CLASSICAL THEOREMS OF RAMSEY THEORY VIA COMBINATORIAL AND ULTRAFILTER METHODS

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ABSTRACT

CLASSICAL THEOREMS OF RAMSEY THEORY VIA COMBINATORIAL AND ULTRAFILTER METHODS

In this thesis, the ultimate aim is to present the proofs of the four classical theorems of Ramsey theory: Ramsey's, Schur's, van der Waerden's, and Rado's theorems. We discuss the finite and infinite versions of these theorems, which are equivalent to each other, along with their proofs. Additionally, we introduce the basics of nonstandard analysis tools, so called filters. Furthermore, we present two different proofs of Schur's and a special case of van der Waerden's theorems using ultrafilter methods.

ÖZET

KOMBİNATORİK VE ULTRAFİLTRE YÖNTEMLERİYLE RAMSEY TEORİSİNİN KLASİK TEOREMLERİ

Bu tezde ana amaç, Ramsey teorisinin dört klasik teoremi olan Ramsey, Schur, van der Waerden ve Rado teoreminin ispatını sunmaktır. Bu teoremlerin birbirlerine denk olan sonlu ve sonsuz versiyonlarını ispatlarıyla birlikte ele alıyoruz. Ayrıca, filtreler olarak bilinen, standart olmayan analizin temel araçlarını tanıtıyoruz. Bunun yanı sıra, ultrafiltreler kullanılarak Schur teoreminin ve van der Waerden teoreminin özel bir durumunun iki farklı ispatını sunuyoruz.

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FIP : finite intersection property

$\mathbb{N} = \{1, 2, 3, \ldots\}$: positive integers

k-AP : an arithmetic progression of length k

i.e. : in other words

Wlog : without loss of generality

$[N]$: $\{1, 2, \ldots, N\}$

CHAPTER 1

INTRODUCTION

Ramsey theory is a branch of combinatorics that focuses on how certain properties are maintained under the partitioning of sets. Essentially, it asks whether, for a given set S with a property P, at least one of the subsets will retain property P whenever S is divided into a finite number of subsets.

A classic example of a property P found in the finite partition of any set is the pigeonhole principle, which forms the foundation of much of Ramsey theory.

Theorem 1.1 (Basic Pigeonhole Principle) *If a set with n elements is partitioned into r disjoint subsets where $n > r$, then at least one subset contains more than one element.*

Theorem 1.2 (Generalized Pigeonhole Principle) *If a set with more than ns elements are partitioned into s sets, then some subsets contain more than n elements.*

This type of counting arguments allow us to conclude that, when a set of objects is divided into a finite number of classes, one of these classes must have a certain size. A classic example of this, demonstrating the pigeonhole principle, can be found in solving the following well known problem in Ramsey Theory.

Lemma 1.1 (Party Problem) *If six people are at a party, then either three people who have all met one another or three people who are mutual strangers.*

Using the pigeonhole principle, we can established that six people are sufficient certain situations. For instance, in the party problem, we might inquire whether six is the smallest number of attendees required to ensure a specific property holds. To demonstrate this, consider the case of five people and find an example where neither three are mutual friends nor three are mutual strangers. For example, label the five people as $1, 2, 3, 4$, and 5, where if 1 knows 2, 2 knows 3, 3 knows 4, 4 knows 5, and 5 knows 1, and all other pairs are strangers. In this setup, there are no groups of three who are all mutual friends or all mutual strangers. This example illustrates that five people can be arranged such that the desired property (neither three mutual friends nor three mutual strangers) holds, thereby showing that six is indeed the minimal number where this guarantee breaks down under typical conditions.

Frank Plumpton Ramsey initiated the study of Ramsey Theory, a branch of mathematics focused on finding order within chaos. In 1928, he released his paper titled *On a Problem of Formal Logic* (Ramsey, 1928) wherein he established what later became recognized as Ramsey's theorem. The theorem states that sufficiently large, finitely colored complete graphs must contain a specific monochromatic subgraph. We can reexpress the party problem using graph theory and Ramsey theorem answers this problem. Although Ramsey Theory encompasses many theorems, its essential overarching result is that structure inevitably emerges within sufficiently large groups.

In 1927, the Dutch mathematician Bartel Leendert van der Waerden gave one of the most essential results in Ramsey theory, which concerns arithmetic progressions.

Definition 1.1 *An r-coloring of a set S is a function $\chi : S \to C$, where $|C| = r$.*

An r-coloring χ of a set S partitions S into r subsets $S_1, S_2, \dots, S_r$ where each subset S_i consists of elements $x \in S$ that are assigned the color i by the coloring function χ.

Definition 1.2 *A coloring χ is monochromatic on a set S, if χ is constant on S.*

Definition 1.3 *A k-term nontrivial arithmetic progression is a sequence of the form*

$$a, a + d, \dots, a + (k - 1)d,$$

where $a \in \mathbb{Z}$ and $d \neq 0$.

Van der Waerden showed that any finite coloring of the positive integers contains arbitrarily long nontrivial (i.e. d is different from 0) monochromatic arithmetic progressions (van der Waerden, 1927). This theorem has finite and infinite versions and finite version says that there exists a number which is called van der Waerden's number. We can find exact results and bounds for these numbers. We refer the reader to *Ramsey Theory on the Integers* (Landman and Robertson, 2014) book for more details about van der Waerden's numbers.

In 1916, the first result of this kind emerged from a combinatorial lemma attributed to Issai Schur which he utilized to establish the existence of nontrivial solutions to Fermat equations $x^n + y^n = z^n$ modulo p for sufficiently large primes p. Specifically, Schur's theorem asserts that in any finite coloring (partition) of the positive integers, there exists a monochromatic triple $a, b, a + b$ (Schur, 1916). Once Schur's theorem was proven, even though we did not know Ramsey's theorem yet, we can then give a nice graph theory proof of Schur's theorem using Ramsey's theorem. The original proof of Schur's theorem can be found in *Über die Kongruenz* $x^m + y^m \equiv z^m \pmod{p}$. This result was followed by Richard Rado's Ph.D. thesis *Studien zur Kombinatorik* (Rado, 1933). Under the supervision of Schur, Rado proved a theorem that beautifully extended the classical theorems of Schur and van der Waerden. He provided a complete characterization of partition regular systems of linear Diophantine equations over the positive integers, which means it has the monochromatic solution to linear diophantine equations. He isolated a straightforward sufficient and necessary condition on the coefficients known as the column condition. We refer the reader to *Note on Combinatorial Analysis* (Rado, 1945) for a summary of Rado's theorem. Rado did not give the condition of being the partition regular for nonlinear or nonhomogenous equations. After his theorem, we can provide lots of nonhomogenous and nonlinear equations which are partition regular using nonstandard analysis methods. For more examples and details about these types of examples of partition regular equations and a kind introduction to nonstandard methods concerning Ramsey types theorems, we direct the reader to *Nonstandard Methods in Ramsey Thoery and Combinatorial Number Theory* (Di Nasso et al., 2019).

While each theorem is distinctly different, they collectively aim to uncover order within chaos. For more information about these theorems and more history of the Ramsey theoreticians, we refer to the beautiful book *Ramsey Theory: Yesterday, Today, and Tomorrow* (Sofier, 2011) by Alexander Sofier.

1.0.1. From Ramsey's Theorem to Szemerédi's Theorem

After van der Waerden's theorem, in 1936, Erdős and Turán conjectured a stronger statement: any subset of the positive integers with positive upper density contains arbitrarily long nontrivial arithmetic progressions.

Definition 1.4 *Let $A \subseteq \mathbb{N}$. The upper density of A, denoted by $\bar{d}(A)$, is defined by*

$$\bar{d}(A) = \limsup_{N\to\infty} \frac{|A \cap \{1, 2, \ldots, N\}|}{N}.$$

Note that $0 \leq \bar{d}(A) \leq 1$.

The Erdős and Turán conjecture suggest that the real reason for van der Waerden's theorem is not merely the finiteness of colors, as in Ramsey's theorem, but rather that some color classes must inevitably have positive upper density. In 1953, Roth prove the Erdős and Turán conjecture for 3-APs (Roth, 1953).

Theorem 1.3 (Roth's Theorem) *Every subset of the positive integers with positive upper density contains a nontrivial* 3*-AP.*

Later the Erdős and Turán's conjecture for the case $k = 3$ became known as Roth's theorem. Roth employed Fourier analysis for his proof. For further details about Roth's theorem, reader are encouraged to refer to Roth's original article (Roth, 1953).

In 1969, Szemerédi gave an advance for Erdős and Turán's conjecture for the case $k = 4$. In 1975, Szemerédi proved his seminal theorem which states the following theorem.

Theorem 1.4 (Szemerédi's Theorem) *((Szemerédi, 1975)) Every subset of the positive integers with positive upper density contains arbitrarily long arithmetic progressions.*

It took another twenty years for Szemerédi to fully resolve the conjecture in 1975, in what is considered a combinatorial *tour de force*. Roth's and Szemerédi's theorems are considered landmark achievements in additive combinatorics.

Szemerédi's theorem led to many improvements in additive combinatorics. Various proofs of this theorem have been found, each giving rise to rich areas of mathematical research. In 1977, Furstenberg proved Szemerédi's using theorem the ergodic theory approach (Furstenberg, 1977). In 2001, Gowers proved this by higher-order Fourier analysis (Gowers, 2001). Although prime numbers have zero upper density, in 2008, Green-Tao (Green and Tao, 2008) proved that the set of prime numbers contains arbitrarily long nontrivial arithmetic progressions. All of these theorems have the same aim:

"dichotomy between structure and pseudo-randomness".

1.0.2. Filters and Ultrafilters

Filters and ultrafilters, introduced in *Filtres et ultrafiltres* and *Théoride des filters* (Cartan, 1937) by Cartan in the 1930s are essential concepts in mathematics and appear in various contexts. A filter on a set S is a subset of its power set containing S and not $\emptyset$, and is closed under finite intersections and supersets. Ultrafilters are special filters that are equivalent to maximal filters. A classic reference to ultrafilters is the book *Ultrafilters* by Comfort and Negropontis (Comfort and Negropontis, 1974).

The set of all ultrafilters on S in denoted by βS. There is a natural topology on βS and this topology yields an incredible space with many properties. We will discuss these properties in the last chapter. Besides of topological properties of βS, it has algebraic properties. For instance, βS is a semigroup with a certain operation. The theory of the Stone-Čech compactification allows for the extension of any operation on the semigroup to its compactification. We will discuss the extension of an operation on S to βS. Furthermore, we will introduce idempotent ultrafilters which are essential role in the algebra of the Stone-Čech compactification. Because they allow for proofs of Ramsey type theorems such as Schur's theorem and van der Waerden's theorem. To give this elegant proof, we need to know the Ellis-Numakura lemma which states that there exist idempotent ultrafilters in βS. This lemma was proved for topological semigroups by Numakura and Wallace (Numakura, 1952). In 1958, Ellis proved this for semitopological semigroups (Ellis, 1958). We will use this lemma, and we will prove Schur's theorem and a special case of ($k = 3$) van der Waerden's theorem.

In summary, we will have the following outcomes. In Chapter 2, we will give the proofs of Ramsey's theorems and Schur's theorem, and we will mention some of the bounds for these theorems. We will also prove the equivalence of two versions of Schur's theorem. In Chapter 3, we will give the main tools for proving van der Waerden's theorem, which is called the color-focused idea. We follow the proof of *Proof of van der Waerden's Theorem in Nine Figures* which was presented by Blondal and Jungic (Blondal and Jungic, 2018). Then, we will also prove the equivalence of van der Waerden's theorem. In Chapter 4, we will present partition regular equations and we will give the main characterization of these types of equations. For the first 3 chapters, the main reference is *Ramsey Theory on the Integers* (Landman and Robertson, 2014). In the last chapter, we will introduce the main tools of the nonstandard analysis. We will give the proofs of the fundamental theorems about filters and introduce the definition of a limit on filters. Then using these main tools, we will give two different proofs of Schur's and van der Waerden's theorem ($k = 3$) using ultrafilter methods. During this chapter, we will follow the book *Algebra in the Stone-Čech Compactification* (Hindman and Strauss, 1998) which is the keystone book of these subjects.

CHAPTER 2

RAMSEY'S THEOREM AND SCHUR'S THEOREM

In this chapter, we will prove Ramsey's theorem and Schur's theorem using techniques from graph theory. We will also exhibit some of the bounds for Ramsey's and Schur's numbers.

2.0.1. Preliminaries

This introductory section covers concepts related to graphs. In this section, we will focus on simple graphs, which are undirected and do not have loops or double edges. For more detailed information on graph theory, the reader can refer to (Bondy and Murty, 2008).

Definition 2.1 *A graph G is a tuple $G = (V, E)$, where V is a set whose elements are vertices and $E \subseteq \binom{V}{2} = \{\{a, b\} : a, b \in V, a \neq b\}$ is a set whose elements are called edges.*

Let us see a simple graph in the following example.

Example 2.1 *Let G be a graph with consists of vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_4\}\}$. Then, the graph $G = (V, E)$ can be visualized as:*

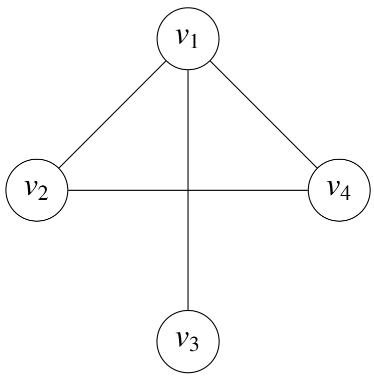


Figure 2.1. An example of simple graph.

Definition 2.2 *A subgraph $G' = (V', E')$ of a graph $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.*

Let us see a subgraph in the following example.

