INITIAL-BOUNDARY VALUE PROBLEM FOR THE HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION ON THE HALF-LINE

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Aykut ALKIN

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İZMİR

We approve the thesis of **Aykut ALKIN**

Examining Committee Members:

Assoc. Prof. Dr. Ahmet BATAL
Department of Mathematics, İzmir Institute of Technology

Prof. Dr. Oğuz YILMAZ
Department of Mathematics, İzmir Institute of Technology

Prof. Dr. Başak KARPUZ
Department of Mathematics, Dokuz Eylul University

Doz. Dr. Konstantinos KALIMERIS
Department of Mathematics, Research Center Academy of Athens

Assist. Prof. Dr. Faruk Temur
Department of Mathematics, İzmir Institute of Technology

31 May 2024

Assoc. Prof. Dr. Ahmet BATAL
Supervisor, Department of Mathematics
İzmir Institute of Technology

Assoc. Prof. Dr. Türker ÖZSARI
Co-Supervisor, Department of Mathematics
İhsan Doğramacı Bilkent University

Prof. Dr. Başak AY SAYLAM
Head of the Department of
Mathematics

Prof. Dr. Mehtap EANES
Dean of the Graduate School of
İzmir Institute of Technology

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ABSTRACT

INITIAL-BOUNDARY VALUE PROBLEM FOR THE HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION ON THE HALF-LINE

We establish local well-posedness in the sense of Hadamard for the higher-order nonlinear Schrödinger equation with a general power nonlinearity formulated on the half-line $\{x > 0\}$. We consider separately the two different scenarios of associated coefficients such that only one boundary condition is required, or exactly two boundary conditions are required. We assume a general nonhomogeneous boundary datum of Dirichlet type at $x = 0$ for the former case, and we add the Neumann type for the latter case. Our functional framework centers around fractional Sobolev spaces $H_x^s(\mathbb{R}_+)$ with respect to the spatial variable. We treat both high regularity ($s > \frac{1}{2}$) and low regularity ($s < \frac{1}{2}$) solutions: in the former setting, the relevant nonlinearity can be handled via the Banach algebra property; in the latter setting, however, this is no longer the case and, instead, delicate Strichartz estimates must be established. This task is especially challenging in the framework of nonhomogeneous initial-boundary value problems, as it involves proving boundary-type Strichartz estimates that are not common in the study of Cauchy (initial value) problems.

The linear analysis, which forms the core of this work, crucially relies on a weak solution formulation defined through the novel solution formulae obtained via the Fokas method (also known as the unified transform) for the associated forced linear problem. In this connection, we note that the higher-order Schrödinger equation comes with an increased level of difficulty due to the presence of more than one spatial derivatives in the linear part of the equation. This feature manifests itself via several complications throughout the analysis, including (i) analyticity issues related to complex square roots, which require careful treatment of branch cuts and deformations of integration contours; (ii) singularities that emerge upon changes of variables in the Fourier analysis arguments; (iii) complicated oscillatory kernels in the weak solution formula for the linear initial-boundary value problem, which require a subtle analysis of the dispersion in terms of the regularity of the boundary data.

ÖZET

YÜKSEK MERTEBEDEN DOĞRUSAL OLMAYAN SCHRÖDİNGER DENKLEMİ İÇİN YARI DOĞRUDA BAŞLANGIÇ-SINIR DEĞER PROBLEMİ

Yüksek mertebeden doğrusal olmayan Schrödinger denklemi için yarı doğru $\{x > 0\}$ üzerinde genel kuvvet tipinde doğrusal olmayan terim ile birlikte Hadamard anlamında lokal iyi konulmuşluğu sağlamaktayız. İlgili katsayıların iki farklı senaryosuna göre bir sınır koşulu yeterli olan veya tam olarak iki sınır koşulu gerektiren durumları ayrı ayrı ele alıyoruz. İlk durum için $x = 0$ durumunda homojen olmayan genel Dirichlet tipinde sınır verisini, sonraki için ise ilaveten Neumann tipini kabul etmekteyiz. Fonksiyonel çerçevemiz, uzaysal değişken açısından $H_x^s(R_+)$ kesirli Sobolev uzayları etrafında dönmektedir. Hem yüksek düzenlilikli ($s > 1/2$) hem de düşük düzenlilikli ($s < 1/2$) çözümleri ele alıyoruz: ilk durumda ilgili doğrusal olmayan terim Banach cebiri özelliği aracılığıyla ele alınabilmektedir; ancak ikinci durumda bu durum geçerli değildir ve bunun yerine hassas Strichartz kestirimleri elde edilmelidir. Bu görev, başlangıç-sınır değer problemleri çerçevesinde özellikle zordur, çünkü başlangıç değerli (Cauchy) problemlerinin çalışılmasında yaygın olmayan sınır tipi Strichartz tahminlerini ispatlamayı içermektedir.

Bu çalışmanın temelini oluşturan doğrusal analiz, ilişkili zorlanmış doğrusal problem için Fokas yöntemi (aynı zamanda birleşik dönüşüm olarak da bilinir) aracılığıyla elde edilen yenilikçi çözüm formülleri ile tanımlanan zayıf bir çözüm formülasyonuna kritik bir şekilde dayanmaktadır. Bu bağlamda, yüksek mertebeden Schrödinger denkleminin, denklemin doğrusal kısmında birden fazla uzaysal türev bulunduğu için artan bir zorluk seviyesi ile geldiğini belirtmek gerekir. Bu özellik, analiz boyunca birkaç karmaşıklık olarak kendini göstermektedir, bunlar: (i) Karmaşık kareköklerle ilgili analitiklik sorunları, bu da dal kesimleri ve integral konturlarının deformasyonlarının dikkatli bir şekilde ele alınmasını gerektirir; (ii) Fourier analizi argümanlarındaki değişken değiştirmeler sırasında ortaya çıkan tekillikler; (iii) Doğrusal başlangıç-sınır değer problemi için zayıf çözüm formülündeki karmaşık titreşimli çekirdekler, bu da sınır verilerinin düzenliliği açısından dağılmanın ince bir analizini gerektirir.

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CHAPTER 1

INTRODUCTION

The higher-order nonlinear Schrödinger equation (HNLS) on the half-line $\{x > 0\}$ is modeled by the partial differential equation

$$iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = \kappa |u|^p u, \qquad (x,t) \in \mathbb{R}_+ \times (0,T), \tag{1.1}$$

where $\alpha, \beta, \delta \in \mathbb{R}, \beta \neq 0$, $p > 0$, $T > 0$ and $\kappa \in \mathbb{C}$. The dependent variable $u = u(x,t)$ is a complex-valued function with a domain where the spatial variable x belongs to the right half-line and the temporal variable t belongs to an interval, i.e $u : \mathbb{R}_+ \times (0,T) \to \mathbb{C}$.

The initial condition is imposed as

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}_+. \tag{1.2}$$

Determination of the boundary conditions is ruled by the sign of β, which is the coefficient of the highest order spatial derivative of the main equation (1.1), as follows: If $\beta > 0$, then the (Dirichlet) boundary condition is imposed as

$$u(0,t) = g(t), \qquad t \in (0,T), \tag{1.3}$$

while if $\beta < 0$, then the (Dirichlet and Neumann) boundary conditions are imposed as

$$u(0,t) = h_0(t), \qquad u_x(0,t) = h_1(t), \qquad t \in (0,T). \tag{1.4}$$

The change in the number of the boundary conditions depending on the sign of β occurs naturally in the analysis. Therefore, there are two different initial-boundary value problems for the higher-order Schrödinger equation, one of which is the single-boundary condition case (1.1)-(1.2)-(1.3) that is posed when $\beta > 0$, and the other one is the double-boundary condition case (1.1)-(1.2)-(1.4) that is posed when $\beta < 0$. This difference is first realized in our study Alkın et al. (2024), and also guaranteed by the study Deconinck et al. (2014), which explains the determination of the number of the boundary conditions for the general case of the evolution equations, namely

$$\partial_t u + P(-i\partial_x)u = 0, \tag{1.5}$$

where P denotes any polynomial with order m and with a leading coefficient a. It says that if m is an odd number (note that $m = 3$ in the case of HNLS), then the number of the boundary conditions is given by

$$\begin{cases} \frac{m-1}{2}, & a > 0, \\ \frac{m+1}{2}, & a < 0. \end{cases} \tag{1.6}$$

The case that m is even is also studied therein, but this is out of our context for the higher-order Schrödinger equation.

To improve the idea behind the number of the boundary conditions, we first take the Korteweg-de Vries (KdV) equation, which satisfies (1.5) with $m = 3$ and $a = 1$, into consideration as an example of such argument. The initial-boundary value problem for KdV equation is posed with only one boundary condition Himonas and Yan (2022). As another example, the heat equation is much more convincing about the importance of the sign of the leading coefficient for the spatial derivatives of an evolution equation. It is known that the heat equation, i.e $u_t - u_{xx} = 0$, is well-posed, while its reversed version, i.e $u_t + u_{xx} = 0$, is ill-posed. Many other examples of such partial differential equations can be observed in the sense of the relation between the number of the boundary conditions and the polynomial-behavior of the spatial derivatives.

We take the Dirichlet boundary condition for the case of $\beta > 0$, and the couple of the Dirichlet and the Neumnann boundary conditions for the case $\beta < 0$. In fact, this is not a strict choice for the initial-boundary value problem for the higher-order nonlinear Schrödinger equation. We analyze the aforementioned cases in Chapters 4 and 5, respectively, and the arguments therein can be analogously modified for some different type of boundary conditions.

The effect of the sign of β is observed only on the boundary conditions. When considering the corresponding whole-line problem, as we proceed in Chapter 3, there is no need to separate the cases for β to be positive or negative. Therefore, the initial value problem for the higher-order nonlinear Schrödinger equation can be studied at once for any $\beta \neq 0$. This matches up with the previous studies Carvajal and Linares (2003); Carvajal (2004, 2006); Laurey (1997); Staffilani (1997), where the whole-line problems for the higher-order nonlinear Schrödinger equation are studied under no restriction on β.

After the explanation of the mathematical model for the higher-order nonlinear Schrödinger equation, we give the physical motivation behind this model. For the choices $\beta = \delta = 0$, $\alpha = 1$ and $p = 2$ on the main equation (1.1), the problem reduces to the cubic nonlinear Schrödinger equation (NLS), which is a ubiquitous model in mathematical physics with a broad spectrum of applications. However, NLS is not precise enough for pulses in the femtosecond regime. In this case, a higher-order dispersive term is necessary for a correction. This need let the higher-order nonlinear Schrödinger equation arise originally in the form

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u + i\epsilon\Big(\beta_1 u_{xxx} + \beta_2(|u|^2 u)_x + \beta_3 u|u|^2_x\Big) = 0, \tag{1.7}$$

for modeling the femtosecond pulse propagation in nonlinear fiber optics Kodama (1985); Kodama and Hasegawa (1987). Note that the nonlinear terms of the original model (1.7) and the one we describe in (1.1) has two differences. Firstly, we take the nonlinear terms involving derivatives as an absence (i.e $\beta_2 = \beta_3 = 0$), which causes to read the higher-order nonlinear Schrödinger equation also as *"truncated"* HNLS in the literature. Secondly, we consider a more general power type nonlinear term $|u|^p u$ with $p > 0$, and treat the cubic problem as a special case.

The higher-order nonlinear Schrödinger equation is studied in the well-posedness point of view here. So, after the physical motivation, it would be appropriate to give also the previous results about the well-posedness of the problem. We use a chronological order for this purpose. In Laurey (1997), the local well-posedness of the initial value problem (1.7)-(1.2) is obtained in $H^s(\mathbb{R})$ for $s > \frac{3}{4}$. Then, an improvement for the lower bound for s has been published in Staffilani (1997) for $s \geq \frac{1}{4}$. Later, it is proved in Carvajal (2004) that the initial value problem (1.7)-(1.2) is locally well-posed in $H^s(\mathbb{R})$ for $s > -\frac{1}{4}$, and this result is supported with a global well-posedness for $s > \frac{1}{4}$ in Carvajal (2006). All these results belong to a *whole-line* problem for the higher-order nonlinear Schrödinger equation with a *cubic* nonlinear term.

The treatment for a more general power type nonlinearity is considered recently in Faminskii (2023) for the initial value problem on the whole-line, namely

$$iu_t + au_{xx} + ibu_x + iu_{xxx} + \lambda|u|^p u + i\beta(|u|^p u)_x + i\gamma u|u|^p_x = 0, \tag{1.8}$$

with the initial condition (1.2), and the well-posedness is studied for the case of $p = 1$, therein.

When it comes to introduce the studies on the well-posedness of the initial-*boundary* value problem for HNLS, we remark that a certain third-order model with *cubic* nonlinearity, which is renamed as the *Hirota equation*, is studied for only the single-boundary condition case, see Huang (2020); Guo and Wu (2021); Wu and Guo (2023). However, it is important to emphasize that in the present work we treat the case of a general power nonlinearity, and obtain some regularity results for a larger class of Sobolev space. Recently, for the initial-boundary value problem for the higher-order nonlinear Schrödinger equation with a single boundary condition and a power nonlinearity, the global solutions are considered in an independent area in Faminskii (2024) for the case of either $p = 1$ or the boundary condition to be homogeneous.

After this quick survey in the literature, we introduce now our destination together with some details about the methods and the techniques that we utilize in the way. Even though the higher-order nonlinear Schrödinger equation is considered as an initial-boundary value problem on the half-line, we first obtain some regularity results for the whole-line problems (also known as Cauchy problems) in Chapter 3. This chapter might be considered as an ingredient for the main purpose. However, there are some results important in itself. The multi-term nature of the spatial differential operator creates challenging difficulties in the proofs of the temporal estimates, due to the changes of variables performed in order to extract the desired Sobolev norms. These difficulties are overcome by introducing a proper cut-off function that depends on the polynomial structure of the spatial differential operator.

In Chapter 4, we focus on the initial-boundary value problem for the higher-order nonlinear Schrödinger equation on the half-line with a single boundary condition (the case of $\beta > 0$), namely

$$\begin{aligned} &iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = f(u), \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \\ &u(0,t) = g(t), \quad t \in (0,T), \end{aligned} \tag{1.9}$$

where $\alpha, \delta \in \mathbb{R}$, $\beta > 0$, $f(z) = \kappa|z|^p z$ with $z \in \mathbb{C}$, $\kappa \in \mathbb{C}$, $p > 0$, and $T > 0$. We prove the local well-posedness of this problem in the sense of Hadamard, namely, we prove existence of a unique local-in-time solution that depends continuously on the initial and boundary data in the Sobolev space $H^s(\mathbb{R})$ for $s \geq 0$. All of the results in this chapter are presented in Alkın et al. (2024).

Then, we turn our interest to the double-boundary case ($\beta < 0$) in Chapter 5 and analyze the initial-boundary value problem

$$\begin{aligned} &iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = f(u), \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \\ &u(0,t) = h_0(t), \quad u_x(0,t) = h_1(t) \quad t \in (0,T), \end{aligned} \tag{1.10}$$

where $\alpha, \delta \in \mathbb{R}$, $\beta < 0$, $f(z) = \kappa|z|^p z$ with $z \in \mathbb{C}$, $\kappa \in \mathbb{C}$, $p > 0$, and $T > 0$.

The goal of Chapters 4 and 5 is to establish the local well-posedness theory for the nonlinear initial-boundary value problems (1.9) and (1.10), respectively, at the level of $H_x^s(\mathbb{R}_+)$ spatial regularity for the initial data. We are interested in both the high regularity ($s > \frac{1}{2}$) and the low regularity ($s < \frac{1}{2}$) settings. The main distinction between the two is that, in the low regularity setting, the well-known Banach algebra property of $H_x^s(\mathbb{R}_+)$ is no longer available. Instead, handling the nonlinearity $|u|^p u$ when $s < \frac{1}{2}$ requires use of more advanced tools that revolve around the celebrated Strichartz estimates. Estimates of this type measure the size and temporal decay of solutions in space-time Lebesgue norms and have played a crucial role in the treatment of the Cauchy problem of nonlinear dispersive equations since their introduction in 1977 Strichartz (1977). On the other hand, the use of Strichartz estimates in the analysis of initial-boundary value problems is a more recent advancement. For the Cauchy problem, Strichartz estimates involve certain norms of initial and/or interior data, while for initial-boundary value problems these estimates additionally depend on information related to boundary data, for which temporal regularity also plays a key role.

Our treatment of the nonlinear problem is crucially based on a contraction mapping argument applied to a weak solution formula for the associated forced linear initial-boundary value problem. Therefore, the first contribution of the present paper is the development of a sharp linear theory through the analysis of the solutions of the relevant forced linear initial-boundary value problem. This is accomplished by decomposing this linear problem into three simpler component problems: (i) a homogeneous Cauchy problem associated with (an appropriate extension of) the initial datum; (ii) a nonhomogeneous Cauchy problem associated with (an appropriate extension of) the forcing; (iii) a reduced initial-boundary value problem involving the original boundary datum and the spatial traces of the two aforementioned Cauchy problems.

A major emphasis in this work is placed on the regularity analysis of the solution to the reduced initial-boundary value problem of item (iii) above. This is done in Sections 4.1 and 5.1, respectively for each cases of β. Weak solutions of this reduced initial-boundary value problem are defined via a boundary integral operator whose explicit form is obtained through the Fokas method (also known as the unified transform method Fokas (1997, 2008)), which provides a rigorous treatment for the initial-boundary value problems.

While the Cauchy problem for nonlinear dispersive equations has been broadly explored through a variety of techniques, progress towards the rigorous study of initial-boundary value problems for these equations is more limited. In fact, problems of this latter kind can present significant challenges even at the linear level. For example, while on the whole line linear evolution equations can be easily solved via Fourier transform in the space variable, on domains with a boundary like the half-line no classical spatial transform exists for linear equations of spatial order three or higher. Another important obstacle arises in the case of boundary conditions that are non-separable. Moreover, even when a linear initial-boundary value problem can be solved via classical techniques, the resulting solution formula is not always useful, especially in regard to setting up an effective iteration scheme for proving the well-posedness of associated nonlinear problems.

At the linear level, the Fokas method bridges the gap between the Cauchy problem and initial-boundary value problems by providing the direct analogue of the Fourier transform in domains with a boundary. Indeed, the method provides a fundamentally novel, algorithmic way of solving any linear evolution equation formulated on a variety of domains in one or higher dimensions and supplemented with any kind of admissible boundary conditions. An alternative perspective that further establishes the Fokas method as the natural counterpart of the Fourier transform in the context of linear initial-boundary value problems stems from the nonlinear component of the method, which was developed for completely integrable nonlinear equations and corresponds to the analogue of the inverse scattering transform in domains with a boundary. Then, noting that the linear limit of the inverse scattering transform is nothing but the Fourier transform, it is only reasonable that the linear limit of the nonlinear component of the Fokas method, namely the linear Fokas method, should provide the equivalent of the Fourier transform for linear initial-boundary value problems.

The analogy between the Fokas method and the Fourier transform has been solidified by a new approach introduced in recent years by Himonas and Mantzavinos for the well-posedness of nonlinear initial-boundary value problems. This approach is based on treating the nonlinear problem as a perturbation of its forced linear counterpart, which is of course a classical idea coming from the Cauchy problem. However, the linear formulae produced via the Fourier transform in the case of the Cauchy problem are now replaced by the Fokas method solution formulae (recall that Fourier transform is no longer available). As these novel formulae involve complex contours of integration, new tools and techniques are required in order to obtain the various linear estimates needed for the contraction mapping argument. It should be noted that several of these estimates are specific to initial-boundary value problems and do not typically arise in the study of the Cauchy problem; they are results of particular importance, as they capture the effect of the boundary conditions on the regularity of the solution of both linear and nonlinear problems. The Fokas method based approach for the rigorous study of initial-boundary value problems has already been implemented in several works in the literature: NLS on the half-line and the half-plane Fokas et al. (2017); Himonas and Mantzavinos (2021, 2020), KdV on the half-line and the finite interval Himonas et al. (2019); Himonas and Yan (2022), biharmonic NLS on the half-line Özsarı and Yolcu (2019), fourth order Schrödinger equation on the half-line Özsarı et al. (2022).

Specifically for the higher-order Schrödinger equation, some certain analyticity issues arise in the application of the Fokas method. This is because the method relies on the construction of analytic maps that respect certain spectral invariance properties of the linear dispersion relation. However, for multi-term spatial differential operators, such a construction requires use of complex square root functions which, in many cases, cause the invariance maps to be non-analytic on some parts of the complex spectral plane. We handle this complication via suitable contour deformations around the branch cuts associated with these maps.

The solutions of the fully nonlinear problems will be constructed as fixed points of the solution operator formed by reunifying the respective solution formulae for the three linear problems of items (i)-(iii) above. In the high regularity setting of $s > \frac{1}{2}$, the spatiotemporal estimates established in the linear theory lead to a contraction mapping argument in the Hadamard-type space $C([0, T]; H_x^s(\mathbb{R}_+))$. The uniqueness in this space utilizes the Sobolev embedding $H_x^s(\mathbb{R}_+) \hookrightarrow L_x^\infty(\mathbb{R}_+)$ (which is valid for $s > \frac{1}{2}$). In the low regularity setting of $s < \frac{1}{2}$, the algebra property in $H_x^s(\mathbb{R}_+)$ and the embedding $H_x^s(\mathbb{R}_+) \hookrightarrow L_x^\infty(\mathbb{R}_+)$ are no longer valid and Strichartz estimates assume the key role instead. In that case, the solution space is refined to $C([0, T]; H_x^s(\mathbb{R}_+)) \cap L_t^\mu((0, T); H_x^{s,r}(\mathbb{R}_+))$ with (μ, r) obeying the admissibility criterion (3.29) associated with the underlying evolution operator. However, this only leads to a conditional uniqueness result in the aforementioned space.

CHAPTER 2

PRELIMINARIES

The study on the analysis of partial differential equations requires a remarkable background about the fundamental tools from the functional, real and complex analysis in addition to the theory pf partial differential equations itself. This chapter provides a sufficient and brief collection of such arguments to be used where necessary in the whole of this thesis.

We start with some different notions of the Fourier transform of a multi-variable function $f(x,t)$ for the spatial variable x and the temporal variable t both belonging to $\mathbb{R}$. All the arguments below about Fourier transforms can be found in many fundamental books and sources such as Strichartz (2003). We take the spatial Fourier transform of $f(x,t)$ as

$$\hat{f}(k,t) = \int_{\mathbb{R}} e^{-ikx} f(x,t)dx, \tag{2.1}$$

and the temporal Fourier transform as

$$\hat{f}(x,\tau) = \int_{\mathbb{R}} e^{-i\tau t} f(x,t)dt, \tag{2.2}$$

by assigning the burden of understanding which transformation (spatial or temporal) is made to the letters k and τ used as spectral variables. The inversions of the aforementioned Fourier transforms are understood, respectively, as

$$f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixk} \hat{f}(k,t)dk, \tag{2.3}$$

and

$$f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\tau} \hat{f}(x,\tau)d\tau. \tag{2.4}$$

Considering the Fourier transform for a single-variable function $g(x)$, which can be defined by ignoring the temporal variable t in (2.1), the well-known Plancherel theorem says that

$$\int_{\mathbb{R}} |g(x)|^2 dx = \int_{\mathbb{R}} |\hat{g}(k)|^2 dk. \tag{2.5}$$

We should also recall the rule for the Fourier transform of the derivative of a function, since we use this transform on a partial differential equation. For this purpose, let P be any polynomial and $h(x) = P\left(\frac{d}{dx}\right) f(x)$. Then,

$$\hat{h}(k) = P(-ik)\hat{f}(k). \tag{2.6}$$

Fourier transform provides an important definition for the fractional Sobolev spaces, which are one of the most suitable settings to analyze a partial differential equation, since it is not usually possible to make good enough analytic estimates on the solutions constructed for the partial differential equation. They provide a balance when comprising the functions which have some, but not too great, smoothness properties. Following definitions on the fractional Sobolev spaces can be found in Di Nezza et al. (2012) and many other partial differential equations sources.

The fractional Sobolev space on the real line for $s \geq 0$ is defined by

$$H^s(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) \quad \middle| \quad \left(1 + k^2\right)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}) \right\} \tag{2.7}$$

and, in general for any $s \in \mathbb{R}$, it is defined by

$$H^s(\mathbb{R}) = \left\{ u \in \mathcal{S}'(\mathbb{R}) \quad \middle| \quad \left(1 + k^2\right)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}) \right\}, \tag{2.8}$$

where $\mathcal{S}'(\mathbb{R})$ is the set of tempered distribution, which is known as the dual space of $\mathcal{S}(\mathbb{R})$, which consists of the infinitely differentiable functions that decrease rapidly together with all their derivatives.

The fractional Sobolev norm is defined for all $s \in \mathbb{R}$ by

$$||u||_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(1 + k^2\right)^s |\hat{u}(k)|^2 dk. \right)^{\frac{1}{2}} \tag{2.9}$$

The space $H^s(\Omega)$ for an open interval Ω in $\mathbb{R}$ is considered as the set of the restrictions to Ω of the elements belonging to the space $H^s(\mathbb{R})$ with the norm

$$||u||_{H^s(\Omega)} = \inf_{v \in H^s(\mathbb{R})} ||v||_{H^s(\mathbb{R})}, \qquad \text{where} \quad v\Big|_{\Omega} = u. \tag{2.10}$$

The linear extension operator $E : H^s(\Omega) \to H^s(\mathbb{R}^n)$ for $\Omega \subset \mathbb{R}^n$ is defined to satisfy the following statements:

(i) $Eu = u$ in Ω,

(ii) if Ω is bounded, then Eu is compactly supported,

(iii) E is continuous, i.e.

$$||Eu||_{H^s(\mathbb{R}^n)} \leq C||u||_{H^s(\Omega)}, \tag{2.11}$$

where C is the constant depending on n, s and Ω.

Trace theory plays also a crucial role in the analysis for the local well-posedness of partial differential equations, therefore we recall the following theorem:

Theorem 2.1 *Tartar (2007) Let $s > \frac{1}{2}$. Then, any function $u \in H^s(\mathbb{R}^n)$ has a trace, say Tu, on the hyperplane $\{x_n = 0\}$ such that $Tu \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. Also, the trace operator T is surjective from $H^s(\mathbb{R}^n)$ onto $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.*

The critical value $s = \frac{1}{2}$ for the trace theory appears also in the Banach algebra property of Sobolev spaces. The Sobolev space $H^s(\mathbb{R})$ is a Banach algebra if and only if $s > \frac{1}{2}$, in other words, for $u, v \in H^s(\mathbb{R})$, the identity $||uv||_{H^s(\mathbb{R})} \leq ||u||_{H^s(\mathbb{R})}||v||_{H^s(\mathbb{R})}$ holds only when $s > \frac{1}{2}$.

Sobolev embedding theory is also important in the regularity analysis of the solutions to a partial differential equation. Here are the statements for some embedding rules.

Theorem 2.2 *(Gagliardo-Nirenberg-Sobolev inequality) Assume $1 \leq p < n$. There exists a constant C, depending only on p and n, such that*

$$||u||_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leq C||Du||_{L^p(\mathbb{R}^n)}, \tag{2.12}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Theorem 2.3 *(General Sobolev inequalities in* $\mathbb{R}$*) Let* Ω *be a bounded open subset of* $\mathbb{R}$*, with a* C^1 *boundary. Assume* $u \in H^s(\Omega)$.

(i) If $s < \frac{1}{2}$*, then* $u \in L^q(\Omega)$*, where* $\frac{1}{q} = \frac{1}{2} - s$*. We have in addition the estimate*

$$||u||_{L^q(\Omega)} \leq C||u||_{H^s(\Omega)}, \tag{2.13}$$

where the constant C depends only on s and Ω.

(ii) If $s > \frac{1}{2}$*, then* $u \in L^\infty(\mathbb{R})$ *with the bound*

$$||u||_{L^\infty(\mathbb{R})} \leq C||u||_{H^s(\mathbb{R})}. \tag{2.14}$$

where the constant C depends only on s.

After this important overview on Sobolev and Fourier theory, we continue with some complex analytical tools. We utilize Fokas' unified transform method in some different parts of this study, and this method requires two basic results from complex analysis, which are Cauchy's theorem and Jordan's lemma. These two important requirements can be found in any fundamental complex analysis book such as Brown and Churchill (2009).

Theorem 2.4 *(Cauchy's) Let* U *be an open simply connected domain in* $\mathbb{C}$ *and* γ *be a closed curve in* U*. If a function* $f : U \to \mathbb{C}$ *is analytic, then*

$$\int_\gamma f(z)dz = 0. \tag{2.15}$$

Lemma 2.1 *(Jordan's) Let V be open set in* $\mathbb{C}$*.* $\forall R > 0$*,* $V \cap B(0, R)$ *is the union of finitely many simply connected regions. Let* $C_R^\pm := \overline{C_R \cap V \cap \mathbb{C}^\pm}$ *and if* $f : \overline{V} \to \mathbb{C}$ *is continuous with*

$$\lim_{R\to\infty} \Big(max\{f(z) : \ z \in C_R^\pm\} \Big) = 0. \tag{2.16}$$

Then, $\forall a > 0$,

$$\lim_{R\to\infty} \int_{C_R^{\pm}} e^{\pm iaz} f(z)dz = 0. \tag{2.17}$$

We now state some well-known theorems that are widely recognized from real and functional analysis. These can be found in Jones (2001) and/or Kreyszig (1991).

Theorem 2.5 *(Dominated Convergence Theorem) Assume the functions* $\{f_j\}_{j=1}^{\infty}$ *are summable and* $f_j \to f$ *a.e. Suppose also* $|f_j| \leq g$ *a.e. for some summable function* g*. Then,*

$$\int_{\mathbb{R}^n} f_j\, dx \to \int_{\mathbb{R}^n} f\, dx. \tag{2.18}$$

Theorem 2.6 *(Inverse Function Theorem) Assume* $f \in C^1(\Omega;\mathbb{R})$ *and* $f'(x_0) \neq 0$*. Then, there exists an open set* $V \subset \Omega$*, with* $x_0 \in V$*, and an open set* $U \subset \mathbb{R}$*, with* $f(x_0) \in U$*, such that*

(i) the mapping $f : V \to U$ *is one-to-one and onto, and*

(ii) the inverse function $f^{-1} : U \to V$ *is* C^1.

(iii) if $f \in C^j$*, then* $f^{-1} \in C^j$ *for* $j \in \mathbb{Z}$.

Theorem 2.7 *(Fubini's) Assume that* $f \in L^1(\mathbb{R}^n)$*. Let* $n = m + l$*, then for a.e.* $y \in \mathbb{R}^m$ *the function* $f_y \in L^1(\mathbb{R}^l)$*, and thus there exists*

$$F(y) = \int_{\mathbb{R}^l} f_y(x)\, dx. \tag{2.19}$$

Furthermore, $F \in L^1(\mathbb{R}^m)$*, and*

$$\int_{\mathbb{R}^m} F(y)\, dy = \int_{\mathbb{R}^n} f(z)\, dz. \tag{2.20}$$

Theorem 2.8 *(Riesz-Thorin interpolation) Let* (U_1,μ_1) *and* (U_2,μ_2) *be two measurable spaces. Assume that* $p_1 \neq p_2$*,* $q_1 \neq q_2$*, and that*

$$T\ :\ L^{p_1}(U_1) \to L^{q_1}(U_2) \tag{2.21}$$

is bounded with norm M_1, and that

$$T \;:\; L^{p_2}(U_1) \to L^{q_1}(U_2) \tag{2.22}$$

is also bounded with norm M_2. Then,

$$T \;:\; L^{p}(U_1) \to L^{q}(U_2) \tag{2.23}$$

is bounded with norm $M \leq M_1^{1-\theta} M_2^{\theta}$ provided that $0 < \theta < 1$ *and* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

It is vital to use some elementary inequalities, which can be found in Evans (2022), in the estimations of the solutions to partial differential equations. Here we list these inequalities:

(i) **Cauchy's inequality with ϵ:** Let $a, b \in \mathbb{R}$ and $\epsilon > 0$.

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \tag{2.24}$$

(ii) **Young's inequality with ϵ:** Let $a, b, \epsilon > 0$ and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$ab \leq \epsilon a^p + \frac{b^q}{C(\epsilon)}, \tag{2.25}$$

where $C(\epsilon) = q(\epsilon p)^{\frac{q}{p}}$.

