GEOMETRIC PROPERTIES OF COUPLER-CURVE EQUATION OF PLANAR SLIDER-CRANK AND FOUR-BAR LINKAGES

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Talha ERAZ

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ABSTRACT

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This thesis study focuses on coupler-curve of planar slider-crank and four-bar mechanisms. The geometric properties of the coupler-curve equations are investigated. The coupler-curve equations of both slider-crank and four bar mechanisms are shown to consist of quadratic and linear components. The quadratic components that appear in the coupler-curve equations are circles which determine the area the coupler-curve may be located. The path generation problem of the slider-crank mechanism is another aspect of this thesis. A limited solution to the path generation problem is introduced and tested numerically. A method that is a combination of the discovered geometric properties of the coupler-curve and numerical approximation methods is introduced. The solution approach works for the task of fitting a coupler-curve on a cluster of points and five precision points problem.

ÖZET

DÜZLEMSEL KRANK-BİYEL VE DÖRT ÇUBUK MEKANİZMALARININ BİYEL EĞRİLERİNİN GEOMETRİK ÖZELLİKLERİ

Bu tez çalışması, düzlemsel dört-çubuk ve krank-biyel mekanizmalarının biyel eğrilerine odaklanmaktadır. Biyel eğrilerinin geometrik özellikleri incelenmiştir. Hem krank-biyel hem de dört çubuk mekanizmalarının biyel eğrisi denklemlerinin ikinci derece ve doğrusal bileşenlerden oluştuğu gösterilmiştir. Biyel eğrisi denklemlerinde görülen ikinci derece bileşenler, biyel eğrisinin bulunabileceği alanın sınırlarını çizen çember ifadeleridir. Krank-biyel mekanizmasının yörünge sentezi problemi bu tezin bir diğer çalışma konusudur. Yörünge sentezi problemine kısıtlı bir çözüm sunulmuş ve sayısal olarak sınanmıştır. Biyel eğrisi denkleminin keşfedilen geometrik özellikleriyle sayısal yakınsama yöntemlerinin birleşimi olan bir yöntem sunulmuştur. Çözüm yaklaşımı, bir nokta bulutuna biyel eğrisi uydurmaya ve 5 hassasiyet noktası problemine çözüm sunmaktadır.

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CHAPTER 1

INTRODUCTION

Kinematics is the field of theoretical mechanics that focuses on motion geometry without relating the motion to any aspect that effects the motion (IFToMM dictionary, 2014). A mechanism is defined as a set of rigid bodies restricted to move together such that motion of one or more rigid bodies or forces and moments on them is transferred to other rigid bodies (IFToMM dictionary, 2014). Kinematic analysis is the analysis of the motion of a mechanism by means of kinematics only. Kinematic synthesis deals with systematic design of mechanisms when the degree-of-freedom of the mechanism is lower than the given task-space dimension. The procedure of kinematic synthesis is divided in two steps: 1) structural (type and number) synthesis, and 2) dimensional synthesis. Type synthesis is the part where the general structure of the mechanism is chosen including the type of links and joints that is suitable for the task. Then the number of links and joints for each type determined in number synthesis is set to obtain the required workspace that meets the demands of the task. Finally, the dimensional synthesis is to determine link lengths, inner angles etc. for the mechanism to perform the task (Hartenberg and Denavit, 1964).

The coupler link is defined as the link of the mechanism that has no common joint with the coordinate frame (IFTOMM Dictionary, 2014, Structure of Machines and Mechanisms). The coupler link may also be defined theoretically as a floating plane on the coordinate frame (Bottema and Roth, 1979). The trajectory of a chosen point on the coupler plane is called the coupler curve and the specified point is called coupler point. This thesis study focuses on the coupler-point path equation of planar slider-crank and 4-bar mechanisms. Next section presents a review of the existing studies on the subject.

1.1. Coupler Curve Analysis in the Literature

Guidance of points or drawing curves via mechanisms have drawn attention of many researchers since ancient times. Archimedes (287-212 B.C.) used a simple double-slider mechanism as an ellipsograph to draw ellipses (Cundy and Rollett, 1961). Watt

(1784) mentions that he is prouder of his invention of the approximate straight line tracing 4-bar linkage than any other mechanical invention he has ever made (Fergusson, 1962). Vincent (1836) was the first to study the general 4-bar coupler curve equation (Verstraten, 2012). In the second half of the 19th century generation of approximate straight lines and other curves has attracted the interest of many famous mathematicians, such as Sylvester, Chebyshev, Cayley, Kempe, Hart and Roberts. Morley (1919), Koetsier (1983) and Verstraten (2012) shed light on the developments in those years.

Roberts (1869) worked on linkages as tools to draw quartic and cubic curves, where one of the linkages he works on describes a slider-crank linkage and the coupler curve is a circular quartic with two double points[1]. He mentions that a 4-bar coupler curve has a general degree six, but if it has kite proportions, the curve degenerates to a quartic curve. Then, Roberts (1871) worked on point paths of a moving plane where the path, which he calls as point directrices, of two of the points on the plane are known. In particular, he works on cases where both of the directrices are circles (corresponds to a 4-bar linkage), one directrix is circle, the other is a straight line (corresponds to a slider-crank linkage) and both directrices are straight lines (corresponds to a double-slider linkage). In case of a 4-bar linkage, he mentions that the coupler curve is a sextic (a sixth degree polynomial curve), has circular points for triple points and has three other finite double points (trinodal). The curve is said to be tricircular, because the sixth order terms are $(x^2 + y^2)^3$, and the fifth and fourth order terms have the factors $(x^2 + y^2)^2$ and $(x^2 + y^2)$, respectively (Bottema and Roth, 1979).

