Four-Bar Function Generation Using Excel Solver

Eres Söylemez[1[0000-0003-2324-959X]] and Gökhan Kiper[2[0000-0001-8793-724X]]

[1] Middle East Technical University, Ankara 06800, Türkiye
[2] İzmir Institute of Technology, Urla, İzmir 35430, Türkiye
eres@metu.edu.tr

Abstract. The Chapter presents a simple and efficient way of approximating a function with a four-bar mechanism using four or five design parameters including one or both of the initial crank angles. The method only involves solution of linear set of equations and evaluating determinants, whereas nonlinear equations are numerically solved using a simple program such as Excel. So, the method is easy to explain and can be taught in an undergraduate course along with the well-known linear three precision point synthesis problem. Precision point synthesis, order synthesis, mixed order synthesis, least squares approximation and extreme point synthesis can all be treated using the same method. The proposed method is illustrated with numerical examples for all mentioned synthesis problems and shown to be quite efficient with very low amount of structural error values.

Keywords: Kinematic synthesis, Function generation, Four-bar mechanism.

1 Introduction

Approximating a function by means of a four-bar mechanism has drawn the attention of many kinematicians over the years. The problem was first treated geometrically as "position coordination" utilizing Burmester theory and curvature theory [1]. Following the approximation methods developed by Chebyshev, Levitskii [2] treated the problem analytically. Svoboda [3] prepared monograms and developed "overlay method" for the graphical correlation of crank angles. The main breakthrough on function generation utilizing a four-bar mechanism was by F. Freudenstein [4-7] where he treated three, four and five precision points and third and fourth order approximations.

Following Freudenstein, the work on four-bar function generators followed different paths. Algebraic solutions for four and five precision points were sought in different ways [8-12] or different methods for higher order approximations were investigated [13-14]. Sandor and Erdman [15] use "dyad formulation" for function generation. Chmielewski [16] describes a numerical method through which the five equations obtained for five precision point synthesis is solved. With the advance of parametric design software packages, Kinzel et al. [17] describe "geometric constraint programming" method for function generation. Another way of improving function approximation was the selection of precision points and respacing for the reduction of the structural error [7, 18]. The function approximation as a constrained or unconstrained nonlinear optimization problem using different numerical optimization methods attracted the attention of many researchers. The aim is to minimize the “***structural error***”, which is the difference between the generated and the required function ($\varepsilon_y = y_{generated} - y_{required}$) or ($\varepsilon_\phi = \phi_{generated} - \phi_{required}$), throughout a defined range. The objective function is usually formulated as the minimization of the sum of the squares of the structural error ($S = \Sigma\varepsilon_{yj}^2$) over the range or the minimization of the maximum error within the range ($\min(\max(\varepsilon_{yj}))$). Developments made since 1970 are given in [19].

Basically, four-bar function generator design is to find the dimensions of a four-bar mechanism (Fig. 1) to approximate a function $y = f(x)$ within a range $x_{in} \le x \le x_{fin}$ as best as possible. We first correlate x and y with input and output angles θ and ϕ linearly by selecting the range Δθ, Δϕ corresponding to $\Delta x = |x_{fin} - x_{in}|$ and $\Delta y = |y_{max} - y_{min}|$ via $k_\phi = \Delta\phi/\Delta y$, $k_\theta = \Delta\theta/\Delta x$ to obtain conversion equations from x, y to θ and ϕ as:

$$\phi = \phi_{in} + k_\phi(y - y_{in}) \quad \text{and} \quad \theta = \theta_{in} + k_\theta(x - x_{in}) \tag{1}$$

where the initial crank angles θ_{in} and ϕ_{in} can be designed, or selected arbitrarily. The ranges Δθ and Δϕ are not considered as design parameters, since they are the ranges of motion. Freudenstein states that “in general, ranges up to 120° are feasible” [5].

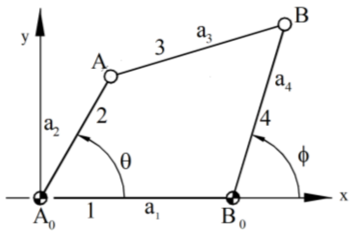


Fig. 1. A four-bar mechanism.

