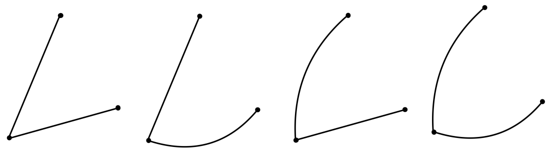
Moreover, let $t(G)$ denote the number of spanning trees of G. Then

$$t(G) = t(G-e) + t(G \backslash e).$$

Proof For a connected graph G and $e \in G$, the spanning trees of G include spanning



(a)



(b)

Figure 2.6. (a) Graph G; (b) Spanning trees of G

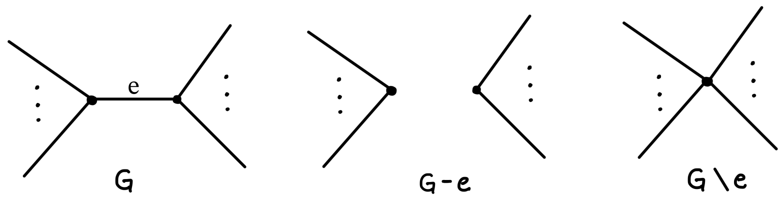


Figure 2.7. The contraction and deletion of a graph of G

trees that consist of the edge e and spanning trees that do not include e. The graph $G-e$ is a subgraph of G, and its spanning trees are the spanning trees of G that do not include e. The set of spanning tree of G that include e is one-to-one correspondence with the set of spanning trees of $G\backslash e$. Thus $|S(G)| = |S(G-e) \sqcup S(G\backslash e)| = |S(G-e)| + |S(G\backslash e)|$, i.e.,

$$t(G) = t(G-e) + t(G\backslash e).$$

$\square$

For a connected graph, we see that there exists a spanning tree. Before presenting the theorem about the number of spanning trees of G, we introduce the concept of the *Laplacian* matrix of G.

Definition 2.10 Let G be a loopless graph with the vertices $v_1, v_2, ..., v_n$. The *degree* matrix $D(G)$ of G is the $n \times n$ diagonal matrix where the i,i-entry is the degree of vertex v_i. The *adjacency* matrix $A(G)$ of G is the $n \times n$ matrix where i, j-entry is the number of edges between the the vertices v_i and v_j, its diagonals are zero. Then the *Laplacian* matrix $L(G)$ is defined as

$$L(G) = D(G) - A(G).$$

Example 2.4 Let G be the graph given in Figure 2.6a. Then the degree matrix of G is

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and the adjacency matrix of G is

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Therefore the Laplacian matrix of G is

$$\begin{pmatrix} 4 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}.$$

Theorem 2.2 (Moore and Mertens, 2011) For a loopless connected graph G, we have

$$det(L^{(ii)}(G)) = t(G),$$

where $L^{(ii)}(G)$ is obtained by deleting i^{th} row and i^{th} column from $L(G)$

Proof We prove by induction on the number of vertices v and the number of edges e of a loopless graph G. Start with $v = 1$ and $e = 0$, in which case $L(G) = (0)$, and $L^{(ii)}(G)$ is a 0×0 matrix where $\det(L^{(ii)}(G)) = 1$. From Proposition 2.1, we have:

$$t(G) = t(G-e) + t(G\backslash e).$$

Since $G-e$ and $G\backslash e$ are hold by induction, $t(G-e) = det(L^{(ii)}(G-e))$ and $t(G\backslash e) = det(L^{(jj)}(G/e))$ for some i, j. Now we want to show that

$$det(L^{(ii)}(G)) = det(L^{(ii)}(G-e)) + det(L^{(ii)}(G/e)).$$

Choose $e = ij \in G$, where the ends of e are the vertices i and j. The Laplacian matrix $L(G)$ can be expressed as:

$$L(G) = \begin{pmatrix} d_i & -1 & r_i^T \\ -1 & d_j & r_j^T \\ r_i & r_j & L' \end{pmatrix},$$

where r_i (r_j) is an $(n-2)\times 1$ row vector representing the adjacency of i (j) with the other vertices and L' is the rest of the matrix.

$$L(G-e) = \begin{pmatrix} d_i - 1 & 0 & r_i^T \\ 0 & d_j - 1 & r_j^T \\ r_i & r_j & L' \end{pmatrix}, \quad \text{and } L(G/e) = \begin{pmatrix} d_i + d_j - 2 & r_i^T + r_j^T \\ r_i + r_j & L' \end{pmatrix}$$

To find the determinants, we delete the first row and column of the above matrices:

$$L^{11}(G) = \begin{pmatrix} d_j & r_j^T \\ r_j & L' \end{pmatrix}, \quad L^{11}(G-e) = \begin{pmatrix} d_j - 1 & r_j^T \\ r_j & L' \end{pmatrix}, \quad L^{11}(G/e) = \begin{pmatrix} L' \end{pmatrix}$$

The determinant of these minors gives:

$$\det\begin{pmatrix} d_j - 1 & r_j^T \\ r_j & L' \end{pmatrix} + \det(L') = (d_j - 1)\det(L') + \sum_{i=2}^{n}(-1)^i L_{1,i}(G-e)\det(L^{1,i}(G-e))$$
$$+\det(L')$$
$$= d_j \det(L') + \sum_{i=2}^{n}(-1)^i L_{1,i}(G-e)\det(L^{1,i}(G-e))$$
$$= \det\begin{pmatrix} d_j & r_j^T \\ r_j & L' \end{pmatrix}$$
$$= \det L^{11}(G)$$

as desired. □

Example 2.5 Let G be the graph given in Figure 2.6a. It has the Laplacian matrix $L(G)$

$$\begin{pmatrix} 4 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix},$$

$L^{11}(G)$ is the matrix obtained by deleting the first row and first column of $L(G)$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $\det L^{11}(G)$ is 4 (see all spanning subgraphs of G in Figure 2.6b).

2.2. On Knot Theory

In mathematics, knot theory emerged in the late 18th century. The first tabulation of knots introduced at the end of 19th century. Whereas there were important contributions in the first half of the 20th century(such as Reidemeister moves and Alexander polynomial), the significant results took place in the second half. Also, Jones, Witten, Dienfeld (1990) and Kontsevich (1998) were awarded their Field medals for their work in knot theory. In this part, we shall present the fundamental definitions and properties of knot theory. The following sources are used Kauffman (2001), Rolfsen (2003), Manturov (2018), Kauffman (1987). We shall give a definition of knot as 3-dimensional object:

Definition 2.11 A smooth knot K is the image of a smooth injective map

$$K : S^1 \hookrightarrow \mathbb{R}^3$$

with $\frac{dK}{dt} \neq 0$. A link in $\mathbb{R}^3$ is the smooth injective image of the disjoint union of finitely many circles:

$$L : S^1 \sqcup \ldots \sqcup S^1 \hookrightarrow \mathbb{R}^3$$

with non-vanishing tangent maps.

The central problem in knot theory is the classification of links (i.e., determining when two knots are the same or different). However, we first consider smooth deformation of on the set of knots and links in $\mathbb{R}^3$.

Definition 2.12 Let K_1 and K_2 be two knots in $\mathbb{R}^3$. An *ambient isotopy* of $\mathbb{R}^3$ that takes K_1 to K_2 is a continuous map

$$f : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

such that

i. $f(x,t_0)$ is a diffeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ for all t_0 in $[0,1]$

ii. $f(x,0) : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity map and $f(K_1,1) = K_2$.

Now we shall define when two knots are considered deformations of each other up to ambient isotopy:

Definition 2.13 Let K_1 and K_2 be two knots in $\mathbb{R}^3$. K_1 and K_2 are called equivalent, denoted by $K_1 \sim K_2$ if there is an ambient isotopy taking K_1 to K_2.

This classification problem can also be extended to oriented links.

Definition 2.14 A smooth *oriented link* L is the image of a smooth injective map of the disjoint union of finitely many oriented circles in $\mathbb{R}^3$ with non-vanishing tangent maps. Similarly, a smooth *oriented knot* K is the image of a smooth injective map of an oriented circle in $\mathbb{R}^3$ with non-vanishing tangent maps.

Definition 2.15 An *ambient isotopy* of $\mathbb{R}^3$ that takes an oriented link $\overrightarrow{L}_1$ to an oriented link $\overrightarrow{L}_2$ is an ambient isotopy of link that is orientation-preserving.

There are several approaches to knot theory. Knots are defined in 3-dimensional space. The approach related to this thesis is combinatorial knot theory. Consider a plane, then project the knot onto it. With respect to the projection, there exist strands that intersect at points (or 4-valent vertices). The preimage of these points does not intersect in $\mathbb{R}^3$. Using this information, we call one strand an overcrossing and the other an undercrossing.

Definition 2.16 A *knot diagram* is a projection of a knot in $\mathbb{R}^2$ with extra under or over information endowed each self-intersection of the projection curve. A *universe* of a knot is a projection of a knot without under and over information, it is a 4-valent graph.

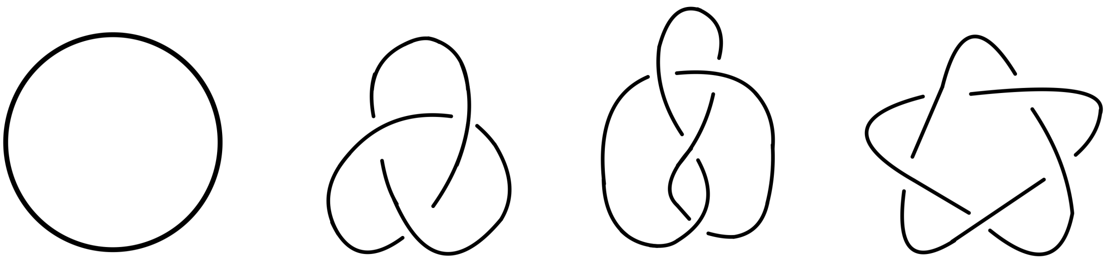


Figure 2.8. The unknot 0_1, Trefoil knot 3_1, Figure-eight knot 4_1 and Cinquefoil knot 5_1

Definition 2.17 An arc of a knot diagram is defined as the portion of the knot that starts from one undercrossing and ends at the next undercrossing.

The following theorem establishes the equivalence between the classification problem of links and the classification problem of link diagrams.

Theorem 2.3 Two knots K_1 and K_2 are equivalent in $\mathbb{R}^3$ if and only if K_1 can be deformed to K_2 with a finite sequence of Reidemeister moves which is shown in Figure 2.9.

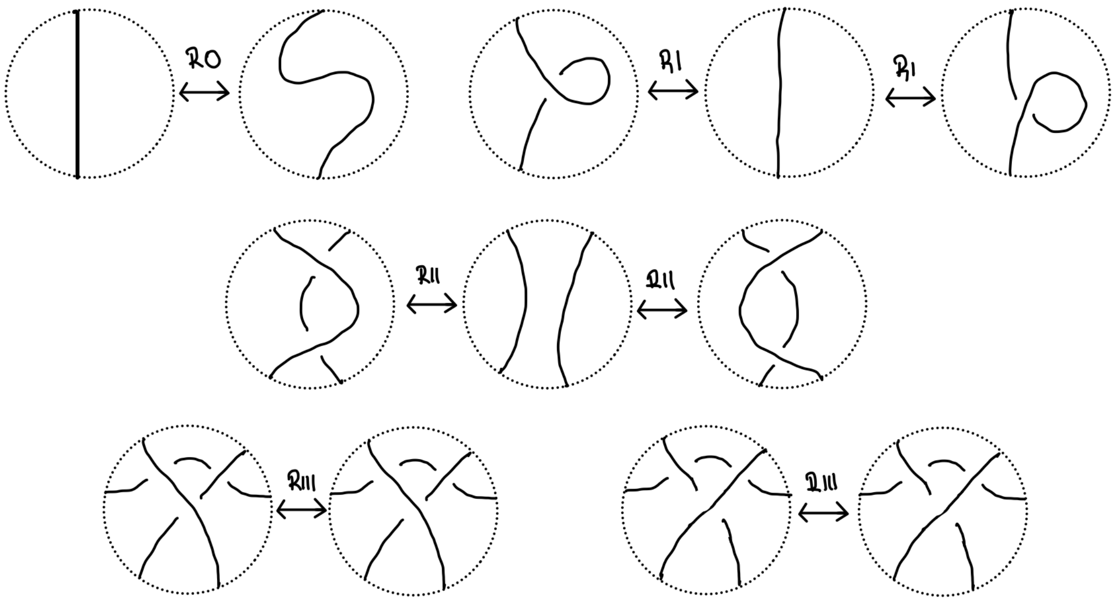


Figure 2.9. Reidemeister Moves

Some definitions need to be given in order to prove the above theorem 2.3. All smooth knots are homeomorphic to the unit circle S^1. Therefore, there exist only one knot, the unknot, up to homeomorphism. Additionally, S^1 is a 1-dimensional smooth manifold.

Theorem 2.4 (Cairns, 1935)(Whitehead, 1940) Every smooth manifold admits an compatible piecewise linear structure.

Now we can introduce polygonal knots, based on the book by Manturov (2018).

Definition 2.18 A *polygonal knot* is a representation of a knot in $\mathbb{R}^3$,consisting of a finite sequence of straight line segments, or edges, without self-intersections, such that each edge is connected to its neighbors at vertices, forming a continuous and closed loop. A *polygonal link* in $\mathbb{R}^3$ is a finite pairwise disjoint union of polygonal knots.

Consider the following example:

Example 2.6 The smooth trefoil knot and the polygonal trefoil knot are given in Figure 2.10.

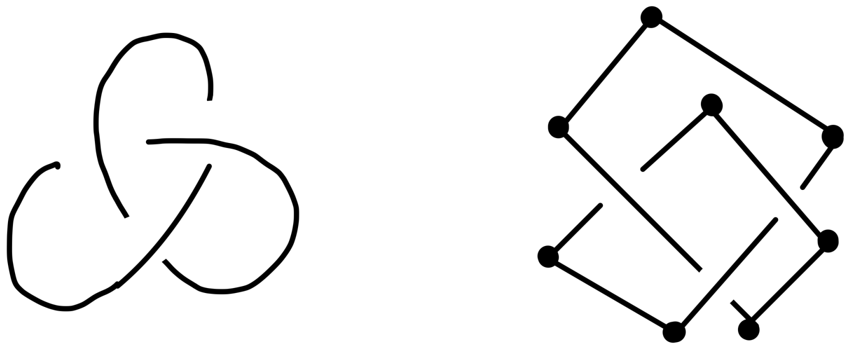


Figure 2.10. Smooth trefoil knot and polygonal trefoil knot

Definition 2.19 Two polygonal links in $\mathbb{R}^3$ are *isotopic* if there exists a finite number of *elementary isotopies* (see Figure 2.11) that transforms one of the polygonal links to the other.