Later, Roberts (1875) focused on the 4-bar coupler curve and proved the famous theorem that there are three distinct 4-bar linkages (cognate linkages) which can exactly generate the same coupler path. Each pair of cognate linkages share a common fixed pivot, whereas all three coupler triangles are similar with each other, as well as the triangle formed by the three fixed pivots. The fixed pivots turn out to be singular foci[2], and the circle through them is called the circle of singular foci (C_{SF}). Roberts also showed that the line segment connecting the two fixed pivots of a 4-bar linkage subtends the same

[1] An algebraic curve in *xy*-plane is said to be circular when the highest order terms of its polynomial is divisible by $x^2 + y^2$. Circularity of a circular curve is the number of times the curve passes through each isotropic point. The two roots $x + iy$ and $x - iy$ are said to be the circular points at infinity (or cyclic points, or isotropic points). A double point (or a node) of a curve is a point where the curve intersects itself.

[2] A focus of an algebraic curve is a point, where both of the lines connecting the the focus to the isotropic points are tangent to the curve. A focus is said to be singular when the isotropic points are on the curve, i.e. the curve is circular (Cayley, 1876a; Bottema and Roth, 1979).

angle at the double points (or nodes) of the coupler curve as the angle subtended at the third fixed pivot, hence all nodes are on C_{SF}.

Cayley (1876a) provided a simple way to construct the cognates and proved that the sum of the angular positions of the three singular foci is equal to the sum of the angular positions of the three nodes of the coupler curve. Also he shows that the focal triangle and the nodal triangle are circumscribed to a parabola with its focus on the circle of singular foci (See fig. 3 of (Blechschmidt and Uicker, 1986) for a nice representation). He presented the conditions for which the nodes are coincident with the singular foci, and one, two or all three nodes are a cusp. Cayley reported that the equation of a tricircular trinodal sextic curve contains 12 constants, whereas when the three nodes lie upon a given curve, the number of constants drop down to 9 (which is the number of parameters to describe a 4-bar linkage with a coupler triangle: the four coordinates of the fixed pivots, the two lengths of the connecting links and the three side lengths of the coupler triangle). Cayley provided a similar representation of the coupler curve to that of Roberts (1875):

$$(QR' - Q'R)^2 + (RP' - R'P)^2 = (PQ' - P'Q)^2 \tag{1}$$

or

$$(R^2 - P^2 - Q^2)(R'^2 - P'^2 - Q'^2) = (RR' - PP' - QQ')^2 \tag{2}$$

where $R = 0$, $R' = 0$ and $PQ' - P'Q = 0$ are circles, P and Q are lines which meet at one of the fixed pivots and P' and Q' are lines which meet at the other fixed pivot. Cayley noted that $QR' - Q'R = 0$ and $RP' - R'P = 0$ are circular cubics, each pass through a fixed pivot and that the nodes are the common intersections of $QR' - Q'R = 0$, $RP' - R'P = 0$ and $PQ' - P'Q = 0$. Cayley presents an expanded form of Eq. (1) as

$$\begin{bmatrix} x^2 + y^2 \\ -(d-a)^2 \end{bmatrix}\begin{bmatrix} x^2 + y^2 \\ -(d-a)^2 \end{bmatrix}\begin{bmatrix} (x-m)^2 + y^2 \\ -(f+h)^2 \end{bmatrix}\begin{bmatrix} (x-m)^2 + y^2 \\ -(f-h)^2 \end{bmatrix} \tag{3}$$
$$= \left\{ \begin{pmatrix} x^2 + y^2 \\ +b^2 - a^2 - f^2 \end{pmatrix}\begin{bmatrix} (x-m)^2 + y^2 \\ +b^2 - d^2 - h^2 \end{bmatrix} + \begin{pmatrix} m^2 + b^2 \\ -a^2 - h^2 \end{pmatrix}\begin{pmatrix} d^2 + f^2 \\ -b^2 \end{pmatrix} - mpy \right\}^2$$

where a, b, d, f, h, m are link lengths of the 4-bar linkage as shown in Fig. 1 and $p = \sqrt{2d^2 f^2 + 2a^2 f^2 + 2b^2 d^2 - b^4 - d^4 - f^4}$.

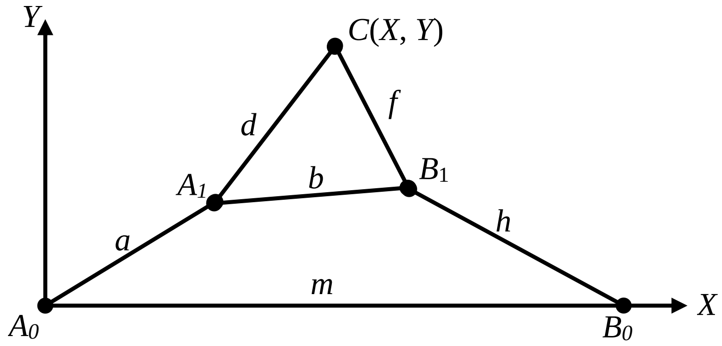

Figure 1. 4-bar linkage $A_0A_1B_1B_0$ with coupler point C

The fully expanded form of Eq. (3) results in a polynomial equation with 15 coefficients. Of course, the 15 coefficients cannot be independent, since the 4-bar linkage comprises 9 independent parameters. Cayley (1876b) also presented an alternative form of the 4-bar coupler curve.