Example 2.2 *Let G be the graph in Example 2.1. Then G' can be a subgraph of G as follows,*

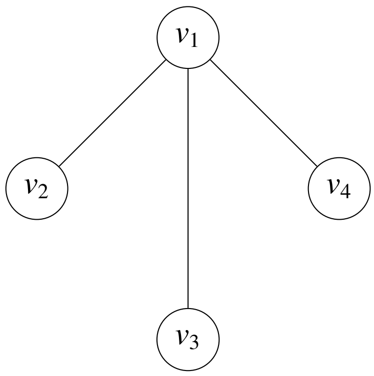


Figure 2.2. An example of a subgraph.

Definition 2.3 *(Bondy and Murty, 2008) A complete graph is a simple graph with the property that every pair of vertices is connected by an edge. We denote the complete graph on n vertices by K_n.*

Example 2.3 *K_3, K_4 are the complete graphs on $3, 4$ vertices, respectively.*

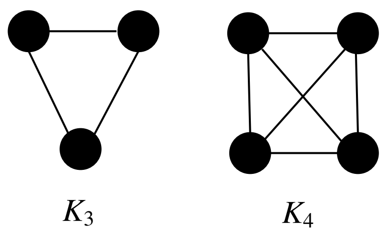


Figure 2.3. The complete graph examples K_3 and K_4.

Definition 2.4 *(Bondy and Murty, 2008) An r-edge-coloring of a graph $G = (V, E)$ is a mapping $c : E \to \{1, \ldots, r\}$, in other words, an assignment of r-colors to the edges of G.*

Example 2.4 *Let us color the edges of K_4 using two colors in the following figure.*

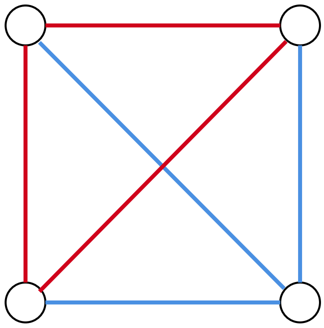


Figure 2.4. An edge-coloring example.

Note that we call *monochromatic graph* of a graph whose all edges have the same color.

2.0.2. Ramsey's Theorem

In this section, we will introduce Ramsey's theorem and Schur's theorem and provide their proofs using various techniques from graph theory.

Now, we will give Ramsey's theorem for two colors by following *Introduction to Combinatorics* (Erickson, 2013).

Theorem 2.1 (Ramsey's theorem for two colors) *(Ramsey, 1928) If $s, t \geq 2$, then there exists a least integer $R = R(s, t)$ such that every 2-edge-coloring of K_R, with the colors red and blue, admits either a red K_s subgraph or a blue K_t subgraph.*

Proof This proof will be by induction on s and t. The base case of induction consists of $R(s, 2) = s$ and $R(2, t) = t$. Edges are colored either with the same color or at least one is colored differently. If we color all edges with the same color, then we obtain monochromatic K_s or K_t. Otherwise, at least a monochromatic K_2 exists.

Now, we are assuming $R(s-1, t) + R(s, t-1)$ exists, and we will show that $R(s, t)$ exists. First, we look at the 2-coloring of a complete graph $G = K_n$ with $n = R(s-1, t) + R(s, t-1)$ vertices. Let us pick an arbitrary vertex v of G. By the finite pigeonhole principle, at least $R(s-1, t)$ red edges or at least $R(s, t-1)$ blue edges are rooted in v. Without loss of generality, suppose that v is joined by red edges to a complete subgraph on $R(s-1, t)$ vertices. Then we have a red K_{s-1} or blue K_t, by the definition of $R(s-1, t)$.

Thus, red K_{s-1} and v and all the edges between the two, we obtain a red K_s. If not, we can obtain a blue K_t. Therefore, $R(s,t)$ exists and satisfies

$$R(s,t) \leq R(s-1,t) + R(s,t-1). \tag{2.1}$$

□

Now, we can extend Ramsey's theorem to multiple colors.

Theorem 2.2 (Ramsey's theorem for multiple colors) *For any $r \geq 2$ and $a_1, \ldots, a_r \geq 2$, there exists a least integer $R = R(a_1, \ldots, a_r)$ with the following property:*

If the edges of the complete graph K_R are colored by $A_1, \ldots, A_r$ colors, then there exists a complete graph K_{a_i} all of whose edges have color A_i.

Proof If $r = 2$, then $R = R(a_1, a_2)$, and we have done by the previous theorem.

Suppose that $R' = R(a_1, \ldots, a_{r-1})$ exists for all $a_1, \ldots, a_{r-1} \geq 2$. Our claim is R exits and $R \leq R(R', a_r)$. We can see the coloring of $K_{R(R',a_r)}$ as a 2-coloring with colors $\{A_1, \ldots, A_{r-1}\}$ and A_r. Thus, we can obtain an A_r-colored complete subgraph K_{a_r} or an $(r-1)$-colored complete subgraph $K_{R'}$, so this holds our induction hypothesis. In either case, we obtain a complete subgraph on the required number of vertices. □

2.0.3. Ramsey Numbers, Bounds and Asymptotic

We presented the existential results of Ramsey's theorem. In this section, we provide several calculations and proofs, along with a summary of the limited knowledge available regarding Ramsey numbers.

The values of $R = R(s,t)$ are referred to as 2-colored Ramsey numbers. Only a few nontrivial Ramsey numbers with $s, t > 2$ have been determined. The fact that we have proven the existence of Ramsey numbers but do not know their specific values highlights one disadvantage of existential proofs.

If $r = 1$, then $R(s, 1) = 1$ because K_1 has no edges and so no edges to color, thus any coloring of K_1 will always contain blue K_1. If $t = 2$, then $R(s, 2) = s$ because if all the edges of K_s are colored red, it will contain a red K_s, but if one edge is colored blue it will contain a blue K_2.

$R = R(s,s)$ are known as the *diagonal Ramsey numbers* since they appear on the main diagonal of a table of Ramsey numbers. The first example yields a bound for the diagonal Ramsey number $R(3,3)$ that yields the *party problem*.

Example 2.5 $R(3,3) = 6$.

Proof First, we show that $R(3,3) > 5$. The complete graph on 5 vertices does not always contain blue K_3 or red K_3. For example, if we color all the outer edges of K_5 blue and color the inner edges of K_5 red, then we cannot obtain a monochromatic subgraph K_3.

Now, we show that when K_6 are 2-colored it always contains a monochromatic subgraph K_3. Consider K_6 and pick any vertex $v \in V(K_6)$. Then we have 5 edges, 2 colors for each edge. By the pigeonhole principle, at least 3 of these edges must have the same color. Wlog, we assume that 3 red edges connecting to v to 3 other vertices.

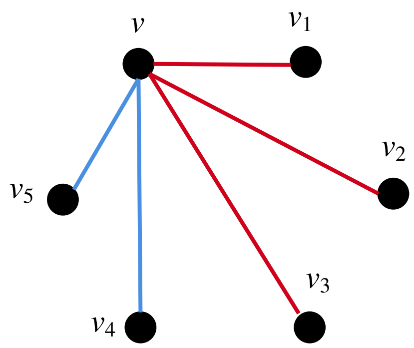


Figure 2.5. $R(3,3) \leq 6$.

Consider the K_3 subgraph generated by the $\{v_1, v_2, v_3\}$. If we color the all edges in K_3 with blue, then we get blue K_3 subgraph.

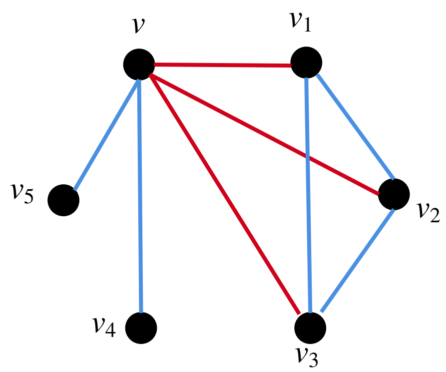


Figure 2.6. The blue K_3 subgraph exists in 2-coloring of K_6.

Otherwise, at least one of the edges must be red. Wlog, we color edge between v_1 to v_2 by red. Then this coloring gives a red K_3 subgraph which is generated by $\{v, v_1, v_2\}$.

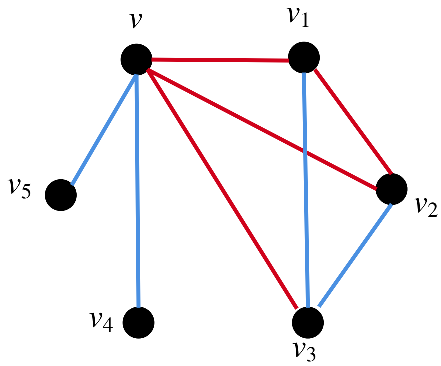


Figure 2.7. The 2-coloring of K_6 contains a red K_3 subgraph.

Therefore, $R(3,3) = 6$. □

Now, we present all of the known nontrivial Ramsey numbers in the following table. In addition, we refer the reader to the survey *Small Ramsey Numbers* (Radziszowski, 2021) to find more results about Ramsey's numbers.

Table 2.1. The Ramsey numbers $R(s,t)$ and some bounds.

$s \setminus t$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$	$t = 8$	$t = 9$	$t = 10$
$s = 3$	6	9	14	18	23	28	36	40-41
$s = 4$		18	25	36-40	49-58	59-79	73-106	92-136
$s = 5$			43-48	59-85	80-133	101-194	133-282	149-381
$s = 6$				102-161	115-273	134-427	183-656	204-949
$s = 7$					205-497	219-840	253-2379	292-2134
$s = 8$						282-1532	329-2683	343-4432
$s = 9$							565-6366	582-9797
$s = 10$								798-17730

Let us now discuss the lower and upper bounds for the Ramsey numbers $R(s,t)$. Another well-known upper bound, attributed to Erdős and Szekeres, is stated in the following theorem.

Theorem 2.3 (Upper bound for ramsey numbers) *For all $s, t \geq 2$, we have*

$$R(s,t) \leq \binom{s+t-2}{s-1}. \tag{2.2}$$

Proof We use induction on s, t. We have that $R(s,2) = \binom{s}{s-1} = s$ and $R(2,t) = \binom{t}{1=t}$, thus the equation (2.2) is satisfied when $s = 2$ or $t = 2$. Now, suppose that the inequality holds for $R(s-1,t)$ and $R(s,t-1)$ for $s, t \geq 3$. Then,

$$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1), \\ &\leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1}, \\ &= \binom{s+t-2}{s-1}, \end{aligned}$$

and the inequality is satisfied. □

One way to find lower bounds for the Ramsey number $R(s,t)$ is by searching graphs for monochromatic K_s and K_t subgraphs. This is very time consuming for even the smaller Ramsey numbers. So we will look at some of the known lower bounds. We will give a lower bound for the diagonal Ramsey number $R(s,s)$, where the proofs use a probabilistic method.

Theorem 2.4 *(Graham et al., 1991) If*

$$\binom{n}{s} 2^{1-\binom{s}{2}} < 1,$$

then $R(s,s) > n$.

Proof We have to show that if $\binom{n}{s}2^{1-\binom{s}{2}} < 1$, then there exists a 2-coloring of K_n with no monochromatic K_s subgraph.

We consider a 2-coloring of K_n with the colors green and blue. The probability of an edge colored by green is $1/2$ and similarly the probability of an edge colored by blue is $1/2$. Since we have $\binom{n}{2}$ edges in K_n, then there are $2^{\binom{n}{2}}$ different possible colorings. The probability of each of these coloring is equal to $\frac{1}{2^{\binom{n}{2}}}$.

Now, if we color K_s with green and blue, the probability of all edges are colored green is $2^{-\binom{s}{2}}$ and similarly the probability of all edges are colored blue is $2^{-\binom{s}{2}}$. This gives that the probability of being monochromatic K_s subgraph is $2^{1-\binom{s}{2}}$. In K_n, there are $\binom{n}{s}$ different K_s subgraphs, thus the probability of monochromatic K_s subgraph in K_n is

$$\binom{n}{s} 2^{1-\binom{s}{2}}$$

Thus, if there was a monochromatic K_s coloring, then we would have $\binom{n}{s}2^{1-\binom{s}{2}} = 1$. Therefore, if $\binom{n}{s}2^{1-\binom{s}{2}} < 1$, then $R(s,s) > n$. □

We refer the reader to the book *Ramsey Theory* (Graham et al., 1991) to find more asymptotic results about Ramsey's numbers.

2.0.4. Proof of Schur's Theorem

In this section, we will give the very early result of Ramsey's theory which is Schur's theorem. We will show bounds for Schur's theorem which is related to Ramsey's numbers. We will first introduce two equivalent versions of Schur's theorem.

Theorem 2.5 (Schur's Theorem Infinite Version) *(Schur, 1916) If the positive integers are colored using finitely many colors, then there is always a monochromatic solution to* $x + y = z$.

The following is an equivalent finitary version. We write $[N] := \{1, 2, \ldots, N\}$.

Theorem 2.6 (Schur's Theorem Finite Version) *(Schur, 1916) For every positive integer r, there exists a positive integer* $N = N(r)$ *such that if each element of* $[N]$ *is colored using one of r colors, then there is a monochromatic solution to* $x + y = z$.

Proof Let $f : [N] \longrightarrow [r]$ be an r-coloring and color the edges of K_{N+1} by giving the edge $\{a, b\}$ with $a < b$ the color $f(b - a)$.

By Ramsey's theorem, if N is large enough, then there exists a monochromatic triangle with vertices $a < b < c$. Then we have

$$f(b - a) = f(c - b) = f(c - a).$$

If we take $b - a = x$, $c - b = y$ and $c - a = z$, then $f(x), f(y)$ and $f(z)$ are monochromatic. Thus, we get a monochromatic solution to $x + y = z$. □

Now, we give proof that the two versions of Schur's theorem are equivalent.

Proposition 2.1 *The finite version of Schur's theorem and the infinite version of Schur's theorem are equivalent.*

Proof The finite version of Schur's theorem easily gives the infinite version of Schur's theorem. We can consider the colorings of the first $N(r)$ integers and use the finitary statement to find a monochromatic solution of the desired equation.

