ON THE RINGS WHOSE INJECTIVEMODULES ARE MAX-PROJECTIVE

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ABSTRACT

ON THE RINGS WHOSE INJECTIVE MODULES ARE MAX-PROJECTIVE

In this thesis, for some classes of rings including, local, semilocal right semihereditary and right Noetherian right nonsingular, we obtain some conditions that equivalent to being right max-QF. For example, for a semilocal right semihereditary ring, we prove that, the ring is right max-QF if and only if it is a direct product of a semisimple ring and a right small ring. A right Noetherian right nonsingular ring is right max-QF if and only if every injective module can be expressed as a direct sum of an injective module with no maximal submodules and a projective module. We show that, for a ring, being max-QF and almost-QF are not left-right symmetric. An example is given in order to show that max-QF and almost-QF rings are not closed under factor rings.

ÖZET

İNJEKTİF MODÜLLERİ MAX-PROJEKTİF OLAN HALKALAR ÜZERİNE

Bu tezde, yerel, yarı yerel sağ yarı kalıtsal ve sağ Noether sağ tekil olmayan dahil olmak üzere halka sınıfları için sağ max-QF olma durumunu sağlayan bazı koşullar elde edilmiştir. Örneğin, yarı yerel sağ yarı kalıtsal bir halka için, halkanın "sağ max-QF" olması için gereken ve yeterli koşulun, yarı basit bir halka ile sağ küçük bir halkanın doğrudan çarpımı olması olduğunu kanıtlıyoruz. Sağ Noether sağ tekil olmayan bir halka, sağ max-QF ise ve ancak her injektif modül, maksimal alt modülleri olmayan bir injektif modülle bir projektif modülün "doğrudan toplamı" olarak ifade edilebiliyorsa, sağ max-QF olur. Bir halka için max-QF ve hemen hemen-QF olma durumunun sol-sağ simetrik olmadığını gösteriyoruz. Max-QF ve hemen hemen-QF halkaların bölüm halkaları" altında kapalı olmadığını göstermek için bir örnek verilmiştir.

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}, \mathbb{N}$	the ring of integers, the set of non-negative integers
$\mathbb{C}$	the set of all complex numbers
$\mathbb{Q}$	the field of rational numbers
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$\mathrm{Ker}(f)$	the kernel of the map f
$\mathrm{Im}(f)$	the image of the map f
$\mathrm{Soc}(M)$	the socle of the R-module M
$\mathrm{Rad}(M)$	the radical of the R-module M
$E(M)$	the injective hull of a module M
$J(R)$	the Jacobson radical of the ring R
$ann_l(X)$	$= \{r \in R \mid rX = 0\}$ = the *left* annihilator of a subset X of a *left* R-module M
$ann_r(X)$	$= \{r \in R \mid Xr = 0\}$ = the *right* annihilator of a subset X of a *right* R-module M
$\mathrm{Ext}_R(C, A) = \mathrm{Ext}^1_R(C, A)$	set of all equivalence classes of short exact sequences starting with the R-module A and ending with the R-module C
$\cong$	isomorphic
$\leq$	submodule
$\ll$	small (=superfluous) submodule
$\trianglelefteq$	essential (=large) submodule

CHAPTER 1

INTRODUCTION

Throughout this thesis, R denotes a ring with an identity element. The modules which are considered here will be unital right modules, unless otherwise stated.

A right module M is said to be R-projective if each homomorphism $f : M \to R/I$ factors through the canonical epimorphism $\pi : R \to R/I$ for any right ideal I of R. This notion generalizes the notion of projectivity. For example, the abelian group $\mathbb{Q}$ is $\mathbb{Z}$-projective, but it is not projective as a $\mathbb{Z}$-module. Sandomierski (F. Sandomierski, 1964) proved that, over a right perfect ring, R-projectivity implies projectivity. More generally, a ring R is said to be right testing if each R-projective right R-module is projective. Faith asked when R-projectivity implies projectivity for all right R-modules. Recently, Trlifaj proved that answer to the Faith's question above is undecidable in ZFC (see, [24]).

In (Y. Alagöz and E. Büyükaşık, 2021), the authors investigate and study a generalization of R-projectivity. Namely, they call a right module max-projective if each homomorphism $f : M \to R/I$ factors through the natural epimorphism $\pi : R \to R/I$ for each maximal right ideal I of R. R-projective and max-projective right modules coincide over the ring of integers and over right perfect rings.

It is well known that, over a QF-ring each injective right R-module is projective. A natural question arose in this context: for what rings injective right R-modules are R-projective (resp. max-projective)? This motivates the following definitions which are studied in ((Y. Alagöz and E. Büyükaşık, 2021)).

A ring R is said to be right almost-QF (respectively, max-QF) if each injective right R-module is R-projective (respectively, max-projective). Some classes of almost-QF and max-QF rings are investigated in ((Y. Alagöz and E. Büyükaşık, 2021)).

In this thesis, we continue the investigation of almost-QF and max-QF rings. We generalize some results of (Y. Alagöz and E. Büyükaşık, 2021). For a semilocal right semihereditary ring we prove that R is right max-QF if and only if $R = S \times T$, where S is a semisimple ring and T is right small. In addition, if the ring is local, then R is max-QF if and only if R is right small or a division ring. For an arbitrary local ring, we show that R is max-QF if and only if R is right small or R is right self-injective and $Ext_R(E, J(R)) = 0$ for each injective right R-module E, where $J(R)$ is the Jacobson radical of R.
It is proved that both of almost-QF and max-QF rings are not left-right symmetric. An

example is given in order to show that max-QF and almost-QF rings are not closed under factor rings.

CHAPTER 2

PRELIMINARIES

This chapter introduces various definitions and characterizations which will be used throughout the study.

Definition 2.1 *A ring is a set R equipped with two binary operations, usually denoted by addition* $(+)$ *and multiplication* $(\cdot)$*, such that the following conditions hold:*

(1) $(R, +)$ *forms an abelian group.*

(2) The operation "·" of multiplication is associative.

(3) The operation "·" is distributive over addition, i.e., for a, b*, and* $c \in R$*, we have:* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ *and* $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$*.*

Example 2.1 *(a)* $R = \{0\}$

(b) $M_n(R)$*, the set of* $n \times n$ *matrices, is a noncommutative ring.*

(c) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ *are all commutative rings with identity.*

Throughout this thesis, each R will be a ring with identity.

Definition 2.2 *An additive abelian group M is said to be a right R-module if there is a binary operation* $(x, a) \mapsto xa$ *from* $M \times R$ *to M, satisfying the following conditions for all* $x, y \in M$ *and* $a, b \in R$*,*

$$\begin{aligned} (x + y)a &= xa + ya \\ x(a + b) &= xa + xb \\ x(ab) &= (xa)b \\ x1 &= x. \end{aligned}$$

Unless otherwise stated, modules will be right R-modules through this thesis.

Definition 2.3 *A nonempty subset N of M is said to be a submodule of M if N is a subgroup of M and $xa \in N$ for $x \in N$ and $a \in R$.*

Definition 2.4 *A submodule N of M is said to be an essential submodule of M for if $N \cap N' = 0$ then $N' = 0$ for each submodule N' of M.*

Definition 2.5 *The right annihilator of an R-module M, denoted r.ann(M) is the set of all elements in R such that $r.ann(M) = \{r \in R | m.r = 0, \forall m \in M\}$.*

Lemma 2.1 *(D. S. Dummit and R. M. Foote, 2003) If M and N are isomorphic right R-modules, then they have the same annihilator.*

Lemma 2.2 *((P. E. Bland, 2011), Proposition 5.1.5) If N is a submodule of an R-module M, then there is a submodule K of M such that K + N is essential in M and the sum is direct.*

Proof Let S be the set of submodules N' of M such that $N \cap N' = \emptyset$. Then, since the zero submodule of M is in S, S is nonempty. By Zorn's Lemma, S has a maximal element, say N_{max}. Since $N_{max} \in S$, then $N \cap N_{max} = 0$, i.e., the sum $N + N_{max}$ is direct. Now claim that $N + N_{max}$ is essential in M. Let X be a nonzero submodule of M, and suppose that $N + N_{max} \cap X = 0$. Since the intersection is empty, then X cannot be contained in N_{max}, i.e., N_{max} properly contained in $X + N_{max}$. Therefore, $N \cap (X + N_{max}) \neq 0$ since N_{max} is the maximal in S. Let $0 \neq z \in N \cap (X + N_{max})$, and choose $x \in X$ and $y \in N_{max}$ such that $z = x + y$. Then $z - y = x \in (N + N_{max}) \cap X = 0$ gives $z = y$ so that $z \in N \cap N_{max} = 0$, a contradiction. Thus $N + N_{max} \cap X \neq \emptyset$, i.e., $N + N_{max}$ is essential in M. □

2.1. Socle And Radical Of Module

Definition 2.6 *An R-module M is said to be a simple module if 0 and M are the only submodules of M.*

Definition 2.7 *The socle of an R-module M is the sum of all simple submodules of M and is denoted by* Soc(*M*).

Definition 2.8 *An R-module is M said to be semisimple if* Soc(*M*) = *M*.

Proposition 2.1 *(F. W. Anderson and K. R. Fuller, 1992) Let R be a ring and M a right R-module. Then following holds for M.*

(1) *For any submodule* N *of* M, $\mathrm{Soc}(N) = N \cap \mathrm{Soc}(M)$.

(2) $\oplus_{i\in I} \mathrm{Soc}(M_i) = \mathrm{Soc}(\oplus_{i\in I} M_i)$.