Sylvester (1875a, b) invented the plagiograph (or the skew pantigraph) which can be used to redraw a curve with a certain scale in a different orientation of any desired angle, but also to divide an angle into any number of equal parts. Hart (1875), Roberts (1875) and Kempe (1877) made use of the plagiograph in their respective works. Kempe (1876) proved that any algebraic curve can be generated by a linkage. The method proposed to generate the linkage has four elements: reversor, multiplicator, additor and translator. These elements are section of linkages and the method put them together to achieve the required trajectory. Kempe concludes that the method would not be practical as the linkage will be too complicated to trace the task trajectory perfectly. Kempe (1877) also wrote a book about linkages invented to draw approximate and exact straight lines (Watt, R. Roberts, Chebyshev, Peucilever-Lipkin, Hart linkages).

The interest on the analysis of the 4-bar coupler curve equation by mathematicians continued with occasionally published papers (Darboux 1879, Hippisley, 1917; Bennett, 1919; Morley, 1920, 1924). However, the applications arose attention of mechanical engineers. Hrones and Nelson (1951) provided a large atlas of 4-bar coupler curves, where every page presents different coupler curves of a 4-bar linkage with a specific link length ratio of crank, coupler and rocker links. Artobolevskii (1964) presented a comprehensive book on mechanisms which can be used to generate well known planar curves.

The aforementioned studies are very useful for a designer to design or select a linkage via analysis of the coupler curve. For further discussion on how the coupler-curve properties can be used for path generation synthesis, first the 4-bar cognate linkages are examined in the next subsection.

1.1.1. Cognate Linkages

The proof of the triple generation of the 4-bar coupler-curve is available in the literature in several publications (Cayley, 1876a; Hart, 1882; Schmid, 1950; De Jonge, 1960; Verstraten, 2012). The theorem is mentioned as Roberts-Chebychev Theorem. Roberts and Chebychev published their work in very close dates and obtained the same result via completely different ways (Hartenberg and Denavit, 1964).

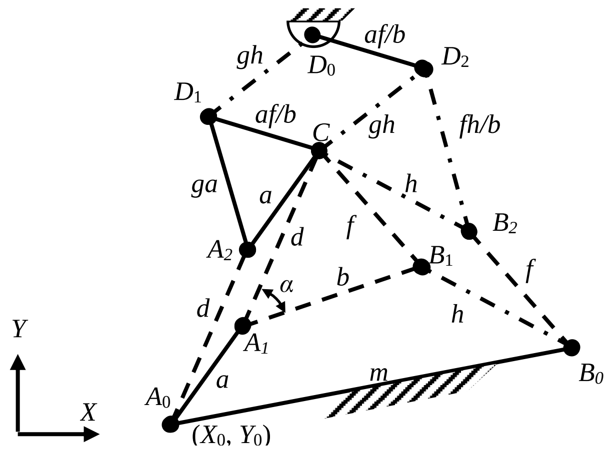

Figure 2. 4-bar linkage cognates

The cognates of the 4-bar mechanism are shown in Figure 2. First, the method to obtain the cognates is explained and then the proof of the third fixed pivot is given in this section.

Starting from the mechanism $A_0A_1B_1B_0$ with the coupler triangle A_1CB_1, first two parallelograms are formed: $A_0A_1CA_2$ and $B_0B_1CB_2$. Then, some link lengths are equal: $|A_0A_1| = |A_2C| = a$, $|B_0B_1| = |B_2C| = h$, $|A_0A_2| = |A_1C| = d$ and $|B_0B_2| = |B_1C| = f$. Next, two triangles, both similar to the coupler triangle A_1CB_1 are formed as A_2D_1C and CD_2B_2, with respective order of corners. In case $b \neq a$ or $b \neq h$, the newly formed triangles have their side lengths as: $|A_2D_1| = ad/b$, $|D_1C| = af/b$, $|CD_2| = dh/b$ and $|D_2B_2| = fh/b$. Finally, one more parallelogram is formed as $D_1CD_2D_0$. The point D_0 is fixed under these conditions and the triangle $A_0D_0B_0$ is similar to the coupler triangle. The proof is as follows:

Forming the loop closure equations: $\overrightarrow{A_0A_1} + \overrightarrow{A_1B_1} + \overrightarrow{B_1B_0} = \overrightarrow{A_0B_0}$ Let $g = d/b$. Then, $\overrightarrow{A_0A_2} = \overrightarrow{A_1C} = ge^{i\alpha}\overrightarrow{A_1B_1}$, $\overrightarrow{A_2D_1} = ge^{i\alpha}\overrightarrow{A_0A_1}$ and $\overrightarrow{CD_2} = \overrightarrow{D_1D_0} = ge^{i\alpha}\overrightarrow{B_1B_0}$, where $e^{i\alpha}$ represents rotation by α. Adding the vectors to obtain position vector of D_0: $\overrightarrow{A_0D_0} = ge^{i\alpha}\overrightarrow{A_1B_1} + ge^{i\alpha}\overrightarrow{A_0A_1} + ge^{i\alpha}\overrightarrow{B_1B_0} = ge^{i\alpha}\overrightarrow{A_0B_0}$, which is $ge^{i\alpha}$ times the loop closure equation. $\overrightarrow{A_0D_0} = ge^{i\alpha}\overrightarrow{A_0B_0}$ proves by side-angle-side similarity that the $A_0D_0B_0$ triangle is similar to the coupler triangle.

The cognate linkages are formed with specific link length ratios and some links being equal in length. The general degree of freedom of the linkage in Figure 2 can be calculated as -1 although the linkage still moves.

There are three initial moving links in the mechanism to obtain the cognates. The remaining links move with the same speed with these three moving links. The binary link A_0A_1 has the same angular speed with the A_2D_1C ternary link due to parallelogram $A_0A_1CA_2$ and the ternary link has the same angular speed with the binary link D_2D_0 due to parallelogram $CD_1D_0D_2$. Therefore, the three links rotate at the same speed and are shown with full lines. Same is valid for the binary link B_0B_1, ternary link CD_2B_2 and binary link D_1D_0 and they are shown with dash dotted lines. Finally, same angular speed is valid for ternary link A_1CB_1 and binary links A_0A_2 and B_0B_2, all are shown with dashed lines.