To prove that the infinite version implies the finite version, fix r. Suppose that the finite version is false so that for every N there is some coloring $f_N : [N] \rightarrow [1, r]$ such that there is no monochromatic solution to $x + y = z$. Take an infinite subsequence of (f_N)

such that for all $t \in \mathbb{N}$ stabilizes to a constant as N increase along this subsequence. Then, the f_N's along this subsequence converge pointwise to a coloring $f : \mathbb{N} \to [r]$ avoiding a monochromatic solution to $x + y = z$, but f contradicts the infinite version. □

2.0.5. Schur Numbers and Bounds

In this section, we will discuss Schur's numbers and their bounds using the Ramsey's numbers.

Definition 2.5 (Schur's number) *We call the least positive number $N = N(r)$ that satisfies Schur's theorem the Schur number and it is denoted by $s(r)$.*

Definition 2.6 *A triple $\{x, y, z\}$ that satisfies Schur's theorem is called a Schur triple.*

We have that the Schur numbers are for $r = 1, 2, 3, 4$. We can observe that $s(1) = 2$, $s(2) = 5$, $s(3) = 14$, $s(4) = 45$ and $s(5) = 161$. Now, we explain that $s(2) = 5$.

Example 2.6 $s(2) = 5$

Proof First, we show that $s(2) \geq 5$. If we consider a 2-coloring of $\{1, 2, 3, 4\}$ such that 1 and 4 have the same color and 2 and 3 have the same color, then there is no monochromatic solution to $x + y = z$.

Now, we show $s(2) \leq 5$. Take any 2-coloring of $\{1, 2, 3, 4, 5\}$. Wlog, 1 is colored blue. We assume that there is no monochromatic solution to $x + y = z$. Thus, 2 must be colored red, because if not we get $1 + 1 = 2$ which is what we want. Similarly, 4 must be colored by blue, because if not $2 + 2 = 4$. 3 is colored by red for avoiding monochromatic solution $1 + 3 = 4$. But if 5 is colored by blue, we get monochromatic solution $1 + 4 = 5$ or if 5 is colored by red, we get monochromatic solution $2 + 3 = 5$. Thus $s(2) \leq 5$. □

Schur's original proof did not involve Ramsey numbers, as Ramsey's theorem was not proven until 1928, while Schur established his result in 1916. However, it's evident from our proof that there is a connection between Schur numbers and Ramsey numbers. Now, we prove the following corollary of Schur's theorem. We will denote $R(3, 3, \ldots, 3)$, where we are using r colors by $R_r(3)$.

Corollary 2.1 *For $r \geq 1$, $s(r) \leq R_r(3) - 1$.*

Proof For any r-coloring of $[1, n - 1]$ we can obtain r-coloring of K_n using the same coloring given in the proof of finite version of Schur's theorem (2.6). By Ramsey's theorem for $n = R_r(3)$, we have a monochromatic K_3. We obtain a monochromatic Schur triple using the monochromatic edges of K_3. Hence, if $n = R_r(3)$, then

$$s(r) \leq n - 1 = R_r(3) - 1.$$

□

Since there exists at least one monochromatic Schur triple in any r-coloring of the set $[1, s(r)]$, we conclude that there must be infinitely many Schur triples in any r-coloring of the positive integers. This follows because if $[1, n]$ contains a monochromatics Schur triple, then so does $m[1, n] = \{m, 2m, \ldots, mn\}$ for any positive integer m. This holds because if $x + y = z$, then $mx + my = mz$. Therefore, any r-coloring of the positive integers must include infinitely many monochromatic Schur triples.

CHAPTER 3

VAN DER WAERDEN'S THEOREM

This chapter aims to show two equivalent versions of van der Waerden's theorem on arithmetic progressions. First, we introduce the finite and infinite versions of van der Waerden's theorem and demonstrate the equivalence between of them. Then, we will prove the finite version and present some of the bounds for van der Waerden's numbers.

3.0.1. van der Waerden's Theorem

Theorem 3.1 (van der Waerden's theorem infinite version) *(van der Waerden, 1927) Suppose the positive integers are colored by finitely many colors. Then, there exist arbitrarily long nontrivial monochromatic arithmetic progressions.*

This theorem does not say that there exists infinitely long monochromatic arithmetic progression. For instance, if the positive integers are colored as:

$$1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,\quad 7,\quad 8,\quad 9,\quad 10,\quad 11,\quad 12,\quad 13,\quad 14,\quad 15,\ldots$$

then, there is not any infinitely long monochromatic arithmetic progression.

Next, we give a *"finite version"* of this theorem. We call the finite version since it concerns only finite parts of the positive integers.

Theorem 3.2 (van der Waerden's theorem finite version) *(van der Waerden, 1927) For all positive integers k and r, there exists a natural number $W(k,r)$ such that, if the set of natural numbers $\{1,2,\ldots,W(k,r)\}$ is colored by r colors, then it must contain at least one nontrivial monochromatic k-AP.*

Now, it is necessary to establish the concept of *"color-focused"*, as it will be frequently utilized in proofs across various cases of van der Waerden's theorem.

Definition 3.1 *(Blondal and Jungic, 2018) Let r be a finite coloring of an interval of positive integers $[x,y]$ and k,t be positive integers. We say that k-APs $A_1,A_2,\ldots,A_t$,* *where*

$$A_i = \{a_i + jd_i : j \in [0, k-1], i \in [1, t]\},$$

are color-focused at a positive integer f if the following properties hold:

1. *$A_i \subseteq [x, y]$ for each $i \in [1, t]$,*

2. *All elements of A_i have the same color,*

3. *If $i \neq j$, then A_i and A_j have different color,*

4. *$a_1 + kd_1 = a_2 + kd_2 = \ldots = a_t + kd_t = f$.*

For any $i \in \{1, 2, \ldots, r\}$, the $(k+1)$-arithmetic progression $A_i \cup \{f\}$ is called a spoke.

Example 3.1 *Let us color the first 9 positive integers using 2 colors in the following way:*

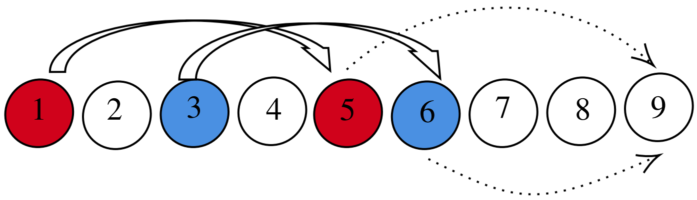


Figure 3.1. An example of color-focused and spoke.

In this example, $\{1, 5\}$ and $\{3, 6\}$ are 2-APs, and they have the same focus which is 9. If we add their focus to these arithmetic progressions, then we get $\{1, 5, 9\}$ and $\{3, 6, 9\}$ are a spoke.

Now, we can start the proof of van der Waerden's theorem by following *Proof of van der Waerden's Theorem in Nine Figures* (Blondal and Jungic, 2018).

Proof (Proof of finite version of van der Waerden's theorem) We will prove the existence of $W(k, r)$ by using double induction.

The Base Case: For any positive integer r, $W(1, r) = 1$ and $W(2, r) = r + 1$ by the pigeonhole principle.

The Inductive Step: Suppose that $k \geq 3$ such that $W(k-1, r)$ exists. Fix $r \geq 2$.

For proving this step, first, we have to prove that the following claim.

Claim 3.1 *For all $t \leq r$, there exists $M = M(t, k, r)$ such that for all r-coloring of $[M]$, there is a monochromatic k-AP or t color-focused $(k-1)$-APs together with their focus.*

Proof We will prove $M = M(t, k, r)$ exists by applying induction on t. For the base case $t = 1$, by the main inductive step, we have $(k-1)$-AP. Observe that so $M \geq W(k-1, r)$. Set $M = 2W(k-1, r)$. In the next figure, we can see that if we have a $(k-1)$-AP in $[1, W(k-1, r)]$, then either it extends to a monochromatic k-AP, or it is a color focused $(k-1)$-AP. Note that the focus of this arithmetic progression can be in $[\frac{M}{2}, M]$.

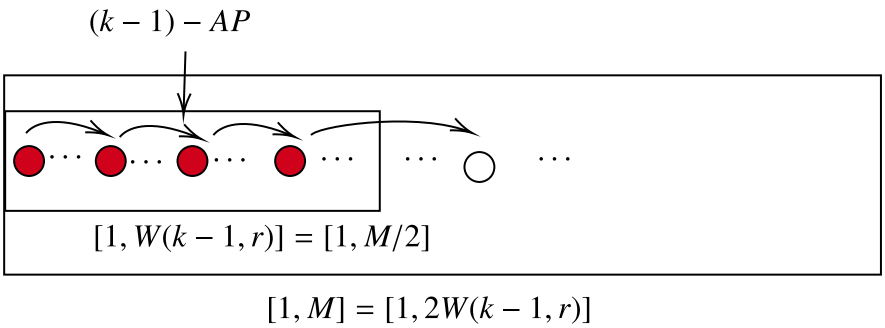


Figure 3.2. The base case when $t = 1$.

Inductive Step of the Claim: Suppose that $t \in [2, r]$ is such that there is an M such that any r-coloring of $[1, M]$ contains or a monochromatic k-AP or at least $t-1$ color-focused $(k-1)$-APs focused at some $f \in [1, M]$. We can say that any set that contains M consecutive positive integers has this property. If we have k-AP, we are done.

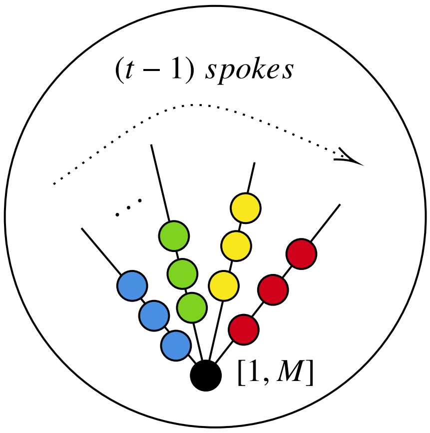


Figure 3.3. Inductive hypothesis for the claim.

If not, then we have $(t-1)$ color-focused $(k-1)$-APs with their focus. Since we do not have a monochromatic k-AP, the focus has a different color from all the previous elements.

We consider the interval $[1, M \cdot W(k-1, r^M)]$ where $M = M(t-1, k, r)$. We will divide this interval into $W(k-1, r^M)$ consecutive blocks of sizes M. We denote these blocks by $B_1, \ldots, B_{W(k-1,r^M)}$ and each block has M elements.

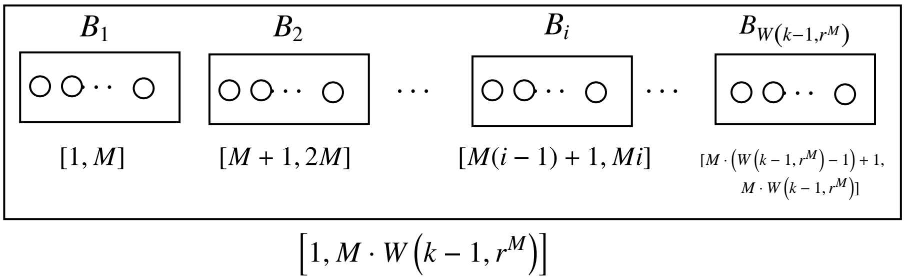


Figure 3.4. There are $W(k-1, r^M)$ consecutive blocks.

Suppose $\chi : [1, M \cdot W(k-1, r^M)] \to [r]$ is an r-coloring of $[1, M \cdot W(k-1, r^M)]$ and there is no monochromatic k-AP. If there is, we are done. Then for each block B_i, $1 \leq i \leq W(k-1, r^M)$, we have r^M ways for the coloring, so by the main inductive step, there is a $(k-1)$-AP in any r^M coloring of $[1, W(k-1, r^M)]$. This means that there exists a $(k-1)$-AP in the number of the blocks, and each corresponding block has has the same color i.e. each block in this AP has M consecutive numbers that are colored in the same way.