(3) $\mathrm{Soc}(M)$ *coincides with the intersection of essential submodules of* M.

Definition 2.9 *The radical of a module* M *is the intersection of all maximal submodules of* M *and is denoted by* $\mathrm{Rad}(M)$. *Also, when* $M = R$ *for a ring* R, *it is also called the Jacobson radical and is denoted by* $J(R)$.

Proposition 2.2 *((P. E. Bland, 2011), Proposition 6.1.8) Let R be a ring. Then following hold.*

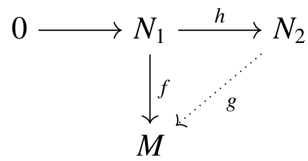
(1) $J(R)$ *is the intersection of the right annihilators of all the simple right R-modules.*

(2) $J(R)=\{x \in R | 1 - xr$ *has right inverse* $\forall r \in R\}$.

2.2. Injectivity And Projectivity of Modules

In this section, we recall the definition of injective and projective modules and give some characterizations.

Definition 2.10 *A right R-module M is injective if every row exact diagram of the form*



where $f : N_1 \to M$, $h : N_1 \to N_2$ *are R-module homomorphisms, and* N_1, N_2 *are R-modules can be completed by an R-module homomorphism* $g : N_2 \to M$.
In particular, suppose M is an injective right R-module and f extends to g. Let g(1)=x for $x \in M$. *Then* $f(a) = g(a) = g(1.a) = g(1)a = xa$ *for every* $a \in N_1$.

Baer showed that it is enough to check injectivity for the right ideals of R.

Theorem 2.1 ***(Baer's Criteria)*** *An R-module M is injective if and only if every R-module homomorphism f from an ideal I of R to M can be extended to an R-module homomorphism g from R to M, i.e.,* $g(a) = f(a)$ *for every* $a \in I$.

This criterion is useful in characterizing injective modules.

Definition 2.11 *A submodule N of M is said to be closed in M if $N \trianglelefteq X \leq M$ where X is a submodule of M, then $N = X$.*

Definition 2.12 *Let R be a ring and M a right R-module. The singular submodule of M is the set $Z(M) = \{m \in M | ann_r(m) \trianglelefteq R\}$. If $Z(M) = M$, M is said to be singular submodule and if $Z(M) = 0$ M is said to be nonsingular.*

Lemma 2.3 *(K. R. Goodearl, 1976) Let N be a submodule of an R-module M. If $Z(M/N) = 0$, then N is closed in M.*

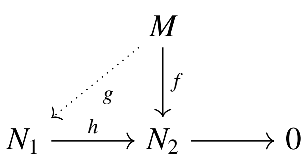
Proof Let N' be an essential extension of N in M, and let $x \in N'$. Then, $I = \{r \in R | xr \in N\}$ is a large ideal, of R and I is a large ideal in R. Since I is large in R and $x + N \in Z(M/N) = 0$, $x \in N$ so that $N = N'$, i.e., N is closed in M. $\square$

Definition 2.13 *Let R be a ring and M a right R-module. M is p-injective if $\forall aR \subseteq R$ the following diagram commutes*

$$\begin{array}{ccc} aR & \longrightarrow & R \\ \downarrow{\scriptstyle f} & & \\ M & & \end{array}$$

where $f : aR \to M$ is a homomorphism.

Definition 2.14 *A right R-module M is projective if every row exact diagram of the form*



$f : M \to N_1$, $h : N_2 \to N_1$ are R-module homomorphisms and N_1, N_2 are R-modules can be completed by an R-module homomorphism $g : M \to N_2$.

Proposition 2.3 *((P. E. Bland, 2011), Proposition 3.2.7) A short exact sequence*

$$0 \longrightarrow N_1 \xrightarrow{f} M \xrightarrow{g} N_2 \longrightarrow 0$$

splits if and only if one of the following three conditions holds.

(1) $\mathrm{Im}(f)$ *is a direct summand of M.*

(2) $\mathrm{Ker}(g)$ *is a direct summand of M.*

(3) $M \cong N_1 \oplus N_2$.

Theorem 2.2 *((P. E. Bland, 2011), Problem Set 5.2(1)) A right R-module M is projective if and only if each short exact sequence of the form*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow M \longrightarrow 0$$

splits.

Definition 2.15 *A ring R is called right p.p. ring if each principal right ideal of R is projective.*

Proposition 2.4 *(T. Y. Lam, 1999) Let R be a ring. Then R is right p.p. ring if and only factors of p-injective right modules are p-injective.*

Proof First, suppose that R is a right p.p. ring, and consider the diagram

$$\begin{array}{ccccc} & & aR & & \\ & & \downarrow f & & \\ E & \xrightarrow{\pi} & E/K & \longrightarrow & 0 \end{array}$$

where π is the canonical epimorphism, and $f : aR \to E/K$.
Since R is a right p.p. ring, there is a homomorphism $g : aR \to E$ such that $\pi \circ g = f$. Then we got the diagram

$$\begin{array}{ccccc} & & aR & \xrightarrow{\iota} & R \\ & \swarrow g & \downarrow f & & \\ E & \xrightarrow{\pi} & E/K & \longrightarrow & 0 \end{array}$$

where $\iota : aR \to R$ is an injection. Since R is projective, there exists $h : R \to E$ such that $\pi \circ g = f$.
Now, we got $h \circ \iota \Rightarrow (\pi \circ h) \circ \iota = \pi \circ g = f$. Thus $\pi \circ h$ extends f, i.e., E/K is p-injective.
For the converse, suppose that factors of p-injective right modules are p-injective, and

consider the diagram

$$\begin{array}{ccccc} & & aR & \xrightarrow{\iota} & R \\ & & \downarrow{\scriptstyle f} & & \\ E & \xrightarrow{\pi} & E/K & \longrightarrow & 0 \end{array} .$$

Since factors of p-injective right modules are p-injective, there is a homomorphism $g : R \to E/K$ such that $g \circ \iota = f$, and since R is projective, there exists a homomorphism $h : R \to E$ such that $\pi \circ h = g$.
Now, $\pi \circ (h \circ \iota) = g \circ \iota = f$, so $h \circ \iota$ lifts f, i.e., aR is projective. □

Lemma 2.4 *A right p.p. ring is right nonsingular.*

Proof Let $x \in Z(R_R) = \{x \in R | r.ann(x) \trianglelefteq R\}$. Then $r.ann(x) = I_R \trianglelefteq R$ so that $xI = 0$. Now, let $f : R \to xR$ be a homomorphism, then $\text{Ker}(f) = \{r \in R | x.r = 0\}$. Consider the sequence

$$0 \longrightarrow I \longrightarrow R \xrightarrow{f} xR \longrightarrow 0 \quad .$$

Since R is a right p.p. ring, the sequence splits so that there exists a homomorphism $g : xR \to R$ and $R = \text{Ker}(f) \oplus \text{Im}(g) = I \oplus \text{Im}(g)$. Then $\text{Im}(g) = 0$ since I is essential so that $g = 0$ and $I = R = ann_r(x)$. Therefore, $x = 0$, i.e., $Z(R_R) = 0$. □

Corollary 2.1 *(F. W. Anderson and K. R. Fuller, 1992) Let R be a right hereditary ring, then R is right nonsingular.*

Proof Proof of the corollary is similar to the Lemma 2.4 □

2.2.1. Injective Hull

This section introduces injective hull of module and give important characterizations which will be used throughout the study.

Definition 2.16 *An injective hull of an R-module M is an injective module $E(M)$ such that $E(M)$ is an essential extension of M. It is the largest essential extension of M and also the smallest injective module containing M.*

Proposition 2.5 *((P. E. Bland, 2011), Proposition 7.1.5) The following properties hold for injective hulls.*

(1) If M is a submodule of an injective R-module E, then $E \cong E(M) \oplus E'$ for some injective submodule E' of E.

(2) If M is an essential submodule of an R-module N, then $E(M) \cong E(N)$.

(3) If $\{M\alpha\}_{\alpha\in\Delta}$ is a family of R-modules, then $\oplus_{\Delta}E(M_{\alpha})$ embeds in $E(\oplus_{\Delta}M_{\alpha})$.

Direct summand of an injective module is injective but direct sum of injective modules is not always injective. Following theorem shows that, on a right Noetherian ring, direct sum of injective modules is injective.

Proposition 2.6 *(Z. Papp, 1959) The following are equivalent for a ring R.*

(1) Every direct sum of injective R-modules is injective.

(2) If $\{M_{\alpha}\}_{\alpha\in\Delta}$ is a family of R-modules, then $\oplus_{\Delta}E(M_{\alpha}) \cong E(\oplus_{\Delta}M_{\alpha})$.

(3) R is a right Noetherian ring.

Definition 2.17 *A submodule N' of an R-module M is small in M if whenever $N' + X = M$, then $X = M$ where X is a submodule of M. Similarly a ring R is called small ring if it is small in its injective hull.*

Proposition 2.7 *((V. S. Ramamurthi, 1982), 3.3) Let R be a ring, and let E(R) be the injective hull of R_R. Then the following conditions are equivalent.*

(1) R is a left small ring.

(2) $\mathrm{Rad}(M) = M$ *for every injective left R-module M.*

(3) $\mathrm{Rad}(E(R)) = E(R)$.

Since injective hull is the essential extension of module, we got the following result for closed submodules of injective modules.