If one of the fixed pivots of a 4-bar linkage is taken to infinity, a slider-crank linkage is obtained. A slider-crank linkage has two cognates as shown in Figure 7 in Chapter 2.

1.2. Path Generation Problem in the Literature

The types of kinematic synthesis are based on the type of the task and listed as: motion generation, function generation and path generation (Sandor and Erdman, 1984). Motion generation is related to moving a rigid body from one position to another. Function generation problem is relating a joint variable to another, usually input and output joints at displacement, velocity, or acceleration level. Path generation is the type of synthesis problem that focuses on the trajectory of a point on the coupler link.

Nolle (1974a, b), Lee and Russel (2018) provide a detailed review of the path generation synthesis methods in the literature. First examples of coupler-curve synthesis or path generation are of James Watt and John Fitch (Ananthasuresh and Kota, 1993). Watt used coupler-curve for the guidance requirements of his steam engine in 1782. Fitch designed crank driven paddles at the stern of a boat in 1788.

Before computers were actively implemented in engineering calculations, graphical methods were dominant in path generation problems. Hrones and Nelson (1951) published an atlas of 4-bar coupler-curves. Each page in the atlas contained a mechanism with specified link length ratios and the coupler link is taken as a line segment, i.e., the coupler point is collinear with and in-between the joints of the coupler Several coupler points are equally distributed on the coupler line. Then, several coupler curves are drawn for each mechanism. The curves in the atlas were made of line segments that connect the corresponding coupler point positions for each 5 degrees of crank rotation. As the purpose was to find a curve shape like the one required by the task, scalability of 4-bar mechanism worked perfectly. Any mechanism with the same set of link length ratios would generate a scaled version of the one in the atlas but the shape looks the same. The curves in the atlas not only gave an idea about the shape of the path to be achieved by given link length ratios but also the timing of the coupler point on the curve for constant crank speed.

Hain (1967) explains several types of path generation problems. One or more well-known geometric shapes (like straight line, circular arc, ellipse, etc.) may partially form the path to be followed. Some number of points may be given and desired path is to be followed with a prescribed timing. A double point may be defined at a specific position or for corresponding crank positions. More points than number of design parameters may be given and a curve that approximately fits these design points can be sought. The common solution at that time was to pick a suitable curve from the atlas as a good initial guess and try to optimize the design parameters.

Currently, the most commonly used methods for path generation synthesis of mechanisms are based on independent synthesis of the two dyads connecting the coupler point to the fixed pivots (Sandor and Erdman, 1984). Sandor and Erdman's book contains graphical solutions for path generation for three positions (with or without prescribed timing) and four positions without prescribed timing. The book also contains an analytical solution for 4-bar path generation for five precision points.

Also path generation for finite line positions is also possible (Kiper & Söylemez, 2019). The problem can be analytically solved for up to 5 homologous positions for a line attached to the coupler link of a 4-bar mechanism.

Some of the recent studies focus on the coupler-curve equation rather than the mechanism itself. The mechanisms studied mostly for path generation synthesis are planar 4-link mechanisms. 4-bar mechanism coupler-curve equation has degree six, whereas the slider-crank mechanism coupler-curve equation has degree four. Simplest coupler-curve and coupler-curve equation among four-link mechanisms belong to double-slider mechanism. The coupler-curve of a double-slider mechanism requires 7 design parameters to be fully described (Figure 3).

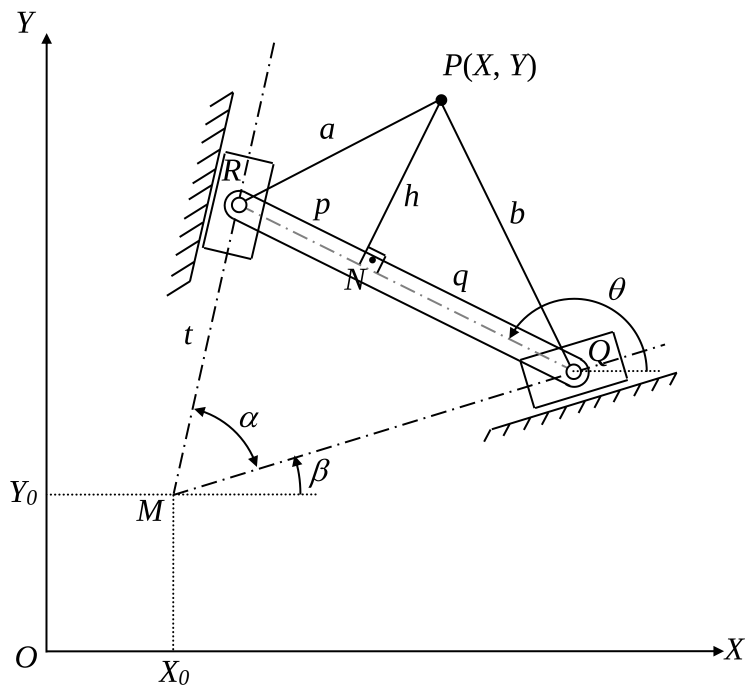

Figure 3. Double-slider mechanism design parameters (Kiper et al., 2016)