(iii) **Cauchy-Schwarz inequality:** If $u, v \in L^2(\Omega)$, then

$$||uv||_{L^1(\Omega)} \leq ||u||_{L^2(\Omega)} ||v||_{L^2(\Omega)}. \tag{2.26}$$

(iv) **Hölder's inequality:** Assume $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, if $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, we have

$$||uv||_{L^1(\Omega)} \leq ||u||_{L^p(\Omega)} ||v||_{L^q(\Omega)}. \tag{2.27}$$

(v) **Minkowski's integral inequality:** Let F be a integrable function on $\mathbb{R}^m \times \mathbb{R}^n$ and $1 \leq p < \infty$. Then,

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m} F(x,y)\,dx\right)^p dy\right)^{\frac{1}{p}} \leq \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^n} |F(x,y)|^p\, dy\right)^{\frac{1}{p}} dx. \tag{2.28}$$

(vi) **Interpolation inequality for L^p-norms:** Assume $1 \leq s \leq r \leq t \leq \infty$ and $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{s}$. Suppose also $u \in L^s(\Omega) \cap L^t(\Omega)$. Then $u \in L^r(\Omega)$, and

$$||u||_{L^r(\Omega)} \leq ||u||^{\theta}_{L^s(\Omega)} ||u||^{1-\theta}_{L^t(\Omega)}. \tag{2.29}$$

We lastly introduce some notations that we use for simplicity throughout the thesis. The first one is $a \lesssim b$, which is used for the inequality $a \leq Cb$, where the constant C is independent of a and b. We also use the notation $a \simeq b$ to indicate that $a = b$ or a is sufficiently close to b. We finish the basic requirements for this thesis with the following notations:

Definition 2.1 *(Big-oh notation) We write*

$$f = O(g) \qquad as \qquad x \to x_0, \tag{2.30}$$

provided there exists a constant C such that $|f(x)| \leq C|g(x)|$ for all x sufficiently close to x_0.

Definition 2.2 *The set-theoretic support of a function $f : X \to \mathbb{R}$, written supp(f), is the set of points in X where f is nonzero:*

$$supp(f) = \{x \in X \quad : \quad f(x) \neq 0\}. \tag{2.31}$$

CHAPTER 3

CAUCHY PROBLEMS

In this chapter, the homogeneous and the nonhomogeneous Cauchy problems for the higher-order Schrödinger equation have been separately analyzed to determine the regularity level in Sobolev spaces. The Fourier transform theory provides quite suitable advantages for this purpose. Firstly, we study the homogeneous problem in Section 3.1, and then we continue with the nonhomogeneous problem in Section 3.2 by taking the advantage of Duhamel's principle, which intuitively acts the nonhomogeneous problem as a set of homogeneous problems each starting afresh at a different time slice.

3.1. Homogeneous Cauchy Problem

We start with the initial value problem

$$\begin{aligned} &iy_t + i\beta y_{xxx} + \alpha y_{xx} + i\delta y_x = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}, \\ &y(x,0) = y_0(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where $\alpha, \beta, \delta \in \mathbb{R}, \beta \neq 0$, and $y_0 \in H^s(\mathbb{R})$.

We have the first result to control the $L_t^\infty((0,T); H_x^s(\mathbb{R}))$ norm of the solution $y(x,t)$ to the homogeneous linear Cauchy problem (3.1) by the $H^s(\mathbb{R})$ norm of the initial datum $y_0(x)$ as follows:

Theorem 3.1 *Let $s \in \mathbb{R}$. The unique solution of the Cauchy problem* (3.1)*, denoted by $y = S[y_0; 0]$, belongs to $C(\mathbb{R}_t; H_x^s(\mathbb{R}))$ and satisfies the conservation law*

$$\|y(\cdot,t)\|_{H_x^s(\mathbb{R})} = \|y_0\|_{H_x^s(\mathbb{R})}, \quad t \in \mathbb{R}. \tag{3.2}$$

Moreover, if $\alpha^2 + 3\beta\delta \geq 0$, then $y \in C(\mathbb{R}_x; H_t^{\frac{s+1}{3}}(-T,T))$ for $T > 0$ and there exists a *constant* $c = c(s, \alpha, \beta, \delta) \geq 0$ *such that*

$$\sup_{x\in\mathbb{R}} \|y(x,\cdot)\|_{H_t^{\frac{s+1}{3}}(-T,T)} \leq c(1+T^{\frac{1}{2}})\,\|y_0\|_{H_x^s(\mathbb{R})}\,, \tag{3.3}$$

while if $\alpha^2 + 3\beta\delta < 0$, *then* $y \in C(\mathbb{R}_x; H_t^{\frac{s+1}{3}}(\mathbb{R}))$ *and there is a constant* $c = c(s, \alpha, \beta, \delta) \geq 0$ *such that*

$$\sup_{x\in\mathbb{R}} \|y(x,\cdot)\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \leq c\,\|y_0\|_{H_x^s(\mathbb{R})}\,. \tag{3.4}$$

Proof Applying the Fourier transform (with respect to the spatial variable x) to (3.1), and then integrating in the temporal variable t, we find $\hat{y}(k,t) = e^{-\omega(k)t}\,\hat{y}_0(k)$, where

$$\omega(k) := -i\beta k^3 + i\alpha k^2 + i\delta k, \tag{3.5}$$

which is purely imaginary for $k \in \mathbb{R}$. Thus $|\hat{y}(k,t)| = |\hat{y}_0(k)|$, and we have

$$\|y(\cdot,t)\|^2_{H_x^s(\mathbb{R})} = \int_{\mathbb{R}} (1+k^2)^s |\hat{y}(k,t)|^2 dk = \int_{\mathbb{R}} (1+k^2)^s |\hat{y}_0(k)|^2 dk = \|y_0\|^2_{H_x^s(\mathbb{R})}, \tag{3.6}$$

which amounts to the conservation law (3.2). The continuity of the map $t \mapsto y(t)$ from $\mathbb{R}_t$ into $H_x^s(\mathbb{R})$ follows from the dominated convergence theorem and the fact that $y_0 \in H_x^s(\mathbb{R})$. To this end, let $t, t_n \in \mathbb{R}$ such that $t_n \to t$ as $n \to \infty$, we have

$$\|y(\cdot,t_n) - y(\cdot,t)\|^2_{H_x^s(\mathbb{R})} = \int_{\mathbb{R}} \left(1+k^2\right)^s \left|e^{-w(k)t_n} - e^{-w(k)t}\right|^2 |\hat{y}_0(k)|^2\, dk, \tag{3.7}$$

which tends to zero as $n \to \infty$, and we also have

$$\int_{\mathbb{R}} \left(1+k^2\right)^s \left|e^{-w(k)t_n} - e^{-w(k)t}\right|^2 |\hat{y}_0(k)|^2 dk \leq \int_{\mathbb{R}} 4\left(1+k^2\right)^s |\hat{y}_0(k)|^2 dk = 4\|y_0\|^2_{H_x^s(\mathbb{R})} < \infty. \tag{3.8}$$

In order to prove the temporal estimates (3.3) and (3.4), we start from the Fourier transform solution representation

$$y(x,t) = S[y_0;0](x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-\omega(k)t}\,\hat{y}_0(k)dk. \tag{3.9}$$

Consider the real-valued map $\tau = i\omega(k) = \beta k^3 - \alpha k^2 - \delta k$, together with its derivative

$$\frac{d\tau}{dk} = i\omega'(k) = 3\beta k^2 - 2\alpha k - \delta = 3\beta\left(\left(k - \frac{\alpha}{3\beta}\right)^2 - \frac{\alpha^2 + 3\beta\delta}{9\beta^2}\right). \tag{3.10}$$

Notice that if $\alpha^2 + 3\beta\delta \leq 0$, then τ is monotone and so $k = (i\omega)^{-1}(\tau)$ is well-defined. In the case of strict inequality $\alpha^2 + 3\beta\delta < 0$, we observe that $i\omega'(k) \neq 0$ for any real k, and so by the inverse function theorem we can change variable from k to τ and rewrite (3.9) as

$$y(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(i\omega)^{-1}(\tau)x + i\tau t}\,\hat{y}_0((i\omega)^{-1}(\tau))\,\frac{d\tau}{i\omega'((i\omega)^{-1}(\tau))}, \tag{3.11}$$

which represents the inverse (temporal) Fourier transform of the function

$$\hat{y}(x,\tau) = \frac{e^{i(i\omega)^{-1}(\tau)x}\,\widehat{y_0}((i\omega)^{-1}(\tau))}{i\omega'((i\omega)^{-1}(\tau))}. \tag{3.12}$$

So, we have

$$\|y(x,\cdot)\|^2_{H_t^{\frac{s+1}{3}}(\mathbb{R})} = \int_{\mathbb{R}} (1+\tau^2)^{\frac{s+1}{3}} \left|\frac{\hat{y}_0((i\omega)^{-1}(\tau))}{i\omega'((i\omega)^{-1}(\tau))}\right|^2 d\tau \tag{3.13}$$

In addition, notice that $\tau = i\omega(k) = O(k^3)$ and $\frac{1}{i\omega'(k)} = O(k^{-2})$ as $|k| \to \infty$. Therefore, by the change of variable $k = (iw)^{-1}(\tau)$, we have

$$\|y(x,\cdot)\|^2_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \lesssim \int_{\mathbb{R}} \left(1+k^2\right)^s |\hat{y}_0(k)|^2 dk = \|y_0\|^2_{H_x^s(\mathbb{R})} \tag{3.14}$$

which amounts to estimate (3.4).

Next, consider the case $\alpha^2 + 3\beta\delta \geq 0$. Let $\theta \in C_c^\infty(\mathbb{R})$ be a function with a range $[0,1]$ whose additional properties will be specified below. Then, we can write $y = y_1 + y_2$, where

$$y_1(x,t) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-\omega(k)t}\theta(k)\hat{y}_0(k)dk,$$
$$y_2(x,t) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-\omega(k)t}(1-\theta(k))\hat{y}_0(k)dk. \tag{3.15}$$

Taking j-th order time derivative of y_1, and using the range of θ together with Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}
\left|\partial_t^j y_1(x,t)\right| &\leq \frac{1}{2\pi}\int_{\mathrm{supp}(\theta)} |\omega(k)|^j|\theta(k)||\hat{y}_0(k)|dk \\
&= \frac{1}{2\pi}\int_{\mathrm{supp}(\theta)} \left(1+k^2\right)^{\frac{-s}{2}} |\omega(k)|^j|\theta(k)| \left(1+k^2\right)^{\frac{s}{2}} |\hat{y}_0(k)|dk \\
&\lesssim \left(\int_{\mathrm{supp}(\theta)} \left(1+k^2\right)^{-s} |\omega(k)|^{2j}dk\right)^{\frac{1}{2}} \|y_0\|_{H_x^s(\mathbb{R})} = c(s,j,\theta)\|y_0\|_{H_x^s(\mathbb{R})}.
\end{aligned} \tag{3.16}$$

We note that this inequality holds for any $s \in \mathbb{R}$. Thus, by the physical space characterization of the Sobolev norm, namely

$$\|f\|_{H_t^{\mu}(-T,T)} = \sum_{j=0}^{\mu} \left\|\partial_t^j f\right\|_{L_t^2(-T,T)}, \quad \mu \in \mathbb{N}_0, \tag{3.17}$$

we obtain

$$\|y_1(x,\cdot)\|_{H_t^{\mu}(-T,T)} \leq c(s,\mu,\theta)T^{\frac{1}{2}}\|y_0\|_{H_x^s(\mathbb{R})} \tag{3.18}$$

for any $\mu \in \mathbb{N}_0$ and any $x, s \in \mathbb{R}$. Note that the term $T^{\frac{1}{2}}$ appears due to the interval $(-T,T)$ of the integral hidden in the L^2-norm definition. Then, since given any $m \in \mathbb{R}$ we can always find $\mu \in \mathbb{N}\cup\{0\}$ such that $m \leq \mu$, estimate (3.18) readily implies

$$\|y_1(x,\cdot)\|_{H_t^{m}(-T,T)} \leq c(s,m,\theta)T^{\frac{1}{2}}\|y_0\|_{H_x^s(\mathbb{R})}, \quad m,s,x \in \mathbb{R}. \tag{3.19}$$

In order to handle y_2, we note that given $\alpha,\beta,\delta \in \mathbb{R}$ such that $\beta \neq 0$ satisfying $\alpha^2+3\beta\delta \geq 0$ one can find $k_j = k_j(\alpha,\delta,\beta) \in \mathbb{R}$, $j = 1,2$, such that (i) the roots $\frac{\alpha\pm\sqrt{\alpha^2+3\beta\delta}}{3\beta}$ of $\omega'(k) = 0$ lie in (k_1,k_2) and (ii) the mapping $\tau = i\omega(k)$ is monotone (increasing when $\beta > 0$, and decreasing when $\beta < 0$) on $\mathbb{R}\setminus(k_1,k_2)$. Now, let $k_3 < k_1$ and $k_4 > k_2$ be any two numbers and fix θ so that it further satisfies the condition

$$\theta(k) = \begin{cases} 1, & k \in [k_1, k_2], \\ 0, & k \notin (k_3, k_4), \end{cases} \tag{3.20}$$

as well as the condition $0 \leq |\theta(k)| \leq 1$, $k \in \mathbb{R}$. Now, to apply the similar arguments that we used to estimate y_1,we can rewrite y_2 as

$$\begin{aligned} y_2(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}\setminus[k_1,k_2]} e^{ikx-\omega(k)t}\,(1-\theta(k))\,\hat{y}_0(k)\,dk \\ &= \frac{1}{2\pi}\int_{(i\omega)(\mathbb{R}\setminus[k_1,k_2])} e^{i(i\omega)^{-1}(\tau)x+i\tau t}\,\frac{1-\theta((i\omega)^{-1}(\tau))}{i\omega'((i\omega)^{-1}(\tau))}\,\hat{y}_0((i\omega)^{-1}(\tau))\,d\tau. \end{aligned} \tag{3.21}$$

Using the definition of the Sobolev norm, for each $x \in \mathbb{R}$ we have

$$\begin{aligned} \|y_2(x,\cdot)\|^2_{H_t^{\frac{s+1}{3}}(\mathbb{R})} &= \int_{\mathbb{R}} (1+\tau^2)^{\frac{s+1}{3}}|\hat{y}_2(x,\tau)|^2 d\tau \\ &= \int_{(i\omega)(\mathbb{R}\setminus[k_1,k_2])} (1+\tau^2)^{\frac{s+1}{3}}\frac{|1-\theta((i\omega)^{-1}(\tau))|^2|\hat{y}_0((i\omega)^{-1}(\tau))|^2}{|i\omega'((i\omega)^{-1}(\tau))|^2}d\tau \\ &\lesssim \int_{\mathbb{R}\setminus[k_1,k_2]} \frac{(1+(i\omega(k))^2)^{\frac{s+1}{3}}}{|i\omega'(k)|}|\hat{y}_0(k)|^2 dk \\ &\lesssim \int_{\mathbb{R}} \left(1+k^2\right)^s |\hat{y}_0(k)|^2 dk = \|y_0\|^2_{H_x^s(\mathbb{R})}, \end{aligned} \tag{3.22}$$

where the last inequality follows from the fact that $i\omega(k) = O(k^3)$ and $\frac{1}{i\omega'(k)} = O(k^{-2})$ as $|k| \to \infty$. Hence, (3.3) follows from (3.19) and (3.22). Continuity in x once again follows from the dominated convergence theorem. □

In addition to the estimates above, we need some information about the spatial derivative of the solution y_x of the homogeneous Cauchy problem (3.1) due to the existence of the Neumann boundary condition in the statement of the original problem for the case $\beta < 0$.

Theorem 3.2 *Let $s \in \mathbb{R}$. The unique solution of the Cauchy problem (3.1), denoted $y = S[y_0; 0]$, satisfies if $\alpha^2 + 3\beta\delta \geq 0$ that, $y_x \in C(\mathbb{R}_x; H_t^{\frac{s}{3}}(-T,T))$ for $T > 0$ and there* *exists a constant* $c = c(s, \alpha, \beta, \delta) \geq 0$ *such that*

$$\sup_{x\in\mathbb{R}} \|\partial_x y(x,\cdot)\|_{H_t^{\frac{s}{3}}(-T,T)} \leq c(1+T^{\frac{1}{2}})\|y_0\|_{H_x^s(\mathbb{R})}; \tag{3.23}$$

while if $\alpha^2+3\beta\delta < 0$, *then* $y_x \in C(\mathbb{R}_x; H_t^{\frac{s}{3}}(\mathbb{R}))$ *and there exists a constant* $c = c(s, \alpha, \beta, \delta) \geq 0$ *such that*

$$\sup_{x\in\mathbb{R}} \|\partial_x y(x,\cdot)\|_{H_t^{\frac{s}{3}}(\mathbb{R})} \leq c\|y_0\|_{H_x^s(\mathbb{R})}. \tag{3.24}$$

Proof Differentiating the Cauchy problem (3.1) with respect to the spatial variable x, we get

$$\begin{aligned} &iY_t + i\beta Y_{xxx} + \alpha Y_{xx} + i\delta Y_x = 0, \quad (x,t) \in \mathbb{R}\times\mathbb{R}, \\ &Y(x,0) = Y_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3.25}$$

where $Y = y_x$ and $Y_0 = (y_0)_x$. Note that $y_0 \in H_x^s(\mathbb{R})$ implies $Y_0 \in H_x^{s-1}(\mathbb{R})$. Using Theorem 3.1 for any $s' \in \mathbb{R}$, we know by (3.3) that

$$\sup_{x\in\mathbb{R}} \|Y(x,\cdot)\|_{H_t^{\frac{s'+1}{3}}(-T,T)} \leq c(1+T^{\frac{1}{2}})\|Y_0\|_{H_x^{s'}(\mathbb{R})}. \tag{3.26}$$

Choosing $s' = s-1$ gives for any $s \in \mathbb{R}$ that

$$\sup_{x\in\mathbb{R}} \|Y(x,\cdot)\|_{H_t^{\frac{s}{3}}(-T,T)} \leq c(1+T^{\frac{1}{2}})\|Y_0\|_{H_x^{s-1}(\mathbb{R})} \leq c(1+T^{\frac{1}{2}})\|y_0\|_{H_x^s(\mathbb{R})}, \tag{3.27}$$

which gives (3.23). Similarly, we can also obtain (3.24) by using (3.4), and the continuity of the maps follows, one more time, from the dominated convergence theorem and the regularity of the initial datum y_0. □

Such estimates on the second or higher-order spatial derivative of y can be easily derived by following the same steps above, but this is out of context for our original initial-boundary value problem. Therefore, we confine ourselves with these two results to correspond them later with the Dirichlet and the Neumann traces of the solution y of the initial value problem (3.1).

We also control the mixed Lebesgue norms $L_t^{\mu}((0,T); H_x^{s,r}(\mathbb{R}))$ of the solution y to the homogeneous Cauchy problem (3.1), where $H^{s,r}(\mathbb{R})$ is the usual Bessel potential space defined with norm

$$\|f\|_{H^{s,r}(\mathbb{R})} := \left\| \mathcal{F}^{-1} \left\{ \left(1 + k^2\right)^{\frac{s}{2}} \mathcal{F}\{f\}(k) \right\} \right\|_{L^r(\mathbb{R})} \tag{3.28}$$

and (μ, r) is any higher-order Schrödinger admissible pair, i.e. any pair (μ, r) satisfying

$$\mu, r \geq 2, \quad \frac{3}{\mu} + \frac{1}{r} = \frac{1}{2}. \tag{3.29}$$

More precisely, we have the following Strichartz estimate:

Theorem 3.3 *Let $s \in \mathbb{R}$ and suppose (μ, r) is higher-order Schrödinger admissible in the sense of* (3.29)*. Then, the solution of the homogeneous linear Cauchy problem* (3.1) *satisfies the Strichartz estimate*

$$\|y\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))} \lesssim \|y_0\|_{H_x^s(\mathbb{R})} . \tag{3.30}$$

Proof By the definition (3.28) of the $H^{s,r}$-norm, we have

$$\|y\|_{L_t^\mu(\mathbb{R};H_x^{s,r}(\mathbb{R}))} = \left\| \mathcal{F}^{-1} \left\{ \left(1 + k^2\right)^{\frac{s}{2}} \hat{y}(k, \cdot) \right\} \right\|_{L_t^\mu((0,T);L_x^r(\mathbb{R}))} . \tag{3.31}$$

Recalling that $\hat{y}(k, t) = e^{-\omega(k)t}\, \hat{y}_0(k)$, we have

$$\mathcal{F}^{-1} \left\{ \left(1 + k^2\right)^{\frac{s}{2}} \hat{y}(k, t) \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - \omega(k)t} \left(1 + k^2\right)^{\frac{s}{2}} \hat{y}_0(k) dk = S\,[\varphi; 0](x, t), \tag{3.32}$$

where $\varphi(x) = \mathcal{F}^{-1} \left\{ \left(1 + k^2\right)^{\frac{s}{2}} \hat{y}_0(k) \right\} (x)$. So, it suffices to prove that

$$\|S\,[\varphi; 0]\|_{L_t^\mu((0,T);L_x^r(\mathbb{R}))} \lesssim \|\varphi\|_{L_x^2(\mathbb{R})} . \tag{3.33}$$

For this, we note that by the definition of the Fourier transform we can write

$$S[\varphi;0](x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} I(x,x',t)\varphi(x')dx', \tag{3.34}$$

where

$$I(x,x',t) := \int_{\mathbb{R}} e^{ik(x-x'-t\delta)+i(t\beta k^3-t\alpha k^2)}dk. \tag{3.35}$$

Then, it is proved (in Lemma 4.2) in Carvajal and Linares (2003) that

$$|I(x,x',t)| \lesssim |\beta t|^{-\frac{1}{3}}, \quad t \neq 0, \tag{3.36}$$

where the inequality constant is independent of x, x', t. Then, the desired estimate (3.33) is inferred (in Theorem 4.1) in Carvajal and Linares (2003). □

3.2. Nonhomogeneous Cauchy Problem

We turn our interest now to the forced initial value problem

$$\begin{aligned} &iz_t + i\beta z_{xxx} + \alpha z_{xx} + i\delta z_x = F, \quad (x,t) \in \mathbb{R}\times(0,T), \\ &z(x,0) = 0, \quad x \in \mathbb{R}, \end{aligned} \tag{3.37}$$

where $F \in L^2_t((0,T);H^s_x(\mathbb{R}))$.

Thanks to Duhamel's principle, the solution of the nonhomogeneous problem (3.37), denoted by $S[0;F]$, can be expressed as

$$\begin{aligned} z(x,t) = S[0;F](x,t) &= -i\int_0^t S[F(\cdot,t');0](x,t-t')dt' \\ &= -\frac{i}{2\pi}\int_0^t\int_{\mathbb{R}} e^{ikx-\omega(k)(t-t')}\hat{F}(k,t')dkdt', \end{aligned} \tag{3.38}$$

where, for each $t' \in [0,t]$, $S[F(\cdot,t');0]$ denotes the solution to the homogeneous Cauchy problem with initial data $F(x,t')$, namely,

$$\begin{aligned} & iy_t^{t'} + i\beta y_{xxx}^{t'} + \alpha y_{xx}^{t'} + i\delta y_x^{t'} = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \\ & y^{t'}(x,0) = F(x,t'), \quad x \in \mathbb{R}. \end{aligned} \tag{3.39}$$

We then have the following result, whose proof is based on the approach that was used for the Korteweg-de Vries equation in Himonas et al. (2019).

Theorem 3.4 *The unique solution of* (3.37) *satisfies the space estimate*

$$\sup_{t\in[0,T]} \|z(\cdot,t)\|_{H_x^s(\mathbb{R})} \leq \|F\|_{L_t^1((0,T);H_x^s(\mathbb{R}))}, \quad s \in \mathbb{R}. \tag{3.40}$$

Moreover, if $-1 \leq s \leq 2$ *with* $s \neq \frac{1}{2}$ *then the following time estimate holds*

$$\sup_{x\in\mathbb{R}} \|z(x,\cdot)\|_{H_t^{\frac{s+1}{3}}(0,T)} \lesssim \max\{T^{\frac{1}{2}}(1+T^{\frac{1}{2}}), T^{\sigma}\}\|F\|_{L_t^2((0,T);H_x^s(\mathbb{R}))}, \tag{3.41}$$

where

$$\sigma = \begin{cases} \frac{1-2s}{6}, & -1 \leq s < \frac{1}{2}, \\ \frac{2-s}{3}, & \frac{1}{2} < s < 2, \\ \frac{1}{2}, & s = 2. \end{cases} \tag{3.42}$$

Remark 3.1 *For* $2 < s < \frac{7}{2}$*, due to the fractional norm* $\|\partial_t z(x,\cdot)\|_{m-1}$ *(see definition* (3.44) *below) the analogue of the time estimate* (3.41) *turns out to be*

$$\sup_{x\in\mathbb{R}} \|z(x,\cdot)\|_{H_t^{\frac{s+1}{3}}(0,T)} \lesssim \max\{T^{\frac{1}{2}}(1+T^{\frac{1}{2}}), T^{\sigma}\}\|F\|_{L_t^2((0,T);H_x^s(\mathbb{R}))} + \sup_{x\in\mathbb{R}} \|F(x,\cdot)\|_{H_t^{\frac{s+1}{3}-1}(0,T)}.$$

The appearance of the space $C(\mathbb{R}_x; H^{\frac{s+1}{3}-1}(0,T))$ *via the relevant norm on the right-hand side has a direct impact on the analysis of the nonlinear problem, as it eventually requires one to establish an appropriate multilinear estimate for the term* $\||u|^p u(x,\cdot)\|_{H_t^{\frac{s+1}{3}-1}(0,T)}$ *(note that the underlying range of s implies* $0 < \frac{s+1}{3} - 1 < \frac{1}{2}$ *and so the algebra property* *is not available). For this reason, a different approach might be preferable for showing well-posedness in this higher range of s. In any case, this task lies outside the scope of the present work, which instead focuses on solutions of lower smoothness and, in particular, towards the low regularity setting* $0 \leq s < \frac{1}{2}$.

Proof In view of the Duhamel representation (3.38), the space estimate (3.40) readily follows from the homogeneous conservation law (3.2).

We proceed to the time estimate (3.41). Restricting $s \geq -1$ allows us to employ the physical space characterization of the Sobolev norm since then the exponent $\frac{s+1}{3}$ is non-negative. In particular, for $-1 \leq s < 2$, setting $m := \frac{s+1}{3}$ and observing that $0 \leq m < 1$, we have

$$\|z(x,\cdot)\|_{H_t^m(0,T)} = \|z(x,\cdot)\|_{L_t^2(0,T)} + \|z(x,\cdot)\|_m\,, \tag{3.43}$$

where the fractional part of the Sobolev norm is zero for $m = 0$ and for $0 < m < 1$ is given by

$$\|z(x,\cdot)\|_m^2 = 2\int_0^T \int_0^{T-t} \frac{|z(x,t+l)-z(x,t)|^2}{l^{1+2m}} dldt. \tag{3.44}$$

For each $x \in \mathbb{R}$, employing Minkowski's integral inequality and subsequently using the homogeneous time estimates (3.3) and (3.4) for $\alpha^2+3\beta\delta \geq 0$ and $\alpha^2+3\beta\delta < 0$ respectively, along with the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\|z(x,\cdot)\|_{L_t^2(0,T)} &\leq \int_0^T \|S[F(\cdot,t');0](x,\cdot-t')\|_{L_t^2(0,T)}\,dt' \\
&\lesssim (1+T^{\frac{1}{2}}) \int_0^T \|F(\cdot,t')\|_{H_x^{-1}(\mathbb{R})}\,dt' \\
&\lesssim T^{\frac{1}{2}}(1+T^{\frac{1}{2}})\,\|F\|_{L_t^2((0,T);H_x^s(\mathbb{R}))}.
\end{aligned} \tag{3.45}$$

For the fractional norm, noting that

$$\begin{aligned}
|z(x,t+l)-z(x,t)|^2 \leq{}& \left| \int_0^t S[F(\cdot,t');0](x,t+l-t') - S[F(\cdot,t');0](x,t-t')dt' \right|^2 \\
&+ \left| \int_t^{t+l} S[F(\cdot,t');0](x,t+l-t')dt' \right|^2
\end{aligned}$$

we have $\|z(x,\cdot)\|_m^2 \lesssim I + J$, where

$$I := \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \Big(\int_0^T \Big| S[F(\cdot,t');0](x,t+l-t') - S[F(\cdot,t');0](x,t-t') \Big| dt' \Big)^2 dldt, \tag{3.46}$$

$$J := \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \Big| \int_t^{t+l} S[F(\cdot,t');0](x,t+l-t')dt' \Big|^2 dldt. \tag{3.47}$$

For I, we proceed as follows. First, we multiply the integrand by the characteristic function $\chi_{[0,T-t]}(l)$ so that $\chi_{[0,T-t]}(l) = 1$ for $0 \leq l \leq T-t$ and $\chi_{[0,T-t]}(l) = 0$ otherwise. This allows us to replace $T - t$ by T in the upper limit of the integral with respect to l. Then, we use Minkowski's inequality for the triple integral and, finally, we use the definition of $\chi_{[0,T-t]}(l)$ once again to switch T by $T - t$ in the limit of the integral taken with respect to l. Performing these steps and employing the homogeneous time estimates (3.3) and (3.4), we find

$$\begin{aligned} I &\leq \left(\int_0^T \left(\int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \Big| S[F(\cdot,t');0](x,t+l-t') - S[F(\cdot,t');0](x,t-t') \Big|^2 dldt \right)^{\frac{1}{2}} dt' \right)^2 \qquad (3.48) \\ &\simeq \left(\int_0^T \|S[F(\cdot,t');0](x,\cdot-t')\|_m \ dt' \right)^2 \lesssim \left(\int_0^T \|F(\cdot,t')\|_{H_x^s(\mathbb{R})} \ dt' \right)^2 \lesssim T \int_0^T \|F(\cdot,t)\|_{H_x^s(\mathbb{R})}^2 dt. \end{aligned}$$

In order to estimate J, we consider the cases $0 < m < \frac{1}{2}$ and $\frac{1}{2} < m < 1$ separately. The range $\frac{1}{2} < m < 1$ corresponds to $\frac{1}{2} < s < 2$ and hence we can employ the Sobolev embedding theorem in x. In particular, substituting for $S[F(\cdot,t');0](x,t+l-t')$ via (3.9) and then using the Sobolev embedding, the Fourier transform characterization of the Sobolev norm, and the fact that $\omega(k)$ is imaginary for $k \in \mathbb{R}$, we have

$$\begin{aligned} J &\leq \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \left\| \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx-\omega(k)(t+l)} \int_t^{t+l} e^{\omega(k)t'} \widehat{F}(k,t')dt'dk \right\|_{L_x^\infty(\mathbb{R})}^2 dldt \\ &\lesssim \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \left\| \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx-\omega(k)(t+l)} \int_t^{t+l} e^{\omega(k)t'} \widehat{F}(k,t')dt'dk \right\|_{H_x^s(\mathbb{R})}^2 dldt \\ &= \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \int_{\mathbb{R}} \left(1+k^2\right)^s \left| \int_t^{t+l} e^{\omega(k)t'} \widehat{F}(k,t')dt' \right|^2 dkdldt. \end{aligned}$$

Thus, by Minkowski's integral inequality between the integrals with respect to t' and k, Cauchy-Schwarz inequality in the t'-integral, and Fubini's theorem between the integrals with respect to t and t',

$$\begin{aligned}
J &\lesssim \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \Big(\int_t^{t+l} \|F(\cdot,t')\|_{H_x^s(\mathbb{R})}\, dt' \Big)^2 dldt \\
&\leq \int_0^T \int_0^{T-t} \int_t^{t+l} \|F(\cdot,t')\|^2_{H_x^s(\mathbb{R})}\; l^{-2m}\; dt' dldt \\
&= \int_0^T \|F(\cdot,t')\|^2_{H_x^s(\mathbb{R})} \int_0^T l^{-2m} \int_{t'-l}^{t'} dtdldt' \\
&\simeq \frac{T^{2-2m}}{2-2m} \int_0^T \|F(\cdot,t')\|^2_{H_x^s(\mathbb{R})}\, dt'.
\end{aligned} \tag{3.49}$$

The range $0 < m < \frac{1}{2}$ corresponds to $-1 < s < \frac{1}{2}$, hence Sobolev embedding is no longer available. However, the fact that $m < \frac{1}{2}$ allows to proceed via the Cauchy-Schwarz inequality in t' as follows:

$$\begin{aligned}
J &\lesssim \int_0^T \frac{1}{l^{2m}} \int_0^{T-l} \int_t^{t+l} |S[F(\cdot,t');0](x,t+l-t')|^2 dt' dtdl \\
&= \int_0^T \frac{1}{l^{2m}} \int_l^T \int_{t-l}^t |S[F(\cdot,t');0](x,t-t')|^2 dt' dtdl \\
&\lesssim \Big(\int_0^T \frac{1}{l^{2m}} dl \Big) \int_0^T \|S[F(\cdot,t');0](x,\cdot-t')\|^2_{L_t^2(t',T)}\, dt' \\
&\lesssim \frac{T^{1-2m}}{1-2m} \int_0^T \|F(\cdot,t')\|^2_{H_x^{-1}(\mathbb{R})}\, dt'.
\end{aligned} \tag{3.50}$$

Note that the equality above is due to the change of variable $t \mapsto t - l$, and the inequality succeeding it follows by extending the range of the integrals with respect to t' and t and then interchanging the resulting integrals. The last inequality follows by Theorem 3.1.