For a four-bar mechanism, the relation between the input-output angles is given by [5]:

$$K_1\cos\phi - K_2\cos\theta + K_3 = \cos(\phi - \theta) \tag{2}$$

where $K_1 = \dfrac{a_1}{a_2}$, $K_2 = \dfrac{a_1}{a_4}$ and $K_3 = \dfrac{a_1^2 + a_2^2 - a_3^2 + a_4^2}{2a_4a_2}$. When we are concerned with the relationship between the input and output angles, the size of the mechanism is not important, so we can select one of the link lengths arbitrarily. Usually, $a_1 = 1$ is selected.

There are five design parameters that can be used when approximating a function within a certain range: a_2, a_3, a_4, θ_{in} and ϕ_{in} ($a_1 = 1$). Instead of these parameters, K_1, K_2, K_3, θ_{in} and ϕ_{in} are preferred, since Eq. (2) gives a linear relationship for K_j.

In this paper a novel simple numerical method of solving Freudenstein’s equation using four and five design parameters is shown. The method can be used for precision point, order and mixed order synthesis, least squares approximation and extreme point synthesis. The advantage of the method over other analytical or numerical methods is its simplicity and that it can be easily applied using a simple tool such as Excel. In this form, the method can be safely taught in undergraduate level courses.

2 Precision Point, Order and Mixed Order Synthesis

The desired function and the function generated by the mechanism can be made to coincide at a number of points known as the “precision points”. Another form of function approximation is that the function may coincide at a few points but higher order derivatives of the desired function coincide with those of the generated function. This is known as the “order synthesis”. We may also have a fewer number of precision points and have first or second order derivatives coincide. This is known as “mixed order synthesis”. Let us denote P-P for two finitely separated points and PP when a point and its first order derivative is to be made coincident for the desired and generated functions.

The number of precision points or the order of the derivatives depend on the number of design parameters. For example, if only K_1, K_2, K_3 are to be considered as the design parameters, then we can have 3-precision points (P-P-P), or one precision point with first and second derivatives coincident (PPP) or two precision points (PP-P) with the first order derivative made coincident at one of the precision points (Case P-PP is the same as PP-P). If we consider θ_{in} or ϕ_{in} as a fourth design parameter, then we can have the following 4 cases: P-P-P-P, PP-P-P, PP-PP, P-PPP, PPPP.

In the solution of four design parameters, at most two different solutions are obtained. These solutions may have branch defect (i.e. the precision points are obtained at different assembly configurations), or the transmission angle or the maximum/minimum link length ratio may be unacceptable. Also, there may be no real solutions.

If both initial crank angles are considered as design parameters, we have five design parameters and one may have the following mixed order synthesis cases (disregarding the order): P-P-P-P-P, P-P-P-PP, P-PP-PP, P-P-PPP, P-PPPP, PP-PPP, PPPPP. For five-parameter approximation, finitely many (usually just one) solutions are obtained.

For all these cases at the precision points, Eq. (2) must be satisfied. For position j:

$$K_1\cos\phi_j - K_2\cos\theta_j + K_3 = \cos(\phi_j - \theta_j) \quad \text{for} \quad j = 1, 2, .., n\ (n \le 5) \tag{3}$$

$$K_1 \sin\phi_j\phi_j' - K_2 \sin\theta_j = \sin(\phi_j - \theta_j)(\phi_j' - 1) \tag{4}$$

$$K_1\left(\cos\phi_j\phi_j'^2 + \sin\phi_j\phi_j''\right) - K_2\cos\theta_j = \sin(\phi_j - \theta_j)\phi_j'' + \cos(\phi_j - \theta_j)(\phi_j' - 1)^2 \tag{5}$$

$$\begin{gathered} K_1\left[\sin\phi_j\left(\phi_j''' - \phi_j'^3\right) + 3\cos\phi_j\phi_j'\phi_j''\right] + K_2\sin\theta_j = \\ \sin(\phi_j - \theta_j)\left[\phi_j''' - (\phi_j' - 1)^3\right] + 3\cos(\phi_j - \theta_j)\phi_j''(\phi_j' - 1) \end{gathered} \tag{6}$$