The double-slider mechanism has a quadratic coupler-curve equation which corresponds to an ellipse:

$$AX^2 + BXY + CY^2 + DX + EY + F = 0 \tag{4}$$

where

$$A = \left[(p+q)\frac{\sin(\alpha+\beta)\sin(\beta)}{\sin(\alpha)} + h \right]^2 + \left[(p+q)\frac{\sin(\alpha+\beta)\cos(\beta)}{\sin(\alpha)} - p \right]^2 \tag{5}$$

$$B = -2\left[(p+q)^2 \frac{\cos(\alpha+\beta)\sin(\alpha+\beta)}{\sin^2(\alpha)} + \frac{p+q}{\sin(\alpha)}\left[\begin{array}{c} \sin(\alpha+2\beta)h \\ -\cos(\alpha+2\beta)p \end{array} \right] \right] \tag{6}$$

$$C = \left[(p+q)\frac{\cos(\alpha+\beta)\sin(\beta)}{\sin(\alpha)} + p \right]^2 + \left[(p+q)\frac{\cos(\alpha+\beta)\cos(\beta)}{\sin(\alpha)} + h \right]^2 \tag{7}$$

$$D = -2AX_0 - BY_0 \tag{8}$$

$$E = -BX_0 - 2CY_0 \tag{9}$$

$$F = AX_0{}^2 + BX_0Y_0 + CY_0{}^2 - \left[p^2 + h^2 - \frac{(p+q)p\sin(\alpha) - h\cos(\alpha)}{\sin(\alpha)} \right]^2 \tag{10}$$

The first design problem is fitting an ellipse to a given set of data points and the solution to this problem exists in the literature (Fitzgibbon et al., 1999). Second part is to find the link lengths of the mechanism to generate the given ellipse equation and this part is recently addressed by Kiper et al. (2016). In his study, Kiper et al. point out that the coupler-curve equation for double-slider mechanism has 5 independent coefficients and that there are infinitely many double-slider linkages that can trace a given ellipse. The same path may be generated by any double-slider mechanism with a set of link dimensions that can be obtained by choosing two of the linkage parameters arbitrarily.

Setting the two sliding axes perpendicular to each other and picking the coupler point on the straight line between two revolute joints of the coupler link, the problem is reduced to Archimedes trammel.

However, nature of the coupler-curve equation is not that simple for slider-crank and 4-bar mechanisms. The slider-crank mechanism has a quartic equation for coupler-curve while the 4-bar has a sextic curve (Figure 4).

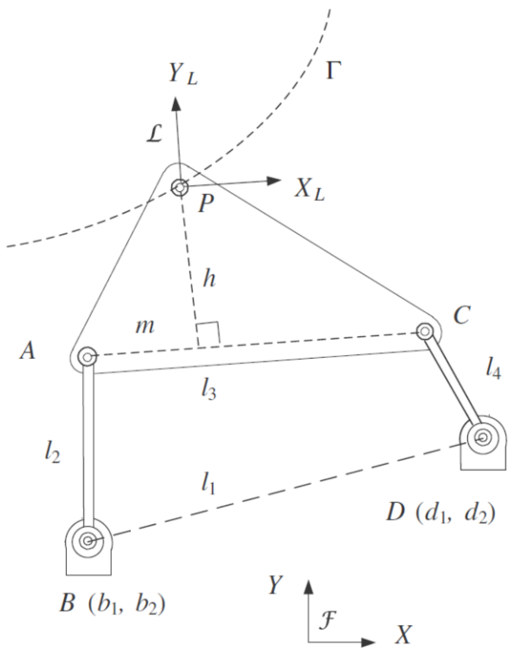

Figure 4. 4-bar mechanism design parameters (Bai & Angeles, 2015)

Bai and Angeles (2015) present the 4-bar coupler-curve equation in the following form:

$$\begin{aligned}(x^2+y^2)^3+(k_1x+k_2y)(x^2+y^2)^2+(k_3x^2+k_4xy+k_5y^2)(x^2+y^2)\\+k_6x^3+k_7x^2y+k_8xy^2+k_9y^3+k_{10}x^2+k_{11}xy+k_{12}y^2+k_{13}x+k_{14}y+k_{15}=0\end{aligned} \quad (11)$$

The slider-crank mechanism has 8 parameters for coupler-curve design and the 4-bar has 9 parameters. Both mechanisms have been studied in the literature by several researchers but still there is no fully analytical solution for the path generation synthesis of the slider-crank mechanism for given 8 points of the coupler-curve or for 4-bar mechanism for given 9 points of the coupler-curve.

The general form of a planar algebraic curve is given as

$$f\left(x,y\right)=\sum_{i,j=0}^{n} A_{i,j}x^{i}y^{j}=0 \tag{12}$$

where n is the degree of the curve. In homogeneous coordinates the general equation of a plane algebraic curve becomes $f(x,y,z)=\sum_{i,j=0}^{n} A_{i,j}x^{i}y^{j}z^{(n-i-j)}=0$. Here, z is the homogeneous coordinate and only two of x, y and z are independent. The coordinates of a point in real space is given by (x/z, y/z, 1) where $z \neq 0$. For points at infinity, z = 0. Circular points, (1, i, 0) and (1, $-i$, 0) are complex points at infinity and called circular points.