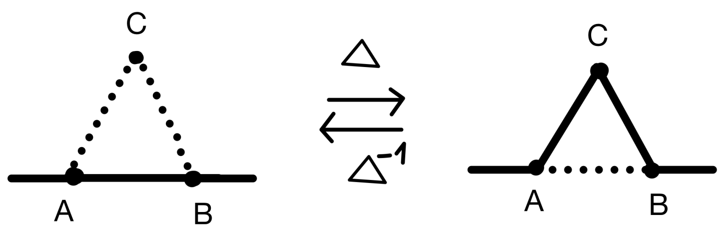


Figure 2.11. The Elementary isotopy

We are now ready to prove Theroem 2.3:

Proof of Theorem 2.3 We prove "only if" part by induction on the number of strands in triangular regions. For one strand, with no strand intersecting the triangular region.

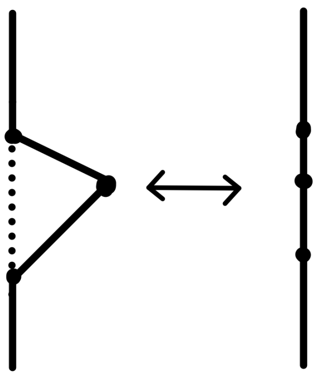


Figure 2.12. The Δ_0 move in $\mathbb{R}^3$

For one strand, with one strand intersecting the triangular region.

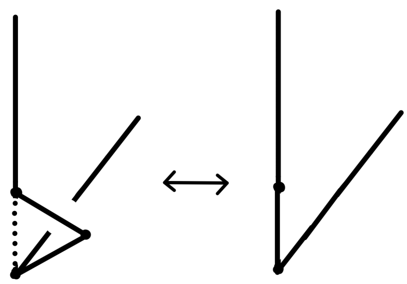


Figure 2.13. The Δ_1 move in $\mathbb{R}^3$

For two strand, with one strand intersecting the triangular region and one vertex intersecting the triangular region.

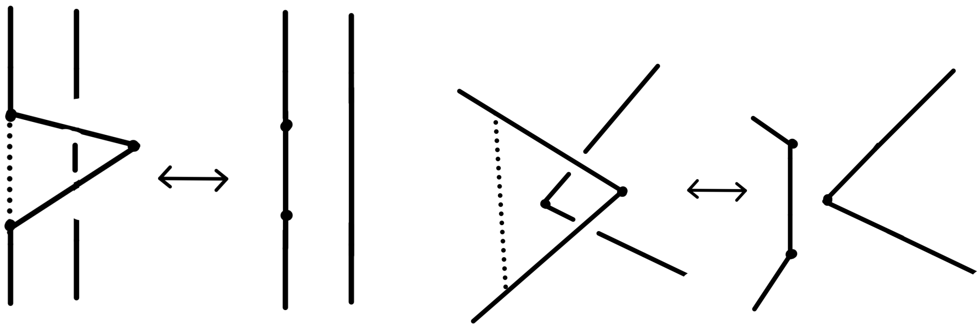


Figure 2.14. The Δ_2 move in $\mathbb{R}^3$

For three strand, with two stands and one vertex in the triangular region.

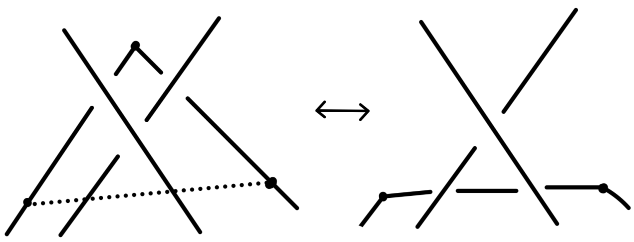


Figure 2.15. The Δ_3 move in $\mathbb{R}^3$

For more complicate we subdivide into smaller tringle (illustrated in Figure 2.16) so that each triangle contains exactly one of the forms show in Figures 2.12, 2.13, 2.14 or 2.15. This method is called a *subdivision*. For n strands, we can use subdivision to transform a link L to L' which is equivalent to L. The Δ_0, Δ_1, Δ_2 and Δ_3 moves correspond to the $R0$, RI, RII and $RIII$, respectively.

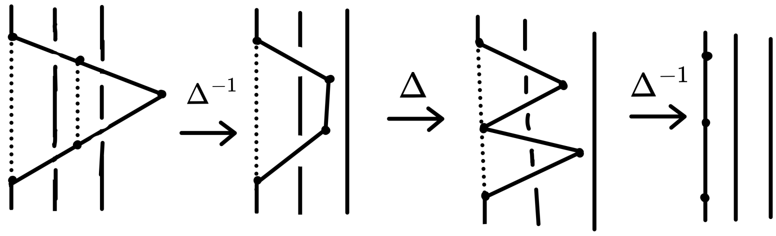
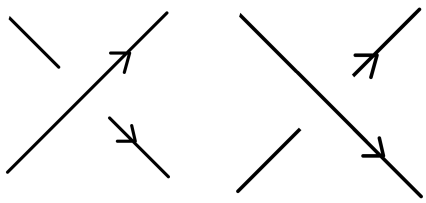


Figure 2.16. The subdivision

Therefore we can extend this to the n strand. This completes the proof. □

Definition 2.20 A link diagram is called oriented if each arc is directed so that oriented crossings have the forms in Figure 2.17.



(a) (b)

Figure 2.17. The oriented crossings: (a) is negative crossing and (b) is positive crossing

We can extend Reidemeister moves to oriented link diagrams. There are four types of oriented $R1$ move:

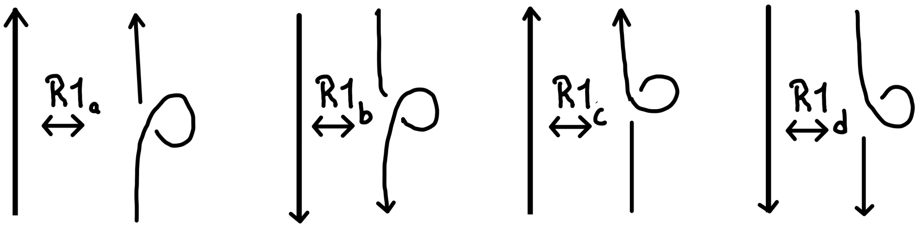


Figure 2.18. The four versions of oriented $R1$ moves

four types of oriented $R2$ move:

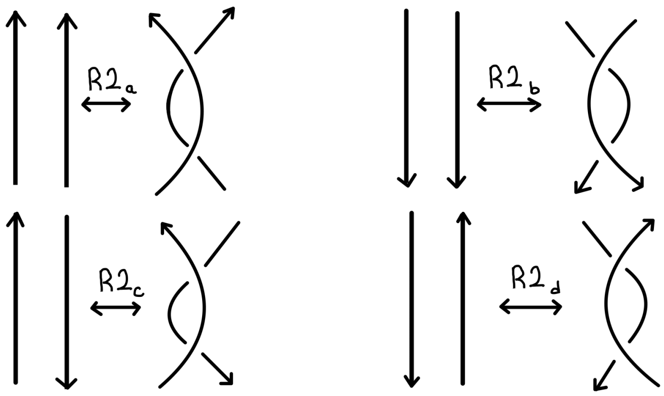


Figure 2.19. The four version of oriented $R2$ moves

and eight types of oriented $R3$ move are given in Figure 2.20. M. Polyak introduce a generating set of four oriented Reidemeister moves, which include $R1_a$, $R1_b$, $R2_a$, and $R3_a$.

Theorem 2.5 (Polyak, 2010) Let $\overrightarrow{L}_1$ and $\overrightarrow{L}_2$ be two oriented link diagrams where $\overrightarrow{L}_1 \sim \overrightarrow{L}_2$. Then there is a finite sequence of four oriented Reidemeister moves, shown in Figure 2.21.

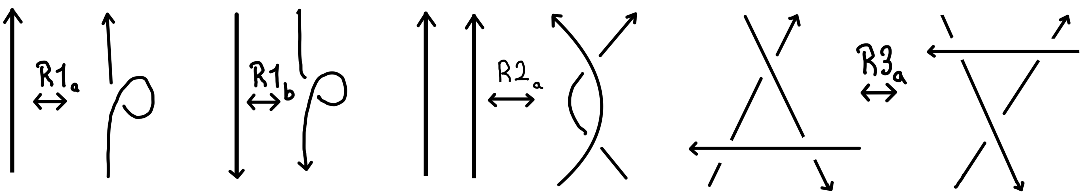


Figure 2.21. Generating set of oriented Reidemeister moves

Proof The $R2_c$ moves are obtained by a sequence of $R1_a$, $R2_a$, $R3_a$, and $R1_a$ and $R2_d$ are obtained by a sequence of $R1_b$, $R2_a$, $R3_a$, and $R1_b$ moves.
The remaining moves are obtained as shown in Figure 2.22, for more details, see Polyak (2010). □

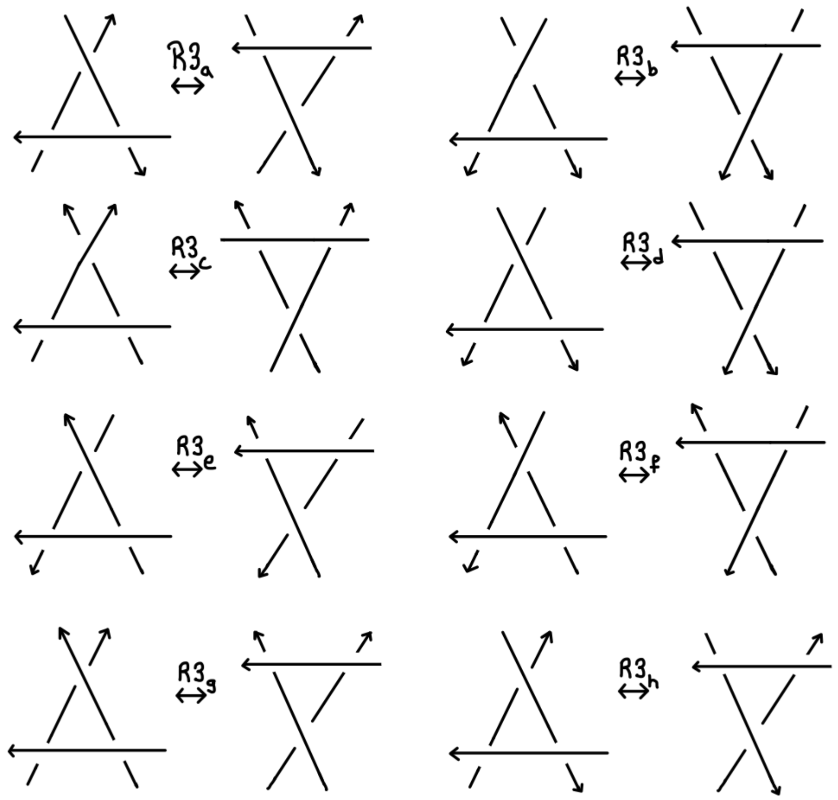


Figure 2.20. The eight version of oriented $R3$ moves

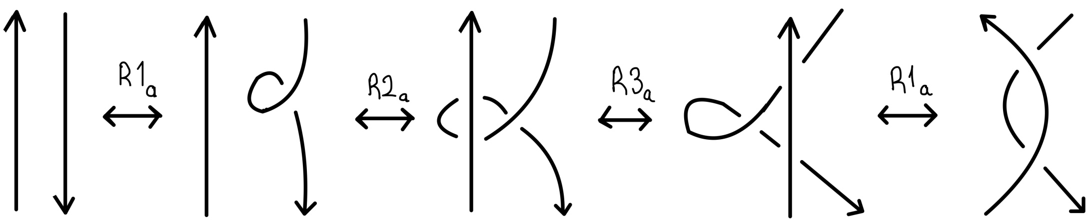
Definition 2.21 A link invariant is a function

$$I : \{Link\ Diagrams\} \to M$$

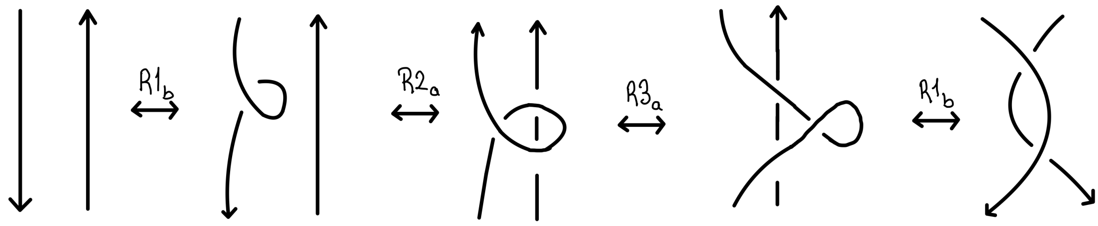
where M is a mathematical set such that if K_1 and K_2 are two knot diagrams that are equivalent to each other, then $I(K_1) = I(K_2)$.

Definition 2.22 Let K be a knot. The *crossing number* $cr(K)$ of K is the minimal number of crossings of K among all of its diagrams.