Proposition 2.8 *(T. Y. Lam, 1999) Closed submodule of an injective module is a direct summand.*

Proof Let X be a closed submodule of an injective module E. Since $E(X)$ is the injective hull of X, $X \trianglelefteq E(X) \leq E$, and since X is closed in E, $X = E(X)$. Hence, X is injective, i.e., X is direct summand of E. □

Definition 2.18 *A ring R is right self-injective if R_R is injective.*

Definition 2.19 *Let $I = E(M)$, and let $H = End(I_R)$. We define*

$$\tilde{E}(M) = \{i \in I : \forall h \in H, h(M) = 0 \Rightarrow h(i) = 0\}.$$

We call $\tilde{E}(M)$ the rational hull of M. We also denote this ring by $Q^r_{max}(M)$ and call it the maximal ring of quotients of R.

Following theorem shows that for a right module M, rational hull of$M$, $Q^r_{max}(M)$, and the injective hull of M, $E(M)$ coincides over right nonsingular ring.

Theorem 2.3 *((T. Y. Lam, 1999), 13.36, Johnson's Theorem) For any ring R, the following are equivalent.*

(1) R is right nonsingular.

(2) $I_R = E(R_R)$ is a nonsingular R module.

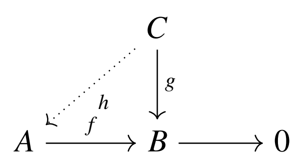
(3) $H = End(I_R)$ is Jacobson semisimple.

(4) $Q = Q^r_{max}(M)$ is Von Neumann regular.

If these conditions hold, then $Q = I$ and $Q \cong H$ are right self injective rings.

Definition 2.20 *An epimorphism $f : A \to B$ is called small epimorphism if* $\mathrm{Ker}(f)$ *is small in A.*

Lemma 2.5 *Let A, B and C be right R-modules and consider the diagram*



where f is a small epimorphism and g is an epimorphism. Then h is an epimorphism.

Proof Let $a \in A$. Then, since g is an epimorphism, there exists $c \in C$ such that $f(a) = g(c)$. The diagram is commutative, so we get $(f \circ h)(c) = f(h(c)) = g(c) = f(a)$ gives that $f(h(c)) = f(a)$, i.e., $f(a - h(c)) = 0$. Then $a - h(c) \in \mathrm{Ker}(f)$, and so $a \in h(c) + \mathrm{Ker}(f)$. Now, $A = h(c) + Ker(f)$, and since $Ker(f)$ is small in A, we get $h(c) = A$ so that h is an epimorphism. □

2.3. Pure submodules, pure-injective modules, and absolutely pure modules

Definition 2.21 *Let R be a ring. A short exact sequence* $0 \to A_R \to B_R \to C_R \to 0$ *of right R-modules is pure if the induced sequence of abelian groups*

$$0 \to A_R \otimes_R E \to B_R \otimes_R E \to C_R \otimes_R E \to 0$$

is exact for every left R-module E.
A submodule A of a right R-module B is a pure submodule of B if the canonical exact sequence $0 \to A \to B \to B/A \to 0$ *is pure.*

Definition 2.22 *A right R-module* M_R *is called pure-injective if the sequence*

$$0 \to C_R \otimes M_R \to B_R \otimes M_R \to A_R \otimes M_R \to 0$$

is exact for every pure exact sequence $0 \to A_R \to B_R \to C_R \to 0$.

Proposition 2.9 *((A. Facchini, 1998), Corollary 1.36) If* $\varphi : R \to S$ *is a ring homomorphism, then every pure injective right S-module is pure-injective as a right R-module.*

The following is an immediate consequence of the Proposition 2.9.

Corollary 2.2 *(A. Facchini, 1998) If R is either commutative or semilocal, then every simple left or right R-module is pure-injective.*

Definition 2.23 *A right R-module M is absolutely pure if it is pure in every module contanining it as a submodule.*

It is easy to see that if a right R-module M is absolutely pure and pure injective, then it is injective.

2.4. Semihereditary and Hereditary Rings

Definition 2.24 *A ring R is said to be right hereditary if every right ideal of R is projective. We call R right semihereditary if every finitely generated right ideal of R is projective.*

Theorem 2.4 *((C. Megibben, 1970), Theorem 2) For a ring R, the following conditions are equivalent.*

(1) R is right semihereditary.

(2) Each finitely generated submodule of a projective right R-module is projective.

(3) The homomorphic image of an absolutely pure right R-module is absolutely pure.

The following theorem shows that over a right hereditary ring, submodules of projective modules are projective and injective modules are closed under factor modules.

Theorem 2.5 *((J. Rotman, 1979), Theorem 4.19) For a ring R the following conditions are equivalent.*

(1) R is right hereditary.

(2) Each submodule of a projective right R-module is projective.

(3) Factor module of an injective R-module is injective.

2.5. Local and Semilocal Rings

Definition 2.25 *A nonzero ring R is local if R has a unique maximal right ideal. Also R is said to be semilocal if $R/J(R)$ is a semisimple ring.*

Definition 2.26 *Let R be a ring. Then U(R) is the group of units of R.*

Theorem 2.6 *((T. Y. Lam, 1991), Theorem 19.1) For any nonzero ring R, the following statements are equivalent.*

(1) R is a local ring.

(2) R has a unique maximal right ideal.

(3) $R/\operatorname{Rad}(R)$ is a division ring.

(4) $R \setminus U(R)$ is an ideal of R.

(5) $R \setminus U(R)$ is a group under addition.

(6) For any n, $a_1 + a_2 + \cdots + a_n \in U(R)$ implies that some $a_i \in U(R)$.

(7) If $a \in R$, then either a or $1 - a$ is a unit.

2.6. Semiperfect and Perfect Rings

Definition 2.27 *A ring R is called semiperfect if R is semilocal, and idempotents of $R/\operatorname{Rad}(R)$ can be lifted to R.*

Example 2.2 ***(a)*** *Local rings are semiperfect.*

(b) *Division rings are semiperfect.*

(c) *Right artinian rings are semiperfect.*

Corollary 2.3 *((F. W. Anderson and K. R. Fuller, 1992), Corollary 27.9) If a ring R is semiperfect, then so is every factor ring of R.*

Definition 2.28 *A projective cover of an R-module M is projective R-module P(M) with an epimorphism $\phi : P(M) \to M$ such that $\operatorname{Ker}(\phi)$ is small in P(M).*

We can characterize right semiperfect rings with projective cover as the following.

Proposition 2.10 *((P. E. Bland, 2011), Definition 7.2.10) A ring R is said to be a semiperfect ring if every finitely generated right R-module has a projective cover.*

Note that semiperfect rings are left-right symmetric.

Definition 2.29 *A subset A of a ring R is called right T-nilpotent if, for any sequence of elements $\{a_1, a_2, \dots\} \subseteq A$, there is an integer $n \geq 1$ such that $a_n \cdots a_2.a_1 = 0$.*

Definition 2.30 *A ring R is called right perfect is $R/J(R)$ is semisimple and $J(R)$ is right T-nilpotent.*

Proposition 2.11 *((H. Bass, 1960), Theorem P) The following are equivalent for a ring R.*

(1) R is a right perfect ring.

(2) $R/J(R)$ is semisimple and every nonzero R-module contains a maximal submodule.

(3) $R/J(R)$ is semisimple and $J(R)$ is right T-nilpotent.

Proposition 2.12 *((H. Bass, 1960), Theorem P) The following are equivalent for a ring R.*

(1) R is a right perfect ring.

(2) R satisfies the descending chain condition on principal left ideals.

(3) Every flat R-module is projective.

(4) R contains no infinite set of orthogonal idempotents and every nonzero left R-module contains a simple submodule.

2.7. Quasi-Frobenius Rings

Definition 2.31 *A ring R is Quasi-Frobenius if R is left or right Noetherian and R is left or right self-injective.*

Proposition 2.13 *((P. E. Bland, 2011), Proposition 10.2.14) A ring R is said to be QF if and only if it satisfies one of the equivalent conditions.*

(1) R is a right Noetherian and satisfies the conditions

(a) $ann_r(ann_l(A)) = A$ *for all right ideals A of R and*

(b) $ann_l(ann_r(A)) = A$ *for all left ideals A of R.*

(2) R is right Noetherian and right self-injective.

(3) R is left Noetherian and right self-injective.

Proposition 2.14 *((T. Y. Lam, 1999), Theorem 15.9) Let R be a ring. Then the following are equivalent.*

(1) Every injective right R-module is projective.

(2) Every projective right R-module is injective.

(3) R is QF-ring.

2.8. Pseudo-Frobenius Rings

Definition 2.32 *A right R-module M is said to be cogenerated by a set of right R-modules* $\{M_\alpha\}_{\alpha\in\Delta}$ *if M can be embedded in* $\prod_\Delta M_\alpha$. *We say that a ring R is cogenerator ring if* R_R *and* ${}_RR$ *are both cogenerators.*

Definition 2.33 *An injective R-module M is an injective cogenerator for the category of right R-modules if every right R-module is cogenerated by M.*

Additionaly, we define Kasch rings in a similar way.

Definition 2.34 *A ring R is said to be a right Kasch if each simple right R-module can be embedded in R.*

If the ring R satisfies the Definition 2.33, then we get the following definition.

Definition 2.35 *R is a right PF ring if* R_R *is an injective cogenerator for the category of right R-modules.*

R is a projective generator for the category of right and left R-modules because every right or left module is an epimorphic image of a free right or left R-module.