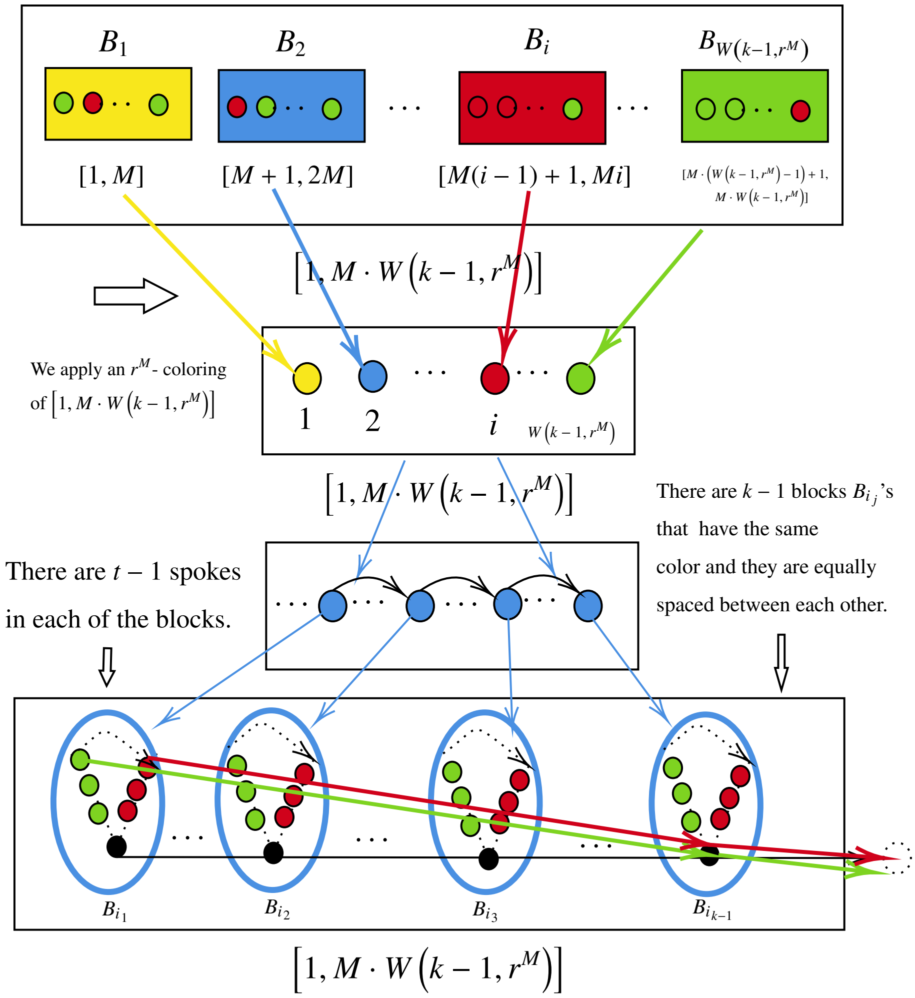
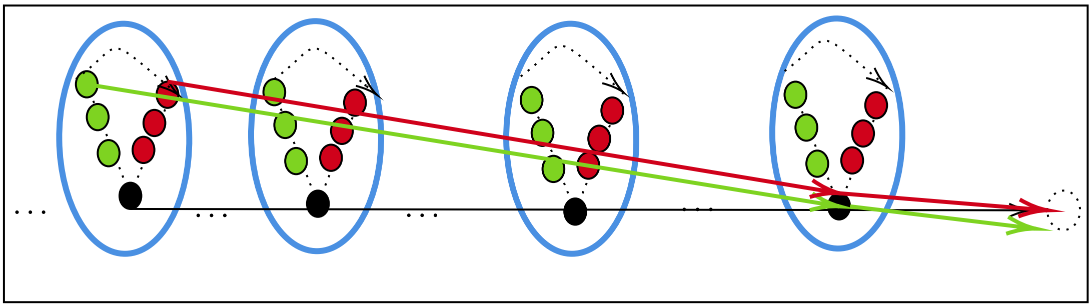


Figure 3.5. The figure of proof of the claim.

The last figure shows that there are identically colored $(k-1)$ blocks concerning χ, and they have the same common difference. When there are $(k-1)$ spokes in each of $t-1$ colors, it generates another spoke which corresponds to the kth term of the AP containing all $k-1$ original focuses. Hence there are r spokes, and we have the claim. □

If $t = r$, then either there is a monochromatic k-AP or there are $(k-1)$-colored focused $(k-1)$-AP in each subblock. Then, we apply the claim, and we are done.



$\left[1, 2M.W\left(k-1, r^M\right)\right]$

Figure 3.6. If $t = r$, then we have $k - 1$ spokes in each of the blocks.

□

3.0.2. Equivalence of van der Waerden's Theorem

In this section, we give the equivalence of two versions of van der Waerden's theorem.

Theorem 3.3 *The infinite version and the finite version of van der Waerden's theorem are equivalent.*

Proof First, we show that the finite version implies the infinite version. Suppose that we have a coloring of $\mathbb{N}$ using r colors. For any k, there exists an integer $W(k, r)$ such that we have a monochromatic nontrivial k-AP in $\{1, \ldots, W(k, r)\}$. Thus, one part of $\mathbb{N}$ contains a monochromatic k-AP for each $k \geq 1$. There are only finitely many partitions, so one of these partitions must contain monochromatic k-APs for infinitely many values of k. This partition contains arbitrarily long monochromatic APs.

Now, we will show the infinite version implies the finite version. Let $r \geq 2$ and assume that for every r-coloring of $\mathbb{N}$, we have arbitrarily long monochromatic APs. We assume for a contradiction that for each $n \geq 1$, there is an r-coloring

$$\chi_n : [1, n] \longrightarrow [1, r-1]$$

with no monochromatic k-APs. Take an infinite subsequence of (χ_n) such that for all $n \in \mathbb{N}$ stabilizes to a constant as n increase along this subsequence. We get a similar contradiction as in Proposition (2.1). Thus, the infinite version of van der Waerden's theorem implies the finite version of van der Waerden's theorem. □

Van der Waerden's theorem can be extended to Szemerédi's theorem (Szemerédi, 1975), which generalizes the concept of finite colorings to subsets possessing a property known as positive upper density. This theorem was later utilized by Green-Tao (Green and Tao, 2008) to demonstrate that arbitrarily long arithmetic progressions exist within the set of prime numbers.

3.0.3. van der Waerden Numbers

Definition 3.2 *The van der Waerden's number $W(k, r)$ is the least positive integer such that for every r-coloring of $[1, W(k, r)]$ there exists a monochromatic arithmetic progression of length k.*

We will apply the idea of proof of finite van der Waerden's theorem for giving upper bound of van der Waerden's number for $k = 3$ and $r = 2$.

Example 3.2 $W(3, 2) \leq 325$.

We will start by partitioning the first 325 numbers into 65 blocks, each containing 5 consecutive numbers. We have 2 colors, so there are $2^5 = 32$ ways to color each block. Therefore, we can find at least two identically colored, i.e., colored in the same way, blocks among the first 33 blocks, by the pigeonhole principle. Let us say the first block B_1, the second block B_2. We can write $B_2 = B_1 + d$ for some $d \leq 325$. Thus, the block $X = B_2 + d = B_1 + 2d$ is a subset of $[1, 325]$.

If there is one element $x \in X$ which has the same color of $x - d \in B_2$, we are done. If is not, we consider the structure in B_1. There are 5 consecutive integers in B_1 and we have 2 colors so within the first 3, there must be a minimum of two with identically colored, by the pigeonhole principle.

Let these elements called that $b_1, b_2 = b_1 + e$ where $e \in \mathbb{N}$. Then, if there is an element $b_3 = b_1 + 2e$ with the same color of b_1 and b_2, we are done. So assume the contrary. Let us look at the B_2. Since B_2 and B_1 has the same color, the first element of B_2 has the same color as the first element of B_1. We say that these elements b_1 and $b_1 + d$. Similarly, b_2 and $b_2 + d$ have the same color. Hence b_1 and $b_1 + d + e$ have the same color. Then also b_3 and $b_3 + d$ have the same color. Consider the following sets:

$$A_1 = \{b_1, b_1 + e + d, b_1 + 2e + 2d\},$$

$$A_2 = \{b_3, b_3 + d, b_3 + 2d\}.$$

Since $b_3 = b_1 + 2e$, we see that $b_1 + 2e + 2d = b_3 + 2d$.

This means that they focus on the same element. There are only two colors, so colors of $b_1 + 2e + 2d$ have the same color as the colors of b_1 and b_3. If it is equal to the color of b_1, we get a monochromatic 3-AP in A_1, if is equal to the color of b_3 we get a monochromatic 3-AP in A_2. Hence, we can find a monochromatic 3-AP in the first 325 numbers.

We can see this in the below figure;

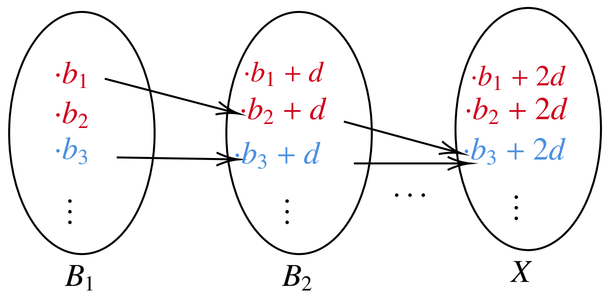


Figure 3.7. $W(3, 2) \leq 325$.

Example 3.3 $W(3, 2) = 9$.

Proof First, we set a lower bound for $W(3, 2)$. Let us consider the first 8 numbers. One such coloring is the following: $\{1, 4, 5, 8\}$ and $\{2, 3, 6, 7\}$. We can see easily, there is no monochromatic 3-AP in this coloring. Thus, $W(3, 2) \geq 8$. We have to find upper bound for $W(3, 2)$. We need to prove that $W(3, 2) \leq 9$. Assume that there exists a 2-coloring of the first nine numbers with no monochromatic 3-APs. We consider the possible ways. Let 3 and 5 are colored by red,

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9.$$

This coloring forces that $1, 4, 7$ must be blue. Because, if one of them is red, then we get a contradiction by the assumption. But if we color by blue, then we get another monochromatic 3-AP. Thus we color 3 and 5 by different colors. Similar to the above situation, we cannot color $4, 6$ and $5, 7$ with the same color. Because if we color $4, 6$ red, then this coloring forces that $2, 5, 8$ must be blue. But this gives a blue 3-AP.

Without loss of generality, we assume the color of 3 is red. By the above observations, we have 2 options. First one is, we have $\{3, 4, 7\}$ and $\{5, 6\}$. This forces that 2 must be blue and 8 must be red because of avoiding of monochromatic $\{2, 5, 8\}$. Then $\{4, 7\}$ forces that 1 must be blue. Now, we have to color 9 by red because of $\{1, 5\}$. But we get a monochromatic $\{7, 8, 9\}$, contradicting our assumption. The other option gives that $\{3, 6, 7\}$ and $\{4, 5\}$. However, in this coloring, we obtain the same contradiction. Thus, every 2-coloring of $[1, 9]$ contains a monochromatic 3-AP. □

Although we know the exact result for only seven van der Waerden numbers, we have bounds for many k and r values as well. The following table shows lots of bounds for van der Waerden numbers.

Table 3.1. van der Waerden's Numbers.

$W(k, r)$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
$k = 3$	9	27	76	> 170	> 223
$k = 4$	35	293	> 1.048	> 2.254	> 9.778
$k = 5$	178	>2173	>17705	>98741	>98748
$k = 6$	1132	>11191	>157209	>786740	>1.555.549
$k = 7$	>3703	>48811	>2284751	>15.993.257	>111.952.799
$k = 8$	>11.495	>238.400	>12,288,155	>86,017,085	>602,119,596
$k = 9$	>41,265	>932,745	>139,847,085	>978,929,595	>6,852,507,165
$k = 10$	>103,474	>4,173,724	>1,189,640,5785	>8,327,484,046	>58,292,388,322
$k = 11$	>193,941	>18,603,731	>3,464,368,083	>38,108,048,913	>419,188,538,043

CHAPTER 4

RADO'S THEOREM AND PARTITION REGULAR EQUATIONS

In this chapter, we will give a generalization of Schur's theorem which is called Rado's theorem. Then, we will introduce the partition regular equations and present some examples.

4.0.1. Partitition Regular Systems

Definition 4.1 *Let $E = E(x_1, \ldots, x_n)$ be a system of linear homogenous equations with variables $x_1, \ldots, x_n$. We call that E is partition regular over A if, for any r-coloring of A, it has a monochromatic solution (not necessarily distinct) to E.*

Example 4.1

1. *We can rewrite a* 3*-AP in the following form:*

$$x_1 + x_2 - 2x_3 = 0. \tag{4.1}$$

   By van der Waerden's theorem, in any finite coloring of $\mathbb{N}$, there exists a monochromatic 3*-AP, and this gives a monochromatic solution to (4.1). Thus (4.1) is a partition regular equation.*

2. *Schur's theorem states that*

$$x_1 + x_2 - x_3 = 0 \tag{4.2}$$

   has a monochromatic solution in any finite coloring of $\mathbb{N}$. Therefore (4.2) is a partition regular equation.

4.0.2. Rado's Theorem for Single Equation

This section will discuss with under which conditions monochromatic solutions for systems of linear homogenous equations with a single equation exists.

Theorem 4.1 *(Rado, 1933) Let $c_1, \ldots, c_n \in \mathbb{Z} \setminus \{0\}$. Then, the equation*

$$c_1x_1 + \cdots + c_nx_n = 0$$

is partition regular if and only if there exists some nonempty set $S \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i \in S} c_i = 0.$$

We can easily see the following examples.

Example 4.2 *1. By Rado's theorem, the above examples (4.1) and (4.2) are partition regular equations, since we get obtain 0 for summation of constant.*

2. $x_1 + x_2 - 4x_3$ is not a partition regular equation because we do not have nonempty subset S such that $\sum_{i \in S} c_i = 0$. We can give the following coloring of $\mathbb{N}$. In this following coloring, we cannot find a monochromatic solution to $x_1 + x_2 - 4x_3 = 0$.

$$\mathbb{N} = \{1\} \cup \{2, 3\} \cup \{4, 5, 6, 7\} \cup \{8, \ldots, 15\} \cup \{16, \ldots, 31\} \cup \ldots$$

Now, we will give two important lemmas for proving Rado's theorem for single equation. Throughout in this chaper, our proof is based on the lecture notes of Leader (Leader, 2000).

Lemma 4.1 *If $k, r, c \in \mathbb{N}$, then there exists an integer $N(k, r, c)$ such that if the set $[1, N(k, r, c)]$ is colored by r colors, then there exist integers a and d with*

$$X = \{a, a + d, \ldots, a + (k - 1)d\} \cup \{cd\} \subseteq [1, N(k, r, c)],$$

and X is monochromatic.

Proof Take any integer k, r, c. We will prove this result by induction on r.

Base Case: If $r = 1$, then this means that all elements have the same color. We can see that $N(k, 1, c) = max\{k, c\}$. For $N(k, 1, c)$, take $a = d = 1$. If $c \leq a + (k - 1)d = 1 + k - 1 = k$, then $c \leq k$. Thus we can take $N(k, 1, c) = k$. If $c > a + (k - 1)d = k$, then $c > k$. Thus, we can choose $N(k, 1, c) = c$. Therefore $N(k, 1, c)$ exists.