Definition 2.23 Let $\overrightarrow{L}$ be an oriented link diagram. A *shared crossing* c of $\overrightarrow{L}$ is the



(a)



(b)

Figure 2.22. $R2_c$ and $R3_d$ moves are obtained by a sequence of generating set of oriented Reidemeister moves in (a) and (b), respectively

crossing that is shared by two components of L. The *linking number* of $\overrightarrow{L}$ is defined by

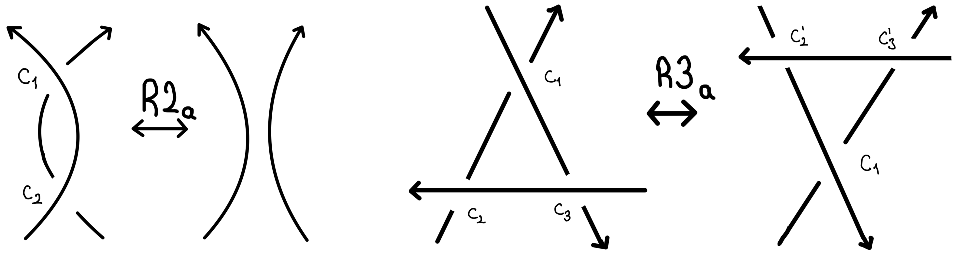
$$Lk(\overrightarrow{L}) = \frac{1}{2} \sum_{c \text{ is shared crossing}} sign(c)$$

where

$$sign(c) = \begin{cases} 1 & \text{if c is + crossing,} \\ -1 & \text{if c is - crossing.} \end{cases}$$

Theorem 2.6 Linking number is an invariant of oriented links.

Proof We need to show that the linking number is preserved under oriented Reidemeister moves. Since both $R1_a$ and $R1_b$ has no shared crossing, they do not change the linking number. We shall now observe the contribution of $R2_a$ and $R3_b$ to the linking number. In figure 2.23a, we see $R2_a$ and suppose that c_1 and c_2 are shared crossings. c_1 is a negative crossing and c_2 is a positive crossing and they both contribute 0. After applying the move, there are no crossing, so there is no contribution under the move $R2_a$. It can be easily checked the linking number preserved under $R3_a$, illustrated in Figure 2.23b. Hence, the linking number is invariant under all oriented Reidemeister moves.



(a) (b)

Figure 2.23. (a) The $R2_a$ move ; (b) The $R3_a$ move

□

CHAPTER 3

POLYNOMIAL INVARIANTS OF KNOTS AND GRAPHS

In this section we introduce the chromatic Polynomial, dichromatic polynomial and the Tutte polynomial. Following this we introduce some knot polynomials: the Bracket polynomial and the Jones polynomial. End of this section we will give the relation between the Tutte polynomial and the Bracket polynomial.

3.1. Graph Polynomials

We introduce the graph invariants in Chapter 2. Now we see Chromatic polynomial, Dichromatic polynomial and the Tutte polynomial in this section. These polynomials are all related the coloring of a graph. First, we give the idea of the vertex coloring of a graph, then we give the construction of graph polynomials related the coloring.

3.1.1. Colorings

Vertex coloring is a significant topic within graph theory, involving the assignment of colors to the vertices of a graph according to specific rules. Vertex coloring requires that each vertex be colored differently from its adjacent vertices, and it has numerous applications. For instance, this method is used in scheduling problems, map coloring issues, and frequency assignment tasks, helping to prevent conflicts and overlaps. This section references Bollobás (2013) and Bondy and Murty (2008).

Definition 3.1 A *coloring* of the vertices of G is an assignment of colors. A *proper coloring* is a coloring such that adjacent vertices are different colors.

In this thesis, we focus the proper coloring. Whereas coloring produces c_n possible colorings for any graph with n vertices and c colors, in proper coloring for the same graph, finding possible coloring is determined by the adjacency relation of vertex with others. While the question of how many colors are necessary to color a graph might not be particularly noteworthy in coloring (because we can color any graph with 1 color), it holds significant importance in proper coloring. This thesis emphasizes proper coloring.

Definition 3.2 The minimal number of colors in a vertex coloring of G is said to be *chromatic number* of G and denoted by $\chi(G)$.

Determining the chromatic number of a graph G is a challenging task, but there are some algorithms to find it. First let's look an example.

Example 3.1 Take a complete graph K_n with n vertices. Since all vertices are adjacent, $\chi(K_n) = n$, i.e., we need to use n colors. For an empty set E_n with n vertices, $\chi(E_n) = 1$., since all vertices are nonadjacent. Note that there are no proper colorings in a graph containing loops.

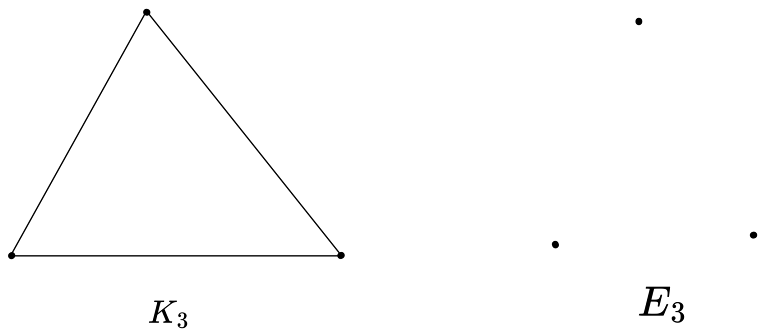


Figure 3.1. The graphs K_3 and E_3

To gain some insight into the chromatic number, a common method is the *greedy algorithm*, although this algorithm may use more colors than necessary. Let $v_1, v_2, ..., v_n$ be vertices of a graph G and let $1, 2, 3, ...$ be colors. This algorithm assigns color 1 to v_1, then assign this color 1 to all vertices that are nonadjacent to the vertex v_1. Otherwise, it assigns the color 2, and so on.

Another approach to produce coloring involves the deletion and contraction of edges in G. For a adjacent vertices v_i and v_j of G, the deletion form is $G - v_iv_j$ and the contraction form is $G \backslash v_iv_j$. (see Figure 3.2). The proper coloring of graph $G - v_iv_j$ includes both v_i and v_j vertices being the same color and different colors. Since graph G has an edge v_iv_j, vertices v_i and v_j must be colored differently. This establishes a one-to-one correspondence between the coloring of G and the coloring of the graph

$G - v_i v_j$ where v_i and v_j are colored differently. Furthermore, since the new vertex v' in $G \setminus v_i v_j$ keeps the adjacency between v_i and v_j, the proper coloring of the graph $G \setminus v_i v_j$ is a one-to-one correspondence with the proper coloring of $G - v_i v_j$ where v_i and v_j are colored identically.

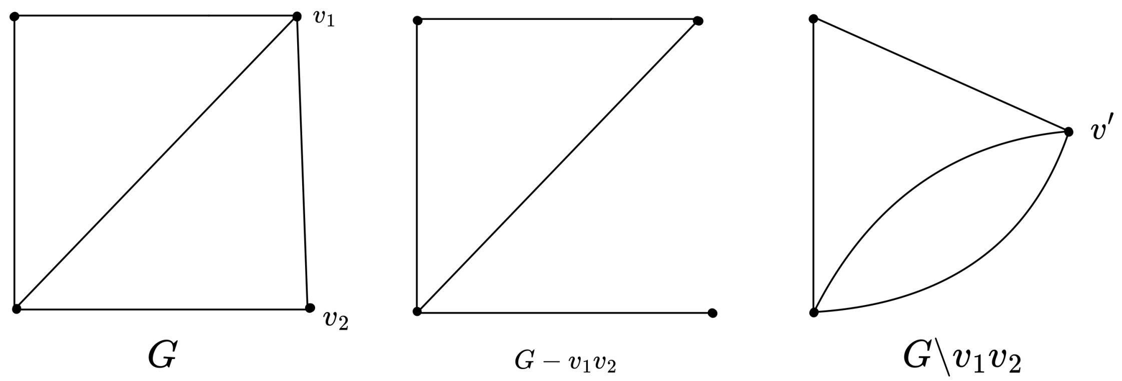


Figure 3.2. The graph G, $G - v_1 v_2$ and $G \setminus v_1 v_2$

Definition 3.3 For a graph G, the number of proper colorings of G with c colors, $c \in \mathbb{N}$ is denoted by $t_G(c)$ such that

$$\begin{cases} t_G(c) \geq 1, & \text{if } c \geq \chi(G) \\ t_G(c) = 0, & \text{if } c < \chi(G). \end{cases}$$

Considering the previous discussion about the relationship among G, $G - v_i v_j$ and $G \setminus v_i v_j$, the number of proper colorings of G, $t_G(c)$, has the following property:

For a graph G with an edge $e \in E(G)$ and c colors, $c \in \mathbb{N}$, we have

$$t_G(c) = t_{G-e}(c) - t_{G \setminus e}(c).$$

3.1.2. The Chromatic Polynomial

The concept of the chromatic polynomial was introduced by George David Birkhoff in 1912. Birkhoff's motivation was to find a more general method to tackle the Four Color Problem, which asserts that any planar map can be colored with no more than four colors so that no two adjacent regions share the same color. The chromatic polynomial $p_G(x)$ is a polynomial with variable x such that for $c \in \mathbb{N}$ (c corresponds to the number of colors), $p_G(c)$ calculates the the number of the proper colorings of a graph G with k colors. In this section, we refer to the source Bollobás (2013).

Definition 3.4 For a graph G with an edge $e = v_i v_j$, the chromatic polynomial is determined by the following properties:

i. $t_G(x) = t_{G-e}(x) - t_{G\backslash e}(x)$

ii. $t_\bullet(x) = x$

iii. Let H be a graph disjoint from G. Then $t_{G\sqcup H}(x) = t_G(x) t_H(x)$

The next theorem gives the existence of the chromatic polynomial:

Theorem 3.1 (Tutte, 1954) Suppose H is a loopless graph with the number edges $e(H)$ and the number of components $k(H)$. Then

$$t_H(x) = \sum_S (-1)^{e(S)} x^{k(S)}, \tag{3.1}$$

where the sum runs over all spanning subgraphs S of H.

Proof Let H be a graph with no edge and a vertex. Then the only spanning subgraph S of H is itself. Therefore $e(S) = e(H) = 0$ and $k(S) = k(H) = 1$ then

$$t_\bullet(x) = (-1)^{e(S)} x^{k(S)} = x.$$

Let H be a graph with an edge $e \in E(H)$. We want to show that

$$t_H(x) = t_{H-e}(x) - t_{H\backslash e}(x).$$

The spanning subgraphs of H can be partitioned into two disjoint subsets: those that include the edge e and those that exclude the edge e. The spanning subgraphs of $H-e$ is one-to-one correspondence with the spanning subgraphs of H that exclude the edge e because the rest of the graph is the same and they produce same spanning graphs. Both have same the number of components and the number of edges. In $H \setminus e$, the new vertex v' is always included in all the spanning subgraphs of $H \setminus e$. Expanding the vertex v' to the edge e give the same spanning subgraphs of H that include e. Therefore, the spanning subgraphs of $H \setminus e$ is one-to-one correspondence with the spanning subgraphs of H that include the edge$e$. However, the only difference is that the spanning subgraphs of $H \setminus e$ have one fewer edge than their corresponding spanning subgraphs of H that include e. Therefore, let S' denote the spanning subgraphs of H that exclude e and S'' denote the spanning subgraphs of H that include e, then

$$\begin{aligned}
t_H(x) &= t_{S' \sqcup S''}(x) \\
&= \sum_{S' \sqcup S''} (-1)^{e(S' \sqcup S'')} x^{k(S' \sqcup S'')} \\
&= \sum_{S'} (-1)^{e(S')} x^{k(S')} + \sum_{S''} (-1)^{e(S'')} x^{k(S'')}
\end{aligned}$$

and also we have

$$\sum_{S'} (-1)^{e(S')x^{k(S')}} = \sum_{H-e} (-1)^{e(H-e)} x^{k(H-e)}$$

$$\sum_{S''} (-1)^{e(S'')} = \sum_{H \backslash e} (-1)^{e(H \backslash e)+1} x^{k(H-e)} = -\sum_{H \backslash e} (-1)^{e(H \backslash e)} x^{k(H-e)}$$

Additionally, we have

$$\begin{aligned}
t_H(x) &= \sum_{H-e} (-1)^{e(H-e)} x^{k(H-e)} - \sum_{H \backslash e} (-1)^{e(H \backslash e)} x^{k(H-e)} \\
&= t_{H-e}(x) - t_{H \backslash e}(x),
\end{aligned}$$

as desired.

Now let H be a graph with n spanning subgraphs $S'_1, S'_2, ..., S'_n$ and G be a graph with m spanning subgraphs $S''_1, S''_2, ..., S''_m$. Then $S \sqcup G$ has nm spanning subgraphs $S_1, S_2, ..., S_{nm}$:

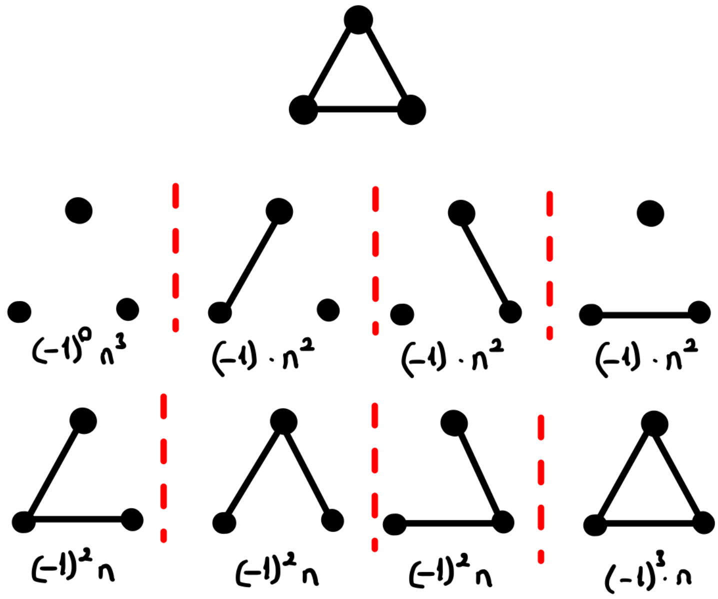
	S'_1	S'_2	S'_3	...	S'_n
S''_1	S_1	S_2	S_3	$\dots$	S_n
S''_2	S_{n+1}	S_{n+2}	$\dots$		S_{2n}
$\vdots$		$\vdots$			$\vdots$
S''_m	$S_{(m-1)n+1}$		...		S_{nm}