Estimates (3.45), (3.48), (3.49), (3.50) combined with the Sobolev norm definition (3.43) imply the desired time estimate (3.41) in the range $-1 \leq s < 2$ with $s \neq \frac{1}{2}$.

Finally, we consider $2 \leq s < \frac{7}{2}$. As this range corresponds to $1 \leq m < \frac{3}{2}$, the Sobolev norm (3.43) must be modified to

$$\|z(x,\cdot)\|^2_{H_t^m(0,T)} = \|z(x,\cdot)\|^2_{H_t^1(0,T)} + \|\partial_t z(x,\cdot)\|^2_{m-1}\,. \tag{3.51}$$

Differentiating (3.38) in t, we have

$$\partial_t z(x,t) = -iS[F(\cdot,t);0](x,0) - i\int_0^t \partial_t[S[F(\cdot,t');0](x,t-t')]dt'. \tag{3.52}$$

We begin by observing that

$$S[F(\cdot,t);0](x,0) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}\widehat{F}(k,t)dk = F(x,t). \tag{3.53}$$

Moreover, by using the Fourier transform property for derivatives, which is given in (2.6), we note that

$$\begin{aligned}\partial_t[S[F(\cdot,t');0](x,t-t')] &= \partial_t\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-\omega(k)(t-t')}\widehat{F}(k,t')dk\right]\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}[-\omega(k)]e^{ikx-\omega(k)(t-t')}\widehat{F}(k,t')dk\\ &= S[(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x)F(\cdot,t');0](x,t-t').\end{aligned} \tag{3.54}$$

Therefore, (3.52) can be rewritten as

$$\partial_t z(x,t) = -iF(x,t) - i\int_0^t S[(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x)F(\cdot,t');0](x,t-t')dt' \tag{3.55}$$

and so

$$\|\partial_t z(x,\cdot)\|_{L_t^2(0,T)} \le \|F(x,\cdot)\|_{L_t^2(0,T)} + \left\|\int_0^t S[(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x)F(\cdot,t');0](x,t-t')dt'\right\|_{L_t^2(0,T)}.$$

The first term on the right-hand side can be handled as follows:

$$\|F(x,\cdot)\|_{L_t^2(0,T)} \le \sup_{x\in\mathbb{R}}\|F(x,\cdot)\|_{L_t^2(0,T)} \le \|F\|_{L_t^2((0,T);H_x^{\frac{1}{2}+}(\mathbb{R}))}. \tag{3.56}$$

For the second term, extending the range of integration in t' and then applying Minkowski's integral inequality in combination with Theorem 3.1, we have

$$\begin{aligned}
&\left\|\int_0^t S\left[(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x)F(\cdot,t'); 0\right](x, t-t')dt'\right\|_{L_t^2(0,T)} \\
&\lesssim (1+T^{\frac{1}{2}})\int_0^T \left\|(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x)F(\cdot,t')\right\|_{H_x^{-1}(\mathbb{R})} dt' \\
&\lesssim (1+T^{\frac{1}{2}})\left[\beta\int_0^T \left\|\partial_x^3 F(\cdot,t')\right\|_{H_x^{-1}(\mathbb{R})} dt' + |\alpha|\int_0^T \left\|\partial_x^2 F(\cdot,t')\right\|_{H_x^{-1}(\mathbb{R})} dt' + |\delta|\int_0^T \|\partial_x F(\cdot,t')\|_{H_x^{-1}(\mathbb{R})} dt'\right] \\
&\lesssim (1+T^{\frac{1}{2}})\left[\beta\int_0^T \|F(\cdot,t')\|_{H_x^2(\mathbb{R})} dt' + |\alpha|\int_0^T \|F(\cdot,t')\|_{H_x^1(\mathbb{R})} dt' + |\delta|\int_0^T \|F(\cdot,t')\|_{L_x^2(\mathbb{R})} dt'\right] \\
&\lesssim (1+T^{\frac{1}{2}})\int_0^T \|F(\cdot,t')\|_{H_x^2(\mathbb{R})} dt'. \qquad (3.57)
\end{aligned}$$

Together, estimates (3.56) and (3.57) imply the bound

$$\|\partial_t z(x,\cdot)\|_{L_t^2(0,T)} \lesssim T^{\frac{1}{2}}(1+T^{\frac{1}{2}})\,\|F\|_{L_t^2((0,T);H_x^2(\mathbb{R}))} \qquad (3.58)$$

which corresponds to the desired estimate (3.41) in the case $s = 2$. □

As for the homogeneous Cauchy problem, we need some information also about the spatial derivative of the solution, namely z_x, of the nonhomogeneous Cauchy problem (3.37) due to the existence of the Neumann boundary condition in the boundary data of the original problem.

Theorem 3.5 *The unique solution of the Cauchy problem (3.37) satisfies if* $0 \le s \le 2$ *with* $s \neq \frac{3}{2}$ *that*

$$\sup_{x\in\mathbb{R}} \|\partial_x z(x,\cdot)\|_{H_t^{\frac{s}{3}}(0,T)} \lesssim \max\{T^{\frac{1}{2}}(1+T^{\frac{1}{2}}), T^{\sigma_1}\}\|F\|_{L_t^2(0,T;H_x^s(\mathbb{R}))}, \qquad (3.59)$$

where

$$\sigma_1 = \begin{cases} \frac{3-2s}{6}, & 0 \le s < \frac{3}{2}; \\ \frac{3-s}{3}, & \frac{3}{2} < s \le 2. \end{cases} \qquad (3.60)$$

Proof We apply the similar arguments that we used to prove Theorem 3.1. Indeed, we differentiate (3.37) with respect to the spatial variable x, and get

$$\begin{aligned} &iZ_t + i\beta Z_{xxx} + \alpha Z_{xx} + i\delta Z_x = G, \quad (x,t) \in \mathbb{R} \times (0,T), \\ &Z(x,0) = 0, \quad x \in \mathbb{R}, \end{aligned} \tag{3.61}$$

where $Z = z_x$ and $G = F_x$. Note that $F \in L_t^2((0,T); H_x^s(\mathbb{R}))$ implies $G \in L_t^2((0,T); H_x^{s-1}(\mathbb{R}))$. Using Theorem 3.4 for any $-1 \le s' \le 2$ with $s' \ne \frac{1}{2}$, and then taking $s' = s-1$ yields the desired estimate. □

The higher-order Schrödinger equation is studied in this paper with the Dirichlet and the Neumann boundary conditions. So, all these estimates on the solutions of Cauchy problems (3.1) and (3.37), together with their first derivatives are sufficient. However, by applying same procedures, these estimates can be treated for any other boundary data couple involving second order derivative by restricting the interval for s to a smaller one.

Regarding $L_t^\mu L_x^r$ Strichartz-type estimates for the nonhomogeneous linear Cauchy problem (3.37), we have the following result which is a consequence of the homogeneous Strichartz estimates given in Theorem 3.3.

Theorem 3.6 *Let $s \in \mathbb{R}$ and suppose (μ, r) is higher-order Schrödinger admissible in the sense of* (3.29). *Then, the solution of the nonhomogeneous linear Cauchy problem* (3.37) *satisfies the Strichartz estimate*

$$\|z\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))} \lesssim \|F\|_{L_t^1((0,T);H_x^s(\mathbb{R}))}. \tag{3.62}$$

Proof Letting $H(x,t,t') := \chi_{\{t' \le t\}}(t') S[F(\cdot,t');0](x,t-t')$, we rewrite $\|z\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))}$ as

$$\|z\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))} = \left\| \int_0^T H(\cdot,\cdot,t')dt' \right\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))}. \tag{3.63}$$

Therefore, in view of the homogeneous Strichartz estimate (3.30), we readily infer

$$
\begin{aligned}
\|z\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))} &\leq \int_0^T \|H(\cdot,\cdot,t')\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))}\, dt' \\
&\leq \int_0^T \|S\left[F(\cdot,t');0\right](\cdot,\cdot - t')\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))}\, dt' \lesssim \int_0^T \|F(\cdot,t')\|_{H_x^s(\mathbb{R})}\, dt'.
\end{aligned}
$$

□

CHAPTER 4

SINGLE-BOUNDARY CONDITION CASE

In this chapter, we analyze the initial-boundary value problem for the higher-order nonlinear Schrödinger on the half-line for the case which the problem is needed to be stated with only one boundary condition. So, we focus on the nonlinear initial-boundary value problem

$$\begin{aligned} &iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = f(u), \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \\ &u(0,t) = g(t), \quad t \in (0,T), \end{aligned} \tag{4.1}$$

where $\alpha, \delta \in \mathbb{R}, \beta > 0$, $f(z) = \kappa|z|^p z$ with $z \in \mathbb{C}$, $\kappa \in \mathbb{C}$, $p > 0$, and $T > 0$.

Analysis of this nonlinear partial differential equation is proceed here by combining a remarkable work on the linear theory and the fixed point argument to relate the results from the linear theory to the nonlinear problem. Therefore, we first consider the linear forced initial-boundary value problem

$$\begin{aligned} &iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = f, \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \\ &u(0,t) = g(t), \quad t \in (0,T), \end{aligned} \tag{4.2}$$

where $\alpha, \delta \in \mathbb{R}, \beta > 0$ and $T > 0$.

We emphasize that this linear problem is stated with three given data, namely the source f, the initial condition u_0, and the boundary condition g. Each of these data effects naturally the solution u of the problem (4.2), and this situation makes the analysis very hard to be handled in one hand. Instead, we use a *decompose-reunify* technique, which is also used analogously in Özsarı and Yolcu (2019) for the biharmonic Schrödinger equation, to observe the effects of these three data separately. Firstly, we *decompose* the linear forced initial-boundary value problem (4.2) into three rather simpler models to be *reunified* after a brief analysis.

The first *decomposed* model is the homogeneous Cauchy problem

$$\begin{aligned} &iy_t + i\beta y_{xxx} + \alpha y_{xx} + i\delta y_x = 0, \qquad (x,t) \in \mathbb{R}\times\mathbb{R}, \\ &y(x,0) = y_0(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{4.3}$$

where $\alpha, \delta \in \mathbb{R}, \beta > 0$, and $y_0 = E_0 u_0$ denotes an extension of u_0 with respect to a fixed bounded extension operator $E_0 : H_x^s(\mathbb{R}_+) \to H_x^s(\mathbb{R})$, namely, we have

$$\|y_0\|_{H_x^s(\mathbb{R})} \lesssim \|u_0\|_{H_x^s(\mathbb{R}_+)}. \tag{4.4}$$

This problem is studied in Section 3.1 for any initial data $y_0 \in H^s(\mathbb{R})$ and any $\beta \neq 0$.

As the second *decomposed* model, we consider the nonhomogeneous Cauchy problem

$$\begin{aligned} &iz_t + i\beta z_{xxx} + \alpha z_{xx} + i\delta z_x = F, \quad (x,t) \in \mathbb{R}\times(0,T), \\ &z(x,0) = 0, \quad x \in \mathbb{R}, \end{aligned} \tag{4.5}$$

where $\alpha, \delta \in \mathbb{R}$, $\beta > 0$, and $F = E_0 f \in L_t^2((0,T); H_x^s(\mathbb{R}))$ is a spatial extension of $f \in L_t^2((0,T); H_x^s(\mathbb{R}_+))$. This problem is also studied in Section 3.2 for any source $F \in L_t^2((0,T); H_x^s(\mathbb{R}))$ and any $\beta \neq 0$. Therefore, we can utilize directly from the results that we obtained in Chapter 3 for Cauchy problems.

As the last *decomposed* model, we consider a reduced initial-boundary value problem

$$\begin{aligned} &iq_t + i\beta q_{xxx} + \alpha q_{xx} + i\delta q_x = 0, \quad (x,t) \in \mathbb{R}_+ \times (0,T'), \\ &q(x,0) = 0, \quad x \in \mathbb{R}_+, \\ &q(0,t) = g_0(t) := E_b[g - y(0,\cdot) - z(0,\cdot)](t), \quad t \in (0,T'), \end{aligned} \tag{4.6}$$

where $\alpha, \delta \in \mathbb{R}$, $\beta > 0$, $T' > T$, $y(0,t)$ and $z(0,t)$ are the solutions to the homogeneous and nonhomogeneous Cauchy problems (4.3) and (4.5) evaluated at $x = 0$, and $E_b : H_t^{(s+1)/3}(0,T) \to H_t^{(s+1)/3}(\mathbb{R})$ is a fixed bounded extension operator satisfying the additional property that supp $g_0 \subset [0,T')$. It is provided that the traces $y(0,t)$ and $z(0,t)$ are well-defined and belong to $H_t^{\frac{s+1}{3}}(0,T)$ in view of Theorems 3.1 and 3.40. The construction of such an extension is analogous to the one provided in detail in Section 3 of

Himonas and Mantzavinos (2020) in the context of the linear Schrödinger equation. In particular, we note that, for continuous Sobolev data, a compactly supported extension can be constructed thanks to the compatibility between the initial and boundary data at $(x,t) = (0,0)$.

On the contrary to the Cauchy problems, the reduced initial-boundary value problem (4.6) is not studied here yet. Therefore, we need to pause before the meticulous declaration on the *reunification* of these models, and work on the analysis the problem (4.6) in an independent zone. And then we will continue to explain how to bring these models together.

4.1. Reduced initial-boundary value problem

We consider the reduced initial-boundary value problem

$$\begin{aligned} &iq_t + i\beta q_{xxx} + \alpha q_{xx} + i\delta q_x = 0, \quad (x,t) \in \mathbb{R}_+ \times (0,T'), \\ &q(x,0) = 0, \quad x \in \mathbb{R}_+, \\ &q(0,t) = g_0(t), \quad t \in (0,T'), \end{aligned} \tag{4.7}$$

where $\alpha, \delta \in \mathbb{R}, \beta > 0, T' > T$ and $g_0 \in H^{\frac{s+1}{3}}(0,T')$.

Firstly, we apply Fokas' unified transform method to obtain a formula representing the weak solution q, and then using this formula, we analyze the regularity level of these solutions, as we did for the ones of the homogeneous and the nonhomogeneous Cauchy problems in Chapter 3.

4.1.1. Solution formula

We first assume that q is sufficiently smooth up to the boundary of $\mathbb{R}_+ \times (0,T')$ and decays sufficiently fast as $x \to \infty$, uniformly in $[0,T']$. The definition of the standard

Fourier transform on $\mathbb{R}$ applied on the piecewise-defined function

$$F(x) = \begin{cases} f(x), & x > 0, \\ 0, & x < 0, \end{cases} \qquad f \in L^2(0,\infty), \tag{4.8}$$

gives rise to the half-line Fourier transform pair

$$\begin{aligned} \hat{f}(k) &= \int_0^\infty e^{-ikx} f(x) dx, \quad \text{Im}(k) \leq 0, \\ f(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \hat{f}(k) dk, \quad x > 0. \end{aligned} \tag{4.9}$$

Note that the above half-line Fourier transform makes sense for all $\text{Im}(k) \leq 0$ and not just for $k \in \mathbb{R}$ as its whole-line counterpart. Taking the half-line Fourier transform (4.9) of (4.7) and integrating over $(0,t)$, we obtain the following spectral identity known as the global relation:

$$e^{\omega(k)t}\, \hat{q}(k,t) = (-\beta k^2 + \alpha k + \delta)\, \widetilde{g}_0(\omega(k),t) + (i\beta k - i\alpha)\, \widetilde{g}_1(\omega(k),t) + \beta\, \widetilde{g}_2(\omega(k),t), \quad \text{Im} k \leq 0, \tag{4.10}$$

where ω is given by (3.5) and the temporal transforms $\widetilde{g}_j(\omega(k),t)$ are defined by

$$\widetilde{g}_j(k,t) = \int_0^t e^{kt'} \partial_x^j q(0,t') dt', \quad k \in \mathbb{C}, \quad j = 0,1,2. \tag{4.11}$$

Then, by the inversion formula in (4.9),

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx-\omega(k)t} \left[(-\beta k^2 + \alpha k + \delta)\, \widetilde{g}_0(\omega(k),t) + (i\beta k - i\alpha)\, \widetilde{g}_1(\omega(k),t) + \beta\, \widetilde{g}_2(\omega(k),t)\right] dk. \tag{4.12}$$

The transforms $\widetilde{g}_1$ and $\widetilde{g}_2$ involve the unknown boundary values $q_x(0,t)$ and $q_{xx}(0,t)$. In order to eliminate them from (4.12), we proceed as follows. For $D := \{k \in \mathbb{C} :$

$\mathrm{Re}(\omega(k)) < 0\}$, consider the region

$$D^+ := D \cap \{\mathrm{Im}(k) > 0\} = \Big\{\mathrm{Im}(k) > 0 : 3\Big(\mathrm{Re}(k) - \frac{\alpha}{3\beta}\Big)^2 - \mathrm{Im}(k)^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2} < 0\Big\}, \quad (4.13)$$

which is depicted in Figures 4.1, 4.2 and 4.3 for the various signs of the quantity $\alpha^2 + 3\beta\delta$. Then, thanks to analyticity (Cauchy's theorem) and exponential decay, it follows that

$$q(x,t) = \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-\omega(k)t}\Big[(-\beta k^2 + \alpha k + \delta)\,\widetilde{g}_0(\omega(k),t) + (i\beta k - i\alpha)\,\widetilde{g}_1(\omega(k),t) + \beta\,\widetilde{g}_2(\omega(k),t)\Big]\,dk, \quad (4.14)$$

where the contour ∂D^+ is positively oriented, i.e. it is traversed in the direction such that D^+ stays to the left of the contour, as shown in Figures 4.1, 4.2 and 4.3.

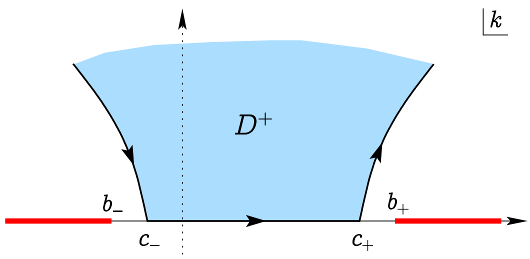

Figure 4.1. The region D^+ for $\alpha^2 + 3\beta\delta > 0$.

In this case, $c_\pm = \frac{\alpha \pm \sqrt{3\beta^2\lambda^2+\alpha^2+3\beta\delta}}{3\beta}$ and the square root branch cut $\mathcal{B} = (-\infty, b_-] \cup [b_+, \infty)$ with branch points $b_\pm = \frac{1}{3\beta}\left(\alpha \pm 2\sqrt{\alpha^2 + 3\beta\delta}\right)$ (shown in red) stays outside the region D^+.

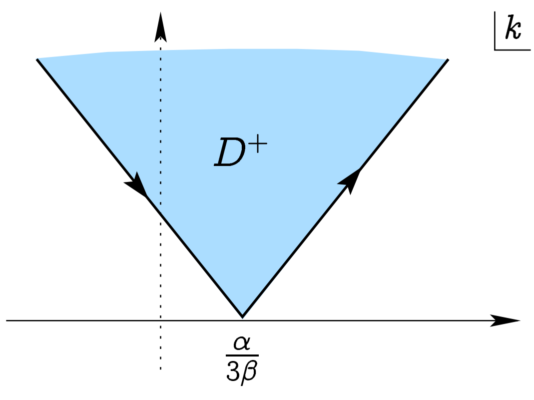

Figure 4.2. The region D^+ for $\alpha^2 + 3\beta\delta = 0$.

In this second case there is no branching.

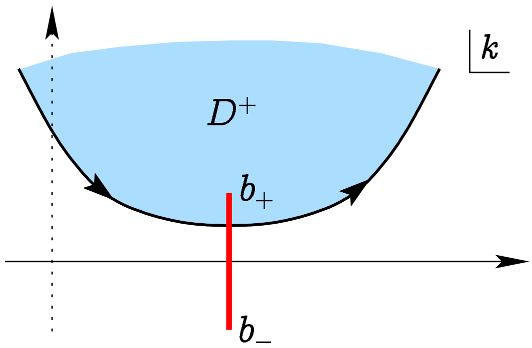

Figure 4.3. The region D^+ for $\alpha^2 + 3\beta\delta < 0$.

In the third case the branch cut $\widetilde{B}$ (shown in red) is taken along the vertical line segment connecting b_+ to b_- and so part of it lies in D^+, thus a local deformation around $\widetilde{B}$ is performed as shown in Figure 4.5 below.

This deformation from the real line to the boundary of D^+ is a direct consequence of Fokas' unified transform method. In Fokas (2008), every details of such deformation can be found for some rather simpler initial-boundary value problems such as the heat equation. In particular, it is also very well explained how such deformation can be handled for Schrödinger equation in Himonas and Mantzavinos (2020). However, for the readers that are not expert about the method, we prefer to explain the details of this deformation for higher-order Schrödinger equation itself without expecting no background about the method. We just kindly demand to be trusted to show these details only for some part of the contours, namely the one occurs on the first quarter plane for the simplest case $\alpha^2 + 3\beta\delta = 0$ as in Figure 4.2. It can be then understood well how to obtain (4.14) from (4.12). To this end, we construct a closed contour $C := C_1 \cup C_2 \cup C_3$, where C_1 is the line segment(or may be considered as a ray) $[c_+, R)$ for $R \to \infty$ on the real axis, and C_3 is the slanted line, with the reverse direction, that occurs on the boundary of D^+ in the first quadrant, and C_2 is the circular arc moving counter-clockwise to connect R with C_3.

Observe that the integrand, say $G_{x,t}(k)$, of (4.12) is analytic in C, therefore

$$\int_C G_{x,t}(k)\, dk = 0, \tag{4.15}$$

by Cauchy's theorem. On the other hand, Jordan's lemma implies that

$$\int_{C_2} G_{x,t}(k)\, dk = 0, \tag{4.16}$$

as well. Combining (4.15) and (4.16), we obtain

$$\int_{C_1} G_{x,t}(k)\, dk = -\int_{C_3} G_{x,t}(k)\, dk. \tag{4.17}$$

So we can deform C_1 to $(-C_3)$, where the minus sign is understood in the directional sense.

Applying this idea to any related part of the deformation, we obtain (4.14) from (4.12). The fact that the integral (4.14) is taken along the deformed contour ∂D^+ will allow us to eliminate the unknown transforms $\widetilde{g}_1$ and $\widetilde{g}_2$ from (4.14) by employing two additional spectral identities emanating from the global relation (4.10) through suitable transformations that keep the spectral function $\omega(k)$ invariant. In particular, both of these identities are valid along ∂D^+ and so we will be able to use them simultaneously. It is important to emphasize that the two additional identities are not valid along $\mathbb{R}$, which is the reason why the deformation from $\mathbb{R}$ to ∂D^+ that leads to (4.14) is necessary.

In order to determine the symmetry transformations, we solve the equation $\omega(\nu) = \omega(k)$ for $\nu = \nu(k)$.

(i) If $\alpha^2 + 3\beta\delta > 0$, then the two nontrivial symmetries are

$$\nu_\pm(k) = -\frac{1}{2}\left(k - \frac{\alpha}{\beta}\right) \pm \frac{\sqrt{3}\, i}{2}\left[\left(k - \frac{\alpha}{3\beta}\right)^2 - \frac{4\left(\alpha^2 + 3\beta\delta\right)}{9\beta^2}\right]^{\frac{1}{2}}. \tag{4.18}$$

The square root term in (4.18) is defined as follows. Denoting the two branch points by $b_\pm := \frac{1}{3\beta}(\alpha \pm 2\sqrt{\alpha^2 + 3\beta\delta})$, we write $k - b_\pm = |k - b_\pm|\, e^{i\theta_\pm}$ with $-\pi < \theta_- \leq \pi$ and $0 \leq \theta_+ < 2\pi$, which correspond to branch cuts along $[b_+, \infty)$ for $(k - b_+)^{\frac{1}{2}}$ and along $(-\infty, b_-]$ for $(k - b_-)^{\frac{1}{2}}$. Then, we associate the square root in (4.18) with the single-valued function

$$\left[\left(k-\frac{\alpha}{3\beta}\right)^2-\frac{4\left(\alpha^2+3\beta\delta\right)}{9\beta^2}\right]^{\frac{1}{2}} = \sqrt{|k-b_+|\,|k-b_-|}\,e^{i(\theta_++\theta_-)/2}, \tag{4.19}$$

which is analytic for all $k \notin \mathcal{B} := (-\infty, b_-] \cup [b_+, \infty)$. In turn, this definition ensures that $\nu_\pm$ are analytic for all $k \in \mathbb{C} \setminus \mathcal{B}$. Importantly, as shown in Figure 4.1, $\mathcal{B} \cap \overline{D^+} = \emptyset$.

(ii) $\alpha^2 + 3\beta\delta = 0$. In this case, the symmetries are the two entire functions

$$\nu_\pm(k) = -\frac{1}{2}\left(k-\frac{\alpha}{\beta}\right) \pm \frac{\sqrt{3}\,i}{2}\left(k-\frac{\alpha}{3\beta}\right), \tag{4.20}$$

as shown in Figure 4.2.

(iii) $\alpha^2 + 3\beta\delta < 0$. In that case, the symmetries are again given by (4.18); however, as the branch points $b_\pm$ are now complex conjugates along the line $\mathrm{Re}(k) = \frac{\alpha}{3\beta}$, we write $k - b_\pm = |k-b_\pm|\,e^{i(\theta_\pm-\pi/2)}$ with $0 \le \theta_\pm < 2\pi$ and corresponding branch cuts along the vertical half-lines from $b_\pm$ to $\frac{\alpha}{3\beta} - i\infty$, so that

$$\left[\left(k-\frac{\alpha}{3\beta}\right)^2-\frac{4\left(\alpha^2+3\beta\delta\right)}{9\beta^2}\right]^{\frac{1}{2}} = \sqrt{|k-b_+|\,|k-b_-|}\,e^{i(\theta_++\theta_--\pi)/2} \tag{4.21}$$

is single-valued and analytic for all $k \in \mathbb{C} \setminus \widetilde{\mathcal{B}}$, where $\widetilde{\mathcal{B}}$ is the finite vertical segment connecting b_+ and b_-, as shown in Figure 4.3. Note that $\widetilde{B} \cap \overline{D^+} \neq \emptyset$ as part of the branch cut $\widetilde{B}$ lies inside the region D^+. For this reason, *before* employing the symmetries $\nu_\pm$ for the elimination of the unknown transforms $\widetilde{g}_2$ and $\widetilde{g}_1$ from (4.14), we use Cauchy's theorem to deform the contour ∂D^+ in (4.14) to the modified contour $\partial\widetilde{D}^+$, which corresponds to the positively oriented boundary of the region $\widetilde{D}^+$ shown in Figure 4.5. This way, the branch cut $\widetilde{\mathcal{B}}$ is avoided *prior to* the use of the symmetries $\nu_\pm$, allowing us to take advantage of analyticity inside the region $\widetilde{D}^+$ later.

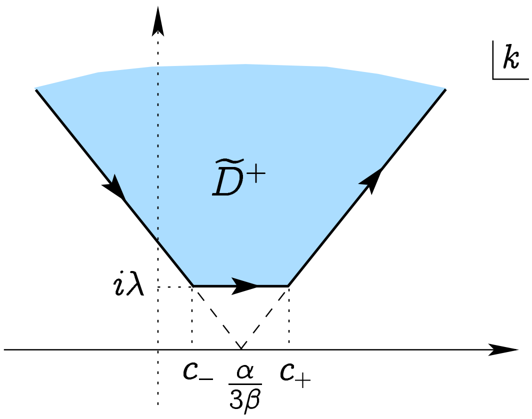

Figure 4.4. Deformation of ∂D^+ to $\partial \widetilde{D}^+$ for $\alpha^2 + 3\beta\delta = 0$.

This deformation is carried out in order to stay away from the point $\frac{\alpha}{3\beta}$, which is a zero of the quantity $\nu_-(k) - \nu_+(k)$.

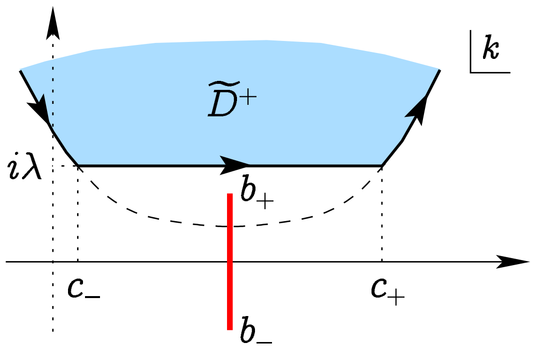

Figure 4.5. Deformation of ∂D^+ to $\partial \widetilde{D}^+$ for $\alpha^2 + 3\beta\delta < 0$.

This second deformation is done in order to avoid crossing the branch cut $\widetilde{B}$ (shown in red).

In view of the above discussion, we rewrite (4.14) as

$$q(x,t) = \frac{1}{2\pi}\int_\Gamma e^{ikx-\omega(k)t}\left[\left(-\beta k^2 + \alpha k + \delta\right)\widetilde{g}_0(\omega(k),t) + i\,(\beta k - \alpha)\,\widetilde{g}_1(\omega(k),t) + \beta\,\widetilde{g}_2(\omega(k),t)\right]dk, \tag{4.22}$$

where the integration contour Γ is given by

$$\Gamma = \begin{cases} \partial D^+, & \alpha^2 + 3\beta\delta > 0, \\ \partial \widetilde{D}^+, & \alpha^2 + 3\beta\delta \leq 0. \end{cases} \tag{4.23}$$

Replacing k by $\nu_\pm(k)$ in the global relation (4.10) and using the fact that $\omega(\nu_\pm(k)) = \omega(k)$, we get the spectral identities

$$\begin{aligned} e^{\omega(k)t}\,\widehat{q}(\nu_\pm(k),t) = \left(-\beta\nu_\pm^2(k) + \alpha\nu_\pm(k) + \delta\right)\widetilde{g}_0(\omega(k),t) + i\left(\beta\nu_\pm(k) - \alpha\right)\widetilde{g}_1(\omega(k),t) \\ + \beta\,\widetilde{g}_2(\omega(k),t), \quad \operatorname{Im}(\nu_\pm(k)) \leq 0. \end{aligned} \tag{4.24}$$

We emphasize that the above identities are valid only for k such that $\operatorname{Im}(\nu_\pm(k)) \leq 0$. Thus, in order to employ them for the elimination of the unknown boundary values from (4.22), we need to ensure that $\Gamma \subseteq \{\operatorname{Im}(\nu_\pm(k)) \leq 0\}$. This is proved in the following lemma.

Lemma 4.1 *Let $\nu_\pm = \nu_\pm(k)$ be the nontrivial (i.e. $\nu_\pm \not\equiv k$) solutions of the equation $\omega(\nu) = \omega(k)$ as given by* (4.18) *or* (4.20)*, depending on the value of $\alpha^2 + 3\beta\delta$. If $k \in \overline{D^+}$, then* $\operatorname{Im}(\nu_\pm) \leq 0$.