$$\begin{gathered} K_1\left[\sin\phi_j\left(\phi_j^{iv} - 6\phi_j'^2\phi_j''\right) + \cos\phi_j\left(3\phi_j''^2 + 4\phi_j'\phi''' - \phi_j'^4\right)\right] + K_2\cos\theta_j = \\ \sin(\phi_j - \theta_j)\left[\phi_j^{iv} - 6\phi_j''(\phi_j' - 1)^2\right] + \cos(\phi_j - \theta_j)\left[4\phi_j'''(\phi_j' - 1) + 3\phi_j''^2 - (\phi_j' - 1)^4\right] \end{gathered} \tag{7}$$

where $\phi_j' = \left.\frac{d\phi}{d\theta}\right|_{\theta=\theta_j}$, $\phi_j'' = \left.\frac{d^2\phi}{d\theta^2}\right|_{\theta=\theta_j}$, $\phi_j''' = \left.\frac{d^3\phi}{d\theta^3}\right|_{\theta=\theta_j}$, $\phi_j^{iv} = \left.\frac{d^4\phi}{d\theta^4}\right|_{\theta=\theta_j}$.

When a problem is to be solved using three design parameters K_1, K_2 and K_3, one will obtain 3 linear equations (for P-P-P Eq. (3) j = 1, 2, 3, for P-PP Eq. (3) j = 1, 2 and Eq. (4) (j = 2 or 1), and for PPP Eq. (3), (4), (5), j = 1). The solution is straightforward.

When the problem is to be solved using 4 design parameters, one can select one of the initial crank angles, θ_{in} or ϕ_{in}, as the fourth design parameter. We can write four equations as a combination of Eqs. (3)-(6) depending on the type of the problem. These equations are linear in terms of K_1, K_2 and K_3, nonlinear in terms of the initial crank angle, say θ_{in}. If we assume that θ_{in} is determined by some means, then the resulting four equations have a solution for K_1, K_2 and K_3 if and only if the augmented matrix formed by the four equations is singular, or if the values of K_1, K_2 and K_3 obtained from the solution of any of the three equations also satisfies the fourth equation.

For the problem with five design parameters, both initial crank angles θ_{in} and ϕ_{in} are design parameters. Again we can write five equations as a combination of Eqs. (3)-(7). The equations are linear in terms of K_1, K_2 and K_3. In order to have a solution, θ_{in} and ϕ_{in} must be such that the rank is three. Hence, the determinant of five 4×4 augmented matrices obtained by eliminating one equation at a time, must all be zero (taking any two of them may be enough). If we square the determinants and sum, the rank of the equations will be three when this sum is equal to zero. Another form of solution is that the values of θ_{in} or ϕ_{in} must be such that when K_1, K_2 and K_3 obtained from any three of the five equations are substituted into the other two equations, these two equations must be satisfied. We determine the error made for assumed values of θ_{in} and ϕ_{in} (say δ_1, δ_2) and evaluate $\Delta = \sqrt{\delta_1^2 + \delta_2^2}$. θ_{in} and ϕ_{in} values must be such that $\Delta = 0$.

Example 1: Consider the generation of $y = \log(x)$ within $1 \le x \le 10$. Let $\Delta\theta = 90°$ and $\Delta\phi = 60°$. This results in $k_\theta = 10°$/unit x and $k_\phi = 60°$/unit y. Selection of precision points according to Chebyshev spacing as a first trial is a good choice for reducing the maximum error [7]. For the independent variable x, precision points are given by:

$$x_j = \frac{1}{2}\left(x_{fin} + x_{in}\right) - \frac{1}{2}\left(x_{fin} - x_{in}\right)\cos\left(\frac{2j-1}{2n}\pi\right) \quad \text{for} \quad j = 1, 2, .., n \tag{8}$$

For order or mixed order synthesis, the precision points can be selected arbitrarily and the derivatives of $y(x)$ and $\phi(\theta)$ are related by:

$$\phi' = \frac{k_\phi}{k_\theta} y',\ \phi'' = \frac{k_\phi}{k_\theta^2} y'',\ \phi''' = \frac{k_\phi}{k_\theta^3} y''' \text{ and } \phi^{iv} = \frac{k_\phi}{k_\theta^4} y^{iv} \tag{9}$$