Bleschmidt and Uicker (1986) proposed a method for 4-bar path generation which requires three double points to be real and known, based on Roberts's theorem. If the two foci are known, the third can be determined. That is, before determining the mechanism and its cognates, one can determine the third fixed pivot for the cognates. The sextic curve that is obtained by relations of double points and circle of singular foci is called $g(x, y)$ and this sextic equation is combined with the equation of the circle of singular foci to obtain the coupler-curve equation. The combination is represented as

$$f(x,y)=g(x,y)+\lambda(h(x,y))^{2}=0 \tag{13}$$

where λ is determined by an arbitrary point that the curve passes through. Letting (x', y') be the known point,

$$\lambda=-g(x',y')/\left[h(x',y')\right]^{2} \tag{14}$$

The method to find the double points and the foci is a fifth order rational transformation. Two polynomials are determined from this transformation and 25 intersections of them contain 8 circular points, 10 common ordinary points and 7 potential double points. Next, 3 double points, without knowing which the right ones are, need to be determined (completely based on trial and error) among these 7 potential ones. Finally, the link dimensions are determined in two steps: "exhaustive search", as mentioned in related paper, is determining a proper set of initial guesses for linkage parameters and "gradient search" to optimize them. In conclusion, they present an algorithm to find the

4-bar coupler-curve equation, if the three nodes, the three singular foci and one ordinary point are known. This method can be used to synthesize a 4-bar linkage for given 4 precision points, 3 of which shall be selected as nodes, and the fixed pivot locations are selected.

Ananthasuresh and Kota (1993) expanded Bleschmidt and Uicker's method, where the form of 4-bar coupler-curve equation is given in a bit different way, still with 15 coefficients:

$$\begin{aligned} &C_1(x^6+y^6+3x^4y^2+3x^2y^4)+C_2(x^5+xy^4+2x^3y^2)+C_3(y^5+x^4y+2x^2y^3) \\ &\quad +C_4(x^3y+xy^3)+C_5(x^4+x^2y^2)+C_6(y^4+x^2y^2)+C_7x^3+C_8y^3 \\ &\quad +C_9x^2y+C_{10}xy^2+C_{11}x^2+C_{12}y^2+C_{13}xy+C_{14}x+C_{15}y+C_0=0 \end{aligned} \tag{15}$$

The proposed method is to fit the equation of coupler-curve to 15 data points, solving set of linear equations for coefficients C_i's and then to check the curve that the equation describes for obvious defects. First mentioned defect is the branching defect. A curve having only one loop is a non-Grashoff mechanism. The coupler point follows the path described by the equation partially in one branch (configuration of the mechanism) and partially in the other. Two partial coverage completely follows the path. The branching defect is not an issue for the paths with a single loop. A Grashoff mechanism has two disconnected real loops and each loop correspond to one of the configurations of the mechanism. The designer must check the loop for having the task points. If all task points are in one loop, then neglecting the other, design may be acceptable. If two loops both cover some task points, then the mechanism cannot cover all task points continuously. This is one type of branching defect. Having more than two disconnected loops on the curve (as the equation allows but the coupler-curve cannot have) is another type of branching defect. Having any type of these defects, curve must be re-created by changing the coefficients via approximate curve fitting methods or simply re-arranging the task points suitable to the task.

After the coupler-curve equation is determined, linkage parameters must be specified. The proposed method for this is to overcome the over-determined system of 15 equations and 9 unknowns by numerical error minimization. The coefficients are functions of linkage parameters. The equations to solve for linkage parameters are in the form $f_i(x, y) - C_i(\hat{a}) = 0$ where $\hat{a}$ is the set of linkage parameters, $C_i(\hat{a})$ is the coefficient formulation of the related term and $f_i(x, y)$ is the term that coefficient is multiplied with in the coupler-curve equation. As the problem is over-determined, the equations will not result in zero on the right-hand side but leave some error. Sum of the square of the errors must be minimized to achieve the linkage parameters.

Via graphical or numerical optimization methods some approximate solutions of the coupler-curve path generation problem were proposed (Cabrera et al., 2002; Bulatović and Djordjevic, 2004; Sancibrian et al., 2004).

Bai (2015) and Bai and Angeles (2015) addressed the over-determined problem by showing that the coefficients of the coupler-curve equation of the 4-bar mechanism are related with each other. The work starts with the parameters defined in Figure 4, avoiding all constant angles to work on simpler equations. The resulting equation is in the form of equation (8), which is the same form as Bleschmidt and Uicker's (1986). The most important note of this work is that it proves the coupler-curve equation is not over-determined. Although the relation between the coefficients of the coupler-curve equation is determined, the solution procedure is so complicated that Bai and Angeles (2015) proposed a trial-error based algorithm for parameter optimisation. The length of the coupler link, i.e. the constant distance between two moving pivots (l_3 in Figure 4), is used to define a new parameter $r = l_3/2$. This parameter is a normalization parameter, originally the coefficient of the tri-circularity term. Therefore, while working with the normalized equation, it is useful to pick the r parameter first. Three normalized coefficients, k_1, k_2 and k_4 are already independent of l_2 and l_4. Three pair of coefficients, when subtracted from one another, drops the terms with l_2 and l_4: $k_3 - k_5$, $k_6 - k_8$ and $k_7 - k_9$. Therefore, 6 equations are obtained for the remaining 6 unknowns. These equations can be solved numerically. Then, l_2 and l_4 are solved linearly from k_5 and k_6. Finally, remaining coefficients of the coupler-curve equation are tested and root mean square (rms) values for k_{10} to k_{15} are calculated. This procedure is repeated for a range of r values and error versus r value is visualized in a plot. The regions of the r value having least error are picked to work with higher precision of r in next iteration of the procedure. Before explaining the iterative solution process, Bai and Angeles (2015) determine all the equations for the proof of determinacy of the coupler-curve equation. All but one equation is used in the procedure. The only equation that is not used is that coefficients k_{10} to k_{15} are all linear in l_2^4, $l_2^2 l_4^2$, l_4^4, l_2^2 and l_4^2. The equations for these coefficients may be rewritten in the form:

$$t_{(i,1)}l_2^4 + t_{(i,2)}l_2^2 l_4^2 + t_{(i,3)}l_4^4 + t_{(i,4)}l_2^2 + t_{(i,5)}l_4^2 + t_{(i,6)} = 0 \text{ for } i = 1, \ldots, 6 \tag{16}$$

This equation has only two unknowns. Setting all 6 equations in matrix form, the coefficient matrix $\boldsymbol{T}$ of all t_i values is obtained. As the number of equations are more than the number of unknowns for this system of equations, the determinant of this matrix must be zero. This is the last equation for the proof of determinacy. However, as the coefficients of the coupler-curve equation from k_{10} to k_{15} are already complicated and lengthy, setting the determinant of the $\boldsymbol{T}$ matrix equal to zero brings an equation very difficult to deal with. Therefore, Bai and Angeles (2015) proposed an iterative method depending on the r value.