Proof For all $k = k_R + ik_I \in \overline{D^+}$ such that $\nu(k) \neq k$ satisfies $\omega(\nu) = \omega(k)$, we must have $\beta(\nu^2 + k\nu + k^2) - \alpha(\nu + k) - \delta = 0$. Writing $\nu = \nu_R + i\nu_I$ and taking real and imaginary parts, this equation is equivalent to the system

$$\widetilde{\nu}_R\widetilde{\nu}_I = ck_I, \quad \widetilde{\nu}_R^2 - \widetilde{\nu}_I^2 = d, \tag{4.25}$$

where $\widetilde{\nu}_R = \nu_R + \frac{k_R}{2} - \frac{\alpha}{2\beta}$, $\widetilde{\nu}_I = \nu_I + \frac{k_I}{2}$, $c = \frac{\alpha}{4\beta} - \frac{3k_R}{4}$ and $d = -\frac{3}{4}k_R^2 + \frac{3}{4}k_I^2 + \frac{\alpha}{2\beta}k_R + \frac{\alpha^2}{4\beta^2} + \frac{\delta}{\beta}$. If $\widetilde{\nu}_I = 0$, then $\nu_I = -\frac{k_I}{2} \leq 0$ as $k \in \overline{D^+}$ and we are done. So let us assume $\widetilde{\nu}_I \neq 0$. Then, combining the two equations in (4.25) we obtain $\widetilde{\nu}_I^4 + d\widetilde{\nu}_I^2 - c^2k_I^2 = 0$, which can be solved for $\widetilde{\nu}_I^2$ to yield $\widetilde{\nu}_I^2 = \frac{-d \pm \sqrt{d^2 + 4c^2k_I^2}}{2} = -\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + c^2k_I^2}$. Note that only the positive sign is acceptable since $\widetilde{\nu}_I \in \mathbb{R} \Rightarrow \widetilde{\nu}_I^2 \geq 0$. That is, $\widetilde{\nu}_I^2 = -\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2k_I^2}$ implying $\widetilde{\nu}_I = \pm\sqrt{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2k_I^2}}$. In turn, from the first of equations (4.25) we get $\widetilde{\nu}_R = \pm\frac{ck_I}{\sqrt{-\frac{d}{2}+\sqrt{\frac{d^2}{4}+c^2k_I^2}}}$ and so

$$\nu_R = \pm\frac{ck_I}{\sqrt{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2k_I^2}}} - \frac{k_R}{2} + \frac{\alpha}{2\beta}, \quad \nu_I = -\frac{k_I}{2} \pm \sqrt{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2k_I^2}}. \qquad (4.26)$$

Observe that the radicand of the outer square root involved in the above expressions is a non-negative number and hence that square root is a real (non-negative) number. In addition, note that expressions (4.26) are consistent with equations (4.18) and (4.20); however, their dependence on k_R and k_I (as opposed to k) is not suitable for discussing the analyticity of the associated expressions for ν, which is why (4.18) and (4.20) were used earlier for that purpose. On the other hand, (4.26) are the forms convenient for proving Lemma 4.1.

The case of the negative square root sign in (4.26) is straightforward as then $\nu_I \leq 0$ for all $k_I \geq 0$ and, in particular, for $k \in \overline{D^+}$ as desired. On the other hand, the case of positive square root sign in(4.26) requires more work. More specifically, by definition (4.13), for $k \in \overline{D^+}$ we have

$$3\left(k_R - \frac{\alpha}{3\beta}\right)^2 - k_I^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2} \leq 0, \qquad (4.27)$$

which can be rearranged to $-\frac{3}{4}k_R^2 + \frac{1}{4}k_I^2 + \frac{\alpha}{2\beta}k_R + \frac{\delta}{4\beta} \geq 0$. For $k_I \neq 0$ (note that $k_I = 0$ implies $\nu_I = 0$ and we are done), this is equivalent to $\frac{k_I^4}{16} + k_I^2\frac{d}{4} \geq c^2k_I^2$ or, after completing the square, $\left(\frac{k_I^2}{4} + \frac{d}{2}\right)^2 \geq \frac{d^2}{4} + c^2k_I^2$. Hence, $\frac{k_I^2}{4} \geq -\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2k_I^2}$ or $\frac{k_I^2}{4} \leq -\frac{d}{2} - \sqrt{\frac{d^2}{4} + c^2k_I^2}$ and, as the second inequality is not possible because it would imply that $k_I^2 \leq 0$, taking the square root of the first inequality and using the fact that $k_I \geq 0$ for $k \in \overline{D^+}$, we obtain $0 \geq -\frac{k_I}{2} + \sqrt{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2k_I^2}} = \nu_I$ as desired.

The proof so far has been under the assumption that $\nu(k) \neq k$; however, although $\nu \not\equiv k$ by hypothesis, there could still be points in $\overline{D^+}$ where $\nu(k) = k$ and hence this scenario must also be considered. In that case, recalling that $\nu_\pm$ satisfy $\beta(\nu_\pm^2 + k\nu_\pm + k^2) - \alpha(\nu_\pm + k) - \delta = 0$, we infer that if $k \in \mathbb{C}$ is such that $\nu_\pm(k) = k$ then $3\beta k^2 - 2\alpha k - \delta = 0$. If $\alpha^2 + 3\beta\delta \geq 0$, then $k = k_\pm = \frac{\alpha}{3\beta} \pm \frac{\sqrt{\alpha^2+3\beta\delta}}{3\beta} \in \mathbb{R}$ i.e. $k_I = \mathrm{Im}(\nu_\pm) = 0$ and we are done. If $\alpha^2 + 3\beta\delta < 0$, then $k = k_\pm = \frac{\alpha}{3\beta} \pm i\frac{\sqrt{-(\alpha^2+3\beta\delta)}}{3\beta}$. Note that $k_- \notin \overline{D^+}$ since $\text{Im}(k_-) < 0$. Also, $k_+ \notin \overline{D^+}$ because if $\alpha^2 + 3\beta\delta < 0$ and $k \in \overline{D^+}$ then by (4.27) we must have $k_I \geq \sqrt{-\frac{\alpha^2+3\beta\delta}{3\beta^2}} = \frac{\sqrt{-(\alpha^2+3\beta\delta)}}{\sqrt{3}\beta} > \frac{\sqrt{-(\alpha^2+3\beta\delta)}}{3\beta} = \text{Im}(k_+)$. This completes the proof of Lemma 4.1. □

Thanks to Lemma 4.1, both of the identities (4.24) are valid for $k \in \overline{D^+}$ and hence can be solved simultaneously as a system for the unknown transforms $\widetilde{g}_1(\omega(k), t)$ and $\widetilde{g}_2(\omega(k), t)$ to yield

$$\widetilde{g}_1(\omega(k), t) = \frac{e^{\omega(k)t}}{i\beta\,[\nu_+(k) - \nu_-(k)]}[\widehat{q}(\nu_+(k), t) - \widehat{q}(\nu_-(k), t)] + ik\widetilde{g}_0(\omega(k), t), \tag{4.28}$$

$$\widetilde{g}_2(\omega(k), t) = \frac{e^{\omega(k)t}}{\beta^2\,[\nu_-(k) - \nu_+(k)]}\left[(\beta\nu_-(k) - \alpha)\,\widehat{q}(\nu_+(k), t) - (\beta\nu_+(k) - \alpha)\,\widehat{q}(\nu_-(k), t)\right] - k^2\widetilde{g}_0(\omega(k), t). \tag{4.29}$$

Substituting these expressions in the integral representation (4.22), we obtain

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\int_\Gamma e^{ikx-\omega(k)t}\left(-3\beta k^2 + 2\alpha k + \delta\right)\widetilde{g}_0(\omega(k), t)dk \\ + \frac{1}{2\pi}\int_\Gamma e^{ikx}\left[\frac{\nu_-(k) - k}{\nu_-(k) - \nu_+(k)}\widehat{q}(\nu_+(k), t) - \frac{\nu_+(k) - k}{\nu_-(k) - \nu_+(k)}\widehat{q}(\nu_-(k), t)\right]dk. \end{aligned} \tag{4.30}$$

Note that the definition (4.23) of Γ in conjunction with the choices of the contour $\partial\widetilde{D}^+$ shown in Figure ensure that $\nu_-(k)-\nu_+(k)$ stays away from zero. Indeed, for $\alpha^2+3\beta\delta > 0$ the solutions of $\nu_-(k) - \nu_+(k) = 0$ occur at the branch points $b_\pm$, which lie on the real axis and outside segment $[\frac{1}{3\beta}(\alpha - \sqrt{\alpha^2 + 3\beta\delta}), \frac{1}{3\beta}(\alpha + \sqrt{\alpha^2 + 3\beta\delta})]$ forming the base of $\Gamma = \partial D^+$ (see Figure 4.1). Moreover, for $\alpha^2 + 3\beta\delta = 0$ the quantity $\nu_-(k) - \nu_+(k)$ vanishes at $\frac{\alpha}{3\beta}$, which is bypassed by $\Gamma = \partial D^+$ as shown in Figure 4.4. Finally, for $\alpha^2 + 3\beta\delta < 0$ the roots of $\nu_-(k) - \nu_+(k) = 0$ are again at the branch points $b_\pm$ and so they stay below the contour $\Gamma = \partial D^+$ depicted on Figure 4.5.

Therefore, using analyticity (Cauchy's theorem) along with exponential decay as $|k| \to \infty$ inside D^+ or $\widetilde{D}^+$, as appropriate, we conclude that the second k-integral on the right-hand side of (4.30) is equal to zero. (To see the decay, note that $|e^{ikx-i\nu_\pm y}| =$

$e^{-\mathrm{Im}(k)x+\mathrm{Im}(\nu_\pm)y}$ and use Lemma 4.1 together with the fact that $x, y > 0$.) Consequently, we deduce the solution formula

$$q(x,t) = -\frac{i}{2\pi}\int_\Gamma e^{ikx-\omega(k)t}\omega'(k)\,\widetilde{g}_0(\omega(k),t)dk. \tag{4.31}$$

In fact, noting that $\left|e^{-\omega(k)(t-t')}\right| = e^{\mathrm{Re}(\omega(k))(t'-t)}$ and recalling that, by definition (4.13), $\mathrm{Re}(\omega(k)) < 0$ inside D^+, we see that the exponential $e^{ikx-\omega(k)(t-t')}$ decays as $|k| \to \infty$ inside D^+ for all $x > 0$, $t' > t$. Thus, combining this decay with analyticity, in the second argument of the time transform $\widetilde{g}_0$ we can replace t by any fixed $T' > t$ and thereby obtain the following equivalent version of the solution formula (4.31), which is more convenient for the purpose of linear estimates as we will see below:

$$q(x,t) = -\frac{i}{2\pi}\int_\Gamma e^{ikx-\omega(k)t}\omega'(k)\,\widetilde{g}_0(\omega(k),T')dk. \tag{4.32}$$

4.1.2. Compatibility between the data

Recall that the initial and boundary data of the initial-boundary value problem (4.2) belong in the L^2-based Sobolev spaces $H_x^s(\mathbb{R}_+)$ and $H_t^{(s+1)/3}(0,T)$, respectively. Moreover, in view of the range of validity of Theorem 3.4 for the nonhomogeneous Cauchy problem established earlier, as well as of Theorem 4.1 for the reduced initial-boundary value problem proved below, we will restrict our attention to the range $0 \le s \le 2$ with $s \neq \frac{1}{2}$.

For $\frac{1}{2} < s \le 2$, continuity becomes relevant to our analysis and it turns out that we need to impose a compatibility condition between the initial and the boundary data. More specifically, note that if $\frac{1}{2} < s \le 2$ then $\frac{1}{2} < \frac{s+1}{3} \le 1$. Therefore, both of the traces $u_0(0)$ and $g(0)$ are well-defined. Furthermore, since $y(0,\cdot)$ and $z(0,\cdot)$ belong to $H_t^{(s+1)/3}(0,T)$ by Theorems 3.1 and 3.4, the traces $y(0,0)$ and $z(0,0)$ are well-defined and equal to $u_0(0)$ and 0, respectively, due to continuity and the initial conditions in problems (3.1) and (3.37).

Thus, using continuity at zero for the function $g_0 \in H_t^{(s+1)/3}(\mathbb{R})$ defined in (4.7), we have

$$g_0(0) = \lim_{t\to 0^+} g_0(t) = \lim_{t\to 0^+} [g(t) - y(0,t) - z(0,t)] = g(0) - y(0,0) - z(0,0) = g(0) - u_0(0)$$

which, upon imposing the (natural) compatibility condition

$$u_0(0) = g(0), \quad \frac{1}{2} < s \leq 2, \tag{4.33}$$

implies that the boundary datum of the reduced problem (4.7) vanishes at $t = 0$, i.e.

$$g_0(0) = g(0) - y(0,0) - z(0,0) = u_0(0) - u_0(0) - 0 = 0, \quad \frac{1}{2} < s \leq 2. \tag{4.34}$$

This feature will turn out to be convenient in the proof of Theorem 4.1 which follows next.

4.1.3. Sobolev-type estimates

We now establish the basic space estimate in the initial-boundary value problem setting. More precisely, we prove the following theorem

Theorem 4.1 *Let $s \geq 0$. Then, the unique solution of the reduced initial-boundary value problem* (4.7) *satisfies*

$$\|q(\cdot,t)\|_{H_x^s(\mathbb{R}_+)} \leq c\left(1 + \sqrt{T'}e^{cT'}\right)\|g_0\|_{H_t^{\frac{s+1}{3}}(0,T')} \tag{4.35}$$

uniformly for $t \in [0,T']$, where $c > 0$ is a constant that only depends on α, β, δ and s.

Proof We employ the Fokas method solution formula (4.32). First, recalling the definition (4.13) of D^+ and the various scenarios depending on the sign of $\alpha^2 + 3\beta\delta$, we parametrize the integration contour in (4.32) as $\Gamma = (-\gamma_1) \cup \gamma_2 \cup \gamma_3$ with

$$
\begin{aligned}
\gamma_1(m) &= \frac{\alpha - \sqrt{3\beta^2 m^2 + \alpha^2 + 3\beta\delta}}{3\beta} + im, && \lambda \le m < \infty, \\
\gamma_2(m) &= m + i\lambda, && c_- < m < c_+, \\
\gamma_3(m) &= \frac{\alpha + \sqrt{3\beta^2 m^2 + \alpha^2 + 3\beta\delta}}{3\beta} + im, && \lambda \le m < \infty,
\end{aligned}
\tag{4.36}
$$

where, $c_\pm = \frac{\alpha \pm \sqrt{3\beta^2\lambda^2 + \alpha^2 + 3\beta\delta}}{3\beta}$ and $\lambda > 0$ is a fixed non-negative real number, behaving like a cursor, such that

$$
\begin{cases}
\lambda = 0, & \alpha^2 + 3\beta\delta > 0, \quad \text{(first panel in Figure)} \\
\lambda > \frac{2\sqrt{-(\alpha^2+3\beta\delta)}}{3\beta}, & \alpha^2 + 3\beta\delta \le 0. \quad \text{(Figure)}
\end{cases}
\tag{4.37}
$$

In view of the above parametrization, for any $j \in \mathbb{N}_0$ we have

$$
\partial_x^j q(x,t) = -\frac{1}{2\pi}\int_\infty^\lambda (i\gamma_1(m))^j e^{i\gamma_1(m)x - (\omega(\gamma_1(m))t} \widetilde{g}_0(\omega(\gamma_1(m)), T') \frac{d[i\omega(\gamma_1(m))]}{dm}\, dm \tag{4.38}
$$

$$
- \frac{1}{2\pi}\int_{c_-}^{c_+} (i\gamma_2(m))^j e^{i\gamma_2(m)x - (\omega(\gamma_2(m))t} \widetilde{g}_0(\omega(\gamma_2(m)), T') \frac{d[i\omega(\gamma_2(m))]}{dm}\, dm \tag{4.39}
$$

$$
- \frac{1}{2\pi}\int_\lambda^\infty (i\gamma_3(m))^j e^{i\gamma_3(m)x - (\omega(\gamma_3(m))t} \widetilde{g}_0(\omega(\gamma_3(m)), T') \frac{d[i\omega(\gamma_3(m))]}{dm}\, dm \tag{4.40}
$$

$$
=: q_1(x,t) + q_2(x,t) + q_3(x,t).
$$

As the terms q_1 and q_3 are analogous, they can be handled in a similar fashion and hence we only provide the details for the estimation of q_1 given by (4.38). To this end, we need to recall an important lemma known as the *boundedness of the Laplace transform* in $L^2(\mathbb{R}_+)$.

Lemma 4.2 *[Lemma 3.2, Fokas et al. (2017)] Suppose that $Q(k) \in L_k^2(\mathbb{R}_+)$. Then, the map*

$$
Q(k) \to \int_0^\infty e^{-kx} Q(k)\, dk \tag{4.41}
$$

is bounded from $L^2_k(\mathbb{R}_+)$ into $L^2_x(\mathbb{R}_+)$ with

$$\left\| \int_0^\infty e^{-kx} Q(k)\, dk \right\|_{L^2_x(\mathbb{R}_+)} \leq \sqrt{\pi} \|Q\|_{L^2_k(\mathbb{R}_+)}. \tag{4.42}$$

After recalling this important lemma, we continue to our current proof. Since,

$$\begin{aligned} \|q_1(\cdot,t)\|^2_{L^2_x(\mathbb{R}_+)} &= \frac{1}{4\pi^2} \int_0^\infty \left| \int_\lambda^\infty (i\gamma_1(m))^j e^{i\gamma_1(m)x - (\omega(\gamma_1(m))t} \widetilde{g}_0(\omega(\gamma_1(m)), T') \frac{d[i\omega(\gamma_1(m))]}{dm}\, dm \right|^2 dx \\ &\lesssim \int_0^\infty \left(\int_0^\infty e^{-mx} |\gamma_1(m)|^j\, |\widetilde{g}_0(\omega(\gamma_1(m)), T')| \left| \frac{d[i\omega(\gamma_1(m))]}{dm} \right| \chi_{(\lambda,\infty)}(m) dm \right)^2 dx, \end{aligned}$$

by the boundedness of the Laplace transform in $L^2(\mathbb{R}_+)$, we have

$$\|q_1(\cdot,t)\|^2_{L^2_x(\mathbb{R}_+)} \lesssim \int_\lambda^\infty |\gamma_1(m)|^{2j} |\widetilde{g}_0(\omega(\gamma_1(m)), T')|^2 \left| \frac{d[i\omega(\gamma_1(m))]}{dm} \right|^2 dm. \tag{4.43}$$

Let $\tau(m) = i\omega(\gamma_1(m))$, $m \in [\lambda, \infty)$. Note that $\tau(m) \in \mathbb{R}$ since $\gamma_1(m) \in \partial D^+$ and $\mathrm{Re}(\omega(k)) = 0$ for $k \in \partial D^+$ and, more precisely, $\mathrm{Range}(\tau) = [i\omega(c_-), \infty)$. Furthermore, since $\tau'(m) \neq 0$ on (λ, ∞) and $\tau \to \infty$ as $m \to \infty$, it follows that $\tau : [\lambda, \infty) \to [i\omega(c_-), \infty)$ is monotone increasing and so $\tau'(m) > 0$. Then, (4.43) becomes

$$\begin{aligned} \|q_1(\cdot,t)\|^2_{L^2_x(\mathbb{R}_+)} &\lesssim \int_\lambda^\infty |\gamma_1(m)|^{2j} |\widetilde{g}_0(-i\tau(m), T')|^2 \left[\tau'(m)\right]^2 dm \\ &= \int_\lambda^\infty |\gamma_1(m)|^{2j} |\widehat{g}_0(\tau(m))|^2 \left[\tau'(m)\right]^2 dm \end{aligned} \tag{4.44}$$

after observing that the time transform (4.11) of g_0 at T' is in fact the Fourier transform of g_0 thanks to the fact that g_0 has compact support inside $(0, T')$, namely

$$\widetilde{g}_0(-i\tau(m), T') = \widehat{g}_0(\tau(m)). \tag{4.45}$$

Next, we have the following auxiliary result.

Lemma 4.3 *There is a constant $c > 0$ depending only on α, β, δ such that*

$$\sup_{m \in [\lambda, \infty)} \frac{|\gamma_1(m)|^{2j} \tau'(m)}{[1 + \tau^2(m)]^{\frac{j+1}{3}}} \leq c < \infty.$$

We prove Lemma 4.3 after the end of the current proof. Employing it in combination with (4.44), we obtain

$$\begin{aligned} \|q_1(\cdot, t)\|^2_{L^2_x(\mathbb{R}_+)} &\lesssim \int_\lambda^\infty [1 + \tau^2(m)]^{\frac{j+1}{3}} |\widehat{g}_0(\tau(m))|^2 \tau'(m) dm \\ &= \int_{i\omega(c_-)}^\infty (1 + \tau^2)^{\frac{j+1}{3}} |\widehat{g}_0(\tau)|^2 d\tau = \|g_0\|^2_{H_t^{\frac{j+1}{3}}(\mathbb{R})} \end{aligned} \tag{4.46}$$

uniformly for $t \in [0, T']$, completing the estimation of q_1.

We proceed to the estimation of q_2 given by (4.39).

Case 1: $\alpha^2 + 3\beta\delta > 0$. Then, $\lambda = 0$ and by the definition of γ_2 we can rewrite q_2 as

$$q_2(x, t) = -\frac{i}{2\pi} \int_{c_-}^{c_+} (im)^j e^{imx - \omega(m)t} \widetilde{g}_0(\omega(m), T') \, \omega'(m) dm. \tag{4.47}$$

so that $q_2(\cdot, t)$ can be regarded as the inverse (spatial) Fourier transform of the function

$$Q_2(m, t) = \begin{cases} 0, & m \notin (c_-, c_+), \\ -i(im)^j e^{-\omega(m)t} \widetilde{g}_0(\omega(m), T') \, \omega'(m), & m \in (c_-, c_+). \end{cases} \tag{4.48}$$

Note that $|e^{\omega(m)\rho}| = 1$ for $m \in (c_-, c_+), \rho \in \mathbb{R}$. Hence, using the definition of the t-transform (4.11) and the Cauchy-Schwarz inequality, we have

$$|\widetilde{g}_0(\omega(m), T')| \leq \sqrt{T'} \, \|g_0\|_{L^2_t(0, T')}. \tag{4.49}$$

implying via Plancherel's theorem that

$$
\begin{aligned}
\|q_2(\cdot,t)\|^2_{L^2_x(\mathbb{R}_+)} &= \int_{-\infty}^{\infty} |Q_2(m,t)|^2 dm \le T'\|g_0\|^2_{L^2_t(0,T')} \int_{c_-}^{c_+} |m|^{2j}|\omega'(m)|^2 dm \\
&= cT'\|g_0\|^2_{L^2_t(0,T')} \lesssim T'\|g_0\|^2_{H_t^{\frac{j+1}{3}}(0,T')}
\end{aligned}
\tag{4.50}
$$

with the various constants depending on α, β, δ, and j.

Case 2: $\alpha^2 + 3\beta\delta \le 0$. Then, $\lambda > \frac{2\sqrt{-(\alpha^2+3\beta\delta)}}{3\beta} > 0$ and, by the definition of γ_2,

$$
|q_2(x,t)| \le \frac{1}{2\pi} e^{-\lambda x} \int_{c_-}^{c_+} |m^2 + \lambda^2|^{\frac{j}{2}} \left| e^{-(\omega(m+i\lambda)t} \widetilde{g}_0(\omega(m+i\lambda), T')\, \omega'(m+i\lambda) \right| dm. \tag{4.51}
$$

Recall that for $k \in D^+$, we have $\mathrm{Re}(\omega(k)) < 0$, which implies $|e^{\omega(m+i\lambda)\rho}| \le 1$ for $m \in (c_-, c_+)$, $\rho \in [0, T']$. Therefore, similarly to (4.49),

$$
|\widetilde{g}_0(\omega(m+i\lambda), T')| \le \sqrt{T'}\, \|g_0\|_{L^2_t(0,T')}. \tag{4.52}
$$

Combining (4.51) and (4.52), we deduce

$$
|q_2(x,t)| \le c\sqrt{T'} e^{cT'} \|g_0\|_{L^2_t(0,T')} e^{-\lambda x}.
$$

Taking the square of the above inequality, integrating with respect to $x \in (0,\infty)$ (for this step, recall that $\lambda > 0$), and then taking square roots, we obtain

$$
\|q_2(\cdot,t)\|_{L^2_x(\mathbb{R}_+)} \le \frac{c}{\sqrt{2\lambda}} \sqrt{T'} e^{cT'} \|g_0\|_{L^2_t(0,T')} \lesssim \sqrt{T'} e^{cT'} \|g_0\|_{H_t^{\frac{j+1}{3}}(0,T')},
$$

where the constant of the last inequality depends only on α, β, δ and j.

The desired estimate (4.35) has been established for $s \in \mathbb{N}_0$. The proof for $s \ge 0$ follows by interpolation, which is given by the following theorem:

Theorem 4.2 *Lions and Magenes (1972) Let π be a continuous linear operator of X into*

H_X and of Y into H_Y for each defining Hilbert spaces. Then, π is also a continuous linear operator from the intermediate space $[X,Y]_\theta$ into $[H_X,H_Y]_\theta$ for $0<\theta<1$, where $[X,Y]_\theta$ is the domain of the operator $\Pi^{1-\theta}$ for a self-adjoint positive operator Π satisfying $(u,v)_X=(\Pi u,\Pi v)_Y$ for all $u,v\in X$.

$\square$

Now, we prove the auxiliary lemma that we used just before.

Proof [Proof of Lemma 4.3] First, we make a few observations. From the definition (3.5) of ω and the triangle inequality,

$$|\omega(k)|\ge\beta|k|^3-|\alpha k^2+\delta k|\ge\beta|k|^3-\left(|\alpha||k|^2+|\delta||k|\right).$$

In addition, for $|k|\ge\frac{|\alpha|+\sqrt{\alpha^2+2\beta|\delta|}}{\beta}$ we have $|\alpha||k|^2+|\delta||k|\le\frac{1}{2}\beta|k|^3$ and so, noting also that $\mathrm{Re}(\omega(k))=0$ along ∂D^+,

$$|i\omega(k)|\ge\frac{\beta}{2}|k|^3\;\Rightarrow\;\frac{1}{1+[i\omega(k)]^2}\le\frac{1}{1+\frac{\beta^2}{4}|k|^6}\simeq\frac{1}{1+|k|^6}. \tag{4.53}$$

Observe further that $|\gamma_1(m)|\ge m$ thus $|\gamma_1(m)|$ can be made as large as we wish by taking $m\in[\lambda,\infty)$ large enough. Therefore, using (4.53), for large enough m we have

$$\frac{1}{1+\tau^2(m)}=\frac{1}{1+[i\omega(\gamma_1(m))]^2}\lesssim\frac{1}{1+|\gamma_1(m)|^6}.$$

On the other hand, for $|k|\ge1$ we have $|k|^2\ge|k|$ and so by the triangle inequality

$$|\omega'(k)|\le3\beta|k|^2+2|\alpha||k|+|\delta|\le(3\beta+2|\alpha|+|\delta|)(1+|k|^2).$$

From the definition of γ_1, there exist non-negative constants c_1,c_2 depending on α,β,δ such that

$$|\gamma_1(m)|\le c_1m\text{ and }|\gamma_1'(m)|\le c_2,\quad m\in[\lambda,\infty).$$

Hence, there are some constants $c_3 > 0$, $M \geq \lambda$ depending on α, β, δ such that

$$\frac{|\gamma_1(m)|^{2j}| \left|\omega'(\gamma_1(m))\gamma_1'(m)\right|}{[1+\tau^2(m)]^{\frac{j+1}{3}}} \leq c_3 \frac{(|\gamma_1(m)|^2)^j(1+|\gamma_1(m)|^2)}{(1+|\gamma_1(m)|^2)^{j+1}} \leq c_3, \quad m > M.$$

However, by continuity of the function on the left-hand side on the compact interval $[\lambda, M]$, there is also some constant $c_4 > 0$ depending on α, β, δ such that

$$\frac{|\gamma_1(m)|^{2j}| \left|\omega'(\gamma_1(m))\gamma_1'(m)\right|}{[1+\tau^2(m)]^{\frac{j+1}{3}}} \leq c_4, \quad m \in [\lambda, M].$$

Combining the last two inequalities yields the desired estimate with $c = \max\{c_3, c_4\} < \infty$.

□

4.1.4. Strichartz-type estimates

It turns out convenient to reparametrize the contour of integration in the solution formula (4.32) of the reduced initial-boundary value problem (4.7) as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with

$$\begin{aligned}
\Gamma_1(m) &= m + i\sqrt{3\Big(m - \frac{\alpha}{3\beta}\Big)^2 - \frac{\alpha^2+3\beta\delta}{3\beta^2}}, && -\infty < m \leq c_-,\\
\Gamma_2(m) &= m + i\lambda, && c_- < m < c_+,\\
\Gamma_3(m) &= m + i\sqrt{3\Big(m - \frac{\alpha}{3\beta}\Big)^2 - \frac{\alpha^2+3\beta\delta}{3\beta^2}}, && c_+ \leq m < \infty,
\end{aligned} \tag{4.54}$$

where, as before, $c_\pm = \frac{\alpha \pm \sqrt{3\beta^2\lambda^2+\alpha^2+3\beta\delta}}{3\beta}$ and $\lambda > 0$ satisfies (4.37). With this parametrization, formula (4.32) can be expressed as the sum

$$q(x,t) = -\frac{i}{2\pi}\sum_{j=1}^{3}\int_{\Gamma_j} e^{ikx-\omega(k)t}\widetilde{g}_0(\omega(k), T')\omega'(k)dk =: \sum_{j=1}^{3} q_j(x,t).$$

We first consider q_1, which after recalling also (4.45) takes the form

$$\begin{aligned} q_1(x,t) &= -\frac{i}{2\pi}\int_{-\infty}^{c_-} e^{i\Gamma_1(m)x-\omega(\Gamma_1(m))t}\,\widehat{g}_0(i\omega(\Gamma_1(m)))\,\omega'(\Gamma_1(m))\Gamma_1'(m)dm \\ &= \frac{1}{2\pi}\int_{-\infty}^{c_-} e^{i\Gamma_1(m)x-\omega(\Gamma_1(m))t}\Big(\int_{-\infty}^{\infty} e^{-imy}\Psi_1(y)dy\Big)dm \end{aligned} \tag{4.55}$$

where Ψ_1 is the inverse Fourier transform of

$$\widehat{\Psi}_1(m) := \begin{cases} -i\widehat{g}_0(i\omega(\Gamma_1(m)))\,\omega'(\Gamma_1(m))\Gamma_1'(m), & m \le c_-, \\ 0, & m > c_-. \end{cases}$$

Then, introducing the kernel

$$\mathcal{K}(y;x,t) = \int_{-\infty}^{c_-} e^{i\phi(m;x,y,t)}p(m;x)dm \tag{4.56}$$

with amplitude

$$p(m;x) = e^{-x\sqrt{3\left(m-\frac{\alpha}{3\beta}\right)^2-\frac{\alpha^2+3\beta\delta}{3\beta^2}}} \tag{4.57}$$

and phase

$$\begin{aligned} \phi(m;x,y,t) &= m(x-y) + i\omega(\Gamma_1(m))t \\ &= m(x-y) + t\left[-8\beta m^3 + 8\alpha m^2 + 2\left(\delta - \frac{\alpha^2}{\beta}\right)m - \frac{\alpha\delta}{\beta}\right], \end{aligned} \tag{4.58}$$

we can rearrange (4.55) in the form

$$q_1(x,t) = [K_1(t)\Psi_1](x) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{K}(y;x,t)\Psi_1(y)dy. \tag{4.59}$$

This writing provides the starting point for proving the following central estimate of Strichartz type.