2.9. Dual Goldie Torsion Theory

Let $\mathcal{X}$ be the class of right R-modules closed under isomorphism and submodules. Consider the following two classes:

$$\mathbb{F}(\mathcal{X}) = \{M \in Mod - R | Hom(X, M) = 0, \forall X \in \mathcal{X}\}$$

$$\mathbb{T}(\mathcal{X}) = \{N \in Mod - R | Hom(N, M) = 0, \forall M \in \mathbb{F}(\mathcal{X})\}$$

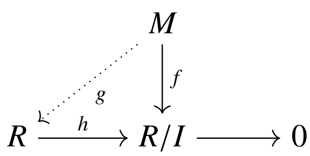
Then the pair $(\mathbb{T}(\mathcal{X}), \mathbb{F}(\mathcal{X}))$ is called a torsion theory.
Let $R - small$ be the class of right small R-modules. For $\mathcal{X} = R - small$, the torsion theory $(\mathbb{T}(R - small), \mathbb{F}(R - small))$ is studied by V.S. Ramamurthi in (V. S. Ramamurthi, 1982). This torsion theory is called dual Goldie torsion theory.
A torsion theory $(\mathbb{T}(\mathcal{X}), \mathbb{F}(\mathcal{X}))$ is said to be splitting if $\mathbb{T}(\mathcal{X}) = Mod - R$ and $\mathbb{F}(\mathcal{X}) = 0$.

CHAPTER 3

MAX-PROJECTIVITY OF MODULES

In this chapter we give some characterizations of max-projective modules.

Definition 3.1 *A right R-module M is R-projective if every row exact diagram of the form*



$f : M \to R/I$, $h : R \to R/I$ *are R-module homomorphisms and I is a right ideal, can be completed by an R-module homomorphism* $g : M \to R$.

If we consider the case for the maximal ideals of R, then we got the following definition.

Definition 3.2 *A right R-module M is max-projective if every epimorphism* $f : R \to R/I$ *with I which is a maximal right ideal and every homomorphism* $g : M \to R/I$*, there exits a homomorphism* $h : M \to R$ *such that* $fh = g$.

Following ones are examples of max-projective modules.

Example 3.1 *(a) Projective modules.*

(b) Modules with $\mathrm{Rad}(M) = M$.

Definition 3.3 *Given modules M and N, M is said to be N-projective if for every epimorphism* $g : N \to T$ *and for every homomorphism* $f : M \to T$*, there exists a homomorphism* $h : M \to N$ *such that* $gh = f$.

This definition generalizes the notion of projectivity. Also, a right R-module M is called projective if M is relative projective for every right R-module N. The following proposition shows that relative projectivity is closed under factor modules, direct sums, and direct summands.

Proposition 3.1 *((F. W. Anderson and K. R. Fuller, 1992), Proposition 16.10) The following statements hold.*

(1) *If M is N-projective and K is a submodule of N, then M is N/K-projective.*

(2) *If $A \cong B$, then M is A-projective if and only if M is B-projective.*

(3) *If M is M_i-projective for all $i = 1, 2, \ldots, n$, then M is $\oplus_{i=1}^{n} M_i$-projective.*

(4) *A direct sum $\oplus_{i=1}^{n} M_i$ of modules is N-projective if and only if each M_i is N-projective.*

(5) *If $A \cong B$, then for any right R-module N, A is N-projective if and only if B is N-projective.*

We define $max - N - projectivity$ as follows.

Definition 3.4 *Given modules M and N, M is said to be max-N-projective if for every epimorphism $g : N \to S$ with S simple and for every homomorphism $f : M \to S$, there exists a homomorphism $h : M \to N$ such that $gh = f$.*

Proposition 3.2 *The following statements hold.*

(1) *If M is max-N-projective and K is a submodule of N, then M is max-N/K-projective.*

(2) *If $A \cong B$, then M is max A-projective if and only if M is max-B-projective.*

(3) *If M is $max - M_i - projective$ for all $i = 1, 2, \ldots, n$ then M is max-$\oplus_{i=1}^{n} M_i$-projective.*

(4) *A direct sum $\oplus_{i=1}^{n} M_i$ of modules is max-N-projective if and only if each M_i is max-N-projective.*

(5) *If $A \cong B$, then for any right R-module N, A is max-N-projective if and only if B is max-N-projective.*

Lemma 3.1 *((Y. Alagöz and E. Büyükaşık, 2021), Lemma 1) The following conditions are true.*

(1) *A direct sum $\oplus_{i \in I} A_i$ of modules is max-projective (resp., R-projective) if and only if each A_i is max-projective (resp., R-projective).*

(2) *If $0 \to A \to B \to C \to 0$ is an exact sequence and M is B-projective, then M is projective relative to both A and C.*

Definition 3.5 *Let M and N be R-modules for ring R. M is said to be N-subprojective if for every homomorphism $f : M \to N$ and for every epimorphism $g : B \to N$, there exists a homomorphism $h : M \to B$ such that $gh = f$.*

Lemma 3.2 *((Y. Alagöz and E. Büyükaşık, 2021), Lemma 2) For an R-module M, the following are equivalent.*

(1) M is max-projective.

(2) M is S-subprojective for each simple R-module S.

(3) For every epimorphism $f : N \to S$ with S simple and homomorphism $g : M \to S$, there exists a homomorphism $h : M \to N$ such that $fh = g$.

Proposition 3.3 *((Y. Alagöz and E. Büyükaşık, 2021), Proposition 1) Let $0 \to A \to B \to C \to 0$ be a short exact sequence. If M is A-subprojective and C-subprojective, then M is B-subprojective.*

Corollary 3.1 *((Y. Alagöz and E. Büyükaşık, 2021), Corollary 2) A $\mathbb{Z}$-module M is max-projective if and only if M is $\mathbb{Z}$-projective.*

Corollary 3.2 *((Y. Alagöz and E. Büyükaşık, 2021), Corollary 3) Let M be an R-module with finite composition length. If M is max-projective, then it is projective.*

Proposition 3.4 *Let R be a right nonsingular ring and E a singular injective right R-module, i.e., $Z(E) = E$. Then E is max-projective if and only if* $\mathrm{Rad}(E) = E$.

Proof Suppose $\mathrm{Rad}(E) \neq E$. Then there is a maximal submodule K of E. Then $E/K \cong R/I$ for some maximal ideal I of R.
Let $f : E \to R/I$ be a nonzero homomorphism. Since E is max-projective, there is an R-module homomorphism $g : E \to R$ such that $\pi \circ g = f$.
Since E is singular, then homomorphic image of E, $g(E)$, is singular too, but $g(E) \subseteq R$ and R is right nonsingular, i.e., $g(E) = 0$. Thus, $f = 0$, which gives contradiction. Thus, there is no maximal submodule K of E, i.e., $\mathrm{Rad}(E) = E$. □

Proposition 3.5 *(P. E. Bland, 2011) Let R be a nonsingular ring. Then R is finite dimensional if and only if for every nonsingular injective right R-module is a direct sum of indecomposable modules.*

Lemma 3.3 *Let R be a right nonsingular ring and E an indecomposable nonsingular injective right R-module. Then E is max-projective if and only if E is projective and cyclic or* $\mathrm{Rad}(E) = E$.

Proof $(\Rightarrow)$ Suppose $\text{Rad}(E) \neq E$, and let us show that E is projective. Since $\text{Rad}(E) \neq E$, then there exists a maximal submodule K of E and the corresponding simple factor module E/K. Since E/K is simple, then $E/K \cong R/I$ where I is a maximal right ideal of R. Thus, there exists a nonzero homomorphism $f : E \to R/I$. Since E is max-projective, by assumption, there is a nonzero homomorphism $g : E \to R$ such that $f = \pi g$ where $\pi : R \to R/I$ is the canonical epimorphism. By the First Isomorphism Theorem, $E/\text{Ker}(g) \cong Im(g) \subseteq R$, so that $E/\text{Ker}(g)$ is nonsingular since R_R is nonsingular.
As the closed submodules of the injective module E are direct summands, and $\text{Ker}(g)$ is a closed submodule of E, we have $E \cong \text{Ker}(g) \oplus E'$ for some submodule E' of E. Now, since E is indecomposable, $\text{Ker}(g) = 0$ or $E' = 0$. If $E' = 0$, then $\text{Ker}(g) = E$ so that $g = 0$, a contradiction. Thus, $\text{Ker}(g) = 0$, and so $E = E'$, and g is monic. Since g is monic, $g(E) \cong E$ is injective. So, $R = g(E) \oplus J$ for some right ideal J of R. Now, since R is projective and $g(E)$ is direct summand of R, $g(E) \cong E$ is cyclic and projective. This proves the necessity.

$(\Leftarrow)$ Clear. □

Lemma 3.4 *Let R be a right nonsingular ring and E a singular injective right R-module. Then E is max-projective if and only if* $\text{Rad}(E) = E$.