Thus we have

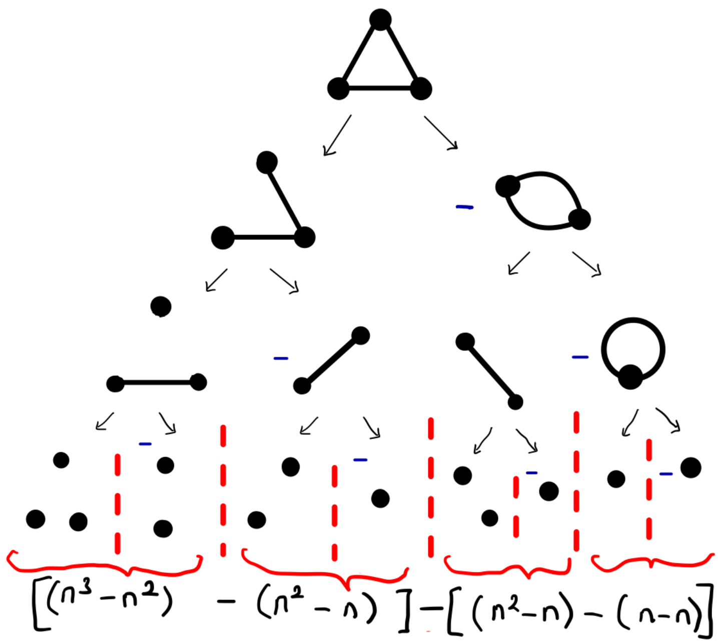
$$\begin{aligned}
t_{G\sqcup H} &= (-1)^{e(S_1)}x^{k(S_1)} + (-1)^{e(S_2)}x^{k(S_2)} + (-1)^{e(S_3)}x^{k(S_3)} + \ldots + (-1)^{e(S_n)}x^{k(S_n)} \\
&+ (-1)^{e(S_{n+1})}x^{k(S_{n+1})} + (-1)^{e(S_{n+2})}x^{k(S_{n+2})} + (-1)^{e(S_{n+3})}x^{k(S_{n+3})} + \ldots + (-1)^{e(S_{2n})}x^{k(S_{2n})} \\
&\vdots \\
&+ (-1)^{e(S_1)}x^{k(S_1)} + (-1)^{e(S_2)}x^{k(S_2)} + (-1)^{e(S_3)}x^{k(S_3)} + \ldots + (-1)^{e(S_n)}x^{k(S_n)} \\
&= \left((-1)^{e(S''_1)}x^{k(S''_1)}\right)\left((-1)^{e(S'_1)}x^{k(S'_1)} + \ldots + (-1)^{e(S'_n)}x^{k(S'_n)}\right) \\
&\vdots \\
&+ \left((-1)^{e(S''_m)}x^{k(S''_m)}\right)\left((-1)^{e(S'_1)}x^{k(S'_1)} + \ldots + (-1)^{e(S'_n)}x^{k(S'_n)}\right) \\
&= \left((-1)^{e(S''_1)}x^{k(S''_1)} + \ldots + (-1)^{e(S''_m)}x^{k(S''_m)}\right)\left((-1)^{e(S'_1)}x^{k(S'_1)} + \ldots + (-1)^{e(S'_n)}x^{k(S'_n)}\right) \\
&= t_G t_H,
\end{aligned}$$

as desired. This completes the proof. □

Example 3.2 For the graph K_3, $p_{K_3}(n)$ is $n(n-1)(n-2)$ using with the formula 3.1 given in Figure 3.3a and in Figure 3.3b, it is calculated using with the contraction/deletion formula given in the 3.4.



(a)



(b)

Figure 3.3. (a) Spanning subgraphs of K_3 and their contribution; (b) The contraction/deletion recursion of K_3

3.1.3. The Dichromatic Polynomial

The dichromatic polynomial is a generalization of the chromatic polynomial. However, the dichromatic polynomial is actually associated with every possible vertex coloring of a graph G. In this section, we utilize the following source (Kauffman, 1989):

Definition 3.5 Let $Col(G)$ denote the set of colorings of G. We define a mapping D

$$D : E(G) \times Col(G) \rightarrow \{0,1\}$$

where a coloring mapping $c : V(G) \rightarrow S(q)$, $S(q)$ with being a set of q distinct colors, and where $D(e,c) = 1$ only if the mapping c assigns the same color for the endpoints of e. The dichromatic polynomial $Z[G]$ can be expressed as the following formula

$$Z[G] = \sum_{c \in \mathrm{Col}(G)} \prod_{e \in E(G)} (1 + vD(e,c)).$$

Lemma 3.1 (Kauffman, 1989) The dichromatic polynomial $Z[G]$ satisfies the following properties:

i. $Z[G] = Z[G - e] + vZ[G \setminus e]$

ii. $Z[G \sqcup H] = Z[G]Z[H]$

iii. $Z[\bullet] = q$

Proof Let G be a graph. Suppose $|\mathrm{Col}(G)| = m$, $c_i \in \mathrm{Col}(G)$ where $i = 1,2,\ldots,m$

$$Z[G] = \sum_{c \in \mathrm{Col}(G)} \prod_{e \in E(G)} (1 + vD(e,c))$$

For an edge $e_n \in E(G)$, we have

$$= (1 + vD(e_n, c_1)) \left[\prod_{e \in E(G) - e_n} (1 + vD(e, c_1)) \right]$$

$$+(1+vD(e_n,c_2))\left[\prod_{e\in E(G)-e_n}(1+vD(e,c_2))\right]$$

$$\vdots$$

$$+(1+vD(e_n,c_m))\left[\prod_{e\in E(G)-e_n}(1+vD(e,c_m))\right]$$

Since the c_i's assign the same colors to the ends of e_n or different colors, say $\mathrm{Col}(G_s)$ assigns the same color and $\mathrm{Col}(G_d)$ assigns the different color

$$\begin{aligned}
&=(1+v)\left[\sum_{\mathrm{Col}(G_s)}\prod_{e\in E(G)-e_n}(1+vD(e,c))\right]\\
&+(1+v0)\left[\sum_{\mathrm{Col}(G_d)}\prod_{e\in E(G)-e_n}(1+vD(e,c))\right]\\
&=\sum_{\mathrm{Col}(G_s)}\prod_{e\in E(G)-e_n}(1+vD(e,c))+\sum_{\mathrm{Col}(G_d)}\prod_{e\in E(G)-e_n}(1+vD(e,c))\\
&+v\sum_{\mathrm{Col}(G_s)}\prod_{e\in E(G)-e_n}(1+vD(e,c))
\end{aligned}$$

Moreover, $\mathrm{Col}(G\text{-}e_n)$ has the colorings that assign either the same or different colors to the ends of e_n, while $\mathrm{Col}(G\setminus e_n)$ has the colorings that assign only the same color to them. Thus, $\mathrm{Col}(G\text{-}e_n) = \mathrm{Col}(G_s)\sqcup\mathrm{Col}(G_d)$ and $\mathrm{Col}(G\setminus e_n) = \mathrm{Col}(G_s)$

$$=\sum_{\mathrm{Col}(G-e_n)}\prod_{e\in E(G)-e_n}(1+vD(e,c))+v\sum_{\mathrm{Col}(G\setminus e_n)}\prod_{e\in E(G)-e_n}(1+vD(e,c))$$

since $E(G-e_n)=E(G)-e_n=E(G\setminus e_n)$, we get

$$\begin{aligned}
&=\sum_{\mathrm{Col}(G-e_n)}\prod_{e\in E(G-e_n)}(1+vD(e,c))+v\sum_{\mathrm{Col}(G\setminus e_n)}\prod_{e\in E(G\setminus e_n)}(1+vD(e,c))\\
&=Z[G-e_n]+vZ[G\setminus e_n],
\end{aligned}$$

as desired.

$$Z[G \sqcup H] = \sum_{c \in \mathrm{Col}(G \sqcup H)} \prod_{e \in E(G \sqcup H)} (1 + vD(e,c))$$
$$= \sum_{c \in \mathrm{Col}(G \sqcup H)} (1 + vD(e_1,c))...(1 + vD(e_n,c)(1 + vD(e'_1,c))...(1 + vD(e'_m,c)))$$

Say $|Col(G)| = g$ and $|Col(H)| = h$, then $|Col(G \sqcup H)| = gh$. Therefore, we have

$$\begin{aligned}
&= (1 + vD(e_1,c_1))...(1 + vD(e_n,c_1))(1 + vD(e'_1,c_1))...(1 + vD(e'_m,c_1)) + \\
&+ (1 + vD(e_1,c_2))...(1 + vD(e_n,c_2))(1 + vD(e'_1,c_2))...(1 + vD(e'_m,c_2)) + ... \\
&\vdots \\
&+ (1 + vD(e_1,c_{gh}))...(1 + vD(e_n,c_{gh}))(1 + vD(e'_1,c_{gh}))...(1 + vD(e'_m,c_{gh}))
\end{aligned}$$

The following table provides an illustration of the elements of $Col(G \sqcup H)$.

	c'_1	c'_2	c'_3	...	c'_h
c''_1	c_1	c_2	c_3	$\dots$	c_h
c''_2	c_{h+1}	c_{h+2}	$\dots$		c_{2h}
$\vdots$		$\vdots$			$\vdots$
c''_g	$c_{(g-1)h+1}$		$\dots$		c_{gh}

Namely, the coloring c_1 of $G \sqcup H$ means that we take the c''_1 coloring of $Col(G)$ and the coloring c'_1 of $Col(H)$. Thus we can rewrite the first h lines of the equation above as follows:

$$\begin{aligned}
&(1+vD(e_1,c_1))\ldots(1+vD(e_n,c_1))(1+vD(e'_1,c_1))\ldots(1+vD(e'_m,c_1))+\\
&\vdots\\
&+(1+vD(e_1,c_h))\ldots(1+vD(e_n,c_h))(1+vD(e'_1,c_h))\ldots(1+vD(e'_m,c_h))\\
&=(1+vD(e_1,c''_1))\ldots(1+vD(e_n,c''_1))(1+vD(e'_1,c'_1))\ldots(1+vD(e'_m,c'_1))+\\
&\vdots\\
&+(1+vD(e_1,c''_1))\ldots(1+vD(e_n,c''_1))(1+vD(e'_1,c'_h))\ldots(1+vD(e'_m,c'_h))\\
&=(1+vD(e_1,c''_1))\ldots(1+vD(e_n,c''_1))[(1+vD(e'_1,c'_1))\ldots(1+vD(e'_m,c'_1))+\\
&\ldots+(1+vD(e'_1,c'_h))\ldots(1+vD(e'_m,c'_h))]
\end{aligned}$$

This means

$$=\sum_{c=c''_1}\prod_{e_1}(1+vD(e,c))\sum_{x\in Col(H)}\prod_{e\in E(H)}(1+vD(e,c)$$

Consequently, we have

$$\begin{aligned}
Z[G\sqcup H] &= \sum_{c\in Col(G)}\prod_{e\in E(H)}(1+vD(e,c))\sum_{x\in Col(H)}\prod_{e\in E(H)}(1+vD(e,c))\\
&= Z[G]Z[H].
\end{aligned}$$

Now suppose $G=\bullet$. We have

$$\begin{aligned}
Z[\bullet] &= \sum_{c\in \mathrm{Col}(G)}\prod_{e\in\emptyset}(1+vD(e,c))\\
&= \sum_{c\in \mathrm{Col}(G)} 1+v0\\
&= \sum_{c\in \mathrm{Col}(G)} 1\\
&= 1+\ldots+1=q,
\end{aligned}$$

since $|Col(G)| = q$. □

Example 3.3 For a graph G given in the figure, $Z[G] = n^3 + 2vn^2 + vn$

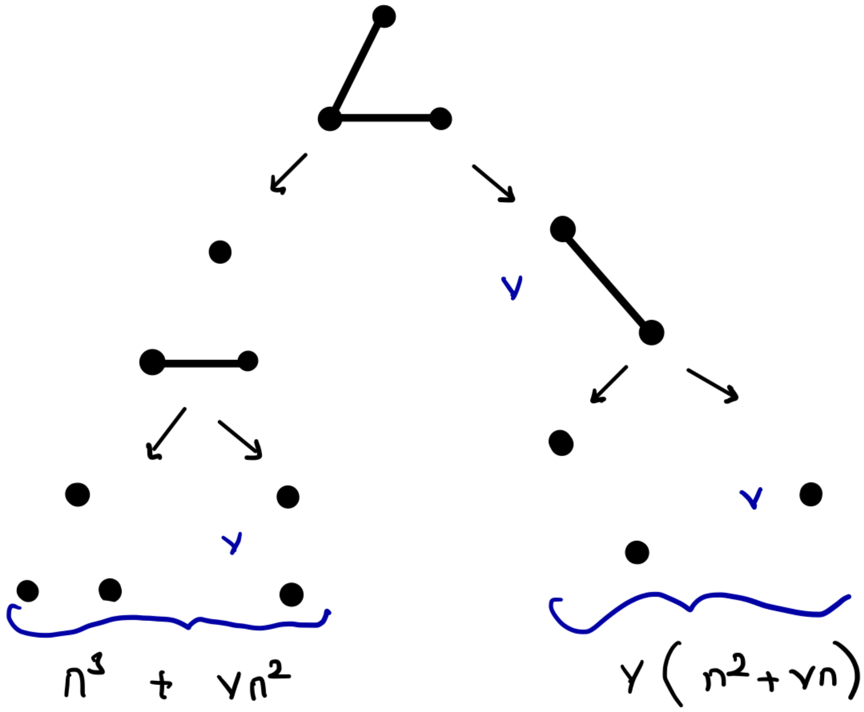


Figure 3.4. The contraction/deletion recursion of G

3.1.4. The Tutte Polynomial

W.T. Tutte defined the Tutte polynomial, $T[G](x,y)$, as the reformulation of the dichromatic polynomial. In this section We rely on Tutte (1954) and Kauffman (1989) in this chapter.

Definition 3.6 Let G be a connected graph and let the edges of G be labelled as $e_1, e_2, e_3, ..., e_n$. Let F be a spanning tree of G. An edge $e_i \in E(F)$, $i = 1,2,...,n$, is *internally active* if $i < j$ for all $e_j \in E(G-F)$ and the endpoints of e_j are both components of $F - e_i$. We say that $e_i \in E(G-F)$ is *externally active* if $i < j$ for all $e_j \in E(F)$ such that e_j is on the unique path in F from one end of the edge e_i to the other.