Theorem 4.3 *Let $s \ge 0$ and (μ, r) be higher-order Schrödinger admissible in the sense of*

(3.29). *Then,*

$$\|q\|_{L_t^{\mu}((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim (1+(T')^{\frac{1}{\mu}+\frac{1}{2}})\|g_0\|_{H_t^{\frac{s+1}{3}}(0,T')} \tag{4.60}$$

where $H_x^{s,r}(\mathbb{R}_+)$ is the restriction on $\mathbb{R}_+$ of the Bessel potential space $H_x^{s,r}(\mathbb{R})$ defined by (3.28) *and the inequality constant depends only r, s.*

Proof We will use a standard duality argument. Let $\eta \in C_c([0,T'];\mathcal{D}(\mathbb{R}_+))$ be an arbitrary function. Then,

$$\begin{aligned}
2\pi\left|\int_0^{T'}\langle K_1(t)\Psi_1,\eta(\cdot,t)\rangle_{L_x^2(\mathbb{R}_+)}\,dt\right| &= \left|\int_0^{T'}\int_0^{\infty}\left(\int_{-\infty}^{\infty}\mathcal{K}(y;x,t)\Psi_1(y)dy\right)\overline{\eta(x,t)}dxdt\right| \\
&= \int_{-\infty}^{\infty}\Psi_1(y)\,\overline{\int_0^{T'}\int_0^{\infty}\overline{\mathcal{K}(y;x,t)}\;\eta(x,t)dxdt}\,dy \\
&\le \|\Psi_1\|_{L^2(\mathbb{R})}\left\|\int_0^{T'}\int_0^{\infty}\overline{\mathcal{K}(y;x,t)}\;\eta(x,t)dxdt\right\|_{L_y^2(\mathbb{R})}.
\end{aligned} \tag{4.61}$$

Set $K_2(y) := \int_0^{T'}\int_0^{\infty}\overline{\mathcal{K}(y;x,t)}\eta(x,t)dxdt$. By the definition of the L^2-norm, we have

$$\begin{aligned}
\|K_2\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty}\left(\int_0^{T'}\int_0^{T'}\int_0^{\infty}\int_0^{\infty}\overline{\mathcal{K}(y;x,t)}\eta(x,t)\mathcal{K}(y;x',t')\overline{\eta(x',t')}dxdx'dtdt'\right)dy \\
&= \int_0^{T'}\int_0^{T'}\int_0^{\infty}\int_0^{\infty}\eta(x,t)\overline{\eta(x',t')}K_3(x,x';t,t')dxdx'dtdt' \\
&= \int_0^{T'}\int_0^{\infty}\eta(x,t)\left(\int_0^{T'}\int_0^{\infty}\overline{\eta(x',t')}K_3(x,x';t,t')dx'dt'\right)dxdt
\end{aligned}$$

where $K_3(x,x';t,t') := \int_{-\infty}^{\infty}\overline{\mathcal{K}(y;x,t)}\mathcal{K}(y;x',t')dy$. Then, by Hölder's inequality in (x,t) and then Minkowski's integral inequality between x and t' we deduce

$$\begin{aligned}
\|K_2\|_{L^2(\mathbb{R})}^2 &\le \|\eta\|_{L_t^{\mu'}((0,T');L_x^{r'}(\mathbb{R}_+))}\left\|\int_0^{T'}\int_0^{\infty}\overline{\eta(x',t')}K_3(x,x';t,t')dx'dt'\right\|_{L_t^{\mu}((0,T');L_x^{r}(\mathbb{R}_+))} \\
&\le \|\eta\|_{L_t^{\mu'}((0,T');L_x^{r'}(\mathbb{R}_+))}\left\|\int_0^{T'}\left\|\int_0^{\infty}\overline{\eta(x',t')}K_3(x,x';t,t')dx'\right\|_{L_x^r(\mathbb{R}_+)}dt'\right\|_{L_t^{\mu}(0,T')}.
\end{aligned} \tag{4.62}$$

We begin with the estimation of the interior $L_x^r(\mathbb{R}_+)$-norm. Using the definition

(4.56) of $\mathcal{K}$, we rewrite K_3 in the form of an oscillatory integral:

$$K_3(x,x';t,t') = \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{c_-} e^{-i\phi(m;x,y,t)}p(m;x)dm\Big)\Big(\int_{-\infty}^{c_-} e^{i\phi(m';x',y,t')}p(m';x')dm'\Big)dy$$
$$= \int_{-\infty}^{c_-} p(m;x)\int_{-\infty}^{\infty} e^{-i\phi(m;x,y,t)}\left(\int_{-\infty}^{c_-} e^{i\phi(m';x',y,t')}p(m';x')dm'\right)dydm.$$

Recalling the definition (4.58) of the phase function ϕ and introducing the function

$$Q(m';x',t') := \begin{cases} e^{im'x'-\omega(\Gamma_1(m'))t'}p(m';x'), & m' \in (-\infty, c_-], \\ 0, & m' \in (c_-, \infty), \end{cases}$$

we have via the Fourier inversion theorem

$$\int_{-\infty}^{\infty} e^{-i\phi(m;x,y,t)}\left(\int_{-\infty}^{c_-} e^{i\phi(m';x',y,t')}p(m';x')dm'\right)dy$$
$$= 2\pi e^{-imx+\omega(\Gamma_1(m))t}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{imy}\left(\int_{-\infty}^{\infty} e^{-im'y}Q(m';x',t')dm'\right)dy$$
$$= 2\pi e^{-imx+\omega(\Gamma_1(m))t}\,Q(m;x',t').$$

Thus, for $m \in (-\infty, c_-]$ we deduce

$$\int_{-\infty}^{\infty} e^{-i\phi(m;x,y,t)}\left(\int_{-\infty}^{c_-} e^{i\phi(m';x',y,t')}p(m';x')dm'\right)dy = 2\pi e^{-i\phi(m;x,x',t-t')}p(m;x')$$

and, consequently,

$$K_3(x,x';t,t') = 2\pi\int_{-\infty}^{c_-} e^{-i\phi(m;x,x',t-t')}p(m;x+x')dm.$$

Next, we employ the following fundamental result.

Lemma 4.4 *Let* $K(x,y,z,t) = \displaystyle\int_{-\infty}^{c_-} e^{i\phi(m;x,y,t)}p(m;z)dm$*, where* $x,z \in \mathbb{R}_+$ *and* $y,t \in \mathbb{R}$*. Then,*

$$|K(x,y,z,t)| \lesssim |t|^{-\frac{1}{3}}, \quad t \neq 0, \tag{4.63}$$

where the constant of the inequality is independent of x, y, z, t.

The proof of Lemma 4.4 relies on the classical van der Corput lemma and is provided after the end of the current proof. Observe that Lemma 4.4 with y, z, t replaced respectively by $x', x + x', t - t'$ yields

$$|K_3(x, x', t, t')| \lesssim |t - t'|^{-\frac{1}{3}}, \quad t \neq t',$$

with inequality constant independent of x, x', t and t'. This dispersive estimate implies

$$\left\| \int_0^\infty \overline{\eta(x', t')} K_3(x, x'; t, t') dx' \right\|_{L_x^\infty(\mathbb{R}_+)} \lesssim |t - t'|^{-\frac{1}{3}} \|\eta(t')\|_{L_x^1(\mathbb{R}_+)}. \tag{4.64}$$

On the other hand, we also have

$$\left\| \int_0^\infty \overline{\eta(x', t')} K_3(x, x'; t, t') dx' \right\|_{L_x^2(\mathbb{R}_+)} \lesssim \|\eta(t')\|_{L_x^2(\mathbb{R}_+)}. \tag{4.65}$$

Indeed, we have

$$\begin{aligned}
&\left\| \int_0^\infty \overline{\eta(x', t')} K_3(x, x'; t, t') dx' \right\|_{L_x^2(\mathbb{R}_+)}^2 = \int_0^\infty \left| \int_0^\infty \overline{\eta(x', t')} K_3(x, x'; t, t') dx' \right|^2 dx \\
&= (2\pi)^2 \int_0^\infty \left| \int_0^\infty \overline{\eta(x', t')} \left(\int_{-\infty}^{c_-} e^{-i\phi(m; x, x', t-t')} p(m; x + x') dm \right) dx' \right|^2 dx \\
&\leq (2\pi)^2 \int_0^\infty \left(\int_{-\infty}^{c_-} e^{-xs(m)} \left(\int_0^\infty e^{-x' s(m)} |\eta(x', t')| dx' \right) dm \right)^2 dx,
\end{aligned}$$

where $s(m) = \sqrt{3(m - \frac{\alpha}{3\beta})^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2}}$. The claimed estimate (4.65) then directly follows by invoking the following lemma, which provides a generalization of the $L^2(\mathbb{R}_+)$-boundedness of the Laplace transform, and is established after the end of the current proof.

Lemma 4.5 *The estimates in* (*i*) *and* (*ii*) *below hold true for* $f \in L_m^2(-\infty, c_-)$ *and* $f \in$

$L_x^2(\mathbb{R}_+)$, *respectively.*

$$\text{(i)} \quad \left\| \int_{-\infty}^{c_-} e^{-xs(m)} f(m) dm \right\|_{L_x^2(\mathbb{R}_+)} \lesssim \|f\|_{L_m^2(-\infty, c_-)},$$
$$\text{(ii)} \quad \left\| \int_0^{\infty} e^{-xs(m)} f(x) dx \right\|_{L_m^2(-\infty, c_-)} \lesssim \|f\|_{L_x^2(\mathbb{R}_+)}.$$

Now, (4.64) and (4.65) together with Riesz-Thorin interpolation theorem yield for any $r \geq 2$ that

$$\left\| \int_0^{\infty} \overline{\eta(x', t')} K_3(x, x'; t, t') dx' \right\|_{L_x^r(\mathbb{R}_+)} \lesssim |t - t'|^{-\frac{2}{\mu}} \|\eta(t')\|_{L_x^{r'}(\mathbb{R}_+)}, \tag{4.66}$$

where $\frac{1}{r'} = 1 - \frac{1}{r}$ and we have also used (3.29). Hence, for any $\eta \in L_t^{\mu'}((0, T'); L_x^{r'}(\mathbb{R}_+))$ we obtain

$$\int_0^{T'} \left\| \int_0^{\infty} \overline{\eta(x', t')} K_3(x, x'; t, t') dx' \right\|_{L_x^r(\mathbb{R})} dt' \lesssim \int_0^{T'} |t - t'|^{-\frac{2}{\mu}} \|\eta(t')\|_{L_x^{r'}(\mathbb{R})} dt'.$$

We handle the right-hand side via Hardy-Littlewood-Sobolev fractional integration, which is given by the following:

Theorem 4.4 (Theorem 1, Stein (1970)) *Let* $0 < \gamma < 1$, $1 \leqslant r < q < \infty$, $\frac{1}{q} = \frac{1}{r} - \gamma$. *If* $f \in L^r(\mathbb{R})$, *then the integral*

$$\left(I_\gamma f \right)(t) = \int_{t' \in \mathbb{R}} |t - t'|^{-1+\gamma} f(t') dt' \tag{4.67}$$

converges absolutely for almost every t. If, in addition, $1 < r$, *then*

$$\left\| I_\gamma f \right\|_{L^q(\mathbb{R}_t)} \leqslant c_{r,q,\gamma} \|f\|_{L^r(\mathbb{R}_t)}. \tag{4.68}$$

We combine the resulting inequality with (4.62), we infer

$$\|K_2\|_{L^2(\mathbb{R})} \lesssim \|\eta\|_{L_t^{\mu'}((0,T');L_x^{r'}(\mathbb{R}_+))}$$

which can be combined with (4.61) to yield

$$\|q_1\|_{L_t^{\mu}((0,T');L_x^{r}(\mathbb{R}_+))} \lesssim \|\Psi_1\|_{L^2(\mathbb{R})}. \tag{4.69}$$

Differentiating the expression (4.55) j times in x and repeating the above arguments, for any $j \in \mathbb{N}_0$ we conclude that

$$\|\partial_x^j q_1\|_{L_t^{\mu}((0,T');L_x^{r}(\mathbb{R}_+))} \lesssim \|\partial_x^j \Psi_1\|_{L^2(\mathbb{R})} \lesssim \|g_0\|_{H_t^{\frac{j+1}{3}}(\mathbb{R})}. \tag{4.70}$$

Observe that the left-hand side of estimate (4.70) is simply the $L_t^{\mu}((0,T'); W^{j,r}(\mathbb{R}_+))$-norm of q_1. In this connection, note that, according to a classical result by Calderón Calderón (1961), for any $j \in \mathbb{N}_0$, $1 < r < \infty$ the Sobolev space $W^{j,r}(\mathbb{R})$ and the Bessel potential space $H^{j,r}(\mathbb{R})$ coincide (i.e. they are equal as sets). Thus, for any $j \in \mathbb{N}_0$, $1 < r < \infty$ we have $\|\cdot\|_{H^{j,r}(\mathbb{R})} \simeq \|\cdot\|_{W^{j,r}(\mathbb{R})}$ and so

$$\|q_1\|_{W^{j,r}(\mathbb{R}_+)} := \inf_{\substack{\tilde{q}_1 \in W^{j,r}(\mathbb{R}) \\ \tilde{q}_1|_{\mathbb{R}_+} = q_1}} \|\tilde{q}_1\|_{W^{j,r}(\mathbb{R})} \simeq \inf_{\substack{\tilde{q}_1 \in H^{j,r}(\mathbb{R}) \\ \tilde{q}_1|_{\mathbb{R}_+} = q_1}} \|\tilde{q}_1\|_{H^{j,r}(\mathbb{R})} =: \|q_1\|_{H^{s,r}(\mathbb{R}_+)}. \tag{4.71}$$

Observing that the left-hand side of estimate (4.70) is simply the $W^{j,r}(\mathbb{R}_+)$-norm of q_1, in view of (4.71) we see that (4.70) is in fact equivalent to

$$\|q_1\|_{L_t^{\mu}((0,T');H_x^{j,r}(\mathbb{R}_+))} \lesssim \|g_0\|_{H_t^{\frac{j+1}{3}}(\mathbb{R})}. \tag{4.72}$$

Finally, by interpolation we deduce

$$\|q_1\|_{L_t^{\mu}((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim \|g_0\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})}, \quad s \geq 0, \tag{4.73}$$

completing the estimation of q_1.

In order to estimate $\|q_2\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))}$, we use (4.47)-(4.49) (note the difference in notation, as q_2 in those expressions now corresponds to $\partial_x^j q_2$) as the portions γ_2 and Γ_2 of the two parametrizations (4.36) and (4.54) coincide. In particular,

$$\|\partial_x^j q_2(\cdot,t)\|_{L_x^r(\mathbb{R}_+)} = \left(\int_{-\infty}^{\infty} |Q_2(m,t)|^r dm\right)^{\frac{1}{r}}$$
$$\leq \left((T')^{\frac{r}{2}} \|g_0\|_{L_t^2(0,T')}^r \int_{c_-}^{c_+} m^{jr} |\omega'(m)|^r dm\right)^{\frac{1}{r}} = c_{j,r}\sqrt{T'}\|g_0\|_{L_t^2(0,T')}.$$

Therefore, for any $j \in \mathbb{N}_0$ we find

$$\|\partial_x^j q_2\|_{L_t^\mu((0,T');L_x^r(\mathbb{R}_+))} = c_{j,r}(T')^{\frac{1}{\mu}+\frac{1}{2}}\|g_0\|_{L_t^2(0,T')} \leq c_{j,r}(T')^{\frac{1}{\mu}+\frac{1}{2}}\|g_0\|_{H_t^{\frac{j+1}{3}}(0,T')},$$

and, using again the equivalence of the Bessel potential and Sobolev norms (4.71) along with interpolation, we conclude that

$$\|q_2\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim (T')^{\frac{1}{\mu}+\frac{1}{2}}\|g_0\|_{H_t^{\frac{s+1}{3}}(0,T')}, \quad s \geq 0. \tag{4.74}$$

As the estimation of q_3 is similar to that of q_1, the proof of Theorem 4.3 is complete. □

We first recall the important Van der Corput lemma to continue with the proofs of the auxiliary lemmas that we used on the way.

Lemma 4.6 (Van der Corput, Stein (1970)) *Suppose that a real-valued function $\Phi(x)$ is smooth in an open interval (a,b), and that $|\Phi^{(k)}(x)| > 1$ for all $x \in (a,b)$. Assume that either $k \geq 2$, or that $k = 1$ and $\Phi'(x)$ is monotone for $x \in \mathbb{R}$. Then there is a constant c_k, which does not depend on Φ, such that*

$$\left|\int_a^b e^{i\lambda\Phi(x)}\, dx\right| \leq c_k \lambda^{-\frac{1}{k}} \tag{4.75}$$

for any $\lambda \in \mathbb{R}$.

Proof [Proof of Lemma 4.4] By the Fundamental Theorem of Calculus, K can be rewritten as

$$K(x,y,z,t) = -\int_{-\infty}^{c_-} \frac{dI}{dm}(m;x,y,t)\, p(m;z)dm,$$

where $I(m;x,y,t) := \int_m^{c_-} e^{i\phi(\xi;x,y,t)}d\xi$. Integrating by parts using the fact that $I(c_-;x,y,t) = 0$ and $p(m;z) \to 0$ as $m \to -\infty$, and noting also that $\frac{dp}{dm}(m;z) > 0$, we get

$$|K(x,y,z,t)| \leq \int_{-\infty}^{c_-} |I(m;x,y,t)|\, \frac{dp}{dm}(m;z)\, dm.$$

Noting that $|\phi^{(3)}(\xi;x,y,t)| = 48\beta|t|$, we can employ Van der Corput lemma for I with $\eta(\xi) = \phi(\xi;x,y,t)$ to infer that $|I(m;x,y,t)| \lesssim |t|^{-\frac{1}{3}}$, $t \neq 0$, where the constant of inequality is independent of m, x, y, t. In turn, for any $t \neq 0$ and $z > 0$ we obtain

$$|K(x,y,z,t)| \lesssim |t|^{-\frac{1}{3}} \int_{-\infty}^{c_-} \frac{dp}{dm}(m;z)dm = |t|^{-\frac{1}{3}} e^{-\lambda z} \leq |t|^{-\frac{1}{3}},$$

which is the desired estimate. □

Proof [Proof of Lemma 4.5] First, we prove part (i). By definition of $s(m)$, we have

$$\frac{ds(m)}{dm} = \frac{3(m-\frac{\alpha}{\beta})}{s(m)} = -\sqrt{3}\,\frac{\sqrt{s^2(m)+c_{\alpha,\beta,\delta}}}{s(m)}$$

with $c_{\alpha,\beta,\delta} = \frac{\alpha^2+3\beta\delta}{3\beta^2}$. Therefore, upon change of variable $s = s(m)$, we get

$$\int_{-\infty}^{c_-} e^{-xs(m)} f(m)dm = \frac{1}{\sqrt{3}} \int_{\lambda}^{\infty} e^{-xs} f(m(s)) \frac{s}{\sqrt{s^2+c_{\alpha,\beta,\delta}}} ds = \int_0^{\infty} e^{-xs} f_\lambda(s)ds,$$

where

$$f_\lambda(s) := \begin{cases} \frac{1}{\sqrt{3}} f(m(s)) \frac{s}{\sqrt{s^2+c_{\alpha,\beta,\delta}}} & s \in (\lambda,\infty), \\ 0, & s \notin (\lambda,\infty). \end{cases}$$

Using the L^2-boundedness of the Laplace transform, we get

$$\left\|\int_0^\infty e^{-xs} f_\lambda(s)ds\right\|_{L_x^2(\mathbb{R}_+)} \lesssim \|f_\lambda\|_{L_s^2(\mathbb{R}_+)}.$$

Finally, note that

$$\begin{aligned}\|f_\lambda\|^2_{L_s^2(\mathbb{R}_+)} &= \frac{1}{3}\int_\lambda^\infty |f(m(s))|^2 \frac{s^2}{s^2+c_{\alpha,\beta,\delta}} ds \\ &= \frac{1}{\sqrt{3}}\int_{-\infty}^{c_-} |f(m)|^2 \frac{s(m)}{\sqrt{s^2(m)+c_{\alpha,\beta,\delta}}} dm \lesssim \int_{-\infty}^{c_-} |f(m)|^2\, dm.\end{aligned}$$

Next, we establish part (ii). Setting $F(m) := \displaystyle\int_0^\infty e^{-xs(m)} f(x)dx$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|F(m)|^2 &= \left|\int_0^\infty e^{-xs(m)} f(x)dx\right|^2 = \left|\int_0^\infty e^{-\frac{xs(m)}{2}} f(x)x^{\frac{1}{4}} \cdot x^{-\frac{1}{4}} e^{-\frac{xs(m)}{2}} dx\right|^2 \\ &\leq \left(\int_0^\infty e^{-xs(m)}|f(x)|^2 x^{\frac{1}{2}} dx\right)\left(\int_0^\infty e^{-xs(m)} x^{-\frac{1}{2}} dx\right).\end{aligned}$$

Then, since the second integral on the right-hand side is equal to $\displaystyle\frac{1}{\sqrt{s(m)}}\int_0^\infty e^{-u} u^{-\frac{1}{2}} du = \frac{\sqrt{\pi}}{\sqrt{s(m)}}$,

$$\int_{-\infty}^{c_-} |F(m)|^2 dm = \sqrt{\pi}\int_{-\infty}^{c_-}\left(\int_0^\infty \frac{1}{\sqrt{s(m)}} e^{-xs(m)}|f(x)|^2 x^{\frac{1}{2}} dx\right) dm.$$

Finally, noting that

$$\int_{-\infty}^{c_-} \frac{1}{\sqrt{s(m)}} x^{\frac{1}{2}} e^{-xs(m)} dm = \frac{1}{\sqrt{3}}\int_\lambda^\infty e^{-xs} s^{-\frac{1}{2}} x^{\frac{1}{2}} \frac{s}{\sqrt{s^2+c_{\alpha,\beta,\delta}}} ds \lesssim \int_\lambda^\infty e^{-xs} s^{-\frac{1}{2}} x^{\frac{1}{2}} ds \leq c\sqrt{\pi},$$

we arrive at the desired estimate

$$\int_{-\infty}^{c_-} |F(m)|^2 dm \lesssim \int_0^\infty |f(x)|^2 dx$$

completing the proof of the lemma. ◻

4.2. Linear reunification

The nonlinear analysis will be performed by using a solution operator $u \mapsto \Phi u$ associated with the original forced linear initial-boundary value problem (4.2). To this end, thanks to the superposition principle we reunify the solution representation formulae corresponding to (i) the homogeneous Cauchy problem (4.3), (ii) the nonhomogeneous Cauchy problem (4.5), and (iii) the reduced initial-boundary value problem (4.6). More precisely, given u, we formally define the map

$$\begin{aligned}\Phi u &:= y|_{Q_T} + z^u|_{Q_T} + q^u|_{(0,T)} \\ &\equiv S\big[E_0 u_0; 0\big]\Big|_{Q_T} + S\big[0; f(Eu)\big]\Big|_{Q_T} - \frac{i}{2\pi}\int_\Gamma e^{ikx-\omega(k)t}\omega'(k)\,\widetilde{g}_0^u(\omega(k), T')dk\Big|_{(0,T)},\end{aligned} \tag{4.76}$$

where $Q_T = \mathbb{R}_+ \times (0, T)$ for some $T > 0$ to be determined and

$$g_0^u(t) := E_b\{g(\cdot) - S\big[E_0 u_0; 0\big](0,\cdot) - S\big[0; f(Eu)\big](0,\cdot)\}(t) \tag{4.77}$$

with the temporal transform $\widetilde{g}_0^u(\omega(k), T')$ defined according to (4.11). The extension operators E_0 and E_b were defined below problems (3.1) and (4.7) respectively; importantly, E_0 satisfies inequality and E_b induces compact support on g_0, namely $\text{supp} g_0 \subset [0, T')$, $T' > T$. Moreover, the operator E is a similar bounded fixed extension operator. In particular, for $s > \frac{1}{2}$ we take $E = E_0$ while for $0 \le s < \frac{1}{2}$ we take E from $H_x^s(\mathbb{R}_+) \cap H_x^{s,r}(\mathbb{R}_+)$ into $H_x^s(\mathbb{R}) \cap H_x^{s,r}(\mathbb{R})$ for a certain $r > 2$ to be specified later.

In view of (4.76), we define the solutions of the nonlinear problem (4.78) as the fixed points of the operator Φ. Thus, our goal will be to prove the existence of a unique such fixed point in a suitable function space. Throughout our analysis, we assume $u_0 \in H_x^s(\mathbb{R}_+)$ and $g \in H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$ with $s \in [0, 2] \setminus \{\frac{1}{2}\}$ and the compatibility conditions (4.33) in place as necessary. We first treat the high regularity case $\frac{1}{2} < s \le 2$ in which we are able to employ the algebra property of $H_x^s(\mathbb{R}_+)$, and then move on to the low regularity case $0 \le s < \frac{1}{2}$ in which we address the lack of the algebra property by refining our solution space motivated by the linear Strichartz estimates.

4.3. High regularity local well-posedness

We consider the nonhomogeneous initial-boundary value problem for the higher-order nonlinear Schrödinger (HNLS) equation on the half line

$$
\begin{aligned}
& iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = f(u), \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\
& u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \\
& u(0,t) = g(t), \quad t \in (0,T),
\end{aligned} \tag{4.78}
$$

where $\alpha, \delta \in \mathbb{R}$, $\beta > 0$, $f(z) = \kappa|z|^p z$ with $z \in \mathbb{C}$, $\kappa \in \mathbb{C}$, $p > 0$, and $T > 0$.

We prove the following result in this section.

Theorem 4.5 (High regularity well-posedness) *Let $\frac{1}{2} < s \leq 2$ and $p > 0$. In addition, if $p \notin 2\mathbb{Z}$, suppose that*

$$
\begin{aligned}
& \text{if } s \in \mathbb{Z}_+, \text{then } p \geq s \text{ if } p \in \mathbb{Z}_+ \text{ and odd}; \lfloor p \rfloor \geq s-1 \text{ if } p \notin \mathbb{Z}_+, \\
& \text{if } s \notin \mathbb{Z}_+, \text{then } p > s \text{ if } p \in \mathbb{Z}_+ \text{ and odd}; \lfloor p \rfloor \geq \lfloor s \rfloor \text{ if } p \notin \mathbb{Z}_+.
\end{aligned} \tag{4.79}
$$

Then, for initial data $u_0 \in H_x^s(\mathbb{R}_+)$ and boundary data $g \in H_{t,\mathrm{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$ satisfying the compatibility condition (4.33)*, there is $T = T(u_0, g) > 0$ such that the initial-boundary value problem* (4.78) *for the higher-order nonlinear Schrödinger equation on the half-line has a unique solution $u \in C([0,T]; H_x^s(\mathbb{R}_+))$. Furthermore, this solution depends continuously on the initial and boundary data.*

Our goal is to establish local well-posedness in the space $X_T := C([0,T]; H_x^s(\mathbb{R}_+))$ for some $T > 0$ to be determined. We consider X_T as a metric space with the metric

$$
d_{X_T}(u_1, u_2) := \|u_1 - u_2\|_{X_T}, \quad u_1, u_2 \in X_T.
$$

Note that any closed ball in X_T is a complete subspace.

Showing that Φ *is into.* The conservation law (3.2) in Theorem 3.1 and the boundedness of the spatial extension operator E_0 imply

$$\|y|_{Q_T}\|_{X_T} \leq \|S[E_0u_0; 0]\|_{C([0,T];H^s_x(\mathbb{R}))} = \|E_0u_0\|_{H^s_x(\mathbb{R})} \lesssim \|u_0\|_{H^s_x(\mathbb{R}_+)}, \tag{4.80}$$

which takes care of the first term in (4.76). For the second term in (4.76), let $u \in X_T$ and combine the nonhomogeneous estimate (3.40) in Theorem 3.4 with the algebra property in $H^s_x(\mathbb{R})$ to yield

$$\begin{aligned} \|z^u|_{Q_T}\|_{X_T} &\leq \left\|S[0; f(E_0u)]\right\|_{C([0,T];H^s_x(\mathbb{R}))} \lesssim \int_0^T \|f(E_0u(\cdot,t))\|_{H^s_x(\mathbb{R})} dt \\ &\lesssim \int_0^T \|E_0u(\cdot,t)\|^{p+1}_{H^s_x(\mathbb{R})} dt \lesssim \int_0^T \|u(\cdot,t)\|^{p+1}_{H^s_x(\mathbb{R}_+)} dt \lesssim T\|u\|^{p+1}_{X_T}. \end{aligned} \tag{4.81}$$

Regarding the third term in (4.76), using estimate (4.35) in Theorem 4.1 and the boundedness of the temporal extension operator E_b, we get (say with $T' = 2T$)

$$\begin{aligned} \|q^u|_{(0,T)}\|_{X_T} \leq \|q\|_{X_{T'}} &\lesssim (1 + \sqrt{T'}e^{cT'})\|g_0^u\|_{H_t^{\frac{s+1}{3}}(0,T')} \\ &\lesssim (1 + \sqrt{T'}e^{cT'})\|g_0^u\|_{H_t^{\frac{s+1}{3}}(0,T)} \lesssim (1 + \sqrt{T}e^{cT})\|g_0^u\|_{H_t^{\frac{s+1}{3}}(0,T)}. \end{aligned} \tag{4.82}$$

By using the definition of g_0 in (4.7) and temporal trace estimates (3.3), (3.4) and (3.41), we obtain

$$\begin{aligned} \|g_0^u\|_{H_t^{\frac{s+1}{3}}(0,T)} &\lesssim \|g\|_{H_t^{\frac{s+1}{3}}(0,T)} + (1 + T^{\frac{1}{2}})\|u_0\|_{H^s_x(\mathbb{R}_+)} \\ &\quad + \max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^{\sigma}\} \|f(E_0u)\|_{L^2_t((0,T);H^s_x(\mathbb{R}))}, \end{aligned} \tag{4.83}$$

with σ given by (3.42). By using the definition of the solution space X_T and the boundedness of the spatial extension operator E_0, we have

$$\|f(E_0u)\|_{L^2_t((0,T);H^s_x(\mathbb{R}))} \lesssim T^{\frac{1}{2}}\|u\|^{p+1}_{X_T}. \tag{4.84}$$

Using the definition (4.76) of Φ and combining estimates (4.80)-(4.84), we deduce

$$\|\Phi(u)\|_{X_T} \leq c_0 \left(c_1(T)\|u_0\|_{H_x^s(\mathbb{R}_+)} + c_2(T)\|g\|_{H_t^{\frac{s+1}{3}}(0,T)} + c_3(T)\|u\|_{X_T}^{p+1} \right), \tag{4.85}$$

where the positive constants c_1, c_2, c_3 are given by $c_1(T) = (1 + \sqrt{T}e^{cT})(1 + T^{\frac{1}{2}})$, $c_2(T) = (1 + \sqrt{T}e^{cT})$, $c_3(T) = T + (1 + \sqrt{T}e^{cT})T^{\frac{1}{2}} \max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^{\sigma}\}$ and c_0 is a non-negative constant independent of T and only depending on fixed parameters such as α, β, δ and s.

In view of estimate (4.85), we set $R(T) := 2A(T)$ with

$$A(T) := c_0\Big(c_1(T)\|u_0\|_{H_x^s(\mathbb{R}_+)} + c_2(T)\|g\|_{H_t^{\frac{s+1}{3}}(0,T)}\Big)$$

and choose T small enough so that $A(T) + c_0c_3(T)R(T)^{p+1} \leq R(T)$ or, equivalently, $c_0c_3(T)R^p(T) \leq \frac{1}{2}$. We note that such a choice is possible because $c_3(T) \to 0^+$ and $R(T)$ remains bounded as $T \to 0^+$. Then, for that choice of T, the map Φ takes the closed ball $\overline{B_{R(T)}(0)} \subset X_T$ into itself. It remains to show that Φ is a contraction on $\overline{B_{R(T)}(0)}$.

Showing that Φ is a contraction. Let $u_1, u_2 \in \overline{B_{R(T)}(0)}$. Then,

$$\begin{aligned} \|\Phi(u_1) - \Phi(u_2)\|_{X_T} &= \left\|z^{u_1}|_{Q_T} - z^{u_2}|_{Q_T}\right\|_{X_T} + \left\|q^{u_1}|_{(0,T)} - q^{u_2}|_{(0,T)}\right\|_{X_T} \\ &\lesssim \left\|S\left[0; f(E_0u_1) - f(E_0u_2)\right]\right\|_{C([0,T];H_x^s(\mathbb{R}))} \\ &\quad + (1 + \sqrt{T}e^{cT})\left\|g_0^{u_1} - g_0^{u_2}\right\|_{H_t^{\frac{s+1}{3}}(0,T)}. \end{aligned} \tag{4.86}$$

We then recall the following difference estimate (e.g. see Batal and Özsarı (2016)).

Lemma 4.7 *Let $s > \frac{1}{2}$, $p > 0$ satisfy* (4.79) *and $\varphi, \varphi_1, \varphi_2 \in H^s(\mathbb{R})$. Then,*

$$\||\varphi_1|^p\varphi_1 - |\varphi_2|^p\varphi_2\|_{H^s(\mathbb{R})} \lesssim \left(\|\varphi_1\|_{H^s(\mathbb{R})}^p + \|\varphi_2\|_{H^s(\mathbb{R})}^p\right)\|\varphi_1 - \varphi_2\|_{H^s(\mathbb{R})}.$$

Employing Lemma 4.7 and the arguments used earlier in (4.81), we deduce

$$\left\|S\left[0; f(E_0u_1) - f(E_0u_2)\right]\right\|_{C([0,T];H_x^s(\mathbb{R}))} \lesssim T(\|u_1\|_{X_T}^p + \|u_2\|_{X_T}^p)\|u_1 - u_2\|_{X_T}. \tag{4.87}$$

Moreover, for the difference of boundary data we have, similarly to (4.83),

$$
\begin{aligned}
\|g_0^{u_1} - g_0^{u_2}\|_{H_t^{\frac{s+1}{3}}(0,T)} &\lesssim \max\{T^{\frac{1}{2}}(1+T^{\frac{1}{2}}), T^{\sigma}\} \, \|f(E_0u_1) - f(E_0u_2)\|_{L_t^2((0,T);H_x^s(\mathbb{R}))} \\
&\lesssim \max\{T^{\frac{1}{2}}(1+T^{\frac{1}{2}}), T^{\sigma}\} T^{\frac{1}{2}} (\|u_1\|_{X_T}^p + \|u_2\|_{X_T}^p) \|u_1 - u_2\|_{X_T},
\end{aligned}
\tag{4.88}
$$

where σ is given by (3.42). Combining (4.87) and (4.88) with (4.86), we obtain

$$
\|\Phi(u_1) - \Phi(u_2)\|_{X_T} \lesssim c_3(T)(\|u_1\|_{X_T}^p + \|u_2\|_{X_T}^p)\|u_1 - u_2\|_{X_T} \lesssim c_3(T) R^p(T) \|u_1 - u_2\|_{X_T}. \tag{4.89}
$$

Note that $c_3(T) \to 0^+$ and $R(T)$ remains bounded as $T \to 0^+$. Therefore, for sufficiently small $T > 0$ the map Φ is a contraction on $\overline{B_{R(T)}(0)}$, and hence Φ has a unique fixed point in $\overline{B_{R(T)}(0)}$ which, as noted earlier, amounts to local existence of a unique solution to the initial-boundary value problem (4.78) for the higher-order nonlinear Schrödinger equation on $\overline{B_{R(T)}(0)}$.