Proof Sufficiency is clear. To prove the necessity, assume that $\text{Rad}(E) \neq E$. Then E contains maximal submodules, and so there is a nonzero homomorphism $f : E \to R/I$ for some right ideal I of R. Since E is singular, and R is nonsingular $\text{Hom}(E, R) = 0$. Thus the map f can non be lifted to a homomorphism from E to R. Hence E is not max-projective. This proves the necessity. □

Lemma 3.5 *Let R be a ring and M a right R-module. If $M/\text{Rad}(M)$ is max projective, then M is max-projective.*

Proof Let $f : M/\text{Rad}(M) \to R/I$ be homomorphism, where I is a maximal right ideal of R. Then $f \circ \eta : M \to R/I$ is a homomorphism, where $\eta : M \to M/\text{Rad}(M)$ is the natural epimorphism. Now $\text{Rad}(M) \subseteq \text{Ker}(f)$, so by the First Isomorphism Theorem, there exists a homomorphism $\bar{f} : M/\text{Rad}(M) \to R/I$ such that $\bar{f} \circ \eta = f$. Since $M/\text{Rad}(M)$ is max projective, there exist a homomorphism $g : M \to R$ such that $\pi \circ g = \bar{f} \circ \eta$ where $\pi : R \to R/I$ is the natural epimorphism. Now, compose $\pi \circ g = \bar{f}$ with η from the right we got $\pi \circ (g \circ \eta) = \bar{f} \circ \eta = f$.
Hence $g \circ \eta$ lifts f, i.e. M is max-projective. □

Theorem 3.1 *((R. D. Ketkar and N. Vanaja, 1981), Theorem 2) Let R be a ring satisfying a.c.c. on left ideals which are direct summands of R. Let Q be a left R-module satisfying (1) every finitely generated factor module of Q has a projective cover, (2) Q is R-projective and (3) J(Q) is small in Q. Then Q is a direct sum of cyclic indecomposable projective modules.*

Proof Let $x \in Q$, $x \notin J(Q)$. Then since $x \notin J(Q)$, there is a maximal submodule M of Q such that $x \notin M$. Then $Q = Rx + M$. By (1), Q/M has a projective cover. Since Q/M is simple, projective cover is cyclic indecomposable. Thus we can write $Q = P \oplus Q_1$ where $P \subseteq Rx$ and P is a cyclic indecomposable projective module. Then $Rx = Ry_1 \oplus Rx_1$ where $x = y_1 + x_1$, $P = Ry_1$ $Rx_1 = Rx \cap Q_1$ and Q_1 also satisfies the conditions (1), (2), and (3). Now if $x_1 \notin Q_1$ we can repeat this process to write $Q_1 = Ry_2 \oplus Rx_2$, Rx_2 cyclic indecomposable projective direct summand of Q contained in Rx_1, $x_1 = y_2 + x$, such that $Rx_1 = Ry_2 \oplus Rx_2$ where $Rx_2 = Rx_1 \cap Q_2$. If this process can be repeated not only for finitely many times, we obtain an infinite direct sum $Ry_1 \oplus Ry_2 \oplus \cdots \oplus Ry_n \oplus \ldots$ inside Rx such that for each n, $Ry_1 + Ry_2 + \cdots + Ry_n$ is cyclic projective generated by $y_1 + y_2 + \cdots + y_n$. Let $g_n : R \to R(y_1 + y_2 + \cdots + y_n)$ be the maps defined by $g_n(l) = y_1 + \cdots + y_n$. These maps split and $\mathrm{Ker}(g_n) = ann_r(y_1 + y_2 + \cdots + y_n)$. Therefore, $\mathrm{Ker}(g_1) \supseteq \mathrm{Ker}(g_2) \supseteq \cdots \supseteq \mathrm{Ker}(g_n) \supseteq \cdots$ form a decreasing sequence of summands of R. Hence we can get an increasing sequence $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n \subseteq \cdots$ of summands of R such that $L_n \cong R(y_1 + y_2 + \cdots + y_n)$. By a.c.c. on these summands, $L_n = L_{n+1}$ for some n. Hence $Ry_1 + Ry_2 + \cdots + Ry_n \cong Ry_1 + Ry_2 + \cdots + Ry_{n+1}$. But this cannot happen since each Ry_1, is a non-zero indecomposable module.
Now let $A = \{y | y \in Q, y \neq 0, Ry$ is cyclic indecomposable projective direct summand of $Q\}$. Then the previous arguments with the fact that $J(Q)$ is small in Q show that $Q = \sum_{y \in A} Ry$. Let $\mathfrak{A}$ be the family of subsets B of A satisfying the conditions: (a)$Q = \sum_{y \in B} Ry$ is a direct sum and (b) for $y_1 \cdots y_n \in B$, $Ry_1 + Ry_2 + \cdots + Ry_n$ is a direct summand of Q. $\mathfrak{A}$ is non-empty and by Zorn's there is a maximal element. Let B_0 be a maximal element in $\mathfrak{A}$. Then $P = \sum_{y \in B_0} Ry = \oplus_{y \in B_0} Ry$ projective. To claim $P = Q$, it is sufficient to prove that $A \subseteq P + J(Q)$ since $Q = \sum_{y \in A} Ry$ and $J(Q)$ is small in Q. Let $y \in A$. We consider two cases:

Case 1. $P \cap Ry = 0$

Then $B_0 \subsetneq B_0 \cup U\{y\} \subseteq A$. By maximality of B_0 we can find $y_1, \ldots, y_n$ in B_0 such that $Ry_1 \oplus Ry_2 \oplus \cdots \oplus Ry_n \oplus Ry$ is not a direct summand of Q. By condition (b) on B_0, we can write $Q = (Ry_1 \oplus Ry_2 \oplus \cdots \oplus Ry_n) \oplus Q_1$. Then $Ry_1 \oplus Ry_2 \oplus \cdots \oplus Ry_n \oplus Ry = (Ry_1 \oplus Ry_2 \oplus \cdots \oplus Ry_n) \oplus ((Ry_1 \oplus Ry_2 \oplus \cdots \oplus Ry_n) \oplus Q_1)$. This implies $(Ry_1 \oplus Ry_2 \oplus \cdots \oplus Ry_n \oplus Ry) \cap Q_1 \cong Ry$. Let $(Ry_1 \oplus Ry_2 \oplus \cdots \oplus Ry_n \oplus Ry) \cap Q_1 \cong Rz$. Then

Rz is cyclic indecomposable submodule of Q_1 and Rz cannot be a direct summand of Q_1. Hence it is clear from the previous arguments that $z \in J(Q_1) \subseteq JQ$. It follows that $y \in P + J(Q)$.

Case 2. $P \cap Ry \neq 0$.

If $y \in P$ then we are done so assume that $y \notin P$. Let $0 \neq sy = x \in P \cap Ry$. $ann_r(y)$ of R since Ry is non-zero and projective so let $ann_r(y) = Rt$. Now, for a finite subset $B \subseteq B_0$ such that $x \in \sum_{z \in B} Rz$, $\sum_{z \in B} Rz$ is a direct summand of Q. Let $h : Q \to \sum_{z \in B} Rz$ be the natural projection and let $y' = h(y)$, then $t(y - y') = 0$. Also, $s(y - y') = 0$ since $sy' = sh(y) = h(sy) = h(x) = x = sy$. Thus $ann_r(y) \subsetneq ann_r(y - y')$. Let us show that $R(y - y')$ does not contain a non-zero projective direct summand. Assume that $R(y - y')$ contain a non-zero projective direct summand and N be the projective direct summand of $R(y - y')$. Since $ann_r(y) \subsetneq ann_r(y - y')$, $y \to (y - y')$ defines an epimorphism $f : Ry \to R(y - y')$. Then $g \circ f : Ry \to N$ is an epimorphism and since Ry is indecomposable, $g \circ f$ is an isomorphism. This implies that $ann_r(y - y') \subsetneq ann_r(y)$ which is a contradiction. Thus $R(y - y')$ does not contain a non-zero projective direct summand and so $y - y' \in J(Q)$. Hence $y \in P + J(Q)$.

Then the proof follows. □

Theorem 3.2 *((R. D. Ketkar and N. Vanaja, 1981), Theorem 1) Let R be a semiperfect ring, and let M be a right R-module such that*

(1) M is R-projective, and

(2) Rad(M) *is small in M.*

Then, M is projective.

Proof Proof follows from Theorem 3.1. □

Over a right perfect ring, every right R-module has a small radical. Since right perfect rings are semiperfect, we have the following corollary.

Corollary 3.3 *Let R be a right perfect ring and M a right R-module. Then the following are equivalent.*

(1) M is projective.

(2) M is R-projective.

(3) M is max-projective.

3.1. Max-Projectivity on Dual-Kasch Rings

Recall that a ring R is right Kasch if every simple right R-module embeds into R. Similarly, in (E. Büyükaşık, C. Lomp and H. B. Yurtsever, 2022), we considered the dual case and defined Dual-Kasch rings.

Definition 3.6 *A ring R is called right dual-Kasch if every simple right R-module is isomorphic to a factor module of an injective module.*

Proposition 3.6 *((E. Büyükaşık, C. Lomp and H. B. Yurtsever, 2022), Proposition 2.13) The following conditions are equivalent for a ring R.*

(1) R is right self-injective.

(2) R is right dual Kasch and $E(R)$ is projective.

Moreover, if R is semilocal, then the following condition is also equivalent:

(3) R is right dual Kasch and $E(R)$ is max-projective.

Corollary 3.4 *((E. Büyükaşık, C. Lomp and H. B. Yurtsever, 2022), Corollary 2.14) The following statements are equivalent for a ring R.*

(1) R is a QF ring.

(2) R is one-sided Noetherian, right dual Kasch and $E(R)$ is projective.

(3) R is one-sided Noetherian, right dual Kasch, semilocal and $E(R)$ is max-projective.

If, moreover, R is commutative, then the following condition is also equivalent:

(4) R is perfect and $E(R)$ is max-projective.

CHAPTER 4

MAX-QF AND ALMOST-QF RINGS

In this chapter we generalize some results from (Y. Alagöz and E. Büyükaşık, 2021) and give new characterizations of max-QF and almost-QF rings. Also we show that being max-QF and almost-QF is not left right symmetric and need not to be closed under factor rings.