Definition 3.7 (Tutte, 1954) Let $\mathcal{F}$ be the collection of spanning trees in a connected graph G. Let $i(H)$ be the number of internally active edges of G with respect to the spanning tree H, and let $e(H)$ be the number of externally active edges of G with respect to the spanning tree H. Then the Tutte polynomial is

$$T[G](x,y) = \sum_{H \in \mathcal{F}} x^{i(H)} y^{e(H)}.$$

Example 3.4 In Figure 3.5, we see the graph $G = K_3$ and it spanning trees F_1, F_2, F_3. For F_1, $E(F_1) = \{e_1, e_2\}$ and $E(G - F_1) = \{e_3\}$. For both e_1 and e_2, e_3 is an edge such that its endpoint is in the both components of $F_1 - e_1$ and also $F_1 - e_2$, and they satisfy $1 < 3$ and $2 < 3$. Therefore, both e_1 and e_2 are internally active. The active edges of the spanning trees are circled. The Tutte polnomial of G is $T[G](x,y) = x^2 + x + y$.

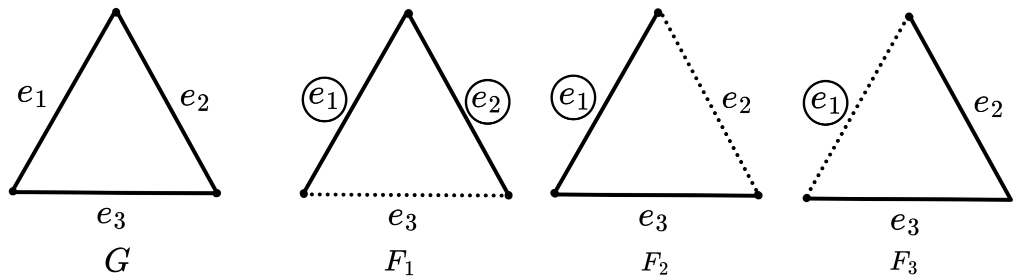


Figure 3.5. The graph G and its spanning trees.

Another example given in Figure 3.6.

Lemma 3.2 (Kauffman, 1989) Let G be a connected planar graph. The Tutte polynomial of G satisfies the following properties

i. if $e \in E(G)$ is neither an isthmus nor a loop, then

$$T[G] = T[G - e] + T[G \setminus e]$$

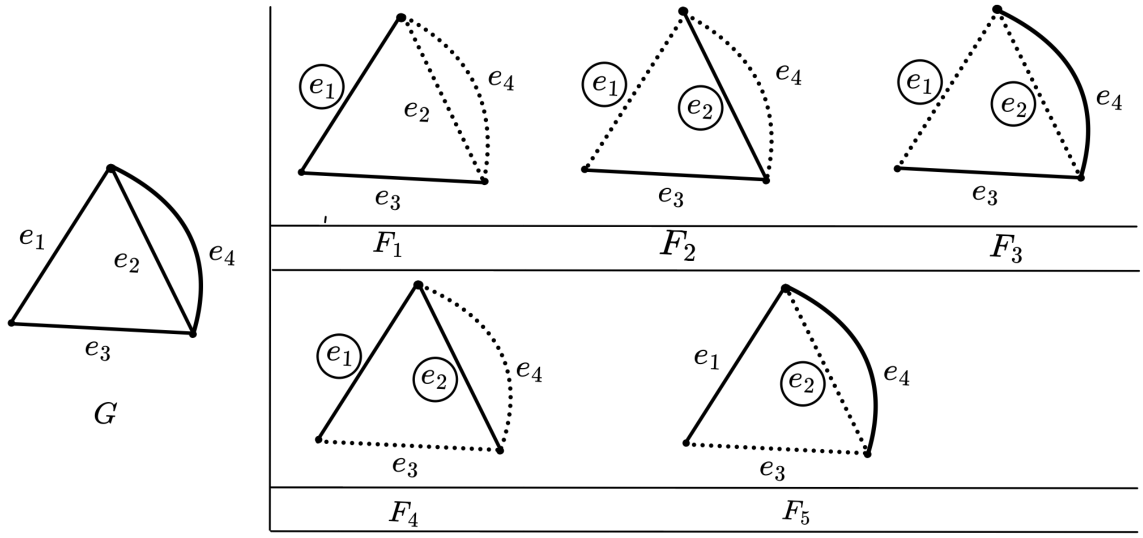


Figure 3.6. For a graph G, the active edges are circled in spanning trees.

ii. if G has only i isthmuses and l loops, then

$$T[G] = x^i y^l.$$

Proof i. Let G be a connected graph with n edges, $e_1, e_2, ..., e_n$ and for an edge e, $G - e$ is the deleted version of the edge e from G and $G \setminus e$ is the graph of G obtained from contracting the edge e. The spanning trees of the graph $G - e$ is the the spanning trees of G such that they do not have this edge e, and The spanning trees of the graph $G \setminus e$ is the the spanning trees of G such that they have this edge e. Suppose $e = e_1$. Then, change the labeling of edges of $G - e$ and $G \setminus e$ as to reduce every indexes by 1. Thus, the contribution of each tree that does not include the edge e to $T[G - e]$ is the same as in $T[G]$ and the other trees contribute to $T[G \setminus e]$ in the same way as $T[G]$.

ii. Suppose G has l loops and i isthmuses and no other edge. If we delete all loops, then we get the only spanning tree F of G. Label the edges of G as the first l loops as $e_1, e_2, ..., e_l$ and then $e_{l+1}, ..., e_{l+i}$ for i isthmuses. All loops are externally actives since they are not in the spanning graph F and their ends only are single vertex which is in F. So the path includes just one vertex that is the end of the corresponding edge. To determine internally activities we look at the edges of $G - F$ which have the endpoints in both component of $F - e_j$, $j = l+1, ..., l+i$. However, the elements of $G - F$ are just loops. Therefore there is no such edge in $G - F$, so all edges are internally active. Hence the number of loops in G, l, and the number of isthmuses, i, are equals to the number of externally active edges and the number of internally active, respectively. Therefore, the Tutte polynomial of this graph

$$T[G] = x^i y^l$$

from Definition 3.6. □

Example 3.5 Let G be the graph given in Example 3.4. Then $T[G](x,y) = x^2 + x + y$. as shown in Figure 3.7.

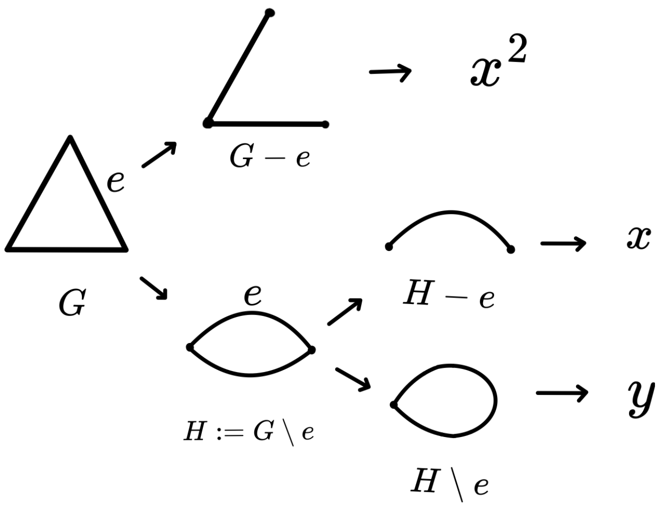


Figure 3.7. Calculating Tutte polynomial via deletion/contraction formula

Let N be the number of vertices of G, i.e., $N = |V(G)|$, and k the number of connected components of G. We have the following relation:

$$Z[G](q,v) = q^k v^{N-k} T[G](1+qv^{-1}, 1+v).$$

Now we present another formula for the Tutte polynomial derived from the Jordan-Euler trail. Prior to advancing to this, we need to introduce *medial graph* of a graph G.

Definition 3.8 The *medial graph* of a graph G is obtained by the putting 4−valent vertex on each edges of G and connecting each of these vertices with other subsequent available crossing via tracing that parallel to edge of G passed, past a vertex. It is denoted by $M(G)$

and 4−valent graph. Replacing the vertices of the medial graph of a graph G with the transversal crossing gives a *universe* U of $M(G)$. (See Figure 3.8)

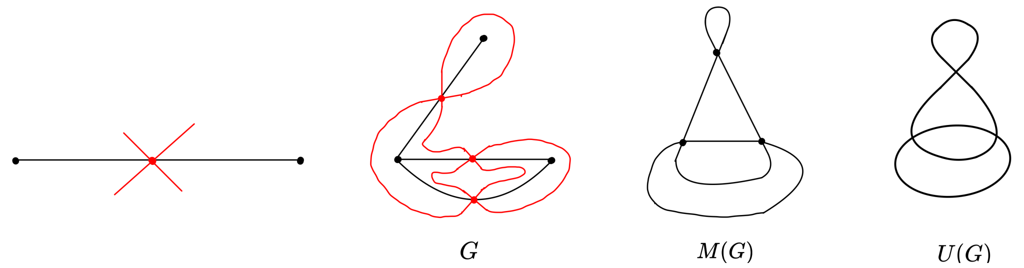


Figure 3.8. Construction of medial graph and universe

Also medial graph $M(G)$ has an inverse. For connected universe U, $M(G(U)) = U$. The universe U is checkerboard shaded such that the unbounded region remains unshaded, and any two adjacent regions have opposite shading. The graph $G(U)$ is then constructed with vertices corresponding to the shaded regions of U. An edge is drawn between two vertices in $G(U)$ if and only if their corresponding regions in U share a crossing, as shown in Figure 3.9.

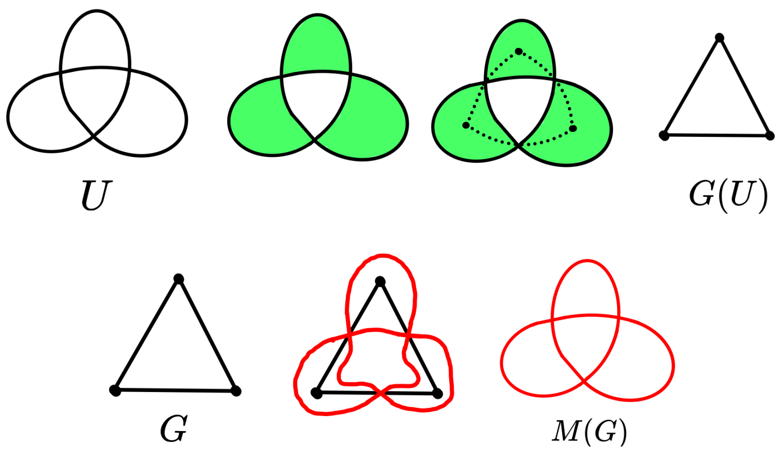
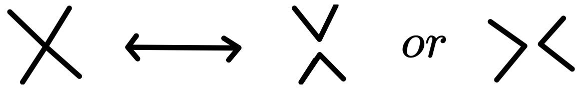


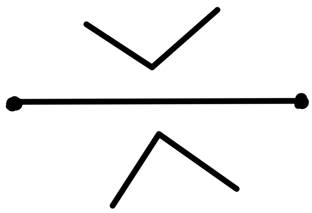
Figure 3.9. The medial graph $M(G)$ and its inverse $G(U)$

Definition 3.9 Let U be a universe of a planar graph G. A Jordan-Euler trail on a universe is an Euler trail that no crosses at a crossing of U. Every crossing is replaced by site (a pair of cusps) :



Proposition 3.1 The collection of Jordan-Euler trails on a universe U of a graph G is in the $1-1$ correspondence with the collection of spanning trees on G.

Proof For a graph G and its universe U, choose the sites appropriate to the following procedure:



Since all vertices are in the shaded regions of U and all vertices are connected in a spanning tree, this procedure connects all shaded regions of U. Therefore spanning trees of G corresponds to a Jordan-Euler trail on U. If we choose the opposite procedure in any site, then the isoloted region is created which is not Jordan-Euler trail on U. □

In Figure 3.10, for a graph G and its universe U, we see the Jordan-Euler trails on U and its corresponding spanning trees on $G(U)$.

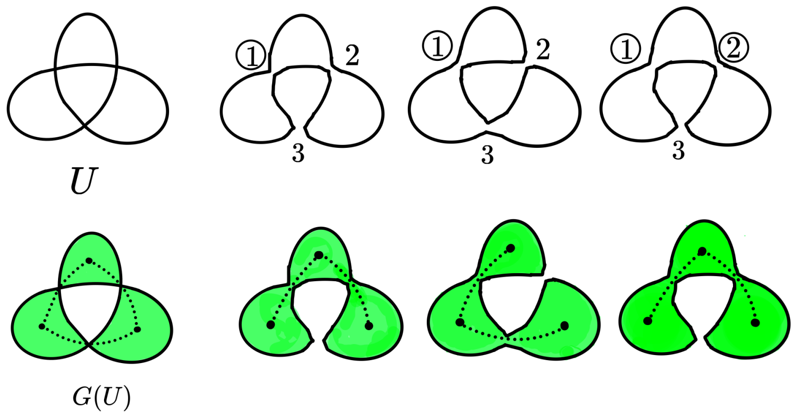


Figure 3.10. Jordan-Euler trails on U and its corresponding spanning trees on $G(U)$

Definition 3.10 Let T be a Jordan-Euler trail of a universe U of a graph G labelled each edge as $1,2,\ldots,n$ and Ti is obtained by changing i^{th} site in the trail T. Then the set of *interactions I* is the set of the sites in Ti having one cusp from both components of Ti.

Definition 3.11 Let T be a Jordan-Euler tail of a universe U of a graph G labelled each edge as $1, 2, ..., n$. The site i is called *active*, then $i < j$ for all j in the interaction set I of Ti. Otherwise, it is called inactive. A site is called *internal* if its cusp point is in the Jordan curve in T. If not, it is called *external*.

Definition 3.12 Let $i(T)$ and $e(T)$ denote the number of internally active sites and the number of externally active sites relative to the Jordan-Euler trail T, respectively. Then

$$T[G](x, y) = \sum_{T \in \text{Trails}(M(G))} x^{i(T)} y^{e(T)}$$

where the sum runs over all Jordan-Euler trails on $M(G)$.

For an example, in Figure 3.10, the active sites are circled. Therefore, $T[G](x, y) = x + y + x^2$.

Lemma 3.3 (Kauffman, 1989) The activities on the spanning tree of a graph G and the corresponding trails of the universe of the medial graph G are equivalent.