For the P-P-P-P case, initially we assume a value of θ_{in} or ϕ_{in}. x_j (j = 1, 2, 3, 4) are determined using Chebyshev spacing and the corresponding y_j values are found. Corresponding crank angles are found using $\theta_j = \theta_{in} + k_\theta(x_j - x_{in})$ and $\phi_j = \phi_{in} + k_\phi(y_j - y_{in})$ and Freudenstein's equation is written for the four positions. When we determine the value of the augmented matrix formed by these four equations, the value of the determinant is not necessarily zero. "Solver" tool in Excel is used to make the value of the determinant zero by changing one of the initial crank angles. In general, there are two possible solutions. If both solutions are sought, the initial guess for the initial crank angle must be altered. Instead of using the augmented matrix, Solver tool can also be used to determine the value of initial crank angle such that the fourth equation is satisfied. In Fig. 2, five different cases are shown ($a_1 = 1$). The results must be analyzed.

P-P-P-P	x_1= 1.34254 x_2= 3.77792 x_3= 7.22208 x_4= 9.65746 θin 45°	a_2= 1.29229 a_3= 0.71422 a_4= 1.57398 ϕin 108.90922°
P-P-P-P	x_1= 1.3425 x_2= 3.7779 x_3= 7.2221 x_4= 9.6575 ϕin 120°	a_2= 2.03603 a_3= 2.04441 a_4= 1.40217 θin 4.73494°
P-P-P-P-P	x_1= 1.22025 x_2= 2.85497 x_3= 5.50000 x_4= 8.14503 x_5= 9.77975	a_2= 1.22025 a_3= 2.85497 a_4= 5.50000 θin 49.87456 ϕin 108.623096°
P-PPP-P	x_1= 2 x_2= 5 x_3= 8	a_2= 1.27717 a_3= 0.56918 a_4= 1.67189 θin 166.987759 ϕin 129.378129
PPPPP	x= 5	a_2= 1.15418 a_3= 4.71008 a_4= 4.83279 θ 94.89724° ϕ 150.57211°

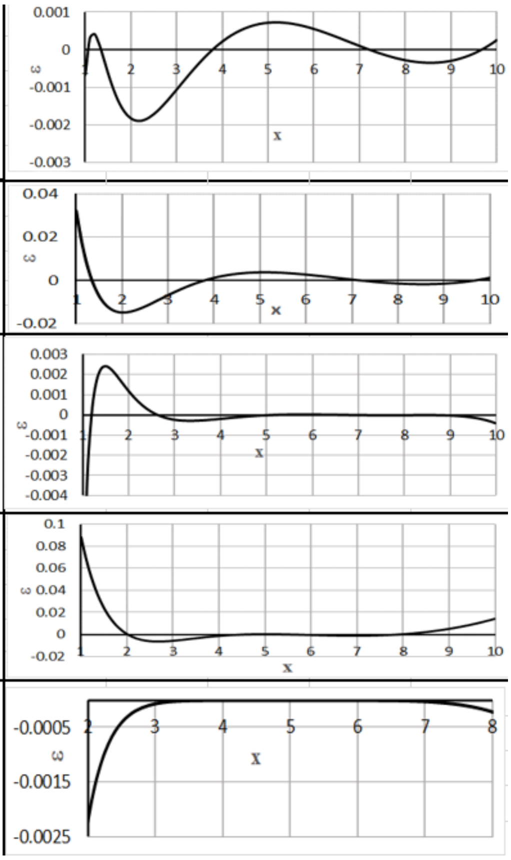


Fig. 2. Approximation of $y = \log(x)$ for $1 \le x \le 10$, $\Delta\theta = 90°$ and $\Delta\phi = 60°$ ($a_1 = 1$).

Excel Solver tool has three optimization algorithms, GRG nonlinear, LP Simplex and Evolutionary. It turns out that GRG nonlinear works quite efficiently in the synthesis problems considered. One can also use Evolutionary engine in the synthesis, provided that limits are placed on all the design parameters.