Definition 4.1 *A ring R is called max-QF if every injective right R-module is max-projective.*

Definition 4.2 *A ring R is called almost-QF if every injective right R-module is R-projective.*

In the following theorem from (Y. Alagöz and E. Büyükaşık, 2021), the authors prove that on a right hereditary and right Noetherian ring, being right almost-QF and right max-QF are equivalent conditions.

Theorem 4.1 *((Y. Alagöz and E. Büyükaşık, 2021), Theorem 1) Let R be a right hereditary and right Noetherian ring. The following statements are equivalent.*

(1) R is right almost-QF.

(2) R is right max-QF.

(3) Every injective right R-module E has a decomposition $E = A \oplus B$ *where* $\mathrm{Rad}(A) = A$ *and B is projective and semisimple.*

(4) $R = S \times T$*, where S is a semisimple Artinian ring and T is a right small ring.*

Generalizing (2), (3) of Theorem 4.1 by replacing right hereditary and right Noetherian with finite dimensional and right nonsingular leads us to the following corollary.

Corollary 4.1 *Let R be a finite dimensional right nonsingular ring. Then the following are equivalent.*

(1) R is right max-QF.

(2) For every injective module E, where $E = E_1 \oplus E_2$*,* $\mathrm{Rad}(E_1) = E_1$ *and* $E_2 = \oplus e_i R$ *is projective where each* e_i *is idempotent.*

Proposition 4.1 *((Y. Alagöz and E. Büyükaşık, 2021), Lemma 4) Let R_1 and R_2 be rings. Then $R = R_1 \times R_2$ is right almost-QF (respectively, right max-QF) if and only if R_1 and R_2 are both right almost-QF(respectively, right max-QF).*

Proposition 4.2 *Let R be a semilocal right semihereditary ring. Then the following statements are equivalent.*

(1) R is right max-QF.

(2) $R = S \times T$, where S is semisimple Artinian, and T is a right small ring.

(3) Every simple injective right module is projective.

(4) Every singular injective right module is max-projective.

(5) Dual Goldie torsion theory splits.

Proof (1) ⇒ (2) Let S be the sum of the injective minimal right ideals of R. The S is an ideal of R. Clearly $S \cap J(R) = 0$ because $J(R)$ does not contain injective submodules. Thus S embeds in $R/J(R)$, and so S is finitely generated and injective. Then $R = S \times T$. Since R is right max-QF, T is right max-QF as well.

Suppose T is not right small. Let K be a maximal submodule of $E = E(T_T)$. Then E/K is absolutely pure by Theorem 2.4. Since R is semilocal, E/K is also pure injective by Corollary 2.2. This implies that E/K is an injective T-module. Then E/K is also injective as a right R-module. As T is right max-QF, E/K is a max-projective T-module. Thus $T = X \oplus Y$ for some right ideals X, Y such that $X \cong E/K$. We obtain that, X is a simple injective right R-module and $S \cap X = 0$, a contradiction. Hence T is a right small ring. This proves (2).

(2) ⇒ (1) by Proposition 4.1.

(2) ⇔ (3) ⇔ (5) clear by ((C. Lomp, 1999), Theorem 4.6).

(1) ⇔ (4) ⇔ (3) is clear by ((Y. Alagöz and E. Büyükaşık, 2021), Theorem 2).

Now, we only need to prove (3) ⇒ (1).

Let E be an injective right R-module and $f : E \to S$ with S simple. If $f = 0$, then there is nothing to prove, so assume that $f \neq 0$. In this case, f is an epimorphism since S is simple. Since R is semihereditary, $f(E) \cong S$ is FP-injective. On the other hand, since R is semilocal, S is pure-injective by Corollary 2.2. Thus, S is injective, and so is projective by (3). Hence π splits, i.e., there exists $\pi' : S \to R$ such that $\pi\pi' = 1_S$. Then, $\pi\pi' f = f$, and so E is max-projective. □

Corollary 4.2 *Let R be an indecomposable semilocal right semihereditary ring. Then R is right max-QF if and only if R is semisimple Artinian or right small. In particular, if R is a local ring, then R is right max-QF if and only if R is right small or a division ring.*

Lemma 4.1 *((Y. Alagöz and E. Büyükaşık, 2021), Proposition 14) Let R be a local right max-QF ring. Then R is either right self-injective or right small.*

Proposition 4.3 *Let R be a local ring. Then the following are equivalent.*

(1) R is right max-QF.

(2) (a) R is right small; or

(b) R is right self injective and $Ext_R(E, J(R)) = 0$, for each injective right R-module E.

Proof (1) ⇒ (2) By Lemma 4.1, R is right small or right self-injective. Suppose R is right self injective and E an injective right R-module. Consider the short exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} R \xrightarrow{\pi} R/J \longrightarrow 0$$

where $J = J(R)$, the Jacobson radical of R. Applying $Hom_R(E, -)$, we obtain

$$0 \longrightarrow Hom_R(E, J) \xrightarrow{\iota^*} Hom_R(E, R) \xrightarrow{\pi^*} ...$$

$$... \qquad Hom_R(E, R/J) \longrightarrow Ext^1_R(E, J) \longrightarrow Ext^1_R(E, R) = 0$$

Since R is right self-injective, $Ext^1_R(E, R) = 0$. We know that R is right max-QF. Thus, the map $\pi^* : Hom_R(E, R) \to Hom_R(E, R/J)$ is onto. Therefore $Ext^1_R(E, J) = 0$. This proves (2).

(2) ⇒ (1) Suppose (a), that is R is right small. Then $\text{Rad}(E) = E$ for each injective right R-module. Since $R/J(R)$ is simple, $Rad(R/J(R)) = 0$. As $f(\text{Rad}(M)) \subseteq \text{Rad}(N)$ for each right modules M, N, and $f \in Hom_R(M, N)$, we have $Hom_R(E, R/J(R)) = 0$. Therefore, E is trivially max-projective, and so R is right max-QF.

Now, assume (b). Then for each injective right R-module E, the short exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} R \xrightarrow{\pi} R/J \longrightarrow 0$$

induces the sequence

$$0 \longrightarrow Hom_R(E,J) \xrightarrow{\iota^*} Hom_R(E,R) \xrightarrow{\pi^*} Hom_R(E,R/J) \longrightarrow Ext^1_R(E,J)$$

where $J = J(R)$. By (b) we have $Ext^1_R(E,J) = 0$. Thus π^* is onto, and so E is max-projective. Hence R is right max-QF. □

Proposition 4.4 *Let R be a local nonsmall right max-QF ring. Then R is a right self-injective ring and R satisfies one of the following conditions:*

(1) (i) R is a right PF ring, and (ii) every injective right module E has a decomposition $E = E_1 \oplus E_2$, where E_1 has essential socle, $\mathrm{Rad}(E_2) = E_2$ *and* $\mathrm{Soc}(E_2) = 0$.

(2) R is right self-injective, and every injective right R-module has a decomposition as $E = E_1 \oplus E_2$, where $\mathrm{Soc}(E_1)$ *is essential in* E_1, $\mathrm{Rad}(E_1) = E_1$, *and* $\mathrm{Soc}(E_2) = 0$.

Proof Since R is right max-QF and nonsmall, it is right self injective by Lemma 4.1.
(1) Assume $\mathrm{Soc}(R_R) \neq 0$. As R is local, it is indecomposable. Thus, R is right uniform as it is right self-injective. Therefore, $\mathrm{Soc}(R_R)$ is simple, and so R is a right PF ring. Let E be an injective right R-module. Let $E_1 = E(\mathrm{Soc}(E))$. Then $E = E_1 \oplus E_2$ with $\mathrm{Soc}(E_2) = 0$. Let us show that $\mathrm{Rad}(E_2) = E_2$.
Assume that $\mathrm{Rad}(E_2) \neq E_2$. Let $f : E_2 \to R/J(R)$ be a nonzero homomorphism. Since E_2 is max-projective, there is a homomorphism $g : E \to R$ such that $f = \pi \circ g$, where $\pi : R \to R/J(R)$ is the natural epimorphism. As f is injective and π is a small epimorphism, g is an epimorphism. Then g splits because R is projective. So $E_2 \cong R \oplus \mathrm{Ker}(g)$. This isomorphism and $\mathrm{Soc}(R_R) \neq 0$ imply that $\mathrm{Soc}(E_2) \neq 0$, a contradiction. Thus, $\mathrm{Rad}(E_2) = E_2$, and so E has the desired decomposition.
This proves (1).
(2) $\mathrm{Soc}(R_R) = 0$. As in the first case, for any injective right module E, we have $E = E_1 \oplus E_2$, where $E_1 = E(\mathrm{Soc}(E))$. Clearly E_1 has essential socle, because every module is essential in its injective hull. Also, $\mathrm{Soc}(E_2) = E_2$. Let us show that $\mathrm{Rad}(E_1) = E_1$. Assume that $\mathrm{Rad}(E_1) \neq E_1$. Let $f : E_1 \to R/J(R)$ be a nonzero homomorphism. By

similar arguments as in Case I, we have $E_1 \cong R \oplus \operatorname{Ker}(g)$ for some homomorphism $g : E_1 \to R$. This isomorphism implies that E_1 has a nonzero submodule, say X, isomorphic to R. Since $\operatorname{Soc}(R) = 0$, we also have $\operatorname{Soc}(X) = 0$. Thus $X \cap \operatorname{Soc}(E_1) = 0$. This contradicts with the fact that $\operatorname{Soc}(E_1)$ is essential in E_1. Therefore, we must have $\operatorname{Rad}(E_1) = E_1$.
This proves (2). □

Recall that, for every right module M, if $M/\operatorname{Rad}(M)$ is max-projective, then M is max-projective for Hereditary rings, so this leads us to the following theorem.