Extending uniqueness to X_T. To prove uniqueness over the entire space X_T and not just the closed ball $\overline{B_{R(T)}(0)}$, we suppose that $u_1, u_2 \in X_T$ are two solutions associated with the same pair of initial and boundary data (u_0, g). At first, we consider the case of u_1, u_2 being sufficiently smooth and, along with their derivatives, decaying sufficiently fast as $x \to \infty$. This allows us to proceed via energy estimates. In particular, we note that the difference $w := u_1 - u_2$ solves the following problem:

$$
\begin{aligned}
& iw_t + i\beta w_{xxx} + \alpha w_{xx} + i\delta w_x = f(u_1) - f(u_2), \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\
& w(x,0) = 0, \quad x \in \mathbb{R}_+, \\
& w(0,t) = 0, \quad t \in (0,T).
\end{aligned}
\tag{4.90}
$$

Multiplying the main equation by $\overline{w}$, integrating in x, taking imaginary parts, and using

Lemma 4.7 and the embedding $H_x^s(\mathbb{R}_+) \hookrightarrow L_x^\infty(\mathbb{R}_+)$, which is valid for $s > \frac{1}{2}$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L_x^2(\mathbb{R}_+)} &= -\frac{\beta}{2}|w_x(0,t)|^2 + \text{Im}\int_0^\infty [f(u_1(x,t)) - f(u_2(x,t))]\bar{w}(x,t)dx \\ &\lesssim \int_0^\infty (|u_1(x,t)|^p + |u_2(x,t)|^p)|w(x,t)|^2 dx \\ &\lesssim (\|u_1(t)\|^p_{H_x^s(\mathbb{R}_+)} + \|u_2(t)\|^p_{H_x^s(\mathbb{R}_+)})\|w(t)\|^2_{L_x^2(\mathbb{R}_+)} \\ &\lesssim (\|u_1\|^p_{X_T} + \|u_2(t)\|^p_{X_T})\|w(t)\|^2_{L_x^2(\mathbb{R}_+)}.\end{aligned}$$

Setting $y(t) := \|w(t)\|_{L_x^2(\mathbb{R}_+)}$, the above energy estimate is satisfied provided that $y'(t) - cy(t) \leq 0$, $t \in (0,T)$ for some non-negative constant c. Solving this differential inequality alongside the condition $y(0) = \|w(0)\|_{L_x^2(\mathbb{R}_+)} = 0$ (note that $w(x,0) \equiv 0$), we obtain $y \equiv 0$ i.e. $w = u_1 - u_2 \equiv 0$. The case of rough u_1, u_2 can be treated via mollification.

Continuous dependence on the data. For $(u_0, g) \in H_x^s(\mathbb{R}_+) \times H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$, let

$$T_{\max} := \sup\{\, T > 0 \,|\, \text{there is a solution associated to the data } (u_0, g) \text{ on } [0,T]\}.$$

Then, either $T_{\max} = \infty$ or else $T_{\max} < \infty$ and there is no solution $u \in X_{T_{\max}}$ since otherwise the lifespan of u could be extended beyond $T_{\max}$ by starting with initial datum equal to $u(T_{\max})$. Therefore, we may let $u \in C([0,T_{\max}); H_x^s(\mathbb{R}_+))$ be the maximal solution associated to the data (u_0, g); then, for $T < T_{\max}$, in particular, $u|_{[0,T]}$ is the unique solution in X_T established above.

Let $T < T_{\max}$ be small enough that Φ is a contraction on $\overline{B_{R(T)}(0)}$ for any solution associated with data $(v_0, h) \in H_x^s(\mathbb{R}_+) \times H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$ and satisfying

$$\|v_0\|_{H_x^s(\mathbb{R}_+)} + \|h\|_{H_t^{\frac{s+1}{3}}(0,T)} \leq 2\left(\|u_0\|_{H_x^s(\mathbb{R}_+)} + \|g\|_{H_t^{\frac{s+1}{3}}(0,T)}\right).$$

If follows that if $\delta > 0$ is small enough, for (v_0, h) satisfying

$$\|v_0 - u_0\|_{H_x^s(\mathbb{R}_+)} + \|g - h\|_{H_t^{\frac{s+1}{3}}(0,T)} < \delta$$

the associated solution v belongs to $\overline{B_{R(T)}(0)}$. Therefore, u and v are both fixed points of Φ on $\overline{B_{R(T)}(0)}$ associated with the pairs of data (u_0, g) and (v_0, h), respectively. Then, the corresponding nonlinear estimates from the contraction argument imply

$$\|u - v\|_{X_T} = \|\Phi u - \Phi v\|_{X_T} \lesssim c(T)\Big(\|u_0 - v_0\|_{H_x^s(\mathbb{R}_+)} + \|g - h\|_{H_t^{\frac{s+1}{3}}(0,T)}\Big) \lesssim \delta c(T).$$

which amounts to continuity of the data-to-solution map. The proof of Theorem 4.5 for well-posedness in the high regularity setting is complete.

4.4. Low regularity local well-posedness

In this section, we prove the following theorem:

Theorem 4.6 (Low regularity well-posedness) *Suppose*

$$0 \le s < \frac{1}{2}, \quad 1 \le p \le \frac{6}{1-2s}, \quad \mu = \frac{6(p+1)}{p(1-2s)}, \quad r = \frac{2(p+1)}{1+2sp}. \tag{4.91}$$

Then, for initial data $u_0 \in H_x^s(\mathbb{R}_+)$ *and boundary data* $g \in H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$, *with the additional assumption that if* $p = \frac{6}{1-2s}$ *(critical case) then* $\|u_0\|_{H_x^s(\mathbb{R}_+)}$ *is sufficiently small, there is* $T = T(u_0, g) > 0$ *such that the initial-boundary value problem* (4.78) *for the higher-order nonlinear Schrödinger equation on the half-line has a unique solution* $u \in C([0,T]; H_x^s(\mathbb{R}_+)) \cap L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}_+))$. *Furthermore, this solution depends continuously on the initial and boundary data.*

The various linear estimates established in here will now be combined with a contraction mapping argument in order to establish local well-posedness in the sense of Hadamard for the nonlinear initial-boundary value problem (4.78). In view of these linear results, the solution space will change as we transition from the setting of high regularity ($\frac{1}{2} < s \le 2$) to the one of low regularity ($0 \le s < \frac{1}{2}$). More specifically, in the former case well-posedness will be established in the space $C([0,T]; H_x^s(\mathbb{R}_+))$ for a appropriate choice of $T > 0$ (see Theorem 4.5), while in the latter case that space will be refined by intersecting it with the Strichartz-inspired space $L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}_+))$ for an admissible choice of exponents (μ, r) in terms of the nonlinearity order p and the Sobolev exponent s according to (3.29) (see Theorem 4.6).

In this setting, we work under the assumptions (4.91). The lack of the algebra property brings in the need for the various Strichartz estimates established previously and hence motivates the solution space

$$Y_T := C([0,T]; H_x^s(\mathbb{R}_+)) \cap L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}_+)).$$

It is convenient to also consider the associated space on the whole spatial line, namely

$$\widetilde{Y}_T := C([0,T]; H_x^s(\mathbb{R})) \cap L_t^\mu((0,T); H_x^{s,r}(\mathbb{R})).$$

The following lemma will serve as the low regularity analogue of the algebra property and Lemma 4.7.

Lemma 4.8 *Let* (s,p)*,* (μ,r) *satisfy* (4.91) *and suppose* $\varphi, \varphi_1, \varphi_2 \in L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))$*. Then,*

$$\left\| |\varphi|^p \varphi \right\|_{L_t^1((0,T);H_x^s(\mathbb{R}))} \lesssim T^{\frac{\mu-p-1}{\mu}} \|\varphi\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))}^{p+1}, \tag{4.92}$$

$$\left\| |\varphi_1|^p \varphi_1 - |\varphi_2|^p \varphi_2 \right\|_{L_t^1((0,T);H_x^s(\mathbb{R}))} \lesssim T^{\frac{\mu-p-1}{\mu}} \Big(\|\varphi_1\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))}^p + \|\varphi_2\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))}^p \Big) \cdot \|\varphi_1 - \varphi_2\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))}. \tag{4.93}$$

Lemma 4.8 corresponds to the one-dimensional analogue of inequality (6.17) for the two-dimensional nonlinear Schrödinger equation proved in Himonas and Mantzavinos (2020). Note, importantly, that the admissibility conditions (4.91) are different than those in Himonas and Mantzavinos (2020) due to the third-order dispersion of the higher-order nonlinear Schrödinger equation. Thus, the proof of Lemma 4.8 does not follow from Himonas and Mantzavinos (2020).

We need the following two results about the fractional derivatives to prove Lemma 4.8.

Proposition 4.1 (Fractional chain rule, Proposition 3.1, Christ and Weinstein (1991)) *Suppose that $F \in C^1(\mathbb{C})$, $\alpha \in (0,1)$, $1 < p, q, r < \infty$, and $r^{-1} = p^{-1} + q^{-1}$. If $u \in L^\infty(\mathbb{R})$, $D^\alpha u \in L^q(\mathbb{R})$, and $F'(u) \in L^p(\mathbb{R})$. Then $D^\alpha F(u) \in L^r(\mathbb{R})$ and*

$$\|D^\alpha F(u)\|_{L^r(\mathbb{R})} \leq C\|F'(u)\|_{L^p(\mathbb{R})}\|D^\alpha u\|_{L^q(\mathbb{R})}. \tag{4.94}$$

Proposition 4.2 (Fractional product rule, Proposition 3.3, Christ and Weinstein (1991)) *Let $\alpha \in (0,1)$, $1 < p_1, p_2, q_1, q_2, r < \infty$, and $r^{-1} = p_i^{-1} + q_i^{-1}$ for $i = 1,2$. Suppose that $f \in L^{p_1}(\mathbb{R})$, $D^\alpha f \in L^{p_2}(\mathbb{R})$, $g \in L^{q_2}(\mathbb{R})$, $D^\alpha g \in L^{q_1}(\mathbb{R})$. Then $D^\alpha(fg) \in L^r(\mathbb{R})$ and*

$$\|D^\alpha(fg)\|_{L^r(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|D^\alpha g\|_{L^{q_1}(\mathbb{R})} + C\|g\|_{L^{q_2}(\mathbb{R})}\|D^\alpha f\|_{L^{p_2}(\mathbb{R})}. \tag{4.95}$$

Proof [Proof of Lemma 4.8] By Hölder's inequality,

$$\left\||\varphi|^p\varphi\right\|_{L^1_t((0,T);H^s(\mathbb{R}))} \leq T^{\frac{\mu-p-1}{\mu}}\left(\int_0^T \left\||\varphi(t)|^p\varphi(t)\right\|^{\frac{\mu}{p+1}}_{H^s_x(\mathbb{R})}dt\right)^{\frac{p+1}{\mu}}.$$

On the other hand, $\|\varphi\|^{p+1}_{L^\mu_t(0,T_0;H^{s,r}_x(\mathbb{R}))} = \left(\int_0^T \|\varphi(t)\|^\mu_{H^{s,r}_x(\mathbb{R})}dt\right)^{\frac{p+1}{\mu}}$. Hence, in order to establish (4.92), it suffices to prove that

$$\|D^\theta(|\varphi(t)|^p\varphi(t))\|_{L^2_x(\mathbb{R})} \lesssim \|\varphi(t)\|^{p+1}_{H^{\theta,r}_x(\mathbb{R})}, \quad t \in (0,T), \tag{4.96}$$

for $\theta = 0$ and $\theta = s$. To this end, we set $F(z) := |z|^p z$, $z \in \mathbb{C}$. If $s \neq 0$, by using the chain rule for fractional derivatives, we have

$$\|D^s F(\varphi(t))\|_{L^2_x(\mathbb{R})} \lesssim \|F'(\varphi(t))\|_{L_x^{\frac{\mu}{3}}(\mathbb{R})}\|D^s\varphi(t)\|_{L^r_x(\mathbb{R})} \tag{4.97}$$

with $\frac{1}{2} = \frac{3}{\mu} + \frac{1}{r}$. Noting that $|F'(\varphi(t))| \leq (p+1)|\varphi(t)|^p$, we further find

$$\|F'(\varphi(t))\|_{L_x^{\frac{\mu}{3}}(\mathbb{R})} \lesssim \|\varphi(t)\|^p_{L_x^{\frac{p\mu}{3}}(\mathbb{R})}, \tag{4.98}$$

while for $\frac{3}{p\mu} = \frac{1}{r} - s$ we also have the embedding

$$\|\varphi(t)\|_{L_x^{\frac{p\mu}{3}}(\mathbb{R})} \lesssim \|\varphi(t)\|_{H_x^{s,r}(\mathbb{R})}. \tag{4.99}$$

Combining (4.98) and (4.99) with (4.97), we obtain (4.96) for $\theta = s \neq 0$. Notice that $r = 2(p+1)$ for $s = 0$. Therefore,

$$\||\varphi(t)|^p\varphi(t)\|_{L_x^2(\mathbb{R})} = \|\varphi(t)\|^{p+1}_{L_x^{2(p+1)}(\mathbb{R})} = \|\varphi(t)\|^{p+1}_{L_x^r(\mathbb{R})},$$

which corresponds to (4.96) for $\theta = 0$.

Regarding inequality (4.93) for the differences, we first consider the case $s = 0$ which implies $r = 2(p+1)$. Using the standard pointwise difference estimate for the power-type nonlinearity and then applying Hölder's inequality in x, we get

$$\begin{aligned}\left\||\varphi_1|^p\varphi_1 - |\varphi_2|^p\varphi_2\right\|_{L_t^1((0,T);L_x^2(\mathbb{R}))} &\lesssim \int_0^T \left(\int_{-\infty}^{\infty}(|\varphi_1(x,t)|^p + |\varphi_2(x,t)|^p)^2|\varphi_1(x,t) - \varphi_2(x,t)|^2 dx\right)^{\frac{1}{2}} dt \\ &\lesssim \int_0^T \left(\|\varphi_1(t)\|^p_{L_x^r(\mathbb{R})} + \|\varphi_2(t)\|^p_{L_x^r(\mathbb{R})}\right)\|\varphi_1(t) - \varphi_2(t)\|_{L_x^r(\mathbb{R})} dt\end{aligned}$$

and the desired estimate (4.93) for $s = 0$ follows via Hölder's inequality in t.

Next, let us consider the case $s \neq 0$, in which $r = \frac{2(p+1)}{1+2sp}$. First, observe that for $z_1, z_2 \in \mathbb{C}$ and $\xi(\rho) = (1-\rho)z_2 + \rho z_1, \rho \in [0,1]$, we have $\xi(0) = z_1, \xi(1) = z_2, \xi'(\rho) = z_1 - z_2,$. Moreover,

$$\begin{aligned}|z_2|^p z_2 - |z_1|^p z_1 &= \int_0^1 \frac{d}{d\rho}\left(|\xi(\rho)|^p\xi(\rho)\right) d\rho \\ &= \frac{(p+2)}{2}(z_1 - z_2)\int_0^1 |\xi(\rho)|^p d\rho + \frac{p}{2}(\bar{z}_1 - \bar{z}_2)\int_0^1 |\xi(\rho)|^{p-2}\xi^2(\rho) d\rho.\end{aligned}$$

Combining this writing with the fractional product rule, we find

$$\begin{aligned}\|D^s F(\varphi_1(t)) - D^s F(\varphi_2(t))\|_{L^2_x(\mathbb{R})} \lesssim \|D^s(\varphi_1(t) - \varphi_2(t))\|_{L^r_x(\mathbb{R})} \sup_{\rho\in[0,1]} \||w(t)|^p\|_{L_x^{\frac{2r}{r-2}}(\mathbb{R})} \\ + \|\varphi_1(t) - \varphi_2(t)\|_{L_x^{\frac{p\mu}{3}}(\mathbb{R})} \Big(\sup_{\rho\in[0,1]} \left\{ \|D^s\left(G(w(t))\right)\|_{L_x^{r_1}(\mathbb{R})} \right\} \Big)\end{aligned}$$

where $\frac{1}{r_1} = \frac{1}{2} - \frac{3}{p\mu}$, $w(t) = (1-\rho)\varphi_2(t) + \rho\varphi_1(t)$ and $G(z) = F'(z) = \frac{p+2}{2}|z|^p + \frac{p}{2}|z|^{p-2}z^2$, $z \in \mathbb{C}$.

Observing that $|G'(w(t))| \leq p(p+1)|w(t)|^{p-1}$ for $p > 1$, we use the fractional chain rule to infer that, for $p > 1$,

$$\begin{aligned}\|D^s\left(G(w(t))\right)\|_{L_x^{r_1}(\mathbb{R})} &\lesssim \||w(t)|^{p-1}\|_{L_x^{r_2}(\mathbb{R})} \|D^s w(t)\|_{L^r_x(\mathbb{R})} \\ &\lesssim \|w(t)\|^{p-1}_{L_x^{\frac{\mu p}{3}}(\mathbb{R})} \|D^s w(t)\|_{L^r_x(\mathbb{R})} \lesssim \|w(t)\|^p_{H_x^{s,r}(\mathbb{R})},\end{aligned}$$

where $\frac{1}{r_2} = \frac{1}{r_1} - \frac{1}{r}$. In the above, the second inequality is due to the fact that, in view of (4.91), $r_2 = \frac{2(p+1)}{(p-1)(1-2s)} = \frac{\mu p}{3(p-1)}$, and the third inequality follows from the embedding (4.99). Furthermore, notice that $\frac{2r}{r-2} = \frac{\mu}{3}$ and so, using once again the embedding (4.99),

$$\||w(t)|^p\|_{L_x^{\frac{2r}{r-2}}(\mathbb{R})} = \|w(t)\|^p_{L_x^{\frac{\mu p}{3}}(\mathbb{R})} \lesssim \|w(t)\|^p_{H_x^{s,r}(\mathbb{R})}.$$

Combining the last three estimates, we deduce

$$\|D^s F(\varphi_1(t)) - D^s F(\varphi_2(t))\|_{L^2_x(\mathbb{R})} \lesssim \left(\|\varphi_1(t)\|^p_{H_x^{s,r}(\mathbb{R})} + \|\varphi_2(t)\|^p_{H_x^{s,r}(\mathbb{R})}\right) \|\varphi_1(t) - \varphi_2(t)\|_{H_x^{s,r}(\mathbb{R})}.$$

Then, integrating over $(0, T)$, applying Hölder's inequality in t, and combining the resulting estimate with the case of $s = 0$, we obtain (4.93) for $s \neq 0$ and $p > 1$.

Finally, for $p = 1$ we note that $\frac{1}{2} = \frac{1+2s}{4} + \frac{1-2s}{4} = \frac{1}{r} + \frac{1-2s}{4} = \frac{1}{r} + \frac{3}{\mu}$. Therefore,

$$\begin{aligned}
\|D^s F(\varphi_1(t)) - D^s F(\varphi_2(t))\|_{L^2_x(\mathbb{R})} &\lesssim \|D^s(\varphi_1(t) - \varphi_2(t))\|_{L^r_x(\mathbb{R})} \sup_{\rho\in[0,1]} \|w(t)\|_{L_x^{\frac{\mu}{3}}(\mathbb{R})} \\
&\quad + \|\varphi_1(t) - \varphi_2(t)\|_{L_x^{\frac{\mu}{3}}(\mathbb{R})} \Big(\sup_{\rho\in[0,1]} \left\{\|D^s (G(w(t)))\|_{L^r_x(\mathbb{R})}\right\} \Big) \\
&\lesssim \left(\|\varphi_1(t)\|^p_{H^{s,r}_x(\mathbb{R})} + \|\varphi_2(t)\|^p_{H^{s,r}_x(\mathbb{R})}\right) \|\varphi_1(t) - \varphi_2(t)\|_{H^{s,r}_x(\mathbb{R})}
\end{aligned}$$

with the last step thanks to the embedding (4.99). □

Now, we are ready to prove Theorem 4.6 for low regularity solutions.

Existence. First, we consider the *subcritical case* $p \neq \frac{6}{1-2s}$ so that $\frac{\mu-p-1}{\mu} > 0$. We work again with the solution operator (4.76), which was obtained via linear reunification. Theorems 3.1 and 3.3 imply

$$\|y|_{Q_T}\|_{Y_T} \leq \|y\|_{\widetilde{Y}_T} \lesssim \|E_0 u_0\|_{H^s_x(\mathbb{R})} \lesssim \|u_0\|_{H^s_x(\mathbb{R}_+)}, \tag{4.100}$$

while Theorems 3.4 and 3.6 along with inequality (4.92) and the same argument that was used in (4.81) yield

$$\|z^u|_{Q_T}\|_{Y_T} \leq \|z^u\|_{\widetilde{Y}_T} \lesssim (T + T^{\frac{\mu-p-1}{\mu}})\|u\|^{p+1}_{Y_T}. \tag{4.101}$$

Now, we estimate the last term in (5.61), we separate $g^u_0(t)$ defined in (5.62) as follows:

$$g^u_0(t) = E_b\{g(\cdot) - S[E_0 u_0; 0](0, \cdot)\}(t) - E_b\{S[0; f(Eu)](0, \cdot)\}(t) := g^{u,1}_0(t) - g^{u,2}_0(t), \tag{4.102}$$

which yields to rewrite

$$q^u|_{(0,T)} := q^{u,1}|_{(0,T)} - q^{u,2}|_{(0,T)}. \tag{4.103}$$

By using the arguments in (4.82) and (4.83), we have

$$\left\|q^{u,1}|_{(0,T)}\right\|_{Y_T} \lesssim \left(1 + \sqrt{T}e^{cT}\right)\left((1 + T^{\frac{1}{2}})\|u_0\|_{H^s_x(\mathbb{R}_+)} + \|g\|_{H_t^{\frac{s+1}{3}}(0,T)}\right). \tag{4.104}$$

To estimate $q^{u,2}|_{(0,T)}$, we use a rearrangement argument which is analogously used for the Schrödinger equation on the half-plane in Section 6 of Ref. Himonas and Mantzavinos (2020). Indeed, we observe that

$$\begin{aligned}
q^{u,2}(x,t) &= \frac{1}{2\pi}\int_\Gamma e^{ikx-w(k)t} i\omega'(k)\tilde{g}_0^{u,2}[w(k),T']dk \\
&= \frac{1}{2\pi}\int_\Gamma e^{ikx-w(k)t} i\omega'(k)\left(\int_0^{T'} e^{w(k)\tau} S[0;f(Eu)](0,\tau)d\tau\right)dk \\
&= \frac{-i}{2\pi}\int_\Gamma e^{ikx-w(k)t} i\omega'(k)\left(\int_0^{T'} e^{w(k)\tau}\left(\int_0^{t'} S[f(Eu);0](0,\tau-t')dt'\right)d\tau\right)dk \\
&= -i\int_0^{T'} q^*(x,t-\tau)d\tau,
\end{aligned}$$

by Fubini's theorem and the continuous (since $s<\frac{1}{2}$) extension by zero of $S[f(Eu);0]$ outside $[0,t']$, where q^* denotes the solution of the reduced initial-boundary value problem (5.2) with the boundary data $S[f(Eu);0]$. Hence, Theorems 4.3, 3.1, and 3.4 implies that

$$\left\|q^{u,2}|_{(0,T)}\right\|_{Y_T} \lesssim \left(1+e^{cT}T^{\frac{1}{\mu}+\frac{1}{2}}\right)(1+T^{\frac{1}{2}})\|f(Eu)\|_{L^1_t((0,T);H^s_x(\mathbb{R}_+))}. \tag{4.105}$$

Using (4.92) and combining the above estimates, we obtain

$$\|\Phi(u)\|_{Y_T} \le c_0\Big(c_1(T)\|u_0\|_{H^s_x(\mathbb{R}_+)} + c_2(T)\|g\|_{H_t^{\frac{s+1}{3}}(0,T)} + c_3(T)\|u\|^{p+1}_{Y_T}\Big), \tag{4.106}$$

where the positive constants c_1, c_2, c_3 are given by $c_1(T) = (1+\sqrt{T}e^{cT}+T^{\frac{1}{\mu}+\frac{1}{2}})(1+T^{\frac{1}{2}})$, $c_2(T) = (1+\sqrt{T}e^{cT}+T^{\frac{1}{\mu}+\frac{1}{2}})$, $c_3(T) = (T+T^{\frac{\mu-p-1}{\mu}}) + (1+\sqrt{T}e^{cT}+T^{\frac{1}{\mu}+\frac{1}{2}})T^{\frac{1}{2}}\max\{T^{\frac{1}{2}}(1+T^{\frac{1}{2}}), T^{\sigma}\}$ and c_0 is a non-negative constant independent of T and only depending on fixed parameters such as α, β, δ and s.

For the contraction, given $u_1, u_2 \in Y_T$ we employ inequality 4.93 together with the same arguments that led to (4.89) to infer

$$\|\Phi(u_1)-\Phi(u_2)\|_{Y_T} \lesssim c_3(T)(\|u_1\|^p_{Y_T}+\|u_2\|^p_{Y_T})\|u_1-u_2\|_{Y_T}. \tag{4.107}$$

This estimate implies the existence of a fixed point in Y_T for sufficiently small $T > 0$ via the same arguments that were used in the proof of Theorem 4.5.

Next, we consider the *critical case* $p = \frac{6}{1-2s}$. The difference here compared to the subcritical case is that the limit $c_3(T) \to 0^+$ as $T \to 0^+$ is no longer true; however, Φ is still a contraction provided that the data (and, correspondingly, the radius of the closed ball that depends on the size of the data) are chosen sufficiently small.

Uniqueness. We adapt the method used for the Cauchy problem in the proof of Proposition 4.2 of Cazenave and Weissler (1990) to the framework of initial-boundary value problems.

First, consider the subcritical case $p \neq \frac{6}{1-2s}$. Let $u_1 = \Phi(u_1), u_2 = \Phi(u_2) \in Y_T$ be two solutions associated with the same pair of initial and boundary data. Suppose to the contrary that there is $t \in [0, T]$ for which $u_1(t) \neq u_2(t)$, and let

$$t_{\text{inf}} := \inf \{t \in [0, T] \,|\, u_1(t) \neq u_2(t)\}.$$

Taking $t_n < t_{\text{inf}}$ such that $t_n \to t_{\text{inf}}^-$ as $n \to \infty$, we see that $u_1(t_n) = u_2(t_n)$ by definition of t_{inf}. Thus, in view of the fact that u_1, u_2 are both continuous from $[0, T]$ into $H_x^s(\mathbb{R}_+)$, taking the limit $n \to \infty$ we deduce that $u_1(t_{\text{inf}}) = u_2(t_{\text{inf}}) =: \varphi \in H_x^s(\mathbb{R}_+)$ makes sense. Set $U_1(t) = u_1(t + t_{\text{inf}})$ and $U_2(t) = u_2(t + t_{\text{inf}})$. Then, U_1 and U_2 are both solutions on the temporal interval $[0, T - t_{\text{inf}}]$ that satisfy the same initial and boundary conditions, namely

$$U_1(0) = U_2(0) = \varphi, \quad U_1|_{x=0} = U_2|_{x=0} = g(\cdot + t_{\text{inf}}) =: g_{\text{inf}}.$$

Since U_1 and U_2 are continuous in t, by the definition of t_{inf} there is a $\delta > 0$ such that $U_1 \neq U_2$ for $t \in (0, \delta)$. Let $t = t_{\text{inf}} + \epsilon$ with $\epsilon \in (0, \delta)$ fixed and to be specified below. We have

$$\begin{aligned} \|U_1 - U_2\|_{L_t^\mu((0,\epsilon);H_x^{s,r}(\mathbb{R}_+))} \lesssim c_{\text{inf}}(\epsilon)\big(\|U_1\|^p_{L_t^\mu((0,\epsilon);H_x^{s,r}(\mathbb{R}_+))} + \|U_2\|^p_{L_t^\mu((0,\epsilon);H_x^{s,r}(\mathbb{R}_+))}\big) \\ \cdot \|U_1 - U_2\|_{L_t^\mu((0,\epsilon);H_x^{s,r}(\mathbb{R}_+))}, \end{aligned} \tag{4.108}$$

where $c_{\inf}(\epsilon) := \epsilon^{\frac{\mu-p-1}{\mu}} + \epsilon^{\frac{1}{\mu}+\frac{1}{2}}\epsilon^{\frac{1}{2}} \max\{\epsilon^{\frac{1}{2}}(1+\epsilon^{\frac{1}{2}}), \epsilon^{\sigma}\}$. Let $\epsilon \in (0, \delta)$ be small enough so that

$$c_{\inf}(\epsilon)\big(\|U_1\|^p_{L^\mu_t((0,\epsilon);H^{s,r}_x(\mathbb{R}_+))} + \|U_2\|^p_{L^\mu_t((0,\epsilon);H^{s,r}_x(\mathbb{R}_+))}\big) < 1, \tag{4.109}$$

which is possible because $c_{\inf}(\epsilon) \to 0^+$ as $\epsilon \to 0^+$. Then, (4.108) implies that $U_1 = U_2$ on $(0, \epsilon) \subset (0, \delta)$, leading to a contradiction. Hence, uniqueness follows.

In the critical case $p = \frac{6}{1-2s}$, although the limit $c_{\inf}(\epsilon) \to 0^+$ as $\epsilon \to 0^+$ is no longer true, the uniqueness argument remains valid as (4.109) still holds due to the fact that, due to the dominated convergence theorem, the norms $\|U_1\|_{L^\mu_t((0,\epsilon);H^{s,r}_x(\mathbb{R}_+))}$ and $\|U_2\|_{L^\mu_t((0,\epsilon);H^{s,r}_x(\mathbb{R}_+))}$ can be made arbitrarily small by taking ϵ small enough.

Finally, the continuous dependence of the unique solution in Y_T on the initial and boundary data can be proved as in the high regularity setting, thereby completing the proof of Theorem 4.6.

CHAPTER 5

DOUBLE-BOUNDARY CONDITION CASE

In this chapter, we study the initial-boundary value problem for the higher-order nonlinear Schrödinger equation with double boundary conditions needed. We state the model first.

$$\begin{aligned} &iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = f(u), \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \\ &u(0,t) = h_0(t), \quad u_x(0,t) = h_1(t) \quad t \in (0,T), \end{aligned} \tag{5.1}$$

where $\alpha, \delta \in \mathbb{R}$, $\beta < 0$, $f(z) = \kappa|z|^p z$ with $z \in \mathbb{C}$, $\kappa \in \mathbb{C}$, $p > 0$, and $T > 0$.

The change in the sign of β from plus to minus causes an increment for the number of boundary conditions for the higher-order Schrödinger equation. We feel obligated to explain that this change entails some significant differences when studying the problem separately from the case that $\beta > 0$. Before the higher-order Schrödinger equation, let us consider a much more simpler problem, namely the heat equation. When the sign of the term with the greatest order in the spatial derivative is plus, the initial-boundary value problem for the heat equation is well-posed. On the contrary, when this aforementioned sign is minus, it is well known that the problem is ill-posed. So this gives us the first hint on the importance of the change for the sign in the spatial leading coefficient. As a second motivation, it is clearly explained in Deconinck et al. (2014) that for the evolution problems with an odd order in the spatial derivatives, such as the higher-order Schrödinger equation, the sign of the leading coefficient decides the number of boundary conditions needed for this problem.