For approximation with five design parameters, any one of the seven mixed order cases can be solved. One will obtain five equations which are either Freudenstein's equation or its derivatives. Assuming values for θ_{in} and ϕ_{in}, we solve the values of K_i from any three of the equations. When these values are substituted into the remaining two equations, there are certain errors, say δ_1 and δ_2. We evaluate $\Delta = \sqrt{\delta_1^2 + \delta_2^2}$ and using the Solver tool the values of θ_{in} and ϕ_{in} which make $\Delta = 0$ are found. Note that the maximum error values can be decreased by respacing the precision points [7].

3 Least Squares Approximation of a Function

In the approximation of a function described by Levitskii et al. [2, 8], when Eq. (2) is written for points other than the precision points, there is some error involved, i.e.:

$$K_1\cos\phi_k - K_2\cos\theta_k + K_3 - \cos(\phi_k - \theta_k) = \delta_k \quad \text{for} \quad k = 1, 2, .., N \tag{10}$$

where N is greater than the number of design parameters. The “residual error”, δ_k, is not the structural error, but it is related. Taking the squares of the errors and adding up:

$$S = \sum_{k=1}^{N}\left[K_1\cos\phi_k - K_2\cos\theta_k + K_3 - \cos(\phi_k - \theta_k)\right]^2 \tag{11}$$

For S to be at its minimum, a necessary condition is that the derivative of S with respect to the design parameters must vanish. This is known as “least squares approximation”.

Taking the partial derivative of S with respect to K_1, K_2, K_3, θ_{in} and ϕ_{in}, we have:

$$\frac{1}{2}\frac{\partial S}{\partial K_1} = \sum_{k=1}^{N}\cos\phi_k\left[K_1\cos\phi_k - K_2\cos\theta_k + K_3 - \cos(\phi_k - \theta_k)\right] = 0 \tag{12}$$

$$\frac{1}{2}\frac{\partial S}{\partial K_2} = -\sum_{k=1}^{N}\cos\theta_k\left[K_1\cos\phi_k - K_2\cos\theta_k + K_3 - \cos(\phi_k - \theta_k)\right] = 0 \tag{13}$$

$$\frac{1}{2}\frac{\partial S}{\partial K_3} = \sum_{k=1}^{N}\left[K_1\cos\phi_k - K_2\cos\theta_k + K_3 - \cos(\phi_k - \theta_k)\right] = 0 \tag{14}$$

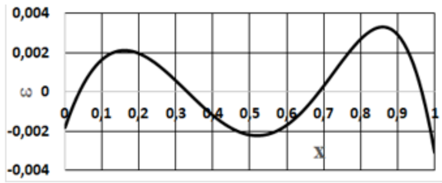
$$\frac{1}{2}\frac{\partial S}{\partial \theta_{in}} = \sum_{k=1}^{N}\left[K_2\sin\theta_k - \sin(\phi_k - \theta_k)\right]\left[K_1\cos\phi_k - K_2\cos\theta_k + K_3 - \cos(\phi_k - \theta_k)\right] = 0 \tag{15}$$

$$\frac{1}{2}\frac{\partial S}{\partial \phi_{in}} = \sum_{k=1}^{N}\left[-K_1\sin\phi_k + \sin(\phi_k - \theta_k)\right]\left[K_1\cos\phi_k - K_2\cos\theta_k + K_3 - \cos(\phi_k - \theta_k)\right] = 0 \tag{16}$$

Although Eqs. (12)-(14) are linear in terms of K_1, K_2 and K_3, Eqs. (15)-(16) are not. Solution using K_i (i = 1, 2, 3) as design parameters only is straightforward and is given in the literature [20]. Proposed solution using four or five design parameters is based on assuming a value for θ_{in} and/or ϕ_{in}. Using four design parameters by taking ϕ_{in} as given, one can assume a value for θ_{in} and solve K_i from Eqs. (12)-(14). When K_i are substituted into Eq. (15) (or Eq. (16) for ϕ_{in}), it will not be satisfied. Then the problem is to determine θ_{in} which will satisfy Eq. (15) (or Eq. (16) for ϕ_{in}). This is basically one parameter search. We have found the Solver tool in Excel to be quite useful. The gradient method (GRG nonlinear) or genetic algorithm (Evolutionary) can be used to determine θ_{in}. For five design parameters case, the same method described in Example 1 is used: Assume values for θ_{in} and ϕ_{in}, determine the errors (δ_1 and δ_2) for Eqs. (15)-(16), evaluate $\Delta = \sqrt{\delta_1^2 + \delta_2^2}$ and use Solver tool to make $\Delta = 0$ by changing θ_{in} and ϕ_{in}. Note that the values of K_i solved from Eqs. (12)-(14) also depend on θ_{in} and ϕ_{in}.