Proposition 4.5 *Let R be a right Hereditary ring. Then following statements are equivalent.*

(1) R is right max-QF.

(2) For every injective right module E, the module $E/\operatorname{Rad}(E)$ *is max-projective.*

(3) Every injective right module E with $\operatorname{Rad}(E) = 0$ *is max-projective.*

4.1. Noetherian almost-QF and max-QF rings

In this section, we will give characterizations of almost-QF and max-QF rings on right Noetherian right nonsingular and right finite-dimensional rings. It is well known that on a right Noetherian ring, injective modules are direct sum of their indecomposable submodules.

Theorem 4.2 *(E. Matlis, 1958) For any ring R, the following statements are equivalent.*

(1) R is right Noetherian.

(2) Any injective module M_R *is a direct sum of indecomposable submodules.*

This well known characterization by E. Matlis yields us some characterizations of max-projective modules and max-QF rings over Noetherian rings.

Proposition 4.6 *(K. R. Goodearl, 1976) Let N be a submodule of a module N′. If N is essential in N′, then N′/N is singular.*

Proof Consider the short exact sequence

$$0 \longrightarrow N \xrightarrow{f} N' \xrightarrow{g} N'/N \xrightarrow{g} 0$$

where $f : N \to N'$ is an essential monomorphism. Then since N is an essential submodule of N', $f(A)$ is essential. Thus, by ((K. R. Goodearl, 1976), Proposition 1.20) N'/N is essential. □

Lemma 4.2 *Let R be a right Noetherian right nonsingular ring. Then, the following are equivalent for an injective right module E.*

(1) E is R-projective.

(2) E is max-projective.

(3) $E = E_1 \oplus E_2$ *where* $\mathrm{Rad}(E_1) = E_1$ *and* E_2 *is projective.*

Proof (1) ⇒ (2) is clear.

(2) ⇒ (3) Since R is right Noetherian, every injective right R-module is a direct sum of indecomposable injective right R-modules. Thus, $E = \oplus_{i \in I} E_i$, where I is an index set, and E_i is indecomposable for each $i \in I$. As E is max-projective and R is right nonsingular, E_i is projective, or $\mathrm{Rad}(E_i) = E_i$ for each $i \in I$ by Proposition 4.2. Let $J = \{i \in I \mid \mathrm{Rad}(E_i) = E_i\}$. Then ,for $E_1 = \oplus_{i \in J} E_i$ and $E_2 = \oplus_{i \in I \setminus J} E_i$, we have $\mathrm{Rad}(E_1) = E_1$ and E_2 is projective. Thus, (3) follows.

(3) ⇒ (1) Let Q be an injective right R-module. Then, by (3), $Q = Q_1 \oplus Q_2$, where $\mathrm{Rad}(Q_1) = Q_1$ and Q_2 is projective. Since R is right Noetherian, R/I is Noetherian for each right ideal I of R. Thus, $\mathrm{Hom}(Q_1, R/I) = 0$ for each right ideal I of R. So, Q_1 is R-projective. Projective modules are trivially R-projective, and so Q_2 is R-projective. Hence $Q = Q_1 \oplus Q_2$ is R-projective as a direct sum of R-projective modules. □

Proposition 4.7 *Let R be a right Noetherian right nonsingular ring. Then the following are equivalent.*

(1) R is almost-QF.

(2) R is max-QF.

(3) Every injective right module E has a decomposition $E = E_1 \oplus E_2$*, where* $\mathrm{Rad}(E_1) = E_1$ *and* E_2 *is projective.*

Proof (1) ⇒ (2) is clear.

(2) ⇒ (3) By Lemma 4.2.

(3) ⇒ (1) Every projective module is R-projective, and each module N with $\mathrm{Rad}(N) = N$ is R-projective over a right Noetherian ring. Hence (3) implies (1). □

Proposition 4.8 *Let R be a right finite-dimensional right nonsingular ring. Then the following are equivalent.*

(1) R is right max-QF.

(2) $E(R_R)$ is max-projective and $\mathrm{Rad}(E) = E$ *for every singular injective right R-module.*

(3) For every injective module E can be decomposed $E = E_1 \oplus E_2$, where $\mathrm{Rad}(E_1) = E_1$ *and E_2 is projective.*

(4) Every nonsingular injective right R-module is max-projective and Rad(E)=E for every singular injective right R-module.

Proof (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Let E be an injective right R-module. Since R is right nonsingular, $E = Z(E) \oplus K$ for some submodule K of E. Note that K is right nonsingular injective right R-module, and $\mathrm{Rad}(Z(E)) = Z(E)$ by Lemma 3.4. Then K is a direct sum of indecomposable injective modules by ((K. R. Goodearl, 1976), Example 3.5), that is $K = \oplus_{i \in I} K_i$, where each K_i is indecomposable and injective. Then for each $i \in I$, K_i is projective or $\mathrm{Rad}(K_i) = K_i$ by Lemma 3.3. Then K can be expressed as $K = K_1 \oplus K_2$, where $\mathrm{Rad}(K_1) = K_1$ and K_2 is projective. For $E_1 = Z(E) \oplus K_1$, and $E_2 = K_2$, E has the desired decomposition in (3).

(3) $\Rightarrow$ (1) is clear.

(4) $\Rightarrow$ (1) Since R is right nonsingular, every injective right module Q can be written as $Q = Z \oplus N$, where Z is the singular submodule of Q and N is nonsingular. By (4), $\mathrm{Rad}(Z) = Z$, hence it is max-projective. As N is nonsingular, N is max-projective again by (4). Hence Q is max-projective, and so R is right max-QF.

(1) $\Rightarrow$ (4) Clearly (1) implies that nonsingular injective right R-modules are max-projective. By Lemma 3.4, we have $\mathrm{Rad}(E) = E$ for every singular right R-module. □

Lemma 4.3 *Let R be a right nonsingular ring and Q an indecomposable nonsingular injective right R-module. Then Q embeds in $E(R_R)$.*

Proof Let $0 \neq x \in Q$. Then $xR \cong R/I$ for some closed right ideal I of R. Let J be a complement of I in R. Then $J \cong (J \oplus I)/I$ is essential in R/I. Thus, xR contains an essential submodule, say K, isomorphic to J. Since Q is indecomposable and injective, it is uniform. Therefore, K is essential in Q. Therefore, $Q = E(K) \cong E(J) \subseteq E(R_R)$. Hence, Q embeds in $E(R_R)$, and this completes the proof. □

For a right R-module M, let $P(M) = \sum\{N \leq M | \operatorname{Rad}(N) = N\}$. Then $\operatorname{Rad}(P(M)) = P(M)$ for every right R-module M.

Lemma 4.4 *Let R be a right nonsingular right max-QF ring with $P(E(R_R)) = 0$. Then every indecomposable nonsingular injective right R-module is projective.*

Proof Let K be an indecomposable nonsingular injective right R-module. Then K is max-projective by the hypothesis, and so K is projective or $\operatorname{Rad}(K) = K$ by Lemma 3.3. On the other hand, K embeds in $E(R_R)$ by Lemma 4.3. As $E(R_R) = 0$ by the hypothesis, $Rad(K) = K$ is not possible. Therefore K is projective. □

We obtain the following corollary by Proposition 4.8 and Lemma 4.4.

Corollary 4.3 *Let R be a right finite dimensional right nonsingular ring with $P(E(R_R)) = 0$. The following are equivalent.*

(1) R is right max-QF.

(2) $E(R_R)$ is projective and $\operatorname{Rad}(E) = E$ for every singular injective right R-module.

(3) For every injective module E, where $E = E_1 \oplus E_2$ where $\operatorname{Rad}(E_1) = E_1$ and E_2 is projective.

(4) Every nonsingular injective right R-module is projective and $\operatorname{Rad}(E) = E$ for every singular injective right R-module.

4.2. Symmetry of Max-QF and Almost-QF Rings

In this section, we will give an example in order to show that being almost-QF and max-QF is not left-right symmetric for a ring R.

Proposition 4.9 *Let R be the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. Then R has the following properties.*

(1) R is right Noetherian.

(2) R is not left Noetherian.

(3) R is right hereditary.

(4) R is not left hereditary.

(5) R is left semihereditary.

Proof The proofs of (1) and (2) follows from ((T. Y. Lam, 1991), 1.22) and the proofs of (3),(4) and (5) follows from ((T. Y. Lam, 1999), 2.33). □

Lemma 4.5 *The map* $\phi : \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} \to \mathbb{Q}$ *given by* $\phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = c$ *is a ring epimorphism, and* $\mathrm{Ker}(\phi) = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$ *is a (two sided) maximal ideal of R.*

Proof Let $x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, y = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. Then

$$\phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} . \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}) = \phi(\begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}) = c.c' = \phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix})\phi(\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix})$$

and also

$$\phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}) = \phi(\begin{pmatrix} a + a' & b + b' \\ 0 & c + c' \end{pmatrix}) = c + c' = \phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) + \phi(\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}).$$

We can also find $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ for every $c \in \mathbb{Q}$ so that ϕ is a ring epimorphism. The first isomorphism theorem and the fact that $\mathbb{Q}$ is a field, implies that $\mathrm{Ker}(\phi)$ is a maximal ideal of R. □

Lemma 4.6 *With the notations in Lemma 4.5, the left R-module* $S = R/\mathrm{Ker}(\phi)$ *is singular and injective.*

Proof Set $L = \mathrm{Ker}(\phi)$. Since L maximal left ideal, it is essential or a direct summand of R. Assume that $L \oplus K = R$ for some left ideal of R. Then $l + k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for some $l \in L$ and $k \in K$. Thus, K contains an element of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Q}$. Then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in K \cap L$, a contradiction. Therefore L is an essential left ideal of R, and so $S = R/L$ is singular by ((K. R. Goodearl, 1976), Proposition 1.20(b)).