Bai's work is extended to a more analytical approach by Bai et al. (2020) and Wu et al. (2021) where the coupler-curve equation for the 4-bar mechanism is obtained with different coefficients and hence the pre-assumed parameter was no longer necessary. The first step in the solution procedure solves 4 out of 9 design parameters in terms of equation coefficients. Then, two more parameters are solved not only in terms of equation coefficients but also an unknown design parameter. Next, last three parameters are solved via equation coefficients. Finally, the two terms solved in terms of the unknown parameter are determined.

Finally, Bai (2021) presented a new determined system of coefficients that has six unknowns only and a univariate degree 9 polynomial equation that is derived from the newly formulated determined system. Angeles and Bai (2022) recently summarized their findings in a book.

1.3. Aim of the Thesis

Bai's methods offer a solution for known coupler-curve equations. However, fitting the equation of the coupler-curve for a given number of design points remains unsolved in the literature. In order to achieve this, one first needs to understand the over-constraint conditions for the coupler-curve equations. This is the first problem addressed in Chapters 2 and 3.

The coupler-curve equation of the 4-bar mechanism is expressed in polynomial form in several studies with different coefficients. The first purpose of this study is to investigate the geometric properties of the coupler-curve through its equation. Slider-crank mechanism is a degenerate case of the 4-bar mechanism, and the double-slider mechanism is the degenerate case of the slider-crank mechanism. The double-slider mechanism is already worked out by Kiper et.al. (2016). The slider-crank and 4-bar mechanism coupler-curves are focus of this study. Ünel proved that any algebraic curve with an even degree (degree 2, 4, 6, etc.) may be represented in a form of combination of quadratic expressions (Ünel & Wolovich, 1999). Although coupler-curve equations are over-constrained, the equations may still be expressed in terms of circles, ellipses or maybe even lines. The quadratic and linear components of the coupler-curve equations provide solid geometrical properties of the linkages. The location and size of these components, as well as the distances between them define some geometric invariants (under coordinate transformation) of the coupler-curves.

When the geometric properties of coupler-curve equations are understood, a path generation method can be developed. This is the second problem addressed in this thesis and some methods are proposed in Chapter 4.

CHAPTER 2

SLIDER-CRANK COUPLER-CURVE EQUATION

A planar slider-crank mechanism can be considered as a degenerate case of a planar 4-bar mechanism, where one of the link lengths becomes infinitely large. The sextic coupler-curve-equation of the 4-bar degenerates to a quartic equation for the slider-crank mechanism.

This Chapter investigates the slider-crank mechanism coupler-curve equation and presents some geometric properties of the coupler-curve. The coupler-curve equation may be expressed in many different forms, but the one best serves the purpose of the study is either a polynomial form with desirable coefficients or the form in which the fourth order equation is expressed in terms of simpler geometric components such as ellipses, circles and lines. Section 2.1 presents the derivation of the form in simpler components and the Section 2.2 presents the derivation of the polynomial form.

2.1. Geometric Components of Slider-Crank Coupler-curve Equation

The path generation problem for the slider-crank mechanism must have 8 independent design parameters. The parameters may be selected as constant link lengths and angles. The parameters to start with are given in Figure 5. The vector equations for two dyads can be expressed as $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MR} + \overrightarrow{RP}$ and $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MQ} + \overrightarrow{QR} + \overrightarrow{RP}$ or the Cartesian coordinates of coupler point $P(X, Y)$ can be expressed as:

$$X = X_0 + a\cos(\beta+\theta) - d\cos(\alpha+\beta+\varphi) \tag{17}$$

$$X = X_0 + q\cos(\beta) - h\sin(\beta) + b\cos(\varphi+\beta) - d\cos(\alpha+\beta+\varphi) \tag{18}$$

$$Y = Y_0 + a\sin(\beta+\theta) - d\sin(\alpha+\beta+\varphi) \tag{19}$$

$$Y = Y_0 + q\sin(\beta) + h\cos(\beta) + b\sin(\varphi+\beta) - d\sin(\alpha+\beta+\varphi) \tag{20}$$

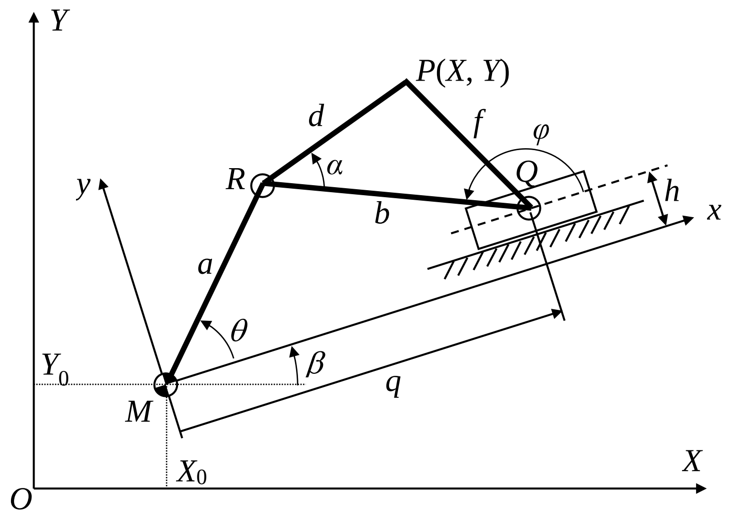

Figure 5. Slider-crank mechanism design parameters

Rearranging equations (17) and (19)

$$\begin{aligned} X - X_0 + d\cos(\alpha+\beta+\varphi) &= a\cos(\beta+\theta) \\ Y - Y_0 + d\sin(\alpha+\beta+\varphi) &= a\sin(\beta+\theta) \end{aligned} \tag{21}$$