Example 2: Consider the generation of $y = e^x$ for $0 \le x \le 1$, $\Delta\theta = 120°$ and $\Delta\phi = 100°$. This results in $k_\theta = 120°$/unit x and $k_\phi = 58.2°$/unit y. Assuming $\theta_{in} = 60°$ and $\phi_{in} = 45°$, we select 11 points for increments of 0.1 in x and determine the corresponding y_k, θ_k, ϕ_k (k = 0, 1, …, 10). K_1, K_2, K_3 are solved from Eqs. (12)-(14) and substituted into Eqs. (15)-(16). $K_1 = –0.16229$, $K_2 = 00.27223$, $K_3 = 0.95160$, $\delta_1 = –0.00062$, $\delta_2 = 0.00545$, $\Delta = 0.00828$ are found. Solver tool is used to make $\Delta = 0$ by changing θ_{in} and ϕ_{in}. When θ_{in} and ϕ_{in} change, K_i also change. The result is shown in Fig. 3 with $\Delta = 3.4808\times10^{-10}$. Five precision point synthesis with Chebyshev spacing is also shown for comparison.

Least Square 5 Design Variables	x_j= 0.1j, j=0,1,2..10	L= -0,00200
		a_2= 8,67551
		a_3= 7,48004
		a_4= 2,37302
		θ_{in} 258,800424°
		ϕ_{in} 206,396663°
5 Precision point Synthesis	x_1= 0,02447	a_2= 3,91277
	x_2= 0,20611	a_3= 1,30980
	x_3= 0,50000	a_4= 3,52821
	x_4= 0,79389	ϕ_{in} 210,613002°
	x_5= 0,97553	θ_{in} 200,006320°



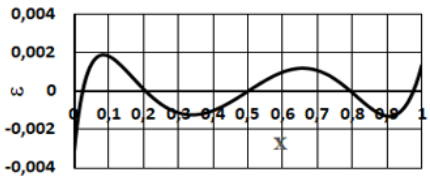


Fig. 3. Approximation of $y = e^x$ for $0 \le x \le 1$, $\Delta\theta = 120°$ and $\Delta\phi = 100°$ ($a_1 = 1$) using least squares method and five precision point approximation using Chebyshev spacing.

4 Extreme Point Synthesis

Although the approximating function generated by the mechanism is not a Chebyshev polynomial, Chebyshev spacing results in a good initial approximation. Another way to use Chebyshev's theory is applying Chebyshev's alternation theorem, where the local extrema of the error function should be equal in magnitude and alternating in sign. If there are n design parameters, there must be n + 1 local extrema [2]. Two of the design points are the terminal points of the interval. The remaining n – 1 design points are within the interval. At these design points we want the errors to be equal to an unknown value ±L. Using the correlation of x with θ and y with ϕ and selecting Δϕ, Δθ, θ_{in}, ϕ_{in} as before, we can determine the crank angles θ_i, ϕ_i to be correlated at the design points using a four-bar mechanism. This approximation method is called Best or Chebyshev approximation in the literature, but in order to emphasize the aim, we shall call it "***Extreme Point Synthesis***". This procedure can be used for four, five or six extrema. Design parameters are K_1, K_2, K_3, θ_{in} and ϕ_{in}, but L also needs to be determined.