Now let us prove that S is an injective left R-module. For this purpose, we shall use the Baer's Criteria for injectivity. First note that by ((T. Y. Lam, 1999), 2.33) the left ideals of R are of the following form:

$$N_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} \qquad (n \neq 0)$$

$$N_2 = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$$

$$N_V = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : (x, y) \in V \quad (\text{ a subgroup of } \mathbb{Z} \oplus \mathbb{Q})\}$$

I: We claim that $\mathrm{Hom}(N_1, S) = 0$. Suppose there is a nonzero homomorphism $f : N_1 \to S$. Since S is a simple left ideal and f is nonzero, $\mathrm{Im}(f) = S$. Then $\mathrm{Ker}(f)$ is a maximal submodule of N_1. It is easy to check that, maximal submodules of N_1 are of the form:

$$K_p = \begin{pmatrix} pn\mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$$

$$K = \begin{pmatrix} n\mathbb{Z} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$$

If $\mathrm{Ker}(f) = K_p$, then $N_1/K_p \cong S$. But

$$ann_l(N_1/K_p) = \begin{pmatrix} p\mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} \neq \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & 0 \end{pmatrix} = ann_l(S).$$

This is a contradiction. Therefore, $\mathrm{Hom}(N_1, S) = 0$.

If $\mathrm{Ker}(f) = K$, then for $\begin{pmatrix} nk & a \\ 0 & b \end{pmatrix} \in N_1$ we have

$$f\begin{pmatrix} nk & a \\ 0 & b \end{pmatrix} = f\begin{pmatrix} nk & a \\ 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = f\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \cdot f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $f\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} + \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. Then, it is easy to check that

$$f\begin{pmatrix} nk & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} nk & a \\ 0 & b \end{pmatrix} \cdot f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

That is f is the right multiplication by $f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, the map $g : R \to S$ defined by $g\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ extends f.

In conclusion, we see that each homomorphism $f : N_1 \to S$ extends to a homomorphism $g : R \to S$.

II: Consider the left ideal $N_2 = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, and let $f : N_2 \to S$ be a homomorphism. Since

$$_RR = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix},$$

N_2 is a direct summand of R. Let $\pi : R \to N_2$ be the projection homomorphism and $i : N_2 \to R$ the inclusion homomorphism. Then the homomorphism $g = f \circ \pi. : R \to S$, clearly extends f.

III: Consider the left ideal $N_V = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : (x, y) \in V$ (V is a subgroup of $\mathbb{Z} \oplus \mathbb{Q}$)$\}$.

Since $\begin{pmatrix} n & q_1 \\ 0 & q_2 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} nx & ny \\ 0 & 0 \end{pmatrix}$, the left multiplication on N_V is determined by $\mathbb{Z}$. Therefore the lattice of left submodules of N_V and that of V are isomorphic. Therefore, for each maximal submodule K of N_V, $N_V/K \cong \mathbb{Z}_p$ for some prime integer p. Hence, as S is infinite, $\mathrm{Hom}(N_V, S) = 0$. This means that, any homomorphism from $N_V \to S$ extends trivially to a homomorphism $R \to S$.

Thus, by Baer Criteria, S is an injective simple left R-module. □

Lemma 4.7 *R=$\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is not left small in $M_2(\mathbb{Q})$=$\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$.*

Proof Consider $X = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$. X is a proper left submodule of $M_2(\mathbb{Q})$ and $R+X = M_2(\mathbb{Q})$. Thus, by Definition 0.6, R is not left small in $M_2(\mathbb{Q})$. □

Lemma 4.8 *$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right small in T_R=$\begin{pmatrix} \mathbb{Q}_\mathbb{Z} & \mathbb{Q}_\mathbb{Q} \\ \mathbb{Q}_\mathbb{Z} & \mathbb{Q}_\mathbb{Q} \end{pmatrix}$.*

Proof Right R-submodules of T are of the form:

$$N_U = \{\begin{pmatrix} x & \mathbb{Q} \\ y & \mathbb{Q} \end{pmatrix} : (x, y) \in U(U \text{ is a subgroup of } \mathbb{Q} \oplus \mathbb{Q})$$

$$\begin{pmatrix} \mathbb{Q}_{\mathbb{Z}} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \mathbb{Q}_{\mathbb{Z}} & \mathbb{Q} \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Q} \end{pmatrix}.$$

Thus, $R + X \neq T$ for each proper submodule X of T. Hence R is right small submodule of T. □

Lemma 4.9 *Let* $R=\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. *Then* R_R *is essential in the right R-module* $W=\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$.

Proof Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a nonzero element of W.

Case I: If $a \neq 0$ or $c \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$ is a nonzero element of R.

Case II: If $b \neq 0$ or $d \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ is a nonzero element of R.

Therefore, R_R is essential in W. □

Lemma 4.10 *Let* $R=\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. *Then* $E(R_R)$, *is the injective hull of R as a right R module over itself, is* $M_2(\mathbb{Q}) = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$.

Proof By Lemma 4.9, R_R is essential in $M_2(\mathbb{Q})$. Also $M_2(\mathbb{Q})$ is a semisimple ring. Then, $M_2(\mathbb{Q}) = Q^r_{max}(R)$. Since R is right nonsingular, $Q^r_{max}(M) = E(R_R)$ by Theorem 2.3. Therefore, $E(R_R) = M_2(\mathbb{Q})$. □

Proposition 4.10 *Let* $R=\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. *Then the following statements hold.*

(1) R is right max-QF, but not left max-QF.

(2) R is right almost-QF, but not left almost-QF.

Proof (1) By, Lemma 4.8 and Lemma 4.10, R is a right small ring. Thus, R is right max-QF. Consider the simple left module $S = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} / \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$. Then S is singular and injective by Lemma 4.6. Since R is left nonsingular, the identity map $1_S : S \to S$ can not be lifted to a homomorphism $S \to R$, that is there is no homomorphism $g : S \to R$

such that $\pi g = 1_S$, where $\pi : R \to S$ is the natural epimorphism. Thus, the injective left R-module S is not max-projective. Therefore, R is not a left max-QF ring.
(2) Since R is right Hereditary and right Noetherian, being right almost-QF and right max-QF coincide. Thus R is right almost-QF by (1). Again by (1), R is not left max-QF, so it is not left almost-QF. □

4.3. Super max-QF rings

In this section, we show that max-QF and almost,QF need not to be closed under factor rings and define super almost-QF and super max-QF rings.
In the following example, we show that max-QF rings are not closed under factor rings.

Example 4.1 *((T. Y. Lam, 1999), page 420, Ex. 5) The ring $R = k[x, y]/(x^2, y^2)$ is a local QF ring. For the ideal $I = (x^2, xy, y^2)/(x^2, y^2)$, the ring*

$$S = R/I \cong k[x, y]/(x^2, xy, y^2)$$

is not QF because it is not self-injective.

The ring R is max-QF. But its factor ring S is not max-QF. Because S is Artinian as a factor ring of an Artinian ring and Artinian ring is max-QF iff it is QF.

Thus, quotient rings of max-QF rings need not be max-QF.

Note that the example above also shows that factor rings of almost-QF rings need not be almost-QF.

Definition 4.3 *R is said to be right super almost-QF (respectively, super max-QF) if every quotient ring of R is right almost-QF (respectively, max-QF).*

Recall that a ring R is super QF if every factor ring of R is QF. Since being QF is left-right symmetric, a ring R is left super-QF if and only if R is right super QF. Every super QF ring is left-right super almost-QF, and every right super almost-QF ring is super max-QF.

Lemma 4.11 *Let R be a PID which is not a field. Then, R is super almost-QF, but not QF.*

Proof Every domain is small, and so almost-QF. Every proper factor ring of PID is QF. Thus every PID is super almost-QF. As *R* is a domain which is not a field, *R* is not QF. □

Over a right Artinian ring, the notions of projectivity, *R*-projectivity and max-projectivity coincide. Hence, we have the following:

Proposition 4.11 *Let R be right Artinian ring. The following are equivalent.*

(1) R is right super almost-QF.

(1') R is right super max-QF.

(2) R is super QF.

(3) R is left super almost-QF.

(3') R is right super max-QF.

Proposition 4.12 *If R is a commutative ring, then R/P is a max-QF ring for each prime ideal P.*

Proof Note that, an ideal of *P* of a commutative ring *R* is prime if and only if the factor ring R/P is an integral domain. Every integral domain is max-QF. Thus, the proof follows. □

CHAPTER 5

CONCLUSION

This thesis is motivated by the paper (Y. Alagöz and E. Büyükaşık, 2021) and the aim was to give further characterizations of max-QF rings and address some questions that are mentioned in (Y. Alagöz and E. Büyükaşık, 2021). We obtain some new characterizations of max-QF rings over right nonsingular, right Hereditary, right finite dimensional and right Noetherian rings.

We prove that being max-QF and almost-QF rings are not left-right symmetric. We also show that almost-QF and max-QF rings are not closed under factor rings. This motivates us to define super almost-QF and super max-QF rings as the rings whose factor rings are almost-QF and max-QF, respectively.

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