Inductive hypothesis: Assume that the lemma holds for $r-1$, i.e., $N(k, r-1, c) = N$ exists. Take any r-coloring of the integers. By van der Waerden's theorem, we know that there exists monochromatic $(kN + 1)$-AP, and call it

$$A = \{a, a + d, \ldots, a + (kN)d\}.$$

Without loss of generality, this $(kN + 1)$-AP is colored blue. Now, we have two choices:

1. If there exists blue tsd for $1 \le t \le N$. Let us consider the following set.

$$A' = \{a, a + td, \dots, a + (k - 1)td\}.$$

Each element of A' has the form $a + itd$ for $i \le k - 1$. Hence $a + itd < a + dkN$. This yields that $A' \subset A$ and all elements of A' is blue. Thus, $A' \cup \{tcd\}$ is monochromatic. Therefore, $N(k, r, c) = W(kN + 1, r)$.

2. If there is no blue tcd for any $1 \le t \le N$. We can consider the set

$$A'' = \{cd, 2cd, \dots, Ncd\}.$$

We know that there is no blue element in A''. Therefore, A'' is $r - 1$-colored. We may define $r - 1$ coloring of $[1, N(k, r - 1, c)]$. We take $x \in [1, N(k, r - 1, c)]$, and the color of x is equal to the color of $xcd \in A''$. By our assumption, there exists a and d such that

$$\{a, a + d, \dots, a + (k - 1)d\} \cup \{cd\} \subseteq [1, N]$$

is monochromatic for any $r - 1$ coloring. Therefore, the elements of following set

$$\{cd \cdot a, cd \cdot a + d, \dots, cd \cdot (a + (k - 1)d)\} \cup \{cd \cdot cd\} \subseteq cd \cdot [1, N]$$

have the same color.

□

We can prove the following lemma using Lemma (4.1).

Lemma 4.2 *For all $c, t \in \mathbb{N}$,*

$$cx + ty = cz \tag{4.3}$$

is partition regular.

Proof Let us take any $c, t \in \mathbb{N}$, and consider any r-coloring of $\mathbb{N}$. By Lemma (4.1), there exist a, d such that

$$X = \{a, a + d, \dots, a + td\} \cup \{cd\} \subseteq [1, N(t + 1, r, c)],$$

and X is monochromatic. Thus, we can say that $a, a + td, cd$ have the same color. If we substitute $x = a, y = cd$ and $z = a + td$ into the $cx + ty = cz$, then we obtain that

$$c \cdot a + t \cdot cd = c \cdot (a + td).$$

Thus, $a, a + td, cd$ are monochromatic solution for $cx + ty = cz$. Therefore, $cx + ty = cz$ is partition regular. □

Proof [Proof of Rado's theorem for Single Equations] Suppose that there exists a set of c_i summing to 0. Without loss of generality, we can take $c_1 + \cdots + c_k = 0$ for some $k \geq 1$.

If $k = n$, we can set $x_i = x_1$ for all $1 \leq i \leq k$, and we get

$$\begin{aligned} c_1x_1 + \cdots + c_nx_1 &= x_1(c_1 + c_2 + \cdots + c_n), \\ &= x_1(0), \\ &= 0. \end{aligned}$$

Suppose $k < n$, we set a monochromatic solution. Let

$$x_i = \begin{cases} x, & for\, i = 1 \\ z, & for\, 2 \leq i \leq k \\ y, & for\, i \geq k+1. \end{cases}$$

Then, the equation can be written as

$$c_1x + (c_1 + \cdots + c_k)z + (c_{k+1} + \cdots + c_n)y = 0. \tag{4.4}$$

We know that $c_2 + \cdots + c_k = -c_1$. Thus, the equation becomes

$$c_1x + (-c_1)z + (c_{k+1} + \cdots + c_n)y = 0, \tag{4.5}$$

and if we get $c_1 = s$, $c_{k+1} + \cdots + c_n = t$, then we arrive that

$$sx + (-s)z + ty = 0. \tag{4.6}$$

By Lemma (4.2), $sx + ty = sz$ is partition regular.

Suppose that $\sum_{i=1}^{n} c_ix_i = 0$ is partition regular. Take a prime p such that $p > \sum_{i=1}^{n} |c_i|$ and consider the following coloring:

$$\chi_p : \mathbb{N} \to \{1, 2, \ldots, p-1\}$$

- if p divides n, then we can write $n = p^k m$ where p does not divide m and $\chi_p(n) = m \pmod{p}$,
- if p does not divide n, then $\chi_p(n) = n \pmod{p}$.

By our assumption, the equation is partition regular, so there must exist a monochromatic solution $s_1, s_2, \ldots, s_n$ with respect to χ_p. Let us divide through by p^l where p^l is the greatest power dividing each s_i, such that $s_i = p^l \cdot s_i'$. The s_i''s are all of the same color with s_i's by the construction of χ_p. Then, if we divided this equation by p^l, then we obtain

$$\frac{c_1}{p^l} \cdot \frac{s_1}{p^l} + \cdots + \frac{c_n}{p^l} \cdot \frac{s_n}{p^l} = \frac{0}{p^l} = 0. \tag{4.7}$$

Thus $(s'_i)_i$ is a solution to equation. Wlog, we may assume that p does not divide $s'_1+\cdots+s'_k$ and p divides $s'_{k+1}+\cdots+s'_n$. We know that there exists s'_i such that p does not divide s'_i since otherwise we could have divided out a larger power of p.

Since $(s'_i)_i$ is a solution to the equation, we have

$$c_1 s'_1 + \cdots + c_n s'_n \equiv 0 \pmod{p}. \tag{4.8}$$

We know that $s'_1, \ldots, s'_k$ have the same color say it m' and $s'_{k+1}, \ldots, s'_n$ have the same color, so

$$m'(c_1 + \cdots + c_k) + 0 + \cdots + 0 \equiv 0 \pmod{p}. \tag{4.9}$$

This means that p divides $c_1+\cdots+c_k$. However, we have choosen $p > \sum_{i=1}^{n} |c_i|$. Therefore, if p will divide $c_1+\cdots+c_k$, then $c_1+\cdots+c_k$ must be equal to zero as desired. □

Example 4.3

$$x_1 + x_2 - 3x_3 = 0$$

is not a partition regular equation.

Proof We assume that there is a monochromatic solution to $x_1 + x_2 - 3x_3 = 0$ with respect to the following coloring. Consider the coloring

$$\chi_5 : \mathbb{N} \to \{1, 2, 3, 4\},$$

such that

- If 5 divides n, then we write $n = 5^k m$ where 5 does not divid m, $\chi_5(n) = m \pmod{5}$,
- If 5 does not divide n, then $\chi_5(n) = n \pmod{5}$.

We assume that a monochromatic solution exists, i.e. that there exist $s_1, s_2, s_3 \in \mathbb{N}$ monochromatic with respect to χ_5 such that

$$s_1 + s_2 - 3s_3 = 0. \tag{4.10}$$

As we divide s_i's by 5, we can assume that at least one of s_i's is coprime to 5. Now, we consider the equation

$$s_1 + s_2 - 3s_3 \equiv 0 \pmod{5}. \tag{4.11}$$

This means that s_1, s_2, s_3 are monochromatic and each is either equal to 0 modulo 5 or equal to a common value $m \in \{1, 2, 3, 4\}$ modulo 5. Since at least one of the s_i's is coprime to 5, at least one is equal to m modulo 5. Assume that s_1 and s_2 are coprime to 5 and colored by m. We can write $s_1 = 5t+m$, $s_2 = 5s+m$ and $s_3 = 5^k m$ for some $k, t, s \in \mathbb{N}$. As

$$s_1 + s_2 - 3s_3 \equiv 0 \pmod 5, \tag{4.12}$$

we get

$$m + m \equiv 0 \pmod 5. \tag{4.13}$$

Since m is coprime to 5, we may divide it out, yielding an equation of the form,

$$2 \equiv 0 \pmod 5, \tag{4.14}$$

this is a contradiction. Therefore, there is no monochromatic solution with respect to χ_5, so $x_1 + x_2 - 3x_3 = 0$ is not a partition regular equation. □

4.0.3. Generalization of Rado's Theorem

Definition 4.2 (Columns Condition) *Let A be an $m \times n$ matrix $A = (a_1 a_2 \cdots a_n)$ where $a_i \in \mathbb{Z}^m$ for $1 \leq i \leq n$. A is said to regular to satisfy the columns condition if its column can be partitioned as $A_1 \cup \cdots \cup A_m$ where each A_m is a set of column vector from A such that the following conditions hold:*

1. $\sum_{a_i \in A_1} a_i = 0$ *and,*
2. *The sum* $\sum_{a_i \in A_j} a_i$ *for all $j > 1$, can be written as a linear combination of $a_1, \ldots, a_{j-1}$ from the set $A_1 \cup \cdots \cup A_{j-1}$.*

We can write arithmetic progressions as systems of equations. For instance, let $x_1, x_2, \ldots, x_n$ be an arithmetic progression. This progression yields the following system of equations:

$$\begin{aligned} x_2 + x_1 &= x_3 - x_2, \\ x_3 + x_2 &= x_4 - x_3, \\ &\vdots \\ x_{n-1} + x_{n-2} &= x_n - x_{n-1}. \end{aligned}$$

We can write the above system of linear equations in the following matrix form:

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \ldots \\ 0 & 1 & -2 & 1 & 0 & \ldots \\ \vdots \\ 0 & 0 & \ldots & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = 0.$$

To understand this condition, we look at the following example.

Example 4.4 *The equation* $x_1 + x_2 - x_3 = 0$ *can be written as*

$$A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

We can put the name of the columns of the vector as a_1, a_2, a_3*. Then* $A = A_1 \cup A_2$ *where* $A_1 = \{a_1, a_3\}$ *and* $A_2 = \{a_2\}$*. Thus, we get 0 using* $a_1 + a_3$ *and* $a_2 = a_1$ *so the column condition is satisfied.*

Example 4.5 *The system*

$$B = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

is not partition regular. We cannot obtain 0 using any summation of the coefficients and coefficents of B cannot be written as a linear combination between them. Therefore, B does not satisfy the column condition. In the following coloring of $\mathbb{N}$*, we cannot find a monochromatic solution to this linear homogenous system,*

$$\mathbb{N} = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \ldots$$

Now, we will give the Rado's theorem for system of linear homogenous equations with no proof.

Theorem 4.2 (Rado's Theorem Full Theorem) *Let E be a system of linear homogenous equations and write E as* $Ax = 0$*. E is partition regular if and only if A satisfies the columns condition.*

Example 4.6 *Let us examine the following system in the variables* $x_1, \ldots, x_6, y$

$$x_1 + x_2 = x_3,$$
$$x_4 - x_5 = y,$$
$$x_5 - x_6 = y.$$

We can write it in matrix form where

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

Then we have

$$\begin{bmatrix} a_1 & a_2 & \dots & a_7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \end{bmatrix}$$

Then $A_1 = \{a_1, \dots, a_5\}$ *and* $\sum_{a_i \in A_1} a_i = 0$*, and* $A_2 = \{a_6\}$ *and* $A_3 = \{a_7\}$*. Thus, we have* $A_2 = \{a_1\}$ *and* $A_3 = a_4 + 2a_5$*. So this matrix satisfies the columns condition.*

Thus, $x_1 + x_2 = x_3$ *has a monochromatic solution, so Schur's theorem is satisfied. And* x_6, x_5, x_4 *is a monochromatic* 3*-AP with common difference* y*. Hence, this ensures that van der Waerden's theorem for* 3*-APs.*

CHAPTER 5

RAMSEY'S THEORY VIA ULTRAFILTERS

This chapter presents the basic theory of filters. We will give ultrafilters proof of theorems of Ramsey theory such as Schur's and van der Waerden's theorems for $k = 3$. We may refer the reader to *Nonstandard Methods in Ramsey Theory and Combinatorial Number Theory* (Di Nasso et al., 2019) for a nice introduction to the nonstandard analysis and to provide an overview of its most prominent applications in Ramsey theory and combinatorial number theory. Also, we direct the reader to *Algebra in Stone-Čech Compactification* (Hindman and Strauss, 1998) for more details about Stone-Čech compactification and more details about the algebraic and topological properties of βS.

5.0.1. A Short Introduction to Ultrafilters

In this section, we give the main definitons and properties of filters. Throughout this section, we let S denote an infinite set. Our main references during this section are *Nonstandard Methods in Ramsey Theory and Combinatorial Number Theory* (Di Nasso et al., 2019) and *Algebra in Stone-Čech Compactification* (Hindman and Strauss, 1998).

Definition 5.1 *A proper filter on S is a set $\mathcal{F}$ of subsets of S (that is, $\mathcal{F} \subseteq \mathcal{P}(S)$) such that*

- $\emptyset \notin \mathcal{F}$, $S \in \mathcal{F}$,
- *if* $A, B \in \mathcal{F}$, *then* $A \cap B \in \mathcal{F}$,
- *if* $A \in \mathcal{F}$ *and* $A \subseteq B$, *then* $B \in \mathcal{F}$.

Example 5.1 *(1)* $\mathcal{F} = \{S\}$ *is a basic filter.*

(2) $Fr = \{A \subseteq S | S \setminus A \, is \, finite\}$ *is an example of a filter on S, called the Frechét or cofinite filter on S.*

Definition 5.2 *A nonempty family $\mathcal{D}$ of subsets of S has* finite intersection property *(FIP) if for every natural number n, for every $D_1, \ldots, D_n \in \mathcal{D}$, the intersection*

$$D_1 \cap \cdots \cap D_n$$

is nonempty.

By definition, every filter has the finite intersection property.