Proof Let G be a graph with labelled edge as $1, 2, 3, .., n$ and H be a spanning tree of G and T be a Jordan-Euler trails of the universe of G. From the proposition 3.1, Hi is equivalent to Ti for $i = 0, 1, 2.., n$. Therefore, a site j in the set I defined in the definition 3.11 is the edge j where its ends are both components of Hi. Hence the internal activities on a spanning tree are the same with the internal activities on the corresponding trail. An external sites in T corresponds to the edges of $G - H$. For an external sites i in T, Ti has two components, one inside the other. Since every sites corresponds to the edges of G, a site j in the set I is the corresponding edge j which is on the unique path from the ends of the edge i. □

3.2. Knot Polynomials

Knot polynomials are powerful invariants used in the study of knot theory. In this section, we shall give the definitions, properties, and applications of these knot polynomials. We have utilized the sources (Kauffman, 2001) (Kauffman, 1987) in this section.

3.2.1. Bracket Polynomials

For a crossing, we rotate the over strand counterclockwise until it reaches to the other strands. The shaded regions are called A regions and the others are called B regions,

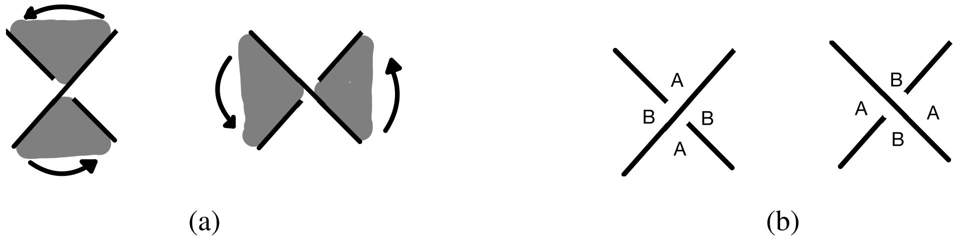


Figure 3.11. (a): Determining the regions; (b) A,B labelled version

see Figure 3.11 We smooth this crossing as to unit A regions or B regions.

Definition 3.13 A *state* of a knot K is a collection of disjoint union of circles obtained by applying A and B smoothings of each crossing of K.

For a knot K with n crossings, applying these smoothings to all crossings of K yield 2^n states of K, since we have two option for all crossings.

Definition 3.14 For a state S of a knot K, $< K|S >$ is the product of the smoothing types applied to the state S. Also, $||S||$ is one less than the number of circles in S, i.e.,

$$||S|| = (\#\text{circles in S}) - 1.$$

Definition 3.15 Let K be a knot. The bracket polynomial is defined by

$$< K > (A, B, d) = \sum_{s \in S(K)} < K|s > d^{||s||}$$

where $S(K)$ is set of states of K, also A, B and d are commutative variables.

Example 3.6 For the Trefoil knot T, we have the scheme given in the Figure 3.12. Then computing the contribution of every states yields

$$< T > = A^3 d + 3A^2 B + 3AB^2 d + B^3 d^2.$$

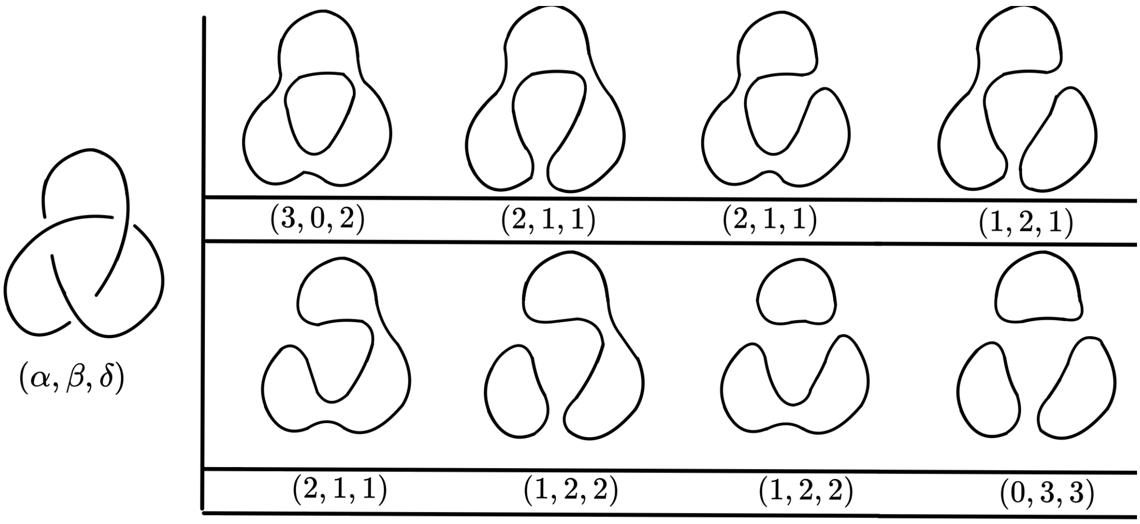


Figure 3.12. States of the Trefoil knot

Proposition 3.2 The bracket polynomial of a knot is an invariant under RII, $RIII$ with the equalities $B = A^{-1}$ and $d = -A^2 - A^{-2}$.

Under RI moves, the bracket polynomial contributes $-A^3$ or $-A^{-3}$ depending on the types of RI. To see their contribution, we first show that the skein relation of the bracket polynomial.

Proposition 3.3 The bracket polynomial is satisfied the relation in the Figure 3.13 which is called skein relation.

In the Figure 3.14, the contribution of one type of RI moves is calculated and second one can be calculated with the same way. Let see an example about computing bracket polynomial via using skein relation.

Example 3.7 The bracket polynomial of Trefoil given in the Figure 3.12 can be computed using the skein relation of the bracket: See the Figure 3.15.

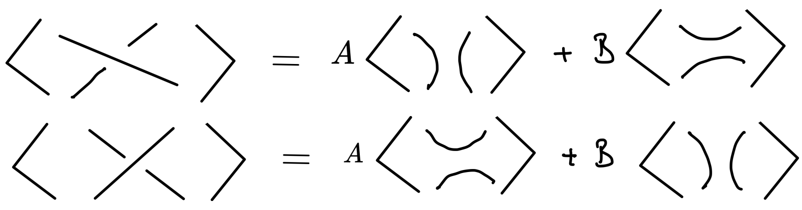


Figure 3.13. The Skein Relation

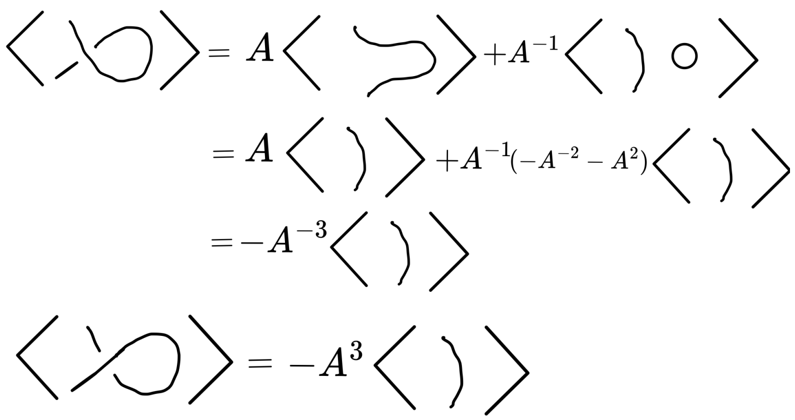


Figure 3.14. The contribution of RI moves

Definition 3.16 The *writhe* of an oriented knot $\overrightarrow{K}$ is defined by

$$w(\overrightarrow{K}) = \sum_{\text{all crossings in } K} sgn(c).$$

Proposition 3.4 The writhe of an oriented knot $\overrightarrow{K}$ is an invariant under RII and $RIII$.

Definition 3.17 For an oriented knot K, the normalized bracket polynomial is defined by

$$f_{\overrightarrow{K}}(A) = (-A^3)^{-w(\overrightarrow{K})} < K > .$$

Proposition 3.5 The normalized bracket polynomial $f_{<\overrightarrow{K}>}$ of an oriented knot $\overrightarrow{K}$ is an invariant under RI, RII, $RIII$.

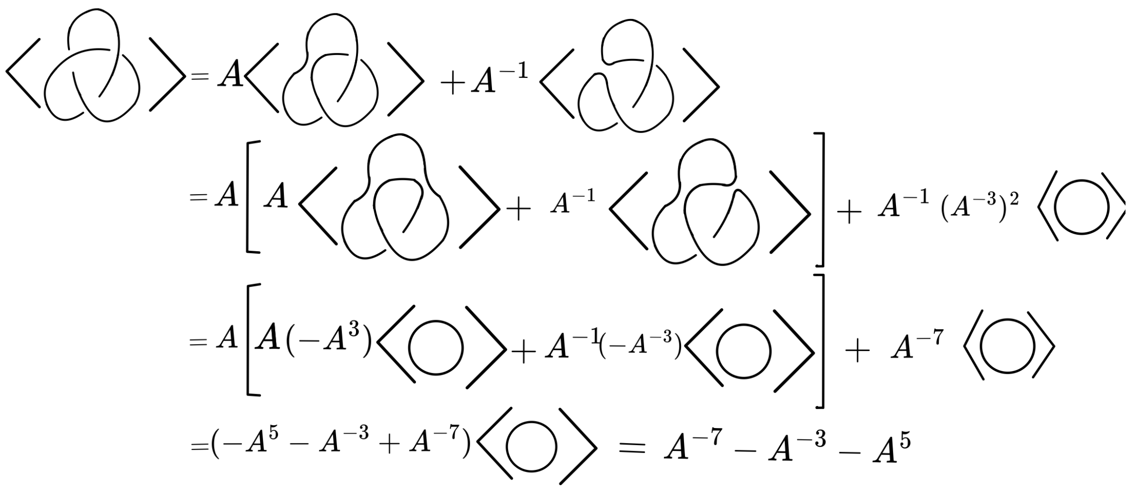


Figure 3.15. Computing the bracket polynomial via skein relation for Trefoil

3.2.2. Jones Polynomials

The Jones polynomial, discovered by Vaughan Jones in 1984, is one of the most celebrated invariants in knot theory. It revolutionized the study of knots by providing a powerful tool for distinguishing between different knots and links. The polynomial is a Laurent polynomial in a variable t, with integer coefficients, and it plays a crucial role in understanding the topological properties of knots and links.

In this section, we will explore the definition and properties of the Jones polynomial. We will begin with the foundational axioms that uniquely determine the Jones polynomial. We will also explore its connections to the bracket polynomial, highlighting how the Jones polynomial emerges as the normalized bracket polynomial, adjusted by the writhe of the knot.

Definition 3.18 The Jones polynomial $J_{\overrightarrow{K}}$ of an oriented knot $\overrightarrow{K}$ is a Laurent polynomial with integer coefficient such that it is the unique polynomial satisfying the following axioms:

i. If $\overrightarrow{K} \sim \overrightarrow{K'}$, then $J_{\overrightarrow{K}}(t) = J_{\overrightarrow{K'}}(t)$

ii. $J_{\bigcirc}(t) = 1$

iii. $t^{-1}J_{\times}(t) - t\,J_{\times}(t) = (t^{1/2} - t^{(-1/2)})J_{\asymp}(t)$

Theorem 3.2 The normalized bracket polynomial $f_{\overrightarrow{K}}(t^{-\frac{1}{4}})$ of an oriented knot $\overrightarrow{K}$ satisfies the all axioms i, ii, iii in the definition 3.18

3.3. Knots in relation with Graphs

Last chapter we talk about the Tutte polynomial of a graph G and the bracket polynomial of a knot K. We first give the theorem about computing the bracket polynomial $< K >$ of knot K on its Jordan-Euler trails. Then we give formula for the bracket polynomial on spanning trees of the signed graph G obtained by applying the inverse of medial graph construction to the link diagram K. Hence, we define the bracket polynomial for a signed graph and also we give edge/contraction formula for the bracket. At the end of the chapter, we see that the bracket polynomial of a signed graph is the Tutte polynomial under the change of variables.

If we have a graph consisting isthmuses and loops, the by definition all edges are active site and these isthmuses and loops correspond to curls under medial graph. Both the Kauffman bracket and Tutte polynomials have recursion formulae, the skein relation and contraction/deletion formula, respectively. For both polynomials, the link consisting just curls and the graph having only isthmuses and loops are calculating directly. By medial construction, a graph consisting only isthmuses and loops corresponds to a link that has just curls and they correspond to that all sites of the Jordan-Euler trails of corresponding universe are active. Therefore, if we smooth all crossings that are inactive, then we have just active sites, i.e., it has only curls and it produces the coefficient $-A^{-3}$ or $-A^{3}$ depending of types of curls. Therefore, the following ideas based on this.

Proposition 3.6 (Kauffman, 1989) The collection of connected signed planar graphs is in 1-1 correspondence with the collection of connected planar link diagrams.

Proof For a signed graph G, associating crossings and signed edges illustrated in the Figure 3.16 gives a unique crossing to each signed edge. Thus medial graph of G gives a link diagram $K(G)$. The inverse process follows the same steps as the medial construction.