Taking all of these into consideration, we approach the problem (5.1) as an entirely new problem in itself. Once we get into the details of its analysis about the well-posedness point of view, we encounter two possible scenarios, one of which is the similarities with the problem for the case $\beta > 0$, which is studied in Chapter 4, and the second is the differences in details. Therefore, while working on this problem, we will avoid repeating ourselves on one hand, and on the other hand, we will try to maintain the integrity of the problem itself.

We apply the same method that we used in Chapter 4 for the problem (4.1) stated for the case $\beta > 0$. Indeed, we first consider the linear version of the problem, and then use a *decompose-reunify* argument to observe the effect of each data separately, then finally pass from the linear theory to the nonlinear analysis to obtain the results on the well-posedness of the problem. Since the *decomposed* Cauchy problems are studied in Chapter 3 for any $\beta \neq 0$, we directly start with the reduced initial-boundary value problem.

5.1. Reduced initial-boundary value problem

We consider the initial-boundary value problem

$$\begin{aligned}
&iq_t + i\beta q_{xxx} + \alpha q_{xx} + i\delta q_x = 0, \quad (x,t) \in \mathbb{R}_+ \times (0,T'),\\
&q(x,0) = 0, \quad x \in \mathbb{R}_+,\\
&q(0,t) = g_0(t) := E_b^1(h_0 - y(0,\cdot) - z(0,\cdot))(t), \quad t \in (0,T'),\\
&q_x(0,t) = g_1(t) := E_b^0(h_1 - y_x(0,\cdot) - z_x(0,\cdot))(t), \quad t \in (0,T'),
\end{aligned} \tag{5.2}$$

where $\alpha, \delta \in \mathbb{R}, \beta < 0, T' > T$, $y(0,t)$ and $z(0,t)$ are the traces of the solutions of Cauchy problems (3.1) and (3.37), respectively, at $x = 0$, and $E_b^j : H_t^{\frac{s+j}{3}}(0,T) \to H_t^{\frac{s+j}{3}}(\mathbb{R})$ are two fixed bounded extension operators for $j \in \{0,1\}$, satisfying the additional property *supp* $g_j \subset [0,T')$. It is provided that the traces $\partial_x^j y(0,t)$ and $\partial_x^j z(0,t)$ are well-defined and belong to $H_t^{\frac{s+1-j}{3}}(0,T)$ for each $j \in \{0,1\}$ in view of Theorems 3.1, 3.2, 3.40 and 3.5.

We first obtain a representation formula for the weak solution $q(x,t)$ of (5.2) by using Fokas method. To this end, we assume q is smooth and decays sufficiently fast as $x \to \infty$, uniformly in $t \in [0,T']$.

Applying the half-line Fourier transform (4.9) to the main equation in (5.2) and integrating in the temporal variable t, we obtain an identity, which is known as global relation:

$$e^{w(k)t}\hat{q}(k,t) = (-\beta k^2 + \alpha k + \delta)\tilde{g}_0[w(k),t] + (i\beta k - i\alpha)\tilde{g}_1[w(k),t] + \beta\tilde{g}_2[w(k),t], \quad \text{Im}k \leq 0, \tag{5.3}$$

by denoting

$$\tilde{g}_j(k,t) = \int_0^t e^{k\tau}\partial_x^j q(0,\tau)\, d\tau, \quad k \in \mathbb{C}, \quad j = 0,1,2. \tag{5.4}$$

and

$$w(k) = -i\beta k^3 + i\alpha k^2 + i\delta k. \tag{5.5}$$

Multiplying both sides in (5.3) by $e^{-w(k)t}$, and then taking the inverse Fourier transform (4.9), we obtain for $(x,t) \in \mathbb{R}_+ \times (0,T')$ that

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-w(k)t}\tilde{g}(k,t)dk, \tag{5.6}$$

where $\tilde{g}(k,t) = (-\beta k^2 + \alpha k + \delta)\tilde{g}_0[w(k),t] + (i\beta k - i\alpha)\tilde{g}_1[w(k),t] + \beta\tilde{g}_2[w(k),t]$.

Note that the equation (5.6) involves the unknown boundary condition $q_{xx}(0,t)$, which is hidden in the boundary t-transform $\tilde{g}_2$. We eliminate this term by first deforming the real axis to a suitable contour in the complex plane, and then utilizing the invariant properties of the polynomial $w(k)$ in the global relation (5.3).

We introduce the region $D^+ := D \cap \mathbb{C}^+$, where $D \equiv \{k \in \mathbb{C} : \ \text{Re } w(k) < 0\}$ and $\mathbb{C}^+ := \{k \in \mathbb{C} : \ \text{Im } k > 0\}$. Observe that D^+ can be defined explicitly as follows:

$$D^+ \equiv \left\{k \in \mathbb{C}^+ \ : \ 3\left(k_R - \frac{\alpha}{3\beta}\right)^2 - k_I^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2} > 0\right\}, \tag{5.7}$$

where the real and imaginary parts of k are denoted by k_R and k_I, respectively. See Figures 5.1, 5.2 and 5.3.

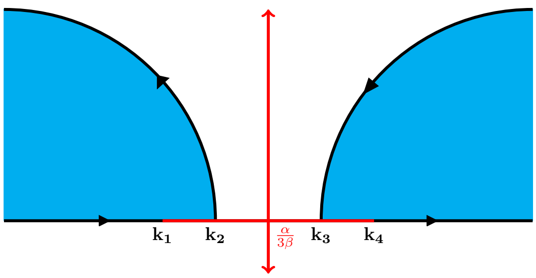

Figure 5.1. The region D^+ for $\alpha^2 + 3\beta\delta > 0$.

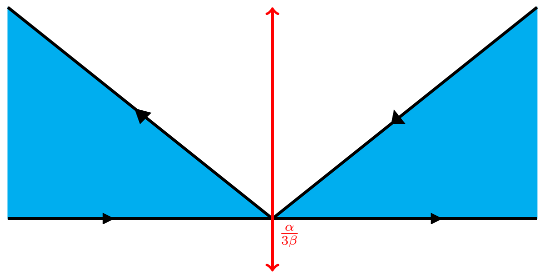

Figure 5.2. The region D^+ for $\alpha^2 + 3\beta\delta = 0$.

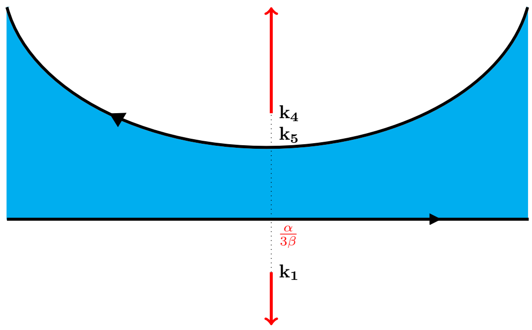

Figure 5.3. The region D^+ for $\alpha^2 + 3\beta\delta < 0$.

Note that in each figures, we define $k_1 = \frac{\alpha+2\sqrt{\alpha^2+3\beta\delta}}{3\beta}$, $k_2 = \frac{\alpha+\sqrt{\alpha^2+3\beta\delta}}{3\beta}$, $k_3 = \frac{\alpha-\sqrt{\alpha^2+3\beta\delta}}{3\beta}$, $k_4 = \frac{\alpha-2\sqrt{\alpha^2+3\beta\delta}}{3\beta}$, and $k_5 = \frac{i\sqrt{-3(\alpha^2+3\beta\delta)}}{-3\beta}$.

Using the analytic behavior and exponential decay of the integral in (5.6), we can deform the real axis to the boundary of ∂D^+ and obtain

$$q(x,t) = \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-w(k)t}\tilde{g}(k)\,dk, \tag{5.8}$$

where ∂D^+ is oriented, as depicted in Figures 5.1, 5.2 and 5.3, in such a way that D^+ stays on the left to its boundary ∂D^+.

Note that the global relation (5.3) is not valid in ∂D^+, since $\operatorname{Im} k \geq 0$ for $k \in \overline{D^+}$. On the other hand, the right hand side of (5.3) involves the terms depending on $w(k)$. So, we try to rewrite this relation by changing k with some $v(k)$, which makes the equation valid for any values of $k \in \overline{D^+}$ and also keeps $w(k)$ stable. In order to determine these invariant maps, we solve the equation $w(v) = w(k)$ for $v = v(k)$. But, this argument should be handled carefully. Since w is a third order polynomial defined on $\mathbb{C}$, the fundamental theorem of algebra ensures that we can find three roots, counting multiplicity, for v in terms of k. The first root is trivially $v(k) = k$. We have two more roots and these roots satisfy

$$v^2 + \left(k - \frac{\alpha}{\beta}\right)v + \left(k^2 - \frac{\alpha}{\beta}k - \frac{\delta}{\beta}\right) = 0 \tag{5.9}$$

Clearly the solution of the equation (5.9) has two solutions, say $v_1(k), v_2(k) \in \mathbb{C}$, counting multiplicity. But, we need to find distinct and nontrivial maps, i.e. $v_1(k) \neq k$, $v_2(k) \neq k$ and also $v_1 \neq v_2$. This can be possible if only if

$$\Delta := \frac{3}{4}\left(k - \frac{\alpha}{3\beta}\right)^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2} = 0$$

Therefore, we need to stay away from the following values of k, depending on $\alpha^2 + 3\beta\delta$:

$$k = \begin{cases} \frac{\alpha \pm 2\sqrt{\alpha^2+3\beta\delta}}{3\beta}, & \alpha^2 + 3\beta\delta > 0, \\ \frac{\alpha}{3\beta}, & \alpha^2 + 3\beta\delta = 0, \\ \frac{\alpha \pm 2i\sqrt{-\alpha^2-3\beta\delta}}{3\beta}, & \alpha^2 + 3\beta\delta < 0. \end{cases} \tag{5.10}$$

For the case $\alpha^2+3\beta\delta < 0$, the given values of k in (5.10) is outside of $\overline{D^+}$. However, when $\alpha^2 + 3\beta\delta \geq 0$, the aforementioned values of k is involved in $\overline{D^+}$. Thanks to Cauchy's theorem, we can deform the contour D^+ to the modified $\tilde{D}^+$ to keep these values of k defined in (5.10) away from the domain of the integral in (5.8). See Figures 5.4 and 5.5.

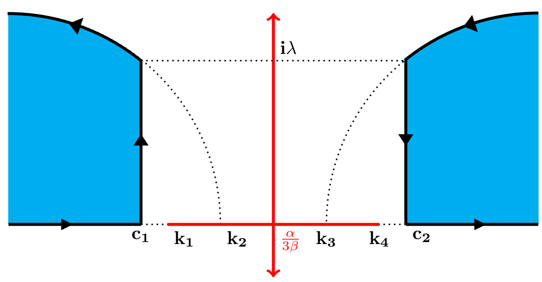

Figure 5.4. Deformation of ∂D^{+} to $\partial\widetilde{D}^{+}$ for $\alpha^2 + 3\beta\delta > 0$.

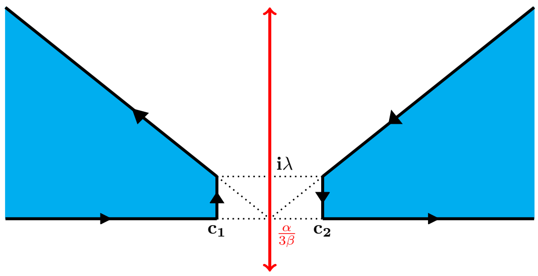

Figure 5.5. Deformation of ∂D^{+} to $\partial\widetilde{D}^{+}$ for $\alpha^2 + 3\beta\delta = 0$.

We define k_j's are defined for $j = \overline{1,4}$ as before, and $c_1 = \frac{\alpha+\sqrt{3\beta^2\lambda^2+\alpha^2+3\beta\delta}}{3\beta}$, $c_2 = \frac{\alpha-\sqrt{3\beta^2\lambda^2+\alpha^2+3\beta\delta}}{3\beta}$ with λ as defined in (5.30).

The above discussion guarantees that the maps v_1 and v_2 are distinct and not identical for the values of $k \in \Gamma$, where

$$\Gamma = \begin{cases} \partial\tilde{D}^{+}, & \alpha^2 + 3\beta\delta \geq 0, \\ \partial D^{+}, & \alpha^2 + 3\beta\delta < 0. \end{cases} \tag{5.11}$$

Therefore we can rewrite (5.8), to use the invariant properties of the maps v_1 and v_2, as follows:

$$q(x,t) = \frac{1}{2\pi}\int_{\Gamma} e^{ikx-w(k)t}\tilde{g}(k)\, dk. \tag{5.12}$$

Let us define these maps explicitly. Solving (5.9), we obtain

$$v_{1,2}(k) = -\frac{k}{2} + \frac{\alpha}{\beta} \pm \left(-\frac{3}{4}(k - \frac{\alpha}{3\beta})^2 + \frac{\alpha^2 + 3\beta\delta}{3\beta^2}\right)^{\frac{1}{2}}, \tag{5.13}$$

where,

$$z^{\frac{1}{2}} := \sqrt{|z|}\, e^{\frac{i\theta}{2}}, \qquad \theta \in [0, 2\pi), \tag{5.14}$$

for any $z \in \mathbb{C}$ with the principal argument θ.

We should emphasize that v_1 and v_2, as defined in (5.13), are analytic for the values of $k \in \overline{D^+}$ when $\alpha^2 + 3\beta\delta < 0$, and of $k \in \overline{\tilde{D}^+}$ when $\alpha^2 + 3\beta\delta \geq 0$. Observe that the intersection of the branch cuts and the corresponding region $\overline{D^+}$, or $\overline{\tilde{D}^+}$, is empty set. Indeed, the values of k defined in (5.10) also correspond to the branch points of the term with complex square root in (5.13). So, the extra deformation $\tilde{D}^+$ also preserves the analyticity of these maps.

Remark 5.1 *Note that v_1 and v_2 can also be defined as single-valued functions depending on k. Even though this has no pros on our process, we refer to the corresponding part in Chapter 4.*

As we mentioned before, we use the invariant properties of these maps on the global relation (5.3), since this relation is not valid for $k \in \Gamma$(Note that Im $k \geq 0$ on Γ). The idea, which is changing k with the maps v_1 and/or v_2, makes sense if and only if Im $v_1(k) \leq 0$ and/or Im $v_2(k) \leq 0$ for $k \in \Gamma$. To this end, we prove the following lemma.

Lemma 5.1 *If $k \in \overline{D^+}$, then Im $v_1(k) \leq 0$, and Im $v_2(k) \geq 0$, where*

$$v_1(k) := -\frac{k}{2} + \frac{\alpha}{\beta} - \left(-\frac{3}{4}(k - \frac{\alpha}{3\beta})^2 + \frac{\alpha^2 + 3\beta\delta}{3\beta^2}\right)^{\frac{1}{2}}, \tag{5.15}$$

and

$$v_2(k) := -\frac{k}{2} + \frac{\alpha}{\beta} + \left(-\frac{3}{4}(k - \frac{\alpha}{3\beta})^2 + \frac{\alpha^2 + 3\beta\delta}{3\beta^2}\right)^{\frac{1}{2}}. \tag{5.16}$$

Proof We have a system of equations from the real and the imaginary parts of the equation (5.9), where v stands for v_1 and v_2. Namely, we have

$$\left(v_R + \frac{k_R}{2} - \frac{\alpha}{2\beta}\right)\left(v_I + \frac{k_I}{2}\right) = \left(\frac{\alpha}{4\beta} - \frac{3k_R}{4}\right)k_I \tag{5.17}$$

$$\left(v_R + \frac{k_R}{2} - \frac{\alpha}{2\beta}\right)^2 - \left(v_I + \frac{k_I}{2}\right)^2 = -\frac{3}{4}k_R^2 + \frac{3}{4}k_I^2 + \frac{\alpha}{2\beta}k_R + \frac{\alpha^2}{4\beta^2} + \frac{\delta}{\beta}. \tag{5.18}$$

Define $\tilde{v}_R := v_R + \frac{k_R}{2} - \frac{\alpha}{2\beta}$, $\tilde{v}_I := v_I + \frac{k_I}{2}$, $c := \frac{\alpha}{4\beta} - \frac{3k_R}{4}$ and $d := -\frac{3}{4}k_R^2 + \frac{3}{4}k_I^2 + \frac{\alpha}{2\beta}k_R + \frac{\alpha^2}{4\beta^2} + \frac{\delta}{\beta}$ and then substitute (5.17) into (5.18). We obtain

$$\tilde{v}_I^4 + d\tilde{v}_I^2 - c^2k_I^2 = 0. \tag{5.19}$$

Solving (5.19) for $\tilde{v}_I^2$, we get

$$v_I = -\frac{k_I}{2} - \sqrt{\frac{-d}{2} + \sqrt{\frac{d^2}{4} + c^2k_I^2}},$$

which is clearly less than or equal to zero, or

$$v_I = -\frac{k_I}{2} + \sqrt{\frac{-d}{2} + \sqrt{\frac{d^2}{4} + c^2k_I^2}},$$

which is greater than or equal to zero by the definition of the domain D_+.

Indeed, by definition (5.7), for $k \in \overline{D^+}$ we have

$$3(k_R - \frac{\alpha}{3\beta})^2 - k_I^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2} \geq 0,$$

which can be arranged to

$$\frac{3}{4}k_R^2 - \frac{1}{4}k_I^2 - \frac{\alpha}{2\beta}k_R - \frac{\delta}{4\beta} \leq 0.$$

For $k_I \neq 0$(note that if $k_I = 0$, then we are done) this is equivalent to

$$c^2k_I^2 \geq \frac{k_I^4}{16} + d\frac{k_I^2}{4}.$$

After completing the square, and then taking square root we have

$$\frac{-k_I}{2} + \sqrt{\frac{-d}{2} + \sqrt{\frac{d^2}{4} + c^2k_I^2}} \geq 0.$$

On the other hand, v_1 corresponds to the one with the negative imaginary part, since

$$\text{Im } v_1(k) = -\frac{k_I}{2} - \sqrt{|k^*|}\sin\left(\frac{\text{Arg}(k^*)}{2}\right) \leq 0,$$

for $k^* := -\frac{3}{4}\left(k - \frac{\alpha}{3\beta}\right)^2 + \frac{\alpha^2+3\beta\delta}{3\beta^2}$, since $\frac{\text{Arg}(k^*)}{2} \in [0,\pi)$ and $k_I \geq 0$. Hence, Im $v_2(k) \geq 0$ follows directly. □

Remark 5.2 *Lemma 5.1 has a significant importance on the discussion about the number of the boundary conditions for the higher-order Schrödinger equation for the case $\beta < 0$. Indeed, the only valid identity, i.e.* (5.20) *below, for $k \in \Gamma$, has three unknowns $\tilde{g}_0, \tilde{g}_1$ and $\tilde{g}_2$. So, we have one equation with three unknowns. This means one variable can be written in terms of the other two, which have to independent. In conclusion, this explains why the problem* (5.1) *need to be studied with two boundary conditions when $\beta < 0$. In general for evolution equations $\partial_t u + P(-i\partial_x)u = 0$, where P denotes any polynomial, Fokas method gives an opportunity to determine the exact number of the boundary condition needed. This idea is explained briefly in Deconinck et al. (2014).*

By Lemma 5.1, we have an identity as

$$e^{w(k)t}\hat{q}(v_1,t) = (-\beta v_1^2 + \alpha v_1 + \delta)\tilde{g}_0[w(k),t] + (i\beta v_1 - i\alpha)\tilde{g}_1[w(k),t] + \beta\tilde{g}_2[w(k),t], \quad (5.20)$$

which is valid for $k \in \overline{D^+}$, since Im $v_1(k) \leq 0$. Solving (5.20) in terms of $\tilde{g}_2[w(k),t]$ gives

$$\beta\tilde{g}_2[w(k),t] = (\beta v_1^2 - \alpha v_1 - \delta)\tilde{g}_0[w(k),t] - (i\beta v_1 - i\alpha)\tilde{g}_1[w(k),t] + e^{w(k)t}\hat{q}(v_1,t).$$

Hence, we rewrite

$$\tilde{g}(k,t) = (k - v_1)(\alpha - \beta k - \beta v_1)\tilde{g}_0[w(k),t] + i\beta(k - v_1)\tilde{g}_1[w(k),t] + e^{w(k)t}\hat{q}(v_1,t),$$

and obtain

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{\Gamma} e^{ikx-w(k)t}\left((k-v_1)(\alpha-\beta k-\beta v_1)\tilde{g}_0[w(k),t] + i\beta(k-v_1)\tilde{g}_1[w(k),t]\right)dk \\ &+ \frac{1}{2\pi}\int_{\Gamma} e^{ikx}\hat{q}(v_1,t)dk. \end{aligned}$$

The second integral and the contribution of the first integral from t to T' above are equal to zero by Cauchy's theorem, and we obtain

$$q(x,t) = \frac{1}{2\pi}\int_{\Gamma} e^{ikx-w(k)t}\left((k-v_1)(\alpha-\beta k-\beta v_1)\tilde{g}_0[w(k),T'] + i\beta(k-v_1)\tilde{g}_1[w(k),T']\right)dk, \tag{5.21}$$

or equivalently, by rewriting v_1 explicitly, we have the representation formula as follows:

$$q(x,t) = \frac{1}{2\pi}\int_{\Gamma} e^{ikx-w(k)t}\left(\phi_0(k)\tilde{g}_0[w(k),T'] + \phi_1(k)\tilde{g}_1[w(k),T']\right)dk \tag{5.22}$$

where

$$\phi_0(k) = -\frac{3\beta}{2}k^2 + \alpha k + \frac{\alpha^2}{4\beta} + \delta + (\beta k-\delta)\left(-\frac{3}{4}\left(k-\frac{\alpha}{3\beta}\right)^2 + \frac{\alpha^2+3\beta\delta}{3\beta^2}\right)^{\frac{1}{2}} \tag{5.23}$$

and

$$\phi_1(k) = \frac{3i\beta}{2}k - i\alpha + i\beta\left(-\frac{3}{4}\left(k-\frac{\alpha}{3\beta}\right)^2 + \frac{\alpha^2+3\beta\delta}{3\beta^2}\right)^{\frac{1}{2}}, \tag{5.24}$$

where $z^{\frac{1}{2}}$ is defined as in (5.14) for any complex z.

Compatibility conditions. Under the regularity results of each decomposed model, we restrict the range for s to $[0,2]-\{\frac{1}{2},\frac{3}{2}\}$. We need to impose the first condition

$$u_0(0) = h_0(0) \text{ (and naturally } g_0(0)=0), \quad \frac{1}{2} < s \leq 2,\ s \neq \frac{3}{2} \tag{5.25}$$

as in Chapter 4. In addition, if $\frac{3}{2} < s \leq 2$ then $\frac{1}{2} < \frac{s}{3} \leq \frac{2}{3}$. Therefore, both of the traces $u_0(0)$ and $h_1(0)$ are well-defined. Furthermore, since $y_x(0,\cdot)$ and $z_x(0,\cdot)$ belong to

$H_t^{\frac{s}{3}}(0,T)$ by Theorems 3.2 and 3.5, the traces $y_x(0,0)$ and $z_x(0,0)$ are well-defined and equal to $u_0'(0)$ and 0, respectively. So, we need to impose also the second condition

$$u_0'(0) = h_1(0) \text{ (and naturally } g_1(0) = 0), \quad \frac{3}{2} < s \leq 2. \tag{5.26}$$

Sobolev-type estimates. We have the following space estimate.

Theorem 5.1 *Let $s \geq 0$. Then, the unique solution q of the reduced initial-boundary value problem* (5.2) *satisfies*

$$\|q(\cdot,t)\|_{H_x^s(\mathbb{R}_+)} \leq c\left(1 + \sqrt{T'}e^{cT'}\right)\sum_{b=0}^{1}\|g_b\|_{H_x^{\frac{s+1-b}{3}}(0,T')} \tag{5.27}$$

uniformly for $t \in [0,T']$, where c depends only on s, α, β, δ.

Proof We utilize the representation formula given in (5.22) obtained by Fokas method for the solution q. Parameterize Γ depending on the different scenarios on the value of $\alpha^2 + 3\beta\delta$ as follows:

$$\Gamma := \begin{cases} \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup (-\gamma_4) \cup (-\gamma_5) \cup \gamma_6, & \text{if } \alpha^2 + 3\beta\delta \geq 0, \\ \gamma_1 \cup \gamma_3 \cup (-\gamma_4) \cup \gamma_6, & \text{if } \alpha^2 + 3\beta\delta < 0. \end{cases} \tag{5.28}$$

where

$$\begin{aligned} &\gamma_1(m) = m, \quad -\infty < m \leq c_1, \\ &\gamma_2(m) = c_1 + im, \quad 0 < m < \lambda, \\ &\gamma_3(m) = \frac{\alpha + \sqrt{3\beta^2 m^2 + \alpha^2 + 3\beta\delta}}{3\beta} + im, \quad \lambda \leq m < \infty, \\ &\gamma_4(m) = \frac{\alpha - \sqrt{3\beta^2 m^2 + \alpha^2 + 3\beta\delta}}{3\beta} + im, \quad \lambda \leq m < \infty, \\ &\gamma_5(m) = c_2 + im, \quad 0 < m < \lambda, \\ &\gamma_6(m) = m, \quad c_2 \leq m < \infty, \end{aligned} \tag{5.29}$$

such that $c_1 = \frac{\alpha + \sqrt{3\beta^2\lambda^2 + \alpha^2 + 3\beta\delta}}{3\beta}$, $c_2 = \frac{\alpha - \sqrt{3\beta^2\lambda^2 + \alpha^2 + 3\beta\delta}}{3\beta}$, and the real number λ, that behaves like a cursor depending on the constants α, β, δ, is defined to satisfy the following:

$$\begin{cases} \lambda = \frac{\sqrt{-3(\alpha^2+3\beta\delta)}}{-3\beta} & \text{if } \alpha^2 + 3\beta\delta < 0, \\ \frac{\sqrt{\alpha^2+3\beta\delta}}{-\beta} < \lambda < \infty & \text{if } \alpha^2 + 3\beta\delta \geq 0. \end{cases} \tag{5.30}$$

Notice that $c_1 = c_2 = \frac{\alpha}{3\beta}$ when $\alpha^2 + 3\beta\delta < 0$.

Using the above parameterization, we can rewrite (5.22) as follows:

$$q(x,t) := \sum_{n=1}^{6} \sum_{b=0}^{1} q_{n,b}(x,t), \tag{5.31}$$

where

$$q_{n,b}(x,t) = \frac{1}{2\pi} \int_{I_n} e^{i\gamma_n(m)x - w(\gamma_n(m))t} \phi_b(\gamma_n(m)) \tilde{g}_b(w(\gamma_n(m)), T')\gamma_n'(m)dm \tag{5.32}$$

together with a special definition

$$q_{2,0} = q_{2,1} = q_{5,0} = q_{5,1} \equiv 0, \qquad \text{if } \alpha^2 + 3\beta\delta < 0, \tag{5.33}$$

for covering the absence of γ_2 and γ_5 in this case. The intervals I_n of the integral (5.32) is defined as

$$\begin{aligned} I_1 &= (-\infty, c_-], \\ I_2 &= (-I_5) = (0, \lambda), \\ I_3 &= (-I_4) = [\lambda, \infty), \\ I_6 &= [c_+, \infty). \end{aligned} \tag{5.34}$$

The minus sign in front of I_4 and I_5 in (5.34) is used in the directional sense. In addition, in (5.32), ϕ_r is given in (5.23) and (5.24), respectively for $r = 0$ and $r = 1$.

We analyze each of twelve elements $q_{n,r}$, defined in (5.32)-(5.33), of the solution

q, separately. Let us start with $q_{1,b}(x,t)$, which can be rewritten explicitly as

$$q_{1,b}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{c_1} e^{imx-w(m)t}\phi_b(m)\tilde{g}_b(w(m),T')dm. \tag{5.35}$$

This integral represents the inverse (spatial) Fourier transform of the function

$$\hat{Q}_{1,b}(m;t) = \begin{cases} e^{-w(m)t}\phi_b(m)\tilde{g}_b(w(m),T'), & \text{if } m \leq c_1; \\ 0, \quad \text{if } m > c_1. \end{cases} \tag{5.36}$$

On the other hand, using the compact support condition on the boundary data g_b, we have

$$\tilde{g}_b(w(m),T') = \hat{g}_b(iw(m)), \tag{5.37}$$

for $b \in \{0,1\}$. Therefore, we get

$$\begin{aligned} \|q_{1,b}(\cdot,t)\|^2_{H^s_x(\mathbb{R}_+)} &\leq \|q_{1,b}(\cdot,t)\|^2_{H^s_x(\mathbb{R})} = \int_{-\infty}^{\infty}(1+m^2)^s|\hat{Q}_{1,b}(m,t)|^2dm \\ &\lesssim \int_{-\infty}^{c_1}(1+m^2)^s|\phi_b(m)|^2|\hat{g}_b(iw(m))|^2dm. \end{aligned}$$

Let us define $\tau(m) := iw(m) = \beta m^3 - \alpha m^2 - \delta m$. The map τ is real-valued and monotonically decreasing if $\alpha^2 + 3\beta\delta \leq 0$. When $\alpha^2 + 3\beta\delta > 0$, τ is still monotonically decreasing on $(-\infty, c_1]$. Therefore, there is no harm to change variable from m to τ. So, at first we have

$$\|q_{1,b}(\cdot,t)\|^2_{H^s_x(\mathbb{R}_+)} \lesssim \int_{-\infty}^{c_1}(1+m^2)^s|\phi_b(m)|^2|\hat{g}_b(\tau(m))|^2dm.$$

And then, note that there exists a constant c depending on α, β, δ such that

$$\sup_{m\leq c_1}\frac{(1+m^2)^s|\phi_b(m)|^2}{(1+\tau^2(m))^{\frac{s+1-b}{3}}\tau'(m)} \leq c < \infty. \tag{5.38}$$

See Lemma 4.3 for an analogous argument in details. So, we get

$$\begin{aligned}\|q_{1,b}(\cdot,t)\|^2_{H^s_x(\mathbb{R}_+)} &\lesssim \int_{-\infty}^{c_1}(1+\tau^2(m))^{\frac{s+1-b}{3}}|\hat{g}_b(\tau(m))|^2\tau'(m)dm \\ &= \int_{\infty}^{iw(c_1)}(1+\tau^2)^{\frac{s+1-b}{3}}|\hat{g}_b(\tau)|^2 d\tau \le \|g_b\|^2_{H_t^{\frac{s+1-b}{3}}(\mathbb{R})} = \|g_b\|^2_{H_t^{\frac{s+1-b}{3}}(0,T')}.\end{aligned}$$

Note that $iw(c_1) \in \mathbb{R}$. Hence, we obtain for $r = 0, 1$ that

$$\|q_{1,b}(\cdot,t)\|_{H^s_x(\mathbb{R}_+)} \lesssim \|g_b\|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.39}$$

Using the same steps, it is straightforward to obtain for $b = 0, 1$ that

$$\|q_{6,b}(\cdot,t)\|_{H^s_x(\mathbb{R}_+)} \lesssim \|g_b\|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.40}$$

Consider $q_{2,b}$, now. Recall that it is defined in (5.33) specially as $q_{2,b} = 0$, when $\alpha^2+3\beta\delta < 0$. So, we assume $\alpha^2+3\beta\delta \ge 0$ for this case. We use the interpolation technique here. For any $j \in \mathbb{N}_0$, we have

$$\partial_x^j q_{2,b}(x,t) = \frac{i}{2\pi}\int_0^{\lambda}(i\gamma_2(m))^j e^{i\gamma_2(m)x - w(\gamma_2(m))t}\phi_b(\gamma_2(m))\tilde{g}_b(w(\gamma_2(m)),T')\,dm, \tag{5.41}$$

and so,

$$\|\partial_x^j q_{2,b}(\cdot,t)\|^2_{L^2_x(\mathbb{R}_+)} = \frac{1}{4\pi^2}\int_0^{\infty}\left|\int_0^{\lambda}(i\gamma_2(m))^j e^{i\gamma_2(m)x-w(\gamma_2(m))t}\phi_b(\gamma_2(m))\tilde{g}_b(w(\gamma_2(m)),T')\,dm\right|^2 dx \tag{5.42}$$

Observe that

$$
\begin{aligned}
&\left|\int_0^\lambda (i\gamma_2(m))^j e^{i\gamma_2(m)x-w(\gamma_2(m))t}\phi_b(\gamma_2(m))\tilde{g}_b(w(\gamma_2(m)),T')\,dm\right| \\
\leq\ & \int_0^\lambda e^{-mx-\beta tm(\lambda^2-m^2)}|\gamma_2(m)|^j|\phi_b(\gamma_2(m))|\ |\tilde{g}_b(w(\gamma_2(m)),T')|\,dm \\
\leq\ & e^{\frac{-2\beta\lambda^3T'}{3\sqrt{3}}}\int_0^\lambda e^{-mx}|\gamma_2(m)|^j|\phi_b(\gamma_2(m))|\ |\tilde{g}_b(w(\gamma_2(m)),T')|\,dm,
\end{aligned}
$$

since $\mathrm{Re}(-w(\gamma_2(m))t) = -\beta tm(\lambda^2 - m^2) \leq \frac{-2\beta\lambda^3T'}{3\sqrt{3}}$ for $0 \leq m \leq \lambda$ and $0 \leq t \leq T'$. So, we have

$$
\|\partial_x^j q_{2,b}(\cdot,t)\|^2_{L^2_x(\mathbb{R}_+)} \leq \frac{e^{cT'}}{4\pi^2}\int_0^\infty\left(\int_0^\infty e^{-mx}|\gamma_2(m))|^j|\phi_b(\gamma_2(m))|\ |\tilde{g}_b(w(\gamma_2(m)),T')|\chi(0,\lambda)\,dm\right)^2 dx, \tag{5.43}
$$

where $c = \frac{-2\beta\lambda^3}{3\sqrt{3}}$. Then, by the boundedness of Laplace transform, we have

$$
\|\partial_x^j q_{2,b}(\cdot,t)\|^2_{L^2_x(\mathbb{R}_+)} \lesssim e^{cT'}\int_0^\lambda |\gamma_2(m))|^{2j}|\phi_b(\gamma_2(m))|^2\ |\tilde{g}_b(w(\gamma_2(m)),T')|^2 dm. \tag{5.44}
$$

Using the definition (5.4) of $\tilde{g}_b$ together with Cauchy-Schwarz inequality, we get

$$
\|\partial_x^j q_{2,b}(\cdot,t)\|^2_{L^2_x(\mathbb{R}_+)} \lesssim e^{cT'}T'\|g_b\|^2_{L^2_t(0,T')}\int_0^\lambda |\gamma_2(m))|^{2j}|\phi_b(\gamma_2(m))|^2 dm \lesssim e^{cT'}T'\|g_b\|^2_{L^2_t(0,T')}. \tag{5.45}
$$

Therefore, we obtain for $b = 0, 1$ that

$$
\|q_{2,b}(\cdot,t)\|_{H^s_x(\mathbb{R}_+)} \lesssim e^{cT'}\sqrt{T'}\|g_b\|_{L^2_t(0,T')} \leq e^{cT'}\sqrt{T'}\|g_b\|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.46}
$$

Using the same steps, it is straightforward to obtain for $b = 0, 1$ that

$$
\|q_{5,b}(\cdot,t)\|_{H^s_x(\mathbb{R}_+)} \lesssim e^{cT'}\sqrt{T'}\|g_b\|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.47}
$$

For the other estimates on $q_{n,b}$ for $n = 3, 4$ and $b = 0$, we use the corresponding part in the case of $\beta > 0$. Although the change of the sign of β sometimes creates differences about the analysis of the higher-order Schrödinger equations, such as the number of the boundary condition, we can also observe similarities in the details. Indeed, the boundaries γ_3 and γ_4, defined in (5.29), coincides with the ones defined for the single boundary condition case. It is also straightforward to apply the arguments therein for $b = 1$, and hence we have

$$\|q_{n,b}(\cdot,t)\|_{H_x^s(\mathbb{R}_+)} \lesssim \|g_b\|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.48}$$

for $n = 3, 4$ and $b = 0, 1$. Combining all estimations above (5.39)-(5.48) in the view of (5.31), we complete the proof. □

Strichartz estimates. We have the theorem.