Proposition 5.1 *Let S be a set and $\mathcal{D}$ is a set of subsets of S. If $\mathcal{D}$ has the finite intersection property, then there exists a filter $\mathcal{F}$ on S such that $\mathcal{D} \subseteq \mathcal{F}$.*

Proof Given the family $\mathcal{D}$ with the *FIP*. Consider

$$\mathcal{F} = \{E \subseteq S : D_1 \cap \cdots D_n \subseteq E \text{ for some } D_1, \ldots, D_n \in \mathcal{D}\}.$$

If $\mathcal{D}$ has the *FIP*, then $D_1 \cap \cdots D_n$ is nonempty. Thus $\emptyset \in \mathcal{F}$. And also, $S \in \mathcal{F}$. Let us take $E_1, E_2 \in \mathcal{F}$. Then there exists $n \in \mathbb{N}$ and $D_1 \cap \cdots \cap D_n \in \mathcal{D}$ with $D_1 \cap \cdots \cap D_n \subseteq E_1$ and E_2. If we intersect E_1 and E_2, then there is nonempty $D_1 \cap \cdots \cap D_n \subseteq E_1 \cap E_2$. There exists $i \in \mathbb{N}$ such that $D_1 \cap \cdots \cap D_i \in \mathcal{D}$ with $D_1 \cap \cdots \cap D_i \subseteq E_1$ and $E_1 \subseteq E_2$. Then, $D_1 \cap \cdots \cap D_i \subseteq E_2$. Thus $E_2 \in \mathcal{F}$. □

Definition 5.3 *A filter $\mathcal{F}$ on S is said to be* maximal *if for any filter $\mathcal{G}$ on S with $\mathcal{F} \subseteq \mathcal{G}$, we have $\mathcal{G} = \mathcal{F}$.*

Definition 5.4 *A filter $\mathcal{F}$ on S is an ultrafilter if for any subset A of S, either $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F}$. We will denote an ultrafilter by $\mathcal{U}$.*

For example, Frechét filter on an infinite set S is not an ultrafilter because there are sets $B \subseteq S$ such that B and $S \setminus B$ are both infinite.

Lemma 5.1 *If $A \cup B \in \mathcal{U}$, then either $A \in \mathcal{U}$ or $B \in \mathcal{U}$.*

Proof Suppose $A \notin \mathcal{U}$ and $B \notin \mathcal{U}$. Then by the definition of ultrafilter, $S \setminus A$ *and* $S \setminus B \in \mathcal{U}$. Then,

$$(S \setminus A) \cap (S \setminus B) = S \setminus (A \cup B) \in \mathcal{U},$$

since $\mathcal{U}$ is a filter. Thus, $A \cup B \notin \mathcal{U}$, a contradiction. □

Proposition 5.2 *Let S be a set and $\mathcal{F}$ be a filter on S. The filter $\mathcal{F}$ on S is an ultrafilter if and only if it is maximal.*

Proof First, we will show that every ultrafilter is maximal. If $\mathcal{F}$ is properly contained in a filter $\mathcal{G}$, then there is a set $A \subseteq S$ such that $A \in \mathcal{G} \setminus \mathcal{F}$, so $S \setminus A \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathcal{G}$, one sees that $S \setminus A \in \mathcal{G}$. However, $A \cap (S \setminus A) = \emptyset \in \mathcal{G}$, this is a contradiction. Thus, $\mathcal{F}$ is maximal.

Now, we will show that every maximal filter is an ultrafilter. Suppose that $\mathcal{F}$ is a maximal filter but it is not an ultrafilter. So, there exists $A \subseteq S$ such that $A \notin \mathcal{F}$ and

$S \setminus A \notin \mathcal{F}$. We will observe that given any element F of $\mathcal{F}$, one has $F \cap A \neq \emptyset$. If $F \cap A = \emptyset$ then $F \subseteq S \setminus A$. Since every filter is closed under supersets, we get $S \setminus A \in \mathcal{F}$. This contradicts the assumption. Thus, $F \cap A \neq \emptyset$.

By the Proposition (5.1), there is a filter $\mathcal{G}$ on S such that $\mathcal{F} \cup \{A\} \subseteq \mathcal{G}$. However, this contradicts the maximality of $\mathcal{F}$. Therefore, $\mathcal{F}$ is an ultrafilter. □

Definition 5.5 *An ultrafilter $\mathcal{U}$ on a set S is called principal if there is an element $s \in S$ such that*

$$\mathcal{U} = \mathcal{U}_s := \{A \subseteq S : s \in A\}.$$

If an ultrafilter is not principal, then we say that it is a nonprincipal ultrafilter.

Proposition 5.3 *Any ultrafilter on a nonempty finite set is principal.*

Proof Let S be a nonempty finite set and $\mathcal{U}$ be an ultrafilter on S.

Consider the set

$$X = \bigcap_{U \in \mathcal{U}} U.$$

Since S is finite, the ultrafilter $\mathcal{U}$ is also finite. It follows the *FIP* of $\mathcal{U}$ that $X \in \mathcal{U}$.

Take $x \in X$. Since $\mathcal{U}$ is ultrafilter, we have that either $\{x\} \in \mathcal{U}$ or $S \setminus \{x\} \in \mathcal{U}$. If $S \setminus \{x\} \in \mathcal{U}$, then $x \in X \subseteq S \setminus \{x\}$ which is a contradiction. Therefore $\{x\} \in \mathcal{U}$.

Now, we indicate that $\mathcal{U}$ is generated by x. Assume the contrary that there is $Y \in \mathcal{U}$ such that $\{x\} \nsubseteq Y$ in which case $\{x\} \cap Y = \emptyset \in \mathcal{U}$, which is a contradiction. Thus, $\mathcal{U}$ is generated by x so $\mathcal{U}$ is principal ultrafilters. □

Now, we can establish the existence of nonprincipal ultrafilters.

Proposition 5.4 *Every filter $\mathcal{F}$ extends to an ultrafilter.*

Proof For a filter $\mathcal{F}$, let $\mathcal{A}$ be the set of all filters $\mathcal{F}'$ on S such that $\mathcal{F} \subseteq \mathcal{F}'$. Since $\mathcal{F} \in \mathcal{A}$, one has that $\mathcal{A}$ is nonempty.

We claim that for each chain in $\mathcal{A}$, $\{\mathcal{F}_\alpha : \alpha \in I\}$,

$$\bigcup_{\alpha \in I} \mathcal{F}_\alpha$$

is a filter in $\mathcal{A}$. It is clear that $\emptyset \notin \cup_{\alpha \in I} \mathcal{F}_\alpha$ and $S \in \cup_{\alpha \in I} \mathcal{F}_\alpha$. For an element $A \in \cup_{\alpha \in I} \mathcal{F}_\alpha$, $A \in \mathcal{F}_\alpha$ for some $\alpha \in I$. Then, for all B such that $A \subseteq B$, $B \in F_\alpha \subset \cup_{\alpha \in I} \mathcal{F}_\alpha$. For $C, B \in \cup_{\alpha \in I} \mathcal{F}_\alpha$, $C \in \mathcal{F}_\alpha$ and $B \in \mathcal{F}_\beta$, for some α, β. Wlog, we assume that $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$. Thus $B \in \mathcal{F}_\alpha$ and $C \cap B \in \mathcal{F}_\alpha \subseteq \cup_{\alpha \in I} \mathcal{F}_\alpha$. By Zorn's lemma, there exists a maximal element in $\mathcal{A}$. Hence it is an ultrafilter by Proposition (5.2), and it contains $\mathcal{F}$. □

Proposition 5.5 *Let S be an infinite set. An ultrafilter $\mathcal{U}$ on S is nonprincipal if and only if $\mathcal{U}$ extends the Frechét filter.*

Proof Assume that $\mathcal{U}$ is a nonprincipal ultrafilter. For any element $s \in S$, then either $\{s\} \in \mathcal{U}$ or $S \setminus \{s\} \in \mathcal{U}$. By our assumption, $S \setminus \{s\} \in \mathcal{U}$. For any set A in S, if A is finite, then the set

$$S \setminus A = \cap_{s \in A} S \setminus \{s\}$$

in $\mathcal{U}$. Therefore, Frechét filter is contained by $\mathcal{U}$.

Conversely, suppose that $\mathcal{U}$ is a principal ultrafilter generated by s and so $\{s\} \in \mathcal{U}$. Since $S \setminus \{s\}$ is infinite $S \setminus \{s\} \in Fr$. Therefore, $S \setminus \{s\} \in \mathcal{U}$. So

$$\{s\} \cap (S \setminus \{s\}) = \emptyset \in \mathcal{U},$$

this is a contradiction. Thus, $\mathcal{U}$ is a nonprincipal ultrafilter on S. □

Since the union of an increasing chain of filters on S containin a filter $\mathcal{F}$ is also filter on S that includes $\mathcal{F}$, the previous two propositions (5.4) and (5.7), along with Zorn's Lemma, lead to the following result.

Corollary 5.1 *There exist nonprincipal ultrafilters on an infinite set S.*

5.0.2. The Set of Ultrafilters as a Topological Space

In this section, we explore ultrafilters on an infinite set S, treating S as a topological space with the discrete topology. We demonstrate that the space of ultrafilters, denoted as βS, serves as the Stone-Čech compactification of the discrete topological space S.

Definition 5.6 *Let S be a discrete topological space.*

1) $\beta S = \{\mathcal{U} : \mathcal{U}\, is\, an\, ultrafilter\, on\, S\}$

2) Given $A \subseteq S$, we set $U_A = \{\mathcal{U} \in \beta S : A \in \mathcal{U}\}$.

We have the following properties.

Lemma 5.2 *Let S be a set and $A, B \subseteq S$.*

1. $U_{A\cap B} = U_A \cap U_B$,

2. $U_{A\cup B} = U_A \cup U_B$,

3. $U_{S\setminus A} = \beta S \setminus U_A$,

4. $U_A = \emptyset$ *if and only if* $A = \emptyset$,

5. $U_A = \beta S$ *if and only if* $A = S$,

6. $U_A = U_B$ *if and only if* $A = B$.

Remark 5.1 *Let $S \neq \emptyset$. We consider the family of sets $\mathcal{B} = \{U_A : A \subseteq S\}$. By the Lemma (5.2) Property* 1*, $\mathcal{B}$ is closed under finite intersections. Consequently, $\{U_A : A \subseteq S\}$ forms a basis for a topology on βS. We define the topology of βS to be the topology which has these sets as a basis.*

Remark 5.2 *Every principal ultrafilter $\mathcal{U}_s$ can be identified with the point s, since the function S to βS defined by $s \to \mathcal{U}_s$ is injective on its image. Thus, S becomes a topological subspace of βS.*

Theorem 5.1 *S is dense in βS.*

Proof Let A be a nonempty set of S. Take $a \in A$. Then $\mathcal{U}_a \in U_A$ and we are done. □

Theorem 5.2 *βS is a compact Hausdorff space.*

Proof Let $\mathcal{U}_1, \mathcal{U}_2 \in \beta S$ with $\mathcal{U}_1 \neq \mathcal{U}_2$. There exists $A \in \mathcal{U}_1$ such that $A \notin \mathcal{U}_2$. Since $\mathcal{U}_2$ is an ultrafilter, we have $S \setminus A \in \mathcal{U}_2$. U_A and $U_{S\setminus A}$ are two open sets containing $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. Moreover U_A and $U_{S\setminus A}$ are disjoint. Thus, βS is Hausdorff.

Assume, for the contradiction, that there is an open cover of βS with no finite subcover. We may assume that cover has the form $\{U_{A_\alpha} : \alpha \in I\}$, for some index set I. We have

$$U_{A_{\alpha_1} \cup \cdots \cup A_{\alpha_n}} = U_{A_{\alpha_1}} \cup \cdots \cup U_{A_{\alpha_n}}$$

is not equal to βS by our assumption. It follows that $A_{\alpha_1} \cup \cdots \cup A_{\alpha_n} \neq S$. In other words, $S \setminus A_{\alpha_n} \neq \emptyset$. Consider the collection of all sets containing at least one of $S \setminus A_{\alpha_i}$ such that $\{S \setminus A_\alpha : \alpha \in I\}$ extends to a filter by finite intersection property. This filter is contained in some ultrafilter $\mathcal{U} \in \beta S$. By our assumption, $\{U_{A_\alpha} : \alpha \in I\}$ covers βS,so $\mathcal{U} \in U_{A_\alpha}$ for some $\alpha \in I$. It follows that $S \setminus A_\alpha$ and A_α are both in $\mathcal{U}$, thus

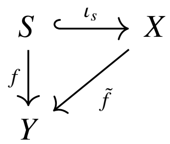
$$(S \setminus A_\alpha) \cap A_\alpha \in \mathcal{U},$$

but this contradicts the filter definition. Therefore, βS is a compact space. □

5.0.3. Stone-Čech Compactification

Next, we will show that βS is the Stone-Čech compactification of S. For this purpose, we will define the *ultralimit* context of filters and we will give a definition of the Stone-Čech compactification.

Definition 5.7 *The Stone-Čech compactification of the topological space S is a compact Hausdorff space X together with a continuous function $\iota_s : S \to X$ that has the following property: any continous function $f : S \to Y$, where Y is a compact Hausdorff space, extends uniquely to a continuous function $\tilde{f} : X \to Y$.*



Before proving the main theorem of this section, we define limit in the context of filters.

Definition 5.8 *Let S be a topological space and $\mathcal{U} \in \beta S$. Let $(x_s)_s \in S$ be an indexed family in a compact Hausdorff space Y, and $y \in Y$. Then $\lim\limits_{s \to \mathcal{U}} x_s = y$ if and only if for any neighborhood U of y, $\{s \in S : x_s \in U\} \in \mathcal{U}$.*

We call the unique y the ultralimit of (x_s) with respect to $\mathcal{U}$, denoted $\lim\limits_{s \to \mathcal{U}} x_s$, or just by $\lim\limits_{\mathcal{U}} x_s$.

Lemma 5.3 *Suppose that Y is a compact Hausdorff space and (y_s) is a family of elements of Y indexed by S. Then for any ultrafilter $\mathcal{U} \in \beta S$, there is a unique $y \in Y$ such that $\lim\limits_{s \to \mathcal{U}} y_s = y$.*

Proof Suppose that there is no such y. Then for every $y \in Y$, there is an open neighborhood U_y of y such that $\{s \in S : y_s \in U_y\} \notin \mathcal{U}$. By compactness, there are $y_1, \dots, y_n \in Y$ such that $Y = U_{y_1} \cup \dots \cup U_{y_n}$. There exists then $i \in \{1, \dots, n\}$ such that $\{s \in S : y_s \in U_{y_i}\} \in \mathcal{U}$, yielding the desired contradiction.