□

Theorem 3.3 (Kauffman, 1989) Let K be a link diagram, and T be a trail on the universe U, underlying K where each crossing is labelled with $1, 2, 3, .., n$ so that the sites of T are determined the activeness and inactiveness. Then the bracket polynomial of K is the

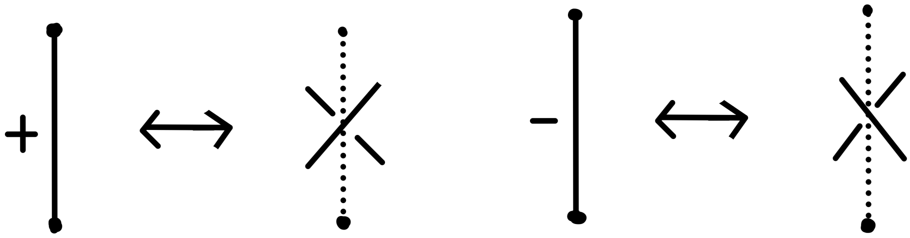
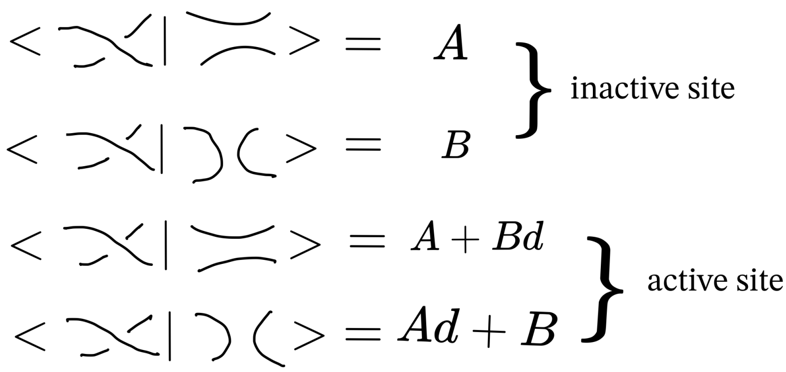


Figure 3.16. Associating signed edges and crossings

following formula

$$< K >= \sum_{T \in \mathcal{T}} < K|T >$$

where $< K|T >$ is the product of the contributions of T from each site with the following scheme



Proof Let K be a link diagram, and T a trail on U, underlying K. If K is a link diagram such that has only curls, then the every site of T is active because of the fact that every curls on $K(U)$ correspond to the isthmuses and loops on its corresponding sign planar graph $G(K)$ and we know that every isthmuses an loops are active. Suppose K is a link diagram. From recursion formula of the bracket polynomial, applying A,B-smoothings to the crossing corresponding to the inactive sites in T, until the deformed form K' of K has all the crossings corresponding to the active sites. Therefore we apply RI to this crossings which contributes $A+Bd$ and $Ad+B$ with respect to the crossing types and also the crossings associated the inactive sites contributes the A or B with respect to the smoothing types. Hence the above scheme gives the contribution of T in each site with respect to the recursion formula of the bracket polynomial. Thus, the sum of the product of the contribution of the sites of T for every $T \in \mathcal{T}$ gives the bracket polynomial of K. $\square$

Definition 3.19 (Kauffman, 1989) Let G be a signed graph. Define $sign(e)$ as

$$\begin{cases} sign(e) = 1 & \text{if } e \text{ is positive,} \\ sign(e) = -1 & \text{if } e \text{ is negative.} \end{cases}$$

Let i^+ be the number of positive isthmuses, i^- be the number of negative isthmuses, l^+ be the number of positive loops, and i^- be the number of negative loops. The deletion/contraction formulas for $Q_G(A, B, d)$ are the followings.

i. If an edge e is neither a loop or an isthmuses in G, then
$Q_G = AQ_{G'} + BQ_{G''}$ if $sign(e) < 0$
$Q_G = BQ_{G'} + AQ_{G''}$ if $sign(e) > 0$

ii. If every edge of a connected graph G are either loops or isthmuses, then
$Q_G = X^{i^+ + l^-} Y^{i^- + l^+}$ where $X = A + Bd$ and $Y = Ad + B$

iii. If G is the disjoint union of graphs G_1 and G_2, then $Q_G = dQ_{G_1}Q_{G_2}$

Proposition 3.7 (Kauffman, 1989) For a signed graph G, let $K(G)$ be a link diagram which is the medial graph of G. Then the bracket polynomial of $K(G)$ equals to the polynomial Q_G.

Proof Let G be signed graph and let $K(G)$ be a corresponding the link diagram of G via medial construction. The recursion formula for Q_G gives the recursion formula for a bracket polynomial $< K(G) >$. □

Proposition 3.8 (Kauffman, 1989) Let $Z[G]$ is dichromatic polynomial of the underlying unsigned of a signed graph G whose edges are positive sign. Let $|V(G)|$ be the number of vertices of G and let c be the number of components of G. Then

$$Z[G](q, v) = q^{(|V(G)|+c)/2} Q_G(q^{-1/2}v, 1, q^{1/2}).$$

CHAPTER 4

AN GENERALIZATIONS OF KNOT THEORY AND GRAPHS: VIRTUAL KNOT THEORY AND RIBBON GRAPHS

4.1. Surfaces

One of the main topics of study in topology is surfaces. This chapter will go over the fundamentals of surfaces, including definitions, classifications, and topological properties. In this section we use the books Massey (1987),Gilbert and Porter (1994).

Definition 4.1 Let X, Y be topological spaces and f be a mapping X to Y. If f and f^{-1} is continuous, then f is called *homeomorphism*.

Definition 4.2 A *surface* S is a compact topological space in which S is Haussdorff such that each point has a neighborhood that is homeomorphic to the open disc ($\{x \in \mathbb{R}^2 | |x| < 1\}$). If a surface has a point that has a neighborhood which is homeomorphic to the closed disc in $\mathbb{R}^2$, then it is called surface with boundary.

Example 4.1 i. A simplest surface is 2-sphere S^2. This is also any topological space that is homeomorphic to the subset

$$\{x \in \mathbb{R}^3 | ||x|| = 1\}$$

of $\mathbb{R}^3$. Or let $I = [0,1] \times [0,1]$, then a 2-sphere is any topological space homeomorphic to the quotient space of I obtained by identifying all boundaries of I to the a single point.

ii. A torus is a surface. A torus is any topological space that is homeomorphic to the $S^1 \times S^1$. Also it can be defined any topological space that is homeomorphic to the subset

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3 | (2-(x^2+y^2)^{1/2})^2 + z^2 = 1\}$$

of $\mathbb{R}^3$. Equivalently, define I as

$$I = \{(x, y) \in \mathbb{R}^2 | x \in [0,1], y \in [0,1]\}$$

which is an unit square in $\mathbb{R}^2$. Identify all opposite sides of unit square I. Therefore, any topological space that is homeomorphic to this identification is called a torus.

iii. (Massey, 1987) A *real projective plane* is any topological space homeomorphic to the quotient space of S^2 obtained by identifying the antipodal points of 2-sphere S^2.

Definition 4.3 A surface S is connected if there exist a path between any two points in S.

Let S be a connected surface. Assign direction of rotation for each point in S. It is called a *local orientation* at a point $p \in S$. Let p and q be two points in S, and C be a path in s between p and q. Since both points have local orientations, we can trace along C with respect to the orientation of p and ends up at the other point. If orientations agree, then C is called *orientation-preserving*. Otherwise, it is called *orientation-reversing*. Therefore, a connected surface has a property related to the orientation:

Definition 4.4 Let S be a surface. If every closed path in S is orientation preserving, then S is *orientable*. If there exists a closed path in S that is orientation reversing, then S is called *nonorientable*.

Hence we can differ two connected surfaces from each other by using their orientability.

Example 4.2 A 2-sphere and a torus are orientable surfaces whilst a projective plane, a Möbius band and a Klein bottle are nonrientable surfaces.

For more additional examples of surfaces, we introduce a binary operation #, which is called *connected sum*.

Definition 4.5 Let S and S' be two compact surfaces. The connected sum of S and S' is obtained by deleting open disc in each surface, and then identifying these created boundaries. It is denoted by $S \# S'$.

The connected sum # has the following properties:

i. # has an identity, 2-sphere S^2. For any surface S, $S\#S^2 = S$.

ii. # is commutative, i.e, $S\#S' = S'\#S$ where S and S' are two surfaces.

iii. # is associative, that means, for surfaces S_1, S_2, and S_3, we have $S_1\#(S_2\#S_3) = (S_1\#S_2)\#S_3$.

However, connected sum has no inverses. Therefore it forms a semigroup.

The surfaces that are connected sum of two surfaces is orientable when these surfaces are orientable. Otherwise, it is nonorientable.

The surface can be divided into a finite number of vertices, edges, and faces. Triangulation provide a significant advantage in that the classification of compact surfaces turn into a finite combinatorial problem.

Definition 4.6 (Massey, 1987) Let T be a finite family of closed subsets

$$T = \{t_1, t_2, ..., t_n\}$$

that covers a compact surface S and let f_i's be homeomorphisms

$$f_i : t_i' \to t_i \text{ for } i = 1, ..., n$$

where t_i''s are triangle in $\mathbb{R}^2$. Under the homeomorphism f_i, the image of edges and vertices in t_i' is called edges and vertices of t_i, respectively, and t_i is called triangle. A *triangulation* of S consists of T and f_i so that any two distinct triangle in T either share a single vertex, share exactly one entire edge, or be disjoint.

The following type of the intersection of two distinct triangles are forbidden:

Definition 4.7 Let T be a triangulation of a compact surface S. The *Euler characteristic* of S is defined by the following formula

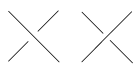
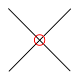
$$\chi(S) = v - e + t$$

where v is the number of vertices, e is the number of edges, t is the number of triangles in the triangulation of S.

4.2. Virtual Knot Theory

Virtual knots lie in a thickened surface $S_g \times I$ with genus g and their diagrams are in S_g, these diagrams contain only *real* crossings. When we project these diagrams into $\mathbb{R}^2$, new crossings called virtual crossings appear. In this section, we use the sources Kauffman (2012),Kauffman (2006):

Definition 4.8 A *virtual knot* is an equivalence class of embeddings of S^1 into thickened surfaces up to ambient isotopy in the thickened surfaces, homeomorphisms of the surfaces and the addition or subtraction of empty handles (i.e. the knot does not go through the handle).

Definition 4.9 A *virtual knot diagram* is an immersion of a circle in $\mathbb{R}^2$ whose double points are either classical crossing  or virtual crossing  .

The next figure is an example of virtual knot diagrams:

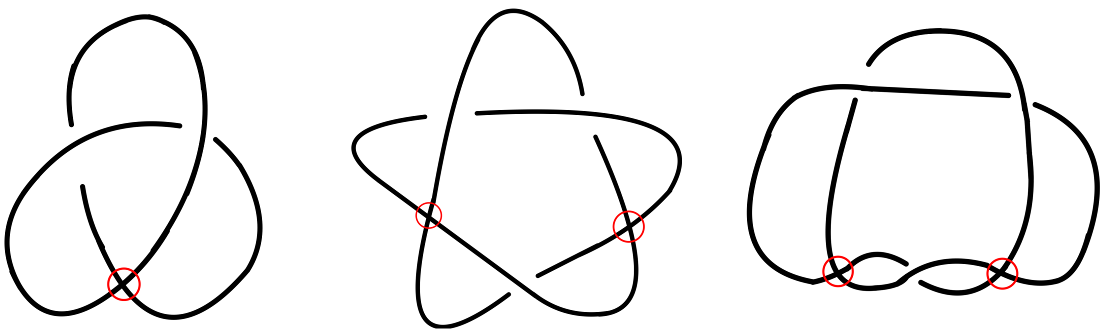
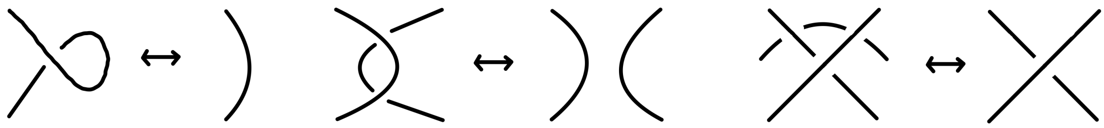


Figure 4.1. Some virtual knots

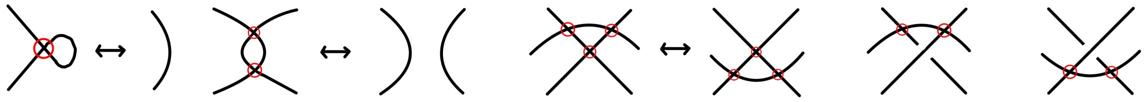
Theorem 4.1 (Kamada and Kamada, 2000) Two virtual knots are equivalent if and only if their corresponding virtual knot diagrams are related by finitely many generalized Reidemeister moves illustrated in the Figure 4.2.

Theorem 4.2 (Goussarov et al., 2000) Two classical knot diagrams are virtually equivalent if and only if they are classically equivalent.

The classical knot theory corresponds to the virtual knot theory on $S^2 \times I$. This theorem shows that virtual knot theory is a generalization of the classical knot theory.



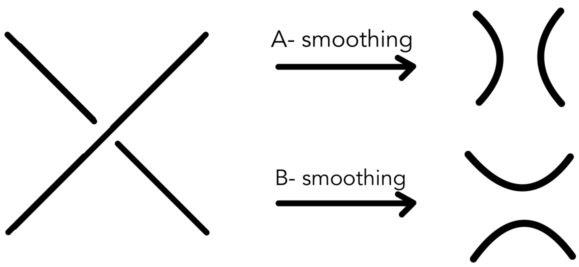
(a) the classical Reidemeister moves



(b) the virtual Reidemeister moves

Figure 4.2. Generalized Reidemeister moves for virtual knot diagrams

L. Kauffman defined the bracket polynomial for virtual knots. To construct this polynomial we again resolve the classical crossings as the bracket polynomial of classical knots. There are two types of resolving crossing: A-smoothing and B-smoothing: Suppose



L is a virtual link diagram. A *state* S of a link diagram L is the resolving of each classical crossing of this diagram. Since every classical crossing has two options for resolving, we have 2^n states for the link diagram L with n classical crossings. The only difference from the classical knots occurs in the state because we allow the virtual crossings in a state of the virtual link diagrams. Now we are ready to define bracket polynomial:

Definition 4.10 The *Kauffman Bracket* of a diagram L is polynomial in three variables A,

B, d defined by the formula

$$< L > (A,B,d) := \sum_{S \in S(L)} A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$$

where $\alpha(S)$ and $\beta(S)$ are the number of A-smoothing and B-smoothing in a state S, respectively, and $\delta(S)$ is the number of components according to the state S.

$< L >$ is not invariant. With the changes $B = A^{-1}$ and $d = -A^2 - A^{-2}$, this version of $< L >$ is invariant under all but RI. With the substitution $A = t^{-1/4}$, $B = t^{1/4}$, and $d = -t^{-1/2} - t^{1/2}$, the *Jones polynomial* $J_L(t)$ is

$$J_L(t) = (-1)^{w(L)} t^{3w(L)/4} < L > (t^{-1/4}, t^{1/4}, -t^{-1/2} - t^{1/2})$$

where $w(L)$ is the writhe of the link L.