Theorem 5.2 *Let $s \geq 0$ and (μ, r) be higher-order Schrödinger admissible in the sense of* (3.29). *Then,*

$$\|q\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim \left(1 + e^{cT'}(T')^{\frac{1}{\mu}+\frac{1}{2}}\right) \sum_{b=0}^{1} \|g_b\|_{H_t^{\frac{s+1-b}{3}}(0,T')} \tag{5.49}$$

where $H_x^{s,r}(\mathbb{R}_+)$ is the restriction on $\mathbb{R}_+$ of the Bessel potential space $H_x^{s,r}(\mathbb{R})$ defined by (3.28) *and the inequality constant depends only r, s, while c depends on α, β, δ and r.*

Proof We use the same definitions (5.28) and (5.34) of q and Γ, respectively. For the estimation of $q_{1,b}(x,t)$, we switch the initial-boundary value problem to a Cauchy problem.

Combining (5.35) and (5.36), we rewrite

$$q_{1,b}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{imx - w(m)t} \hat{Q}_{1,b}(m) dm, \tag{5.50}$$

which is defined only for $x > 0$, but it makes also sense for the negative values of x. Therefore, we extend this equation for all $x \in \mathbb{R}$, and then we observe that (5.50) represents the solution of the Cauchy problem

$$\begin{aligned} & i(q_{1,b})_t + i\beta(q_{1,b})_{xxx} + \alpha(q_{1,b})_{xx} + i\delta(q_{1,b})_x = 0, \qquad x,t \in \mathbb{R}, \\ & q_{1,b}(x,0) = Q_{1,b}(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{5.51}$$

Then, by Theorem 3.3 and the inequality (5.39), we have

$$\left\| q_{1,b} \right\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))} \leq \left\| q_{1,b} \right\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}))} \lesssim \left\| Q_{1,b} \right\|_{H_x^s(\mathbb{R})} \lesssim \| g_b \|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.52}$$

The same idea yields directly the estimate on $q_{6,b}(x,t)$, i.e.

$$\left\| q_{6,b} \right\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim \| g_b \|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.53}$$

Consider $q_{2,b}$, now. Recall that if $\alpha^2 + 3\beta\delta < 0$, then $q_{2,b} \equiv 0$. So, we assume $\alpha^2 + 3\beta\delta \geq 0$ in this case. We first have

$$\left\| q_{2,b} \right\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))} \leq (T')^{\frac{1}{\mu}} \left\| q_{2,b}(\cdot,t) \right\|_{H_x^{s,r}(\mathbb{R}_+))}. \tag{5.54}$$

Using the interpolation inequality for L^p-norm(see Preliminaries), we have for each $j \in \mathbb{N}_0$ that

$$\left\| \partial_x^j q_{2,b}(\cdot,t) \right\|_{L_x^r(\mathbb{R}_+))} \leq \left\| \partial_x^j q_{2,b}(\cdot,t) \right\|_{L_x^2(\mathbb{R}_+))}^{\frac{2}{r}} \left\| \partial_x^j q_{2,b}(\cdot,t) \right\|_{L_x^\infty(\mathbb{R}_+))}^{1-\frac{2}{r}}. \tag{5.55}$$

By (5.45), we already have

$$\left\| \partial_x^j q_{2,b}(\cdot,t) \right\|_{L_x^2(\mathbb{R}_+))}^{\frac{2}{r}} \lesssim e^{cT'} (T')^{\frac{1}{r}} \| g_b \|_{L_t^2(0,T')}^{\frac{2}{r}}. \tag{5.56}$$

On the other hand, we have directly by the definition of $\tilde{g}_b$ and (5.41) that

$$\left\| \partial_x^j q_{2,b}(\cdot,t) \right\|_{L_x^\infty(\mathbb{R}_+))}^{1-\frac{2}{r}} \lesssim e^{cT'} (T')^{\frac{1}{2}-\frac{1}{r}} \| g_b \|_{L_t^2(0,T')}^{1-\frac{2}{r}}. \tag{5.57}$$

Combining all above (5.54)-(5.57), using the equivalence of Bessel and the Sobolev norm (4.71), and then interpolating for the non-integer values of s, we obtain that

$$\left\|q_{2,b}\right\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim e^{cT'}\,(T')^{\frac{1}{\mu}+\frac{1}{2}}\,\|g_b\|_{L_t^2(0,T')} \leq e^{cT'}\,(T')^{\frac{1}{\mu}+\frac{1}{2}}\,\|g_b\|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.58}$$

Similarly, one can obtain

$$\left\|q_{5,b}\right\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim e^{cT'}\,(T')^{\frac{1}{\mu}+\frac{1}{2}}\,\|g_b\|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.59}$$

As we did for the Sobolev-type of estimates, we use our previous results for the terms $q_{n,b}$ for $n = 3, 4$, that is proved as follows:

$$\left\|q_{n,b}\right\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim \|g_b\|_{H_t^{\frac{s+1-b}{3}}(0,T')}. \tag{5.60}$$

for $n = 3, 4$ and $b = 0, 1$. Combining (5.52)-(5.60), we complete the proof. □

5.2. Solution map

It is classical to use a fixed point argument for nonlinear analysis of a partial differential equation. To this end, we define a solution operator $u \to \Phi u$ associated with the forced linear initial-boundary value problem (4.2) as follows:

$$\begin{aligned}\Phi u &:= y|_{Q_T} + z^u|_{Q_T} + q^u|_{(0,T)} \\ &\equiv S\,[E_0u_0; 0]\Big|_{Q_T} + S\,[0; f(Eu)]\Big|_{Q_T} + \frac{1}{2\pi}\sum_{j=0}^{1}\int_\Gamma e^{ikx-w(k)t}\left(\phi_j(k)\tilde{g}_j^u[w(k), T']\right)dk\Big|_{(0,T)},\end{aligned} \tag{5.61}$$

for some $T > 0$ to be determined, $Q_T := \mathbb{R}_+ \times (0, T)$, and

$$g_j^u(t) := E_b\{h_j(\cdot) - \partial_x^j S\,[E_0u_0; 0](0, \cdot) - \partial_x^j S\,[0; f(Eu)](0, \cdot)\}(t). \tag{5.62}$$

Note that all the notations and the letters are defined, explicitly before, as in their corresponding decomposed model.

We use the linear theory as a tool to study the well-posedness of the main nonlinear problem (5.1). To this end, in view of (5.61), we define the solutions of the nonlinear problem (5.1) as the fixed points of the operator Φ. Throughout our work, we assume $u_0 \in H_x^s(\mathbb{R}_+)$, $g_0 \in H_{t,loc}^{\frac{s+1}{3}}(\mathbb{R}_+)$ and $g_1 \in H_{t,loc}^{\frac{s}{3}}(\mathbb{R}_+)$ with $s \in [0,2] - \{\frac{1}{2}, \frac{3}{2}\}$ and the compatibility conditions (5.25) and (5.26) in place as necessary.

5.3. Local well-posedness

In this section, we prove the two main results for both high and low regularity setting for the local well-posedness of the problem (5.1).

Theorem 5.3 (High regularity well-posedness) *Let $s \in (\frac{1}{2}, 2] - \{\frac{3}{2}\}$ and $p > 0$. In addition, if $p \notin 2\mathbb{Z}$, suppose that*

$$\begin{aligned} &\text{if } s \in \mathbb{Z}_+, \text{then } p \geq s \text{ if } p \in \mathbb{Z}_+ \text{ and odd}; \lfloor p \rfloor \geq s-1 \text{ if } p \notin \mathbb{Z}_+, \\ &\text{if } s \notin \mathbb{Z}_+, \text{then } p > s \text{ if } p \in \mathbb{Z}_+ \text{ and odd}; \lfloor p \rfloor \geq \lfloor s \rfloor \text{ if } p \notin \mathbb{Z}_+. \end{aligned} \tag{5.63}$$

Then, for initial data $u_0 \in H_x^s(\mathbb{R}_+)$ and boundary data $h_j \in H_{t,\mathrm{loc}}^{\frac{s+1-j}{3}}(\mathbb{R}_+)$ for $j = 0, 1$, satisfying the compatibility conditions (5.25) *and* (5.26)*, there is $T = T(u_0, h_0, h_1) > 0$ such that the initial-boundary value problem* (4.1) *for the HNLS equation on the half-line has a unique solution $u \in C([0,T]; H_x^s(\mathbb{R}_+))$. Furthermore, this solution depends continuously on the initial and boundary data.*

Proof We claim to establish local well-posedness in the metric space $X_T := C([0,T]; H_x^s(\mathbb{R}_+))$ for some $T > 0$, with the metric

$$d_{X_T}(u_1, u_2) := \|u_1 - u_2\|_{X_T}, \qquad u_1, u_2 \in X_T. \tag{5.64}$$

Note that any closed ball in X_T is a complete subspace.

For local existence, we need to show that the map Φ is onto and is a contraction.

Using the similar arguments as we used before

$$\left\|y|_{Q_T}\right\|_{X_T} \lesssim \|u_0\|_{H_x^s(\mathbb{R}_+)}, \tag{5.65}$$

and

$$\left\|z^u|_{Q_T}\right\|_{X_T} \lesssim T\|u\|_{X_T}^{p+1}. \tag{5.66}$$

Considering the last term in (5.61), we get (for say $T' = 2T$)

$$\left\|q^u|_{(0,T)}\right\|_{X_T} \leq \|q\|_{X_{T'}} \lesssim \left(1 + \sqrt{T'}e^{cT'}\right)\sum_{j=0}^{1}\|g_j^u\|_{H_t^{\frac{s+1-j}{3}}(0,T')} \lesssim \left(1 + \sqrt{T}e^{cT}\right)\sum_{j=0}^{1}\|g_j^u\|_{H_t^{\frac{s+1-j}{3}}(0,T)}. \tag{5.67}$$

By definition of g_j, for $j = 0, 1$, and the temporal estimates of the corresponding Cauchy problems, we obtain that

$$\|g_j\|_{H_t^{\frac{s+1-j}{3}}(0,T)} \lesssim \|h_j\|_{H_t^{\frac{s+1-j}{3}}(0,T)} + (1 + T^{\frac{1}{2}})\|u_0\|_{H_x^s(\mathbb{R}_+)} + c(T)\|f(E_0u)\|_{L_t^2((0,T);H_x^s(\mathbb{R}))}, \tag{5.68}$$

where $c(T) := \max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^{\sigma_j}\}$ together with σ_j defined for $j = 0$ and $j = 1$, separately as in (3.42) and (3.60).

By definition of X_T and the boundedness of E_0, we also have

$$\|f(E_0u)\|_{L_t^2((0,T);H_x^s(\mathbb{R}))} \lesssim T^{\frac{1}{2}}\|u\|_{X_T}^{p+1}. \tag{5.69}$$

Combining all the estimations above, we deduce that

$$\|\Phi(u)\|_{X_T} \lesssim (1 + T^{\frac{1}{2}})\left(1 + \sqrt{T}e^{cT}\right)\|u_0\|_{H_x^s(\mathbb{R}_+)} + c(T)\|u\|_{X_T}^{p+1} + \left(1 + \sqrt{T}e^{cT}\right)\sum_{j=0}^{1}\|h_j^u\|_{H_t^{\frac{s+1-j}{3}}(0,T)}. \tag{5.70}$$

In view of (5.70), we set $R(T) = 2A(T)$ with

$$A(T) := (1 + T^{\frac{1}{2}})\left(1 + \sqrt{T}e^{cT}\right)\|u_0\|_{H_x^s(\mathbb{R}_+)} + \left(1 + \sqrt{T}e^{cT}\right)\sum_{j=0}^{1}\|h_j^u\|_{H_t^{\frac{s+1-j}{3}}(0,T)},$$

and choose $T > 0$ small enough so that $A(T) + c(T)R^{p+1}(T) \leq R(T)$, or equivalently, $c(T)R^p(T) \leq \frac{1}{2}$. Therefore, the map Φ takes the closed ball $\overline{B_{R(T)}(0)} \subset X_T$ into itself.

We show that Φ is a contraction, now. Let $u_1, u_2 \in \overline{B_{R(T)}(0)}$. Then,

$$\begin{aligned}
\|\Phi(u_1) - \Phi(u_2)\|_{X_T} &= \left\| z^{u_1}|_{Q_T} - z^{u_2}|_{Q_T} \right\|_{X_T} + \left\| q^{u_1}|_{(0,T)} - q^{u_2}|_{(0,T)} \right\|_{X_T} \\
&\lesssim \|S[0; f(E_0u_1) - f(E_0u_2)]\|_{C([0,T];H_x^s(\mathbb{R}))} \\
&+ \left(1 + \sqrt{T'}e^{cT'}\right) \sum_{j=0}^{1} \|g_j^{u_1} - g_j^{u_2}\|_{H_t^{\frac{s+1-j}{3}}(0,T)}.
\end{aligned}$$

Using Lemma 4.7, we have

$$\|S[0; f(E_0u_1) - f(E_0u_2)]\|_{C([0,T];H_x^s(\mathbb{R}))} \lesssim T(\|u_1\|_{X_T}^p + \|u_2\|_{X_T}^p)\|u_1 - u_2\|_{X_T}. \tag{5.71}$$

We also have for $j = 0, 1$ that

$$\begin{aligned}
\|g_j^{u_1} - g_j^{u_2}\|_{H_t^{\frac{s+1-j}{3}}(0,T)} &\lesssim c(T)\|f(E_0u_1) - f(E_0u_2)\|_{L_T^2((0,T);H_x^s(\mathbb{R}))} \\
&\lesssim c(T)T^{\frac{1}{2}}(\|u_1\|_{X_T}^p + \|u_2\|_{X_T}^p)\|u_1 - u_2\|_{X_T}.
\end{aligned}$$

Combining all above, we obtain

$$\|\Phi(u_1)-\Phi(u_2)\|_{X_T} \lesssim c(T)T^{\frac{1}{2}}(\|u_1\|_{X_T}^p+\|u_2\|_{X_T}^p)\|u_1-u_2\|_{X_T} \lesssim c(T)T^{\frac{1}{2}}R^p(T)\|u_1-u_2\|_{X_T}. \tag{5.72}$$

So, Φ is a contraction for small $T > 0$, since $R(T)$ remains bounded and $c(T)T^{\frac{1}{2}} \to 0$ as $T \to 0^+$, and therefore Φ has a unique fixed point in $\overline{B_{R(T)}(0)}$, which amounts to local existence of a unique solution to (4.1) on $\overline{B_{R(T)}(0)}$.

For the extension of this result to X_T, and also for the continuous dependence on the data, we refer to our previous work in Chapter 4 with a tiny change in the boundary data. This is straightforward, therefore the proof for Theorem 5.3 is completed. □

Theorem 5.4 (Low regularity well-posedness) *Suppose*

$$0 \le s < \frac{1}{2}, \quad 1 \le p \le \frac{6}{1-2s}, \quad \mu = \frac{6(p+1)}{p(1-2s)}, \quad r = \frac{2(p+1)}{1+2sp}. \tag{5.73}$$

Then, for initial data $u_0 \in H_x^s(\mathbb{R}_+)$ *and boundary data* $g_j \in H_{t,\mathrm{loc}}^{\frac{s+1-j}{3}}(\mathbb{R}_+)$, $j = 0, 1$, *with the additional assumption that if* $p = \frac{6}{1-2s}$ *(critical case) then* $\|u_0\|_{H_x^s(\mathbb{R}_+)}$ *is sufficiently small, there is* $T = T(u_0, g) > 0$ *such that the initial-boundary value problem* (5.1) *for the HNLS equation on the half-line has a unique solution* $u \in C([0,T]; H_x^s(\mathbb{R}_+)) \cap L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}_+))$. *Furthermore, this solution depends continuously on the initial and boundary data.*

Proof We define the space

$$Y_T := C([0,T]; H_x^s(\mathbb{R}_+)) \cap L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}_+)),$$

together with

$$\widetilde{Y}_T := C([0,T]; H_x^s(\mathbb{R})) \cap L_t^\mu((0,T); H_x^{s,r}(\mathbb{R})).$$

First, we consider the sub-critical case $1 \le p < \frac{6}{1-2s}$ so that $\frac{\mu-p-1}{\mu} > 0$. Taking the solution operator, defined via linear unification in (5.61), into consideration, we have the following estimations.

By Theorems 3.1 and 3.3, we obtain

$$\left\| y|_{Q_T} \right\|_{Y_T} \lesssim \|u_0\|_{H_x^s(\mathbb{R}_+)}, \tag{5.74}$$

and by Theorems 3.4 and 3.6, together with (4.92), we also obtain

$$\left\| z^u|_{Q_T} \right\|_{Y_T} \lesssim \left(T + T^{\frac{\mu-p-1}{\mu}} \right) \|u\|_{X_T}^{p+1}. \tag{5.75}$$

To estimate the last term in (5.61), we separate $g_j^u(t)$ defined in (5.62) as follows:

$$g_j^u(t) = E_b\{h_j(\cdot) - \partial_x^j S[E_0u_0; 0](0,\cdot)\}(t) - E_b\{\partial_x^j S[0; f(Eu)](0,\cdot)\}(t) := g_j^{u,1}(t) - g_j^{u,2}(t), \tag{5.76}$$

which yields to rewrite

$$q^u|_{(0,T)} := q^{u,1}|_{(0,T)} - q^{u,2}|_{(0,T)}. \tag{5.77}$$

By using the arguments in (5.67) and (5.68), we have

$$\left\|q^{u,1}|_{(0,T)}\right\|_{Y_T} \lesssim \left(1 + \sqrt{T}e^{cT}\right)\left((1 + T^{\frac{1}{2}})\|u_0\|_{H_x^s(\mathbb{R}_+)} + \sum_{j=0}^{1} \|h_j\|_{H_t^{\frac{s+1-j}{3}}(0,T)}\right). \tag{5.78}$$

To estimate $q^{u,2}|_{(0,T)}$, we use a rearrangement argument which is analogously used for the Schrödinger equation on the half-plane in Section 6 of Ref. Himonas and Mantzavinos (2020). Indeed, we observe that

$$\begin{aligned}
q^{u,2}(x,t) &= \frac{1}{2\pi}\sum_{j=0}^{1}\int_\Gamma e^{ikx-w(k)t}\phi_j(k)\tilde{g}_j^{u,2}[w(k),T']dk \\
&= \frac{1}{2\pi}\sum_{j=0}^{1}\int_\Gamma e^{ikx-w(k)t}\phi_j(k)\left(\int_0^{T'} e^{w(k)\tau}\partial_x^j S[0; f(Eu)](0,\tau)d\tau\right)dk \\
&= \frac{-i}{2\pi}\sum_{j=0}^{1}\int_\Gamma e^{ikx-w(k)t}\phi_j(k)\left(\int_0^{T'} e^{w(k)\tau}\partial_x^j\left(\int_0^{t'} S[f(Eu); 0](0,\tau - t')dt'\right)d\tau\right)dk \\
&= -i\int_0^{T'} q^*(x, t-\tau)d\tau,
\end{aligned}$$

by Fubini's theorem and the continuous (since $s < \frac{1}{2}$) extension by zero of $S[f(Eu); 0]$ outside $[0,t']$, where q^* denotes the solution of the reduced initial-boundary value problem (5.2) with the boundary data $\partial_x^j S[f(Eu); 0]$. Hence, Theorems 4.3, 3.2, and 3.5 implies that

$$\left\|q^{u,2}|_{(0,T)}\right\|_{Y_T} \lesssim \left(1 + e^{cT}T^{\frac{1}{\mu}+\frac{1}{2}}\right)(1 + T^{\frac{1}{2}})\|f(Eu)\|_{L_t^1((0,T);H_x^s(\mathbb{R}_+))}. \tag{5.79}$$

Using (4.92) and combining the above estimates, we obtain

$$\|\Phi(u)\|_{Y_T} \lesssim c_1(T)\|u_0\|_{H_x^s(\mathbb{R}_+)} + c_2(T)\|u\|_{Y_T}^{p+1} + c_3(T)\sum_{j=0}^{1}\|h_j^u\|_{H_t^{\frac{s+1-j}{3}}(0,T)}. \tag{5.80}$$

Rest of the proof directly follows by the arguments that we use in Chapter 4. The proof is completed. □

CHAPTER 6

CONCLUSION

The main target of this thesis was to prove that the initial-boundary value problem for the higher-order nonlinear Schrödinger equation on the half-line is locally well-posed in the Hadamard sense, in other words, we proved the existence of a unique local-in-time solution that depends continuously on the initial and boundary data in the Sobolev space $H^s(\mathbb{R})$ for $s \geq 0$. We figured out in the details of our analysis that the initial-boundary value problem for the higher-order nonlinear Schrödinger equation must be considered in two separate scenario depending on the sign of the coefficient of the term involving highest order spatial derivative. This observation arose naturally thanks to Fokas unified transform method, which plays a crucial role to define a formula that represents the weak solution of the corresponding reduced linear initial-boundary problem. Together with the reduced model, we also analyzed the homogeneous and the nonhomogeneous Cauchy problems for the higher-order nonlinear Schrödinger equation in order to use their restriction on the half-line and then define a solution map for the linear forced initial-boundary value problem.

Linear theory for the higher-order Schrödinger equation created some analytical challenges due to the presence of more than one spatial derivatives in the construction of the model. By handling these challenges, this thesis provided a first, complete treatment via the Fokas method of a nonhomogeneous initial-boundary value problem for a partial differential equation associated with a multi-term linear differential operator. In this point of view, these details can enlighten the analyses of some other evolution equations that have similar multi-spatiodifferential structure.

Nonlinear analysis was treated via a contraction argument for both high ($s > \frac{1}{2}$) and low ($0 \leq s < \frac{1}{2}$) setting. In the former setting, we handled the nonlinearity via the Banach algebra property; while in the latter setting, since this is no longer the case and, instead, we used Strichartz type of estimates. This is especially challenging in the framework of nonhomogeneous initial-boundary value problems, as it involves proving boundary-type Strichartz estimates that are not common in the study of Cauchy problems.

REFERENCES

Alkın, A., D. Mantzavinos, and T. Özsarı (2024). Local well-posedness of the higher-order nonlinear schrödinger equation on the half-line: Single-boundary condition case. *Studies in Applied Mathematics 152*(1), 203–248.

Batal, A. and T. Özsarı (2016). Nonlinear Schrödinger equations on the half-line with nonlinear boundary conditions. *Electron. J. Differential Equations*, Paper No. 222, 20.

Brown, J. W. and R. V. Churchill (2009). *Complex variables and applications*. McGraw-Hill,.

Calderón, A.-P. (1961). Lebesgue spaces of differentiable functions and distributions. In *Proc. Sympos. Pure Math., Vol. IV*, pp. 33–49. American Mathematical Society, Providence, R.I.

Carvajal, X. (2004). Local well-posedness for a higher order nonlinear Schrödinger equation in Sobolev spaces of negative indices. *Electron. J. Differential Equations*, No. 13, 10.

Carvajal, X. (2006). Sharp global well-posedness for a higher order Schrödinger equation. *J. Fourier Anal. Appl. 12*(1), 53–70.

Carvajal, X. and F. Linares (2003). A higher-order nonlinear Schrödinger equation with variable coefficients. *Differential Integral Equations 16*(9), 1111–1130.

Cazenave, T. and F. B. Weissler (1990). The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Anal. 14*(10), 807–836.

Christ, F. M. and M. I. Weinstein (1991). Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. *J. Funct. Anal. 100*(1), 87–109.

Deconinck, B., T. Trogdon, and V. Vasan (2014). The method of fokas for solving linear partial differential equations. *siam REVIEW 56*(1), 159–186.

Di Nezza, E., G. Palatucci, and E. Valdinoci (2012). HitchhikerÊ¼s guide to the fractional sobolev spaces. *Bulletin des sciences mathématiques 136*(5), 521–573.

Evans, L. C. (2022). *Partial differential equations*, Volume 19. American Mathematical Society.

Faminskii, A. V. (2023). The higher order nonlinear Schrödinger equation with quadratic nonlinearity on the real axis. *Adv. Differential Equations 28*(5-6), 413–466.

Faminskii, A. V. (2024). Global weak solutions of an initial-boundary value problem on a half-line for the higher order nonlinear schrödinger equation. *Journal of Mathematical Analysis and Applications 533*(2), 128003.

Fokas, A. S. (1997). A unified transform method for solving linear and certain nonlinear PDEs. *Proc. Roy. Soc. London Ser. A 453*(1962), 1411–1443.

Fokas, A. S. (2008). *A unified approach to boundary value problems*, Volume 78 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

Fokas, A. S., A. A. Himonas, and D. Mantzavinos (2017). The nonlinear Schrödinger equation on the half-line. *Trans. Amer. Math. Soc. 369*(1), 681–709.

Guo, B. and J. Wu (2021). Well-posedness of the initial-boundary value problem for the Hirota equation on the half line. *J. Math. Anal. Appl. 504*(2), Paper No. 125571, 25.

Himonas, A. A. and D. Mantzavinos (2020). Well-posedness of the nonlinear Schrödinger equation on the half-plane. *Nonlinearity 33*(10), 5567–5609.

Himonas, A. A. and D. Mantzavinos (2021). The nonlinear Schrödinger equation on the half-line with a Robin boundary condition. *Anal. Math. Phys. 11*(4), Paper No. 157, 25.

Himonas, A. A., D. Mantzavinos, and F. Yan (2019). The Korteweg–de Vries equation on an interval. *J. Math. Phys. 60*(5), 051507, 26.

Himonas, A. A. and F. Yan (2022). The Korteweg–de Vries equation on the half-line with Robin and Neumann data in low regularity spaces. *Nonlinear Anal. 222*, Paper No. 113008, 31.

Huang, L. (2020). The initial-boundary-value problems for the Hirota equation on the half-line. *Chinese Ann. Math. Ser. A 41*(1), 117–132.

Jones, F. (2001). *Lebesgue integration on Euclidean space*. Jones & Bartlett Learning.

Kodama, Y. (1985). Optical solitons in a monomode fiber. Volume 39, pp. 597–614. Transport and propagation in nonlinear systems (Los Alamos, N.M., 1984).

Kodama, Y. and A. Hasegawa (1987). Nonlinear pulse propagation in a monomode dielectric guide. *IEEE Journal of Quantum Electronics 23*(5), 510–524.

Kreyszig, E. (1991). *Introductory functional analysis with applications*, Volume 17. John Wiley & Sons.

Laurey, C. (1997). The Cauchy problem for a third order nonlinear Schrödinger equation. *Nonlinear Anal. 29*(2), 121–158.

Lions, J.-L. and E. Magenes (1972). *Non-homogeneous boundary value problems and applications. Vol. I*. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg. Translated from the French by P. Kenneth.

Özsarı, T., K. Alkan, and K. Kalimeris (2022). Dispersion estimates for the boundary integral operator associated with the fourth order Schrödinger equation posed on the half line. *Math. Inequal. Appl. 25*(2), 551–571.

Özsarı, T. and N. Yolcu (2019). The initial-boundary value problem for the biharmonic

Schrödinger equation on the half-line. *Commun. Pure Appl. Anal. 18*(6), 3285–3316.

Staffilani, G. (1997). On the generalized Korteweg-de Vries-type equations. *Differential Integral Equations 10*(4), 777–796.

Stein, E. M. (1970). *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J.

Strichartz, R. (1977). Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J. 44*, 705–714.

Strichartz, R. S. (2003). *A guide to distribution theory and Fourier transforms*. World Scientific Publishing Company.

Tartar, L. (2007). *An introduction to Sobolev spaces and interpolation spaces*, Volume 3. Springer Science & Business Media.

Wu, J. and B. Guo (2023). Initial-boundary value problem for the Hirota equation posed on a finite interval. *J. Math. Anal. Appl. 526*(2), Paper No. 127330.

VITA

EDUCATION

2018 - 2024 Doctor of Philosophy in Mathematics

Graduate School, Izmir Institute of Technology,

Izmir -Turkey

Thesis Title: Initial-boundary Value Problem for the Higher-order Nonlinear Schrödinger Equation on the Half-line

Supervisor: Assoc. Prof. Dr. Ahmet Batal

Co-supervisor: Assoc. Prof. Dr. Türker Özsarı

2015 - 2018 Master of Science in Mathematics

Graduate School of Engineering and Sciences, Izmir Institute of Technology,

Izmir -Turkey

Thesis Title: The Dirichlet Problem for the Fractional Laplacian

Supervisor: Assoc. Prof. Dr. Türker Özsarı

2009-2013 Bachelor of Mathematics

Department of Mathematics, Faculty of Science, Dokuz Eylül University, Izmir - Turkey

PROFESSIONAL EXPERIENCE

2016 - Present Research and Teaching Assistant

Department of Mathematics, Izmir Institute of Technology, Izmir -Turkey

PUBLICATIONS

1. Alkın A., Mantzavinos D., Özsarı T. (2024). Local well-posedness of the higher-order nonlinear Schrödinger equation on the half-line: single boundary condition case. Studies in Applied Mathematics.

   DOI: 10.1111/sapm.12642

2. Alkın A., Mantzavinos D., Özsarı T. (preprint). Local well-posedness of the higher-order nonlinear Schrödinger equation on the half-line: the case of double boundary condition.