Now, we show that y is unique. Suppose that there exists $\lim\limits_{s \to \mathcal{U}} y_s = y$ and $\lim\limits_{s \to \mathcal{U}} y_s = y'$ with $y \neq y'$. Since Y is Hausdorff space, there exists open neighorbood U and U' of y and y', respectively, and $U \cap U' = \emptyset$. We consider that

$$A = \{s \in S : y_s \in U\} \in \mathcal{U},$$

$$B = \{s \in S : y_s \in U'\} \in \mathcal{U}.$$

We have $A \cap B$, since $\mathcal{U}$ is an ultrafilter. However, this is a contradiction, as $A \cap B = \emptyset \in \mathcal{U}$. We conclude that $y = y'$. Therefore, limit is unique.

□

Theorem 5.3 *Let S be a discrete topological space and Y be a compact Hausdorff space. Then every function $f : S \longrightarrow Y$ extends uniquely to a continuous function $\tilde{f} = \beta f : \beta S \longrightarrow Y$. In other words, βS is the Stone-Čech compactification of S.*

$$\begin{array}{ccc} S & \hookrightarrow & \beta S \\ {\scriptstyle f}\big\downarrow & \swarrow & {\scriptstyle \tilde{f} =} \lim\limits_{s \to \mathcal{U}} f(s) \\ Y & & \end{array}$$

Before the proof the Theorem (5.3), we will prove two theorems.

Theorem 5.4 *If X is regular, then given a point $a \in X$ and a neighborhood U of a, there is a neighborhoof V of a such that $\overline{V} \subseteq U$.*

Proof Suppose X is regular, a and the neighborhod U of a are given. Let $A = X \setminus U$, then A is a closed set. By the hypothesis, there exist disjoint open sets V and W containing a and A, respectively. The set $\overline{V}$ is disjoint from A, since if $b \in A$, the set W is a neighborhood of b disjoint from V. Therefore, $\overline{V} \subseteq U$. □

Theorem 5.5 *Let $f, f' : X \longrightarrow Y$ be continuous functions with Y Hausdorff. Then the set $\{x \in X : f(x) = f'(x)\}$ is closed.*

Proof Let $B = \{x \in X : f(x) \neq f'(x)\}$. Suppose $x \in B$. Since $f(x) \neq f'(x)$, there are open sets $U, V \subset Y$ so that $f(x) \in U$, $f'(x) \in V$ and $U \cap V = \emptyset$. Let $W = f^{-1}(U) \cap f'^{-1}(V)$. Then W is open and $x \in W$. Moreover, $W \subset B$. Thus B is open so $\{x \in X : f(x) = f'(x)\}$ is closed. □

Proof [Proof of the Theorem (5.3)] Suppose that $f : S \longrightarrow Y$ is a function into a compact Hausdorff space. Define $\tilde{f} : \beta S \longrightarrow Y$ by $\tilde{f}(\mathcal{U}) := \lim\limits_{s \to \mathcal{U}} f(s)$. This is a well defined function and $\tilde{f}$ extends f, since $\tilde{f}(\mathcal{U}) = \tilde{f}(s) = \lim\limits_{s \to \mathcal{U}} f(s) = f(s)$.

Now, we will show $\tilde{f}$ is continuous. Fix $\mathcal{U} \in \beta S$. Let U be an open neighborhood of $\tilde{f}(\mathcal{U})$ in Y. Every Hausdorff space is regular, thus we have that $V \subseteq U$ be an open neighborhood of $\tilde{f}(\mathcal{U})$ in Y such that $\overline{V} \subseteq U$, by the Theorem (5.4). Take $A \in \mathcal{U}$ such that $f(s) \in V$ for $s \in A$. Suppose $\mathcal{V} \in U_A$ so $A \in \mathcal{V}$. Then $\lim\limits_{s \to \mathcal{V}} f(s) \in \overline{V} \subseteq U$, so $U_A \subseteq \tilde{f}^{-1}(U)$. Therefore, $\tilde{f}$ is continuous.

By the Theorem (5.5), if we take two distinct functions $g_1, g_2 : \beta S \longrightarrow Y$, then we obtain the following set

$$\{\mathcal{U} \in \beta S : g_1(\mathcal{U}) = g_2(\mathcal{U})\}$$

is closed. As S is dense in βS, it follows that such a continuous extension is unique. □

Therefore, we have shown that βS is the Stone-Čech compactification of S.

5.0.4. The Algebraic Structure of βS

In this section, we will mention the algebraic structure of βS. We will see that one can extend the operation of discrete semigroup to its Stone-Čech compactification.

Definition 5.9 *A semigroup is a pair $(S, \cdot)$ where S is a nonempty set and $\cdot$ is a binary associative operation on S.*

Definition 5.10 *(Hindman and Strauss, 1998)*

a) A right topological semigroup is triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and for all $x \in S$, $\rho_x : S \longrightarrow S$ is continuous, where $\rho_x(y) = y \cdot x$.

b) A left topological semigroup is triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and for all $x \in S$, $\lambda_x : S \longrightarrow S$ is continuous, where $\lambda_x(y) = x \cdot y$

c) A semitopological semigroup is a right topological semigroup which is also a left topological semigroup.

d) A topological semigroup is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and $\cdot : S \times S \longrightarrow S$ is continuous.

We can extend the semigroup operation of S to βS, and this makes $(\beta S, \cdot)$ a compact topological semigroup.

Theorem 5.6 *(Hindman and Strauss, 1998) Let S be a discrete space and let $\cdot$ be a binary operation defined on S. There is a unique binary operation*

$$* : \beta S \times \beta S \longrightarrow \beta S$$

such that

1. *For every $s, t \in S$, $s * t = s \cdot t$,*
2. *For each $\mathcal{U}, \mathcal{V} \in \beta S$, $\rho_{\mathcal{U}} : \beta S \rightarrow \beta S$ is continuous, where $\rho_{\mathcal{V}}(\mathcal{U}) = \mathcal{U} * \mathcal{V}$,*
3. *For each $s \in S$, $\lambda_s : \beta S \rightarrow \beta S$ is continuous, where $\lambda_s(\mathcal{V}) = s * \mathcal{V}$.*

Proof We first define $*$ on $S \times \beta S$. Given $s \in S$, define $\varphi_s : S \to S \subseteq \beta S$ where $\varphi_s(t) = s \cdot t$. Then by Theorem (5.3), there is a unique continuous function $\lambda_s : \beta S \to \beta S$ such that the restriction of λ_s on S is equal to φ_s.

For $s \in S$, $\mathcal{V} \in \beta S$, define $s * \mathcal{V} = \lambda_s(\mathcal{V})$. This proves (1) and (3) at the same time. Given $\mathcal{V} \in \beta S$, define $\Gamma_{\mathcal{V}} : S \to \beta S$ where $\Gamma_{\mathcal{V}}(s) = s * \mathcal{V}$. There is a unique continuous function $\rho_{\mathcal{V}} : \beta S \to \beta S$ such that the restriction of $\rho_{\mathcal{V}}$ on S is equal to $\Gamma_{\mathcal{V}}$. For $\mathcal{V} \in \beta S \setminus S$, define $\mathcal{U} * \mathcal{V} = \rho_{\mathcal{V}}(\mathcal{U})$ and if $s \in S$, $\rho_{\mathcal{V}}(s) = \Gamma_{\mathcal{V}}(s) = s * \mathcal{V}$. For all $\mathcal{U} \in \beta S$, $\rho_{\mathcal{V}}(\mathcal{U}) = \mathcal{U} * \mathcal{V}$. We observe that (2) holds. By the uniqueness of continuous extensions, this is the only possible definition that satisfies the required conditions. □

Theorem 5.7 *Let S be a discrete topological space and $\mathcal{U} \in \beta S$. Let X and Y be topological spaces, $(x_s)_{s \in S}$ be an indexed family in X, and $f : X \to Y$. If f is continuous and $\lim_{s \to \mathcal{U}} x_s$ exists, then*

$$\lim_{s \to \mathcal{U}} f(x_s) = f(\lim_{s \to \mathcal{U}} x_s).$$

Proof Let $y = \lim\limits_{s \to \mathcal{U}} x_s$ and O be an open neighborhood of $f(y)$. We will show that $\{s \in S : f(x_s) \in O\} \in \mathcal{U}$. By continuity, there is an open neighborhood U of Y such $f(U) \subseteq O$. Also, we have $\{s \in S : x_s \in U\} \in \mathcal{U}$. This gives that $\{s \in S : f(x_s) \in f(U)\} \in \mathcal{U}$ and so

$$\{s \in S : f(x_s) \in f(U)\} \subseteq \{s \in S : f(x_s) \in O\} \in \mathcal{U}.$$

Thus, $\lim\limits_{s \to \mathcal{U}} f(x_s) = f(\lim\limits_{s \to \mathcal{U}} x_s)$. □

Remark 5.3 *Using Theorem (5.7), the operation on βS has a characterization in terms of limits. Let $\cdot$ be a binary operation on S.*

(a) If $s \in S$, $\mathcal{U} \in \beta S$, then $s \cdot \mathcal{U} = \lim_{t \to \mathcal{U}} s \cdot t$.

By continuity of λ_s for every $s \in S$, one sees that

$$\begin{aligned}\lim_{s \to \mathcal{U}} s \cdot t = \lim_{t \to \mathcal{U}} \lambda_s(t) &= \lambda_s(\lim_{t \to \mathcal{U}} t), \\ &= \lambda_s(\mathcal{U}), \\ &= s \cdot \mathcal{U}.\end{aligned}$$

(b) If $\mathcal{U}, \mathcal{V} \in \beta S$, then $\mathcal{U} \cdot \mathcal{V} = \lim\limits_{s \to \mathcal{U}} (\lim\limits_{t \to \mathcal{V}} s \cdot t)$.

By continuity of $\rho_{\mathcal{V}}$ for every $s \in S$, one sees that

$$\begin{aligned}\lim_{s \to \mathcal{U}} (\lim_{t \to \mathcal{V}} s \cdot t) = \lim_{s \to \mathcal{U}} s \cdot \mathcal{V} &= \rho_{\mathcal{V}}(\lim_{s \to \mathcal{V}} s), \\ &= \rho_{\mathcal{V}}(\mathcal{U}), \\ &= \mathcal{U} \cdot \mathcal{V}.\end{aligned}$$

Recall 5.1 *We recall the reader if* $f : X \longrightarrow Y$ *is a continuous function and* $\lim_{s\to\mathcal{U}} f(x_s)$ *and* $\lim_{s\to\mathcal{U}} x_s$ *exist, then* $\lim_{s\to\mathcal{U}} f(x_s) = f(\lim_{s\to\mathcal{U}} x_s)$*, by Theorem (5.7).*

Theorem 5.8 *Let* $(S, \cdot)$ *be a semigroup. Then the extended operation on* βS *is associative.*

Proof Let $\mathcal{U}, \mathcal{V}, \mathcal{D} \in \beta S$. We consider $\lim_{a\to\mathcal{U}} \lim_{b\to\mathcal{V}} \lim_{c\to\mathcal{D}} (a \cdot b) \cdot c$, where $a, b, c \in S$. We have following:

$$\begin{aligned}
\lim_{a\to\mathcal{U}} \lim_{b\to\mathcal{V}} \lim_{c\to\mathcal{D}} (a \cdot b) \cdot c &= \lim_{a\to\mathcal{U}} \lim_{b\to\mathcal{V}} (a \cdot b) \cdot \mathcal{D} && \text{(because } \lambda_{a\cdot b} \text{ is continuous)} \\
&= \lim_{a\to\mathcal{U}} (a \cdot \mathcal{V}) \cdot \mathcal{D} && \text{(because } \rho_{\mathcal{D}} \circ \lambda_a \text{ is continuous)} \\
&= (\mathcal{U} \cdot \mathcal{V}) \cdot \mathcal{D} && \text{(because } \rho_{\mathcal{D}} \circ \rho_{\mathcal{V}} \text{ is continuous).}
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\lim_{a\to\mathcal{U}} \lim_{b\to\mathcal{V}} \lim_{c\to\mathcal{D}} a \cdot (b \cdot c) &= \lim_{a\to\mathcal{U}} \lim_{b\to\mathcal{V}} a \cdot (b \cdot \mathcal{D}) && \text{(because } \lambda_a \circ \lambda_b \text{ is continuous)} \\
&= \lim_{a\to\mathcal{U}} a \cdot (\mathcal{V} \cdot \mathcal{D}) && \text{(because } \lambda_a \circ \rho_{\mathcal{D}} \text{ is continuous)} \\
&= \mathcal{U} \cdot (\mathcal{V} \cdot \mathcal{D}) && \text{(because } \rho_{\mathcal{V}\cdot\mathcal{D}} \text{ is continuous).}
\end{aligned}$$

Therefore, $(\mathcal{U} \cdot \mathcal{V}) \cdot \mathcal{D} = \mathcal{U} \cdot (\mathcal{V} \cdot \mathcal{D})$. □

We have the following proposition as a result of Theorems (5.6) and (5.8).

Proposition 5.6 $(\beta S, \cdot)$ *is a compact right topological semigroup.*

Since the points of βS are ultrafilters, we want to know which subsets of S are members of $\mathcal{U} \cdot \mathcal{V}$.

Definition 5.11 *Let* S *be a semigroup with binary operation* $\cdot$ *. Let* $A \subseteq S$ *and* $s \in S$*,*

1) $s^{-1}A = \{t \in S : st \in A\}$,

2) $As^{-1} = \{t \in S : ts \in A\}$.

Definition 5.12 *Let* S *be a semigroup with binary operation* $+$*. Let* $A \subseteq S$ *and* $s \in S$*,*

1) $-s + A = \{t \in S : s + t \in A\}$,

2) $A - s = \{t \in S : t + s \in A\}$.