Example 4.3 Consider the virtual knot diagram L shown given in the Figure 4.3. Since there are 3 classical crossings, so there are 2^3 states. The Kauffman bracket of L is given

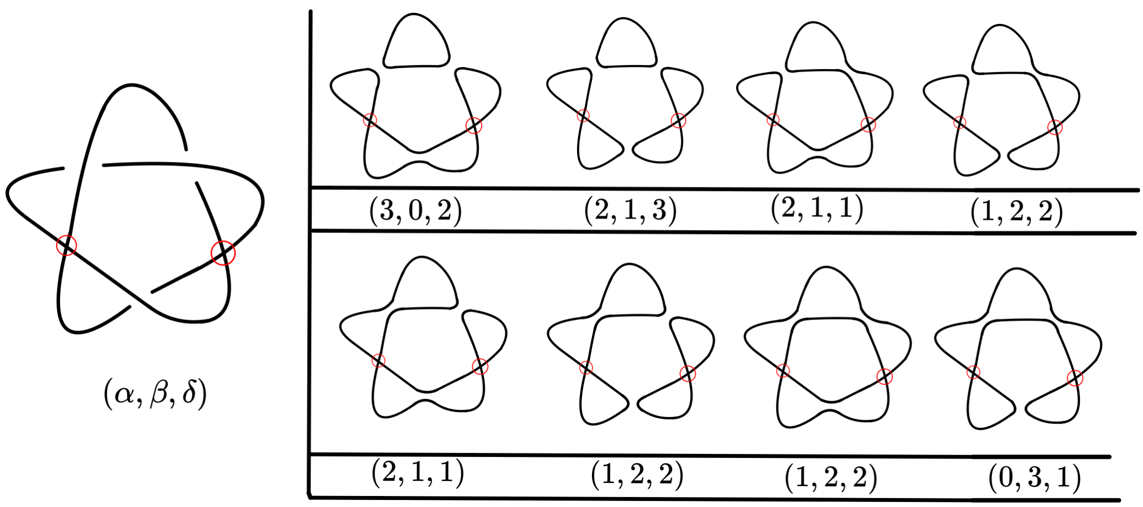


Figure 4.3. The states of the given virtual knot L, $< L > (A,B,d) = A^3d + A^2Bd^2 + 3AB^2d + 2AB^2 + B^3$.

by

$$< L > (A,B,d) = A^3d + A^2Bd^2 + 3AB^2d + 2AB^2 + B^3.$$

4.3. Ribbon Graphs

In this section, we have utilized the sources Chmutov and Pak (2006), Moffatt and Ellis-Monaghan (2013). First, we shall introduce the cellularly embedded graph, then we give the definition of the ribbon graph.

Definition 4.11 A *cellularly embedded graph* G is a graph drawn on a surface S such that for an edge, their only intersections are at their ends, and such that the connected components of $S \setminus G$, which are called *faces*, are homeomorphic to the disc.

Definition 4.12 For cellularly embedded graphs $G \subset S$ and $G' \subset S'$, if there is a homeomorphism from S to S' that sends G to G'. For an orientable surface S, this homeomorphism is an orientation preserving.

There is an example in the Figure 4.4a . Ribbon graphs are equivalent to the cellularly embedded graphs. However, deleting edge or vertices in a ribbon graph gives again ribbon graph whilst deleting edges or vertices in a cellularly embedded graphs may not results a cellularly embedded graphs.

Definition 4.13 A ribbon graph G is a surface with boundary represented as the union of two finite sets of closed discs, corresponding to vertices $V(G)$ and edges $E(G)$ subject to the following restrictions:

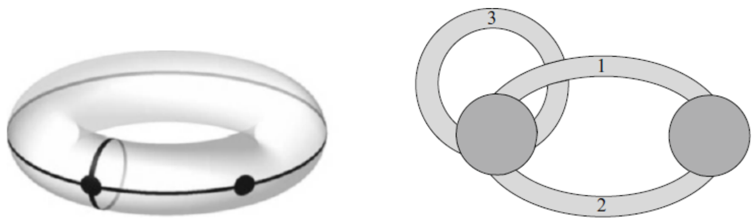
i. These discs and ribbons intersects by disjoint line segments,

ii. Each such line segment lies on the boundary of precisely one vertex and precisely one edge,

iii. Every edge has exactly two such line segments.

In the Figure 4.4b, there is an example of a ribbon graph. For an cellularly graph G in a surface S, taking a small neighborhood of $G \in S$ yields a ribbon graph representation.

Definition 4.14 Two ribbon graph are equivalent if there is a homeomorphism from one to the other that sends vertices to vertices and edges to edges.



(a) (b)

Figure 4.4. (a) A cellularly embedded graph G ;(b) G as a ribbon graph (Moffatt and Ellis-Monaghan, 2013)

CHAPTER 5

A GENERALIZATION OF THE TUTTE POLYNOMIAL: BOLLOBAS RIORDAN POLYNOMIAL

For a ribbon graph G, let $v(G) = |V|$ denote the number of vertices, $e(G) = |E|$ denote the number of edges, and $k(G)$ denote the number of connected components of G. Let $r(G) = v(G) - k(G)$ be the *rank* of G, and $n(G) = e(G) - r(G)$ be the *nullity* of G. Let $bc(G)$ be the number of connected components of the boundary of the surface G. Now we can define the Bollobas-Riordan polynomial.

Definition 5.1 The Bollobas-Riordan polynomial $R_G(x, y, z)$ of a ribbon graph G is defined by the formula

$$R_G(x,y,z) := \sum_{F \in \mathcal{F}(G)} x^{r(G)-r(F)} y^{n(F)} z^{k(F)-bc(F)+n(F)}$$

where $\mathcal{F}(G)$ is the set of spanning subgraphs of the ribbon graph G.

From the definition of the spanning subgraph, each edge of a ribbon graph G has only two options: it appears or not. So the set of all spanning subgraphs of G, $\mathcal{F}(G)$, has $2^{e(G)}$ elements.

Example 5.1 Consider the ribbon graph G illustrated in the Figure 5.1. We have 3 edges, so $\mathcal{F}(G)$ has 2^3 elements. Computing the contributions of all spanning subgraphs yields

$$R_G(x,y,z) = y^3z^2 + 3y + y^2 + 2y^2z^2 + 1$$

5.1. The Bracket Polynomial and The Bollobas-Riordan Polynomial

We construct a ribbon graph G_L from an alternating virtual link diagram L. We

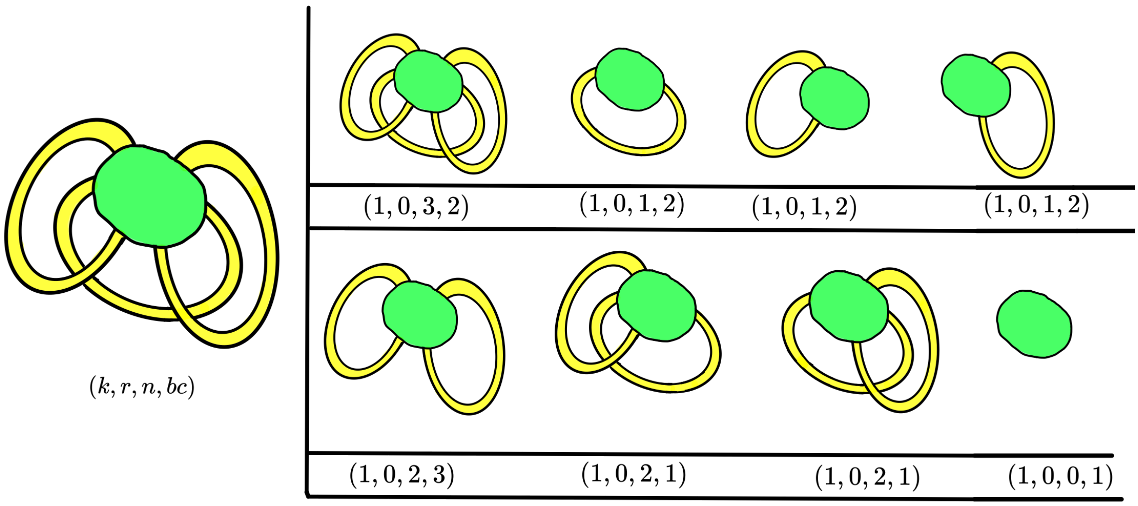
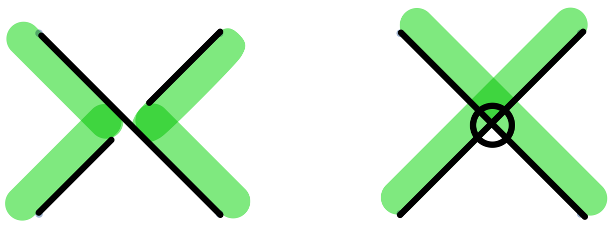


Figure 5.1. The spanning subgraphs of the given ribbon graph

need first define a checkerboard coloring.

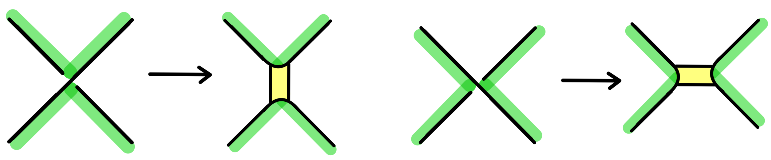
Definition 5.2 A checkerboard coloring of a virtual link diagram is a coloring of one side of the diagram in its small neighborhood which behaves as follows at crossings:



If L is an alternating virtual link diagram, then this coloring is called canonical checkerboard coloring.

Theorem 5.1 (Kamada, 2000) (Kamada, 2004) Every alternating link diagram L has a canonical checkerboard coloring.

We have 3 steps to get a ribbon graph G_L from a virtual link diagram L: We construct a ribbon graph G_L from an alternating link diagram L with the following steps:
Step 1: We color green the angles that are glued together by the A-smoothing.
Step 2: We replace every crossing with an edge-ribbon connecting the corresponding arcs.
Step 3: We glue discs along the circles in the region that are encircled with green.



Example 5.2 *Let L be the link diagram in Example 4.3. Its checkerboard coloring is given in the Figure 5.2 and then its corresponding ribbon graph is illustrated. L corresponds to the ribbon graph G_L given in the Example 5.1.*

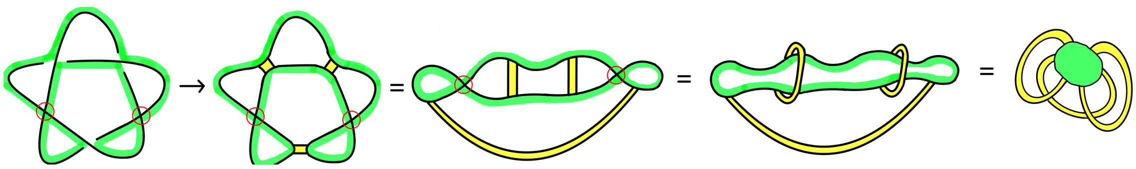


Figure 5.2. Checkerboard coloring of a virtual knot L and its corresponding ribbon graph

Theorem 5.2 (Chmutov and Pak, 2006) Let L be an alternating virtual link diagram and G_L be the corresponding ribbon graph. Then

$$< L > (A,B,d) = A^{r(G)}B^{n(G)}d^{k(G)-1}R_{G_L}(\frac{Bd}{A},\frac{Ad}{B},\frac{1}{d})$$

Proof Let L be an alternating link diagram and G_L is the corresponding ribbon graph. From construction of ribbon graph, crossings of L corresponds to the edges of G_L. Therefore, we have the number of elements of the set of spanning subgraph $\mathcal{F}(G)$ is equal to the number of the set of states $S(L)$. Define $\phi : S(L) \rightarrow \mathcal{F}(G)$. Let an $A-$smoothing of a crossing in a state S means that we keep the corresponding edge in the spanning subgraph $F = \phi(S)$. Let a $B-$smoothing of a crossing in a state S means that wee delete the corresponding edge in $F = \phi(S)$. Then there is natural 1-1 correspondence ϕ between $\mathcal{F}$ and $S(L)$. Since we relate A and B smoothings with the existence of an edge for all $F = \phi(S)$, $e(F) = \alpha(S)$ and $e(G) - e(F) = \beta(S)$ and also $\delta(S) = bc(F)$. By the definition of $R_G(x,y,z)$, we have $x^{r(G)-r(F)}y^{n(F)}z^{k(F)-bc(F)+n(F)}$ for a spanning subgraph $F = \phi(S)$.

Substitution $x = \frac{Bd}{A}$, $y = \frac{Ad}{B}$, $z = \frac{1}{d}$ and multiplication with $A^{r(G)}B^{n(G)}d^{k(G)-1}$ yield

$$\begin{aligned} & A^{r(G)}B^{n(G)}d^{k(G)-1}(A^{-1}Bd)^{r(G)-r(F)}(AB^{-1}d)^{n(F)}d^{-k(F)+bc(F)-n(F)} \\ & = A^{r(F)+n(F)}B^{n(G)+r(G)-(r(F)+n(F))}d^{k(G)-1+r(G)-r(F)-k(F)+bc(F)} \end{aligned}$$

and we have $r(F)+n(F) = e(F)$ and $k(F)+r(F) = v(F) = v(G)$, so we get

$$\begin{aligned} & = A^{e(F)}B^{e(G)-e(F)}d^{bc(F)-1} \\ & = A^{\alpha(S)}B^{\beta(S)}d^{\delta(S)-1} \end{aligned}$$

for $S = \phi^{-1}(F)$. This gives the contribution of the state S to the bracket polynomial, as desired.

□

CHAPTER 6

CONCLUSION

In this thesis, we initially established a one-to-one correspondence between alternating classical knots and signed graphs through the medial construction. We defined the Tutte polynomial for graphs in terms of Jordan-Euler trails. Additionally, using this construction, we demonstrated the definition of the bracket polynomial in terms of Jordan-Euler trails. By examining these cases, we revealed the relationship between the bracket polynomials of alternating knots and the Tutte polynomials of signed graphs. Subsequently, we explored the connection between virtual knots, as introduced by Kauffman as a generalization of classical knot theory, and ribbon graphs. The we showed the link between the Bollobás-Riordan polynomial of ribbon graph, which is a generalization of the Tutte polynomial of graphs, and the Kauffman bracket polynomial of the checkerboard colorable virtual links. Through this thesis, we have shown the bridge between knot theory and graph theory.

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