When the approximating function is a Chebyshev polynomial, the design points are at an extremum point of the function. For n + 1 design points this spacing is given by:

$$x_j = \frac{1}{2}\left(x_{fin} + x_{in}\right) - \frac{1}{2}\left(x_{fin} - x_{in}\right)\cos\left(\frac{j}{n}\pi\right) \quad \text{for} \quad j = 0, 1, 2, .., n \qquad (17)$$

Note that Eq. (8) gives the roots of a Chebyshev polynomial, whereas Eq. (17) gives the values for the extrema of a Chebyshev polynomial. When we assume that the magnitude of the error at the design points are all equal, Freudenstein's equation can be written at the design points in the following form:

$$K_1\cos\phi_j - K_2\cos\theta_j + K_3 - \cos\left(\phi_j - \theta_j\right) = (-1)^j L \quad \text{for} \quad j = 0, 1, 2, .., n \tag{18}$$

Notice that Eq. (18) guarantees that the errors have the same magnitude at the design points, but it does not guarantee that these points correspond to the local extrema of the error function. So, it is a necessary, but not sufficient condition.

If we use 3 design parameters K_i, we can design for 4 design points. When 4 design points are considered, the unknowns are K_1, K_2, K_3 and L. This set of four linear equations in four unknowns can be solved. When we use 4 design parameters, we also include one of the initial crank angle (say θ_{in}) as a design parameter and write Eq. (18) for five design points. The unknowns are K_1, K_2, K_3, L and θ_{in}. Similar to 4 precision point synthesis, we use any four of the five equations by assuming a value for θ_{in}. Then using the Solver tool we change the value of θ_{in} such that either the determinant of 4x4 matrix is zero, or the value of the fifth equation is satisfied with the new value of θ_{in}. With five design parameters, we can write Eq. (18) for six design points. We assume values for θ_{in} and ϕ_{in} and solve for K_1, K_2, K_3 and L from any four of the six equations. The rank of the equations must be 4. Solution for the design parameters may be done using the two methods described for the previous cases. The design points will not necessarily be the points of local extreme values, since the approximating function is not a Chebyshev polynomial. An algorithm developed by Remes [21] can be used to recalculate the design points.

Example 3: For equal extreme local error synthesis functions $y = x^2$ for $0 \leq x \leq 1$, $\Delta\theta = \Delta\phi = 100^o$; y = logx for $1 \leq x \leq 10$, $\Delta\theta = 90^o$, $\Delta\phi = 60^o$; $y = e^x$, $0 \leq x \leq 1$, $\Delta\theta = \Delta\phi = 100^o$ are approximated using 5 or 6 design points. The results are shown in Fig. 4.

5 Discussion

The method proposed in this Chapter is a simple and efficient way of approximating a function with a four-bar mechanism using four or five design parameters. The method only involves solution of linear set of equations and evaluating determinants, whereas numerical solution of nonlinear equations is performed using a simple program such as Excel. So, it is easy to explain and apply even at the undergraduate level. Problems such as precision point synthesis, order synthesis, mixed order synthesis, least squares approximation and extreme point synthesis can all be treated using the same method. The method can be easily extended to function approximation using other planar, spherical or spatial single loop mechanisms. Solver tool in Excel is a widely available tool and if necessary function approximation can also be performed by applying constraints on the link length ratios and/or on the transmission angle deviation as well.