Theorem 5.9 *Let* $(S, \cdot)$ *be a semigroup, and* $A \subseteq S$

a) For any $s \in S$*,* $\mathcal{U} \in \beta S$*,* $A \in s \cdot \mathcal{U}$ *if and only if* $s^{-1}A \in \mathcal{V}$*,*

b) For any $\mathcal{U}, \mathcal{V} \in \beta S$*,* $A \in \mathcal{U} \cdot \mathcal{V}$ *if and only if* $\{s \in A : s^{-1}A \in \mathcal{V}\} \in \mathcal{U}$*, i.e.,*

$$A \in \mathcal{U} \cdot \mathcal{V} \iff \{s \in A : \{t \in S : s \cdot t \in A\} \in \mathcal{V}\} \in \mathcal{U}.$$

5.0.5. Ramsey Type Theorems via the Ellis-Numakura Theorem

This section gives two different proofs of Ramsey type theorems. We will give a crucial theorem which is called the Ellis-Numakura Theorem. We will use this theorem for proving the Schur's theorem and van der Waerden's theorem for $k = 3$.

Definition 5.13 *Let $(S, \cdot)$ be a semigroup. The element $e \in S$ is called idempotent if $e \cdot e = e$.*

Theorem 5.10 (Ellis-Numakura Lemma) *Suppose that $(S, \cdot)$ is a compact semitopological semigroup. Then S has an idempotent element.*

Proof Let Z denote the set of nonempty closed subsemigroups of S. Clearly, $S \in Z$. Also, the intersection of the descending chain of elements of Z is an element of Z by compactness, and the intersection is closed. Therefore, we can find $T \in Z$ which is minimal by Zorn's lemma.

Fix $s \in T$. We claim that s is an idempotent. Let $T_1 = T \cdot s$. As $T \neq \emptyset$ and compact, we see that $T_1 \neq \emptyset$ and T_1 is compact as well. Observe that

$$\begin{aligned} T_1 \cdot T_1 &= (Ts) \cdot (Ts), \\ &\subseteq (T \cdot T)(Ts), \\ &= (T \cdot T \cdot T) \cdot s, \\ &\subseteq T \cdot s = T_1. \end{aligned}$$

Thus T_1 is a semigroup. As $s \in T$, $T_1 \subseteq T$. By minimality, $T = T_1$. So $T_2 = \{t \in T : t \cdot s = s\}$ is a non-empty as $T_1 = T \cdot s = T$, $s \in T \cdot s$ and $s = t \cdot s$ for some $t \in S$.

Note that T_2 is closed and hence T_2 is compact. We show that T_2 is a subsemigroup of S. Let $t, t' \in T_2$. Then $t \cdot t' \in T$ and $(t \cdot t') \cdot s = t \cdot (t' \cdot s) = t \cdot s = s$. By minimality of T, $T_2 = T$. Thus $s \in T_2$ and $s \cdot s = s$. □

Corollary 5.2 *Let S be a semigroup and let T be any nonempty closed subsemigroup of βS. Then T has an idempotent element.*

We have a semigroup S and an idempotent ultrafilter $\mathcal{U} \in \beta S$. Using the definition of the semigroup operation $*$ on βS, we see that, for every $A \in \mathcal{U}$, it is the case that

$$A \in \mathcal{U} * \mathcal{U} = \{s \in S : \{t \in S : s * t \in A\} \in \mathcal{U}\} \in \mathcal{U}. \tag{5.1}$$

This observation will be important in proofs of Schur's theorem and van der Waerden's theorem for $k = 3$ using ultrafilters.

Now, we consider $S = \mathbb{N}$.

Definition 5.14 *Let $\mathcal{U}, \mathcal{V}$ be ultrafilters in $\beta\mathbb{N}$. The sum and product of $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters:*

- $\mathcal{U} + \mathcal{V} = \{A \subseteq \mathbb{N} : \{n \in \mathbb{N} : \{m \in \mathbb{N} : m + n \in A\} \in \mathcal{V}\} \in \mathcal{U}\}$,
- $\mathcal{U} \cdot \mathcal{V} = \{A \subseteq \mathbb{N} : \{n \in \mathbb{N} : \{m \in \mathbb{N} : m \cdot n \in A\} \in \mathcal{V}\} \in \mathcal{U}\}$.

The operations which are defined in Definition (5.14) are associative. Thus, $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$ are also compact right topological semigroups.

In the following theorem, we give the proof of the infinite version of Schur's theorem which represented in *Chapter 2.* Theorem (2.5) using idempotent ultrafilters.

Theorem 5.11 (Schur's theorem) *If the positive integers are colored using finitely many colors, then there is always a monochromatic solution to $x + y = z$.*

Proof Use the Ellis-Numakura Theorem (5.10) and Corollary (5.2) to obtain a nonprincipal idempotent ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $\mathbb{N} = A_1 \cup \cdots \cup A_r$ be an r-coloring of $\mathbb{N}$ and $\mathcal{U}$ be an idempotent ultrafilter on $\mathbb{N}$. As $\mathbb{N} \in \mathcal{U}$, there is $i \in \{1, \ldots, r\}$ such that $A_i = A \in \mathcal{U}$. As $\mathcal{U} = \mathcal{U} + \mathcal{U}$ and $A \in \mathcal{U} + \mathcal{U}$, one has that

$$A_1 = A \cap \{n \in \mathbb{N} : \{m \in \mathbb{N} : n + m \in A\} \in \mathcal{U}\} \in \mathcal{U}.$$

Take $x \in A_1$. Then $x \in A$ and the set

$$A_x = \{m \in \mathbb{N} : x + m \in A\} \in \mathcal{U}.$$

So, we can also take a $y \in A_x \cap A$. Thus, we will have that $y \in A$, and moreover, since $y \in A_x$, we have that $x + t \in A$. Hence, we have shown that $x, y, x + y \in A$ and all of them are monochromatic. □

Now, we will show that van der Waerden's theorem for $k = 3$ has the different proof using the idempotent ultrafilters. Before the proof, we will give following remark.

Remark 5.4 *Let $\mathcal{U}$ be an ultrafilter in $\beta\mathbb{N}$ and $A \subseteq \mathbb{N}$. We can also characterize the operation in terms of limit.*

- $2 \cdot \mathcal{U} = 2 \lim_{n \to \mathcal{U}} n = \lim_{n \to \mathcal{U}} 2 \cdot n.$

Also, we have

$$\begin{aligned} A \in 2\cdot\mathcal{U} &\iff 2\cdot\mathcal{U} \iff A \in \lim_{n\to\mathcal{U}} 2\cdot n = 2\cdot\mathcal{U}, \\ &\iff \{n\in\mathbb{N} : 2n \in U_A\} \in \mathcal{U}, \\ &\iff \{A\in\mathbb{N} : 2n \in A\} \in \mathcal{U}, \\ &\iff \{n\in\mathbb{N} : n \in A/2\} \in \mathcal{U}. \end{aligned}$$

Thus $A/2 = \{n : 2n \in A\}$*. More generally,*

$$A \in x\cdot\mathbb{N} \iff A/x \in \mathcal{U}.$$

In $\beta\mathbb{N}$, the distributive laws are not satisfied. However, a special case does hold.

Lemma 5.4 *(Bergelson et al., 1990) Let* $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N}$ *and* $x \in \mathbb{N}$*. Then* $x\cdot(\mathcal{U}+\mathcal{V}) = x\cdot\mathcal{U} + x\cdot\mathcal{V}$*.*

Proof As both $x\cdot(\mathcal{U}+\mathcal{V})$ and $x\cdot\mathcal{U}+x\cdot\mathcal{V}$ are ultrafilters, it is enough to show

$$x\cdot(\mathcal{U}+\mathcal{V}) \subseteq x\cdot\mathcal{U}+x\cdot\mathcal{V}.$$

Let $A \in x\cdot(\mathcal{U}+\mathcal{V})$ i.e. $A/x \in \mathcal{U}+\mathcal{V}$. Therefore, $B = \{y\in\mathbb{N} : -y + A/x \in \mathcal{V}\} \in \mathcal{U}$. So $x\cdot B \in x\cdot\mathcal{U}$. We claim that $x\cdot B \subseteq \{y \in \mathbb{N} : -y \in A \in x\cdot\mathcal{V}\}$. Let $y \in x\cdot B$ i.e. $y = xz$ for some $z \in B$. Then $-z + A/x \in \mathcal{V}$ so $x\cdot(-z + A/x) \in x\cdot\mathcal{V}$. Note that $x\cdot A/x \subseteq A$. Let $r \in x\cdot A/x$, so $r = x\cdot s$ where $s \in A/x$. Then $r \in A$. Therefore, $-x\cdot z + A \in x\cdot\mathcal{V}$, i.e. $-y + A \in x\cdot\mathcal{V}$. So we proved the claim and we are done. □

Theorem 5.12 *(Bergelson et al., 1990) Let* $\mathcal{U} \in \beta\mathbb{N}$ *with* $\mathcal{U} = \mathcal{U}+\mathcal{U}$*. Then for each* $A \in 2\mathcal{U}+\mathcal{U}$*, there exist* $a, d \in \mathbb{N}$ *with* $\{a, a+d, a+2d\} \subset A$*.*

Proof We know that

$$2\mathcal{U} = 2\cdot(\mathcal{U}+\mathcal{U}) = 2\mathcal{U}+2\mathcal{U}.$$

by Lemma (5.4). Let $A \in 2\cdot\mathcal{U}+\mathcal{U}$ and $B = \{n\in\mathbb{N} : -2n + A \in \mathcal{U}\}$. We observe that given $\mathcal{U} \in \beta\mathbb{N}$ and $A \subseteq \mathbb{N}$, we have the following observation

$$\begin{aligned} A \in 2\cdot\mathcal{U}+\mathcal{U} &\iff \{n\in\mathbb{N} : -n + A \in \mathcal{U}\} \in 2\mathcal{U}, \\ &\iff \{n\in\mathbb{N} : -n + A \in \mathcal{U}\}/2 \in \mathcal{U}. \end{aligned}$$

We denote this set $\{n\in\mathbb{N} : -n + A \in \mathcal{U}\}$ by B'. If $n \in B'/2$, then $2n \in B'$. This gives that $-2n + A \in \mathcal{U}$. Therefore, $A \in 2\mathcal{U}+\mathcal{U}$ if and only if $\{n\in\mathbb{N} : -2n + A \in \mathcal{U}\} \in \mathcal{U}$. Thus, we have $B \in \mathcal{U}$. Put $C = \{n \in \mathbb{N} : -2n + A \in 2\mathcal{U} + \mathcal{U}\}$. Since $A \in 2\mathcal{U} + \mathcal{U}$, $B \in \mathcal{U}$ and since

$$A \in 2\mathcal{U} + \mathcal{U} = \mathcal{U} + 2\mathcal{U} + 2\mathcal{U},$$

we also get $C \in \mathcal{U}$. So $B \cap C \in \mathcal{U}$ and $B \cap C \neq \emptyset$. Pick $n \in B \cap C$. Set

$$D = \{d \in \mathbb{N} : -2n - d + A \in \mathcal{U}\},$$
$$E = \{d \in \mathbb{N} : -2n - 2d + A \in \mathcal{U}\}.$$

As $-2n + A \in \mathcal{U}$ and $\mathcal{U}$ is an idempotent ultrafilter, also we have $-2n + A \in \mathcal{U} + \mathcal{U}$. We get $D \in \mathcal{U}$. Since $-2n + A \in 2\mathcal{U} + \mathcal{U}$, we obtain $E \in \mathcal{U}$. Pick $d \in D \cap E$. Then $-2n + A \in \mathcal{U}$, $-2n - d + A \in \mathcal{U}$, and $-2n - 2d + A \in \mathcal{U}$. Choose b in the following set

$$(-2n + A) \cap (-2n - d + A) \cap (-2n - 2d + A)$$

and set $a = 2n + b$. Hence, $a, a + d, a + 2d \in A$. This completes the proof. □

Corollary 5.3 (van der Waerden's Theorem for $k = 3$) *Let $r \geq 1$ be given. Then for any r-coloring of $\mathbb{N}$ there is a monochromatic 3-AP: there are $a, d \in \mathbb{N}$ such that $a, a + d, a + 2d$ are colored by the same color.*

Proof Use the Ellis-Numakura Theorem (5.10) and Corollary (5.2) to obtain a nonprincipal idempotent ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $\mathbb{N} = A_1 \cup \cdots \cup A_r$. Put $\mathcal{V} = 2\mathcal{U} + \mathcal{U}$, where $\mathcal{U} + \mathcal{U} = \mathcal{U}$. Then there is $i \in \{1, \ldots, r\}$ such that $A_i \in \mathcal{V}$. By the Theorem (5.12), A_i contains a 3-AP. Therefore, we have a monochromatic 3-AP. □

CHAPTER 6

CONCLUSION

This thesis has successfully explored and proven several foundational theorems in additive combinatorics, including Ramsey's theorems, Schur's theorem, and van der Waerden's theorem. The thesis began by presenting proofs and discussing bounds related to Ramsey's and Schur's theorems, as well as demonstrating the equivalence of two versions of Schur's theorem. Through the concept of the color-focused idea, we provided a comprehensive proof of van der Waerden's theorem, following the approach by Blondal and Jungic (Blondal and Jungic, 2018). The equivalence of van der Waerden's theorem was also established.

In the subsequent chapters, we examined partition regular equations, offering a detailed characterization of these equations, with significant reliance on the book *Ramsey Theory on the Integers* (Landman and Robertson, 2014). The final chapter introduced the tools of nonstandard analysis, where we proved key theorems related to filters and limits on filters. Utilizing ultrafilter methods, we presented proofs of Schur's and van der Waerden's theorem (for $k = 3$), and used tools from the book *Algebra in the Stone-Čech Compactification* (Hindman and Strauss, 1998).

Overall, this thesis not only provides rigorous proofs of these classical theorems but also introduces advanced techniques from nonstandard analysis, demonstrating their application to important results in Ramsey theory.

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