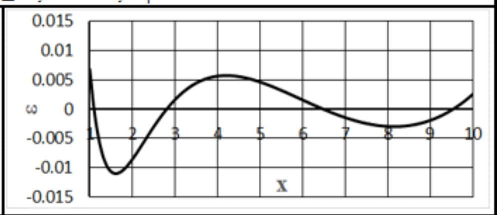
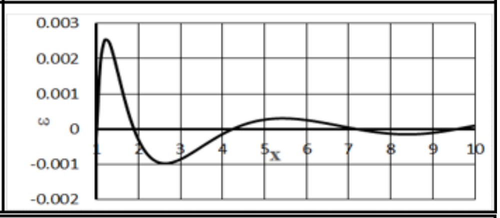
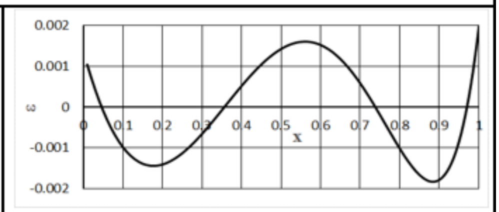
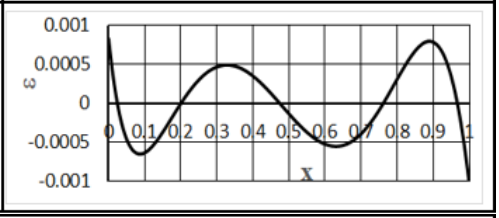
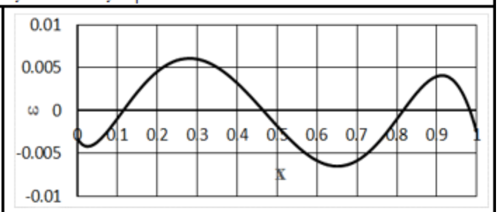
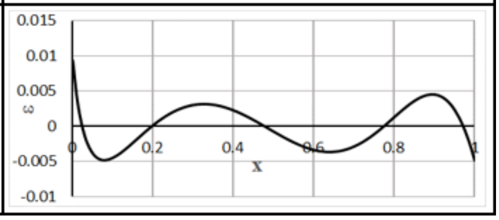
$y=\log x,\ 1\leq x\leq 10,\ \Delta\theta=90^{\circ},\ \Delta\phi=60^{\circ}$			
Extreme Point Syn. 5 Design P	$x_1=$ 1 $x_2=$ 2.318019 $x_3=$ 5.5 $x_4=$ 8.681981 $x_5=$ 10 θ_{in} 15°	L= 0.00259 $a_2=$ 1.44378 $a_3=$ 1.41429 $a_4=$ 1.42801 ϕ_{in} 120.59663°	
Extreme Point Syn. 6 Design P	$x_1=$ 1.00000 $x_2=$ 1.85942 $x_3=$ 4.10942 $x_4=$ 6.89058 $x_5=$ 9.14058 $x_6=$ 10.00000	L= -3.814E-05 $a_2=$ 1.2986421 $a_3=$ 0.68848815 $a_4=$ 1.59704847 ϕ_{in} 47.01375° θ_{in} 108.65274°	
$y=e^{x},\ 0\leq x\leq 1,\ \Delta\theta=\Delta\phi=100^{\circ}$			
Extreme Point Syn. 5 Design P	$x_1=$ 0.00000 $x_2=$ 0.14645 $x_3=$ 0.50000 $x_4=$ 0.85355 $x_5=$ 1.00000 θ_{in} 260°	L= -0.00200 $a_2=$ 2.30440 $a_3=$ 3.70343 $a_4=$ 1.59177 ϕ_{in} 161.94770°	
Extreme Point Syn. 6 Design P	$x_1=$ 0.00000 $x_2=$ 0.09549 $x_3=$ 0.34549 $x_4=$ 0.65451 $x_5=$ 0.90451 $x_6=$ 1.00000	L= 0.00012 $a_2=$ 2.33811 $a_3=$ 4.17746 $a_4=$ 1.89991 ϕ_{in} 232.80716° θ_{in} 215.84132°	
$y=x^{2},\ 0\leq x\leq 1,\ \Delta\theta=100^{\circ},\ \Delta\phi=100^{\circ}$			
Extreme Point Syn. 5 Design P	$x_1=$ 0.00000 $x_2=$ 0.14645 $x_3=$ 0.50000 $x_4=$ 0.85355 $x_5=$ 1.00000 θ_{in} 80°	L= 0.00128 $a_2=$ -2.20461 $a_3=$ 0.97148 $a_4=$ 1.69893 ϕ_{in} 226.25000°	
Extreme Point Syn. 6 Design P	$x_1=$ 0.00000 $x_2=$ 0.09549 $x_3=$ 0.34549 $x_4=$ 0.65451 $x_5=$ 0.90451 $x_6=$ 1.00000	L= -0.00211 $a_2=$ -2.2913747 $a_3=$ 0.8905602 $a_4=$ -1.8272958 ϕ_{in} 257.68292° θ_{in} 77.68292°	

Fig. 4. Approximating functions $y = x^2$ ($0 \leq x \leq 1$, $\Delta\theta = \Delta\phi = 100^o$), $y = \log x$ ($1 \leq x \leq 10$, $\Delta\theta = 90^o$, $\Delta\phi = 60^o$) and $y = e^x$ ($0 \leq x \leq 1$, $\Delta\theta = \Delta\phi = 100^o$) for five and six design points.

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