Taking square of the equations (21) side by side and summing the two equations gives a single equation with only φ as motion parameter in the equation

$$\begin{aligned} a^2 &= (X - X_0)^2 + (Y - Y_0)^2 + d^2 \\ &+2d\left[(X - X_0)\cos(\alpha+\beta+\varphi) + (Y - Y_0)\sin(\alpha+\beta+\varphi)\right] \end{aligned} \tag{22}$$

The motion parameter to be eliminated in equations (18) and (20) is q. Multiplying equation (18) by sin(β) and equation (20) by cos(β) sets parameter q to be multiplied with the same coefficient that can be easily eliminated:

$$\begin{aligned} &(X - X_0)\sin(\beta) + h\sin^2(\beta) - b\sin(\beta)\cos(\varphi+\beta) \\ &+d\sin(\beta)\cos(\alpha+\beta+\varphi) = q\sin(\beta)\cos(\beta) \end{aligned} \tag{23}$$

$$\begin{aligned}(Y-Y_0)\cos(\beta)-h\cos^2(\beta)-b\cos(\beta)\sin(\varphi+\beta)\\ +d\cos(\beta)\sin(\alpha+\beta+\varphi)=q\sin(\beta)\cos(\beta)\end{aligned} \tag{24}$$

Right hand side of the equations (23) and (24) are the same. Equating left hand sides and rearranging lead to the second equation with only φ as motion parameter:

$$(X-X_0)\sin(\beta)-(Y-Y_0)\cos(\beta)+h+b\sin(\varphi)-d\sin(\alpha+\varphi)=0 \tag{25}$$

Trigonometric conversions should be applied to equations (22) and (25) so that the two equations turn into the form Acos(α + φ) + Bsin(α + φ) = C and solved for cosine and sine of angle α + φ to substitute into equation cos2(α + φ) + sin2(α + φ) = 1. First, equation (22) is handled

$$\begin{gathered}(X-X_0)^2+(Y-Y_0)^2+d^2-a^2=\\ 2d\begin{bmatrix}\sin(\alpha+\varphi)\left[(X-X_0)\sin(\beta)-(Y-Y_0)\cos(\beta)\right]\\ +\cos(\alpha+\varphi)\left[-(X-X_0)\cos(\beta)-(Y-Y_0)\sin(\beta)\right]\end{bmatrix}\end{gathered} \tag{26}$$

Then, equation (25) is rearranged as

$$\begin{gathered}[X-X_0]\sin(\beta)-[Y-Y_0]\cos(\beta)+h=\\ \cos(\alpha+\varphi)b\sin(\alpha)+\sin(\alpha+\varphi)\left[d-b\cos(\alpha)\right]\end{gathered} \tag{27}$$

Equations (26) and (27) are linear in terms of cos($\alpha + \varphi$) and sin($\alpha + \varphi$):

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\cos(\alpha+\varphi)\\ \sin(\alpha+\varphi)\end{bmatrix}=\begin{bmatrix}E\\ F\end{bmatrix} \tag{28}$$

where

$$
\begin{aligned}
A &= 2d\left[-\left(X - X_0\right)\cos\left(\beta\right) - \left(Y - Y_0\right)\sin\left(\beta\right)\right] \\
B &= 2d\left[\left(X - X_0\right)\sin\left(\beta\right) - \left(Y - Y_0\right)\cos\left(\beta\right)\right] \\
C &= b\sin\left(\alpha\right) \\
D &= \left[d - b\cos\left(\alpha\right)\right] \\
E &= \left(X - X_0\right)^2 + \left(Y - Y_0\right)^2 + d^2 - a^2 \\
F &= \left[X - X_0\right]\sin\left(\beta\right) - \left[Y - Y_0\right]\cos\left(\beta\right) + h
\end{aligned} \tag{29}
$$

Solving for $\cos(\alpha + \varphi)$ and $\sin(\alpha + \varphi)$ using Cramer's rule:

$$
\cos\left(\alpha + \varphi\right) = \frac{ED - FB}{AD - BC} \quad \text{and} \quad \sin\left(\alpha + \varphi\right) = \frac{AF - CE}{AD - BC} \tag{30}
$$

Finally, substituting in $\cos^2(\alpha + \varphi) + \sin^2(\alpha + \varphi) = 1$:

$$
\begin{gathered}
\left(ED - FB\right)^2 + \left(AF - CE\right)^2 = \left(AD - BC\right)^2 \Rightarrow \\
E^2\left(C^2 + D^2\right) + F^2\left(A^2 + B^2\right) - 2EF\left(AC + BD\right) - \left(AD - BC\right)^2 = 0
\end{gathered} \tag{31}
$$

Equation (31) is the coupler-curve equation and is free of motion parameters. The equation in open form is as follows:

$$
\begin{aligned}
& f^2\left[\left(X - X_0\right)^2 + \left(Y - Y_0\right)^2 - a^2 + d^2\right]^2 \\
& +4d^2\left[\left(X - X_0\right)\sin\left(\beta\right) - \left(Y - Y_0\right)\cos\left(\beta\right) + h\right]^2\left[\left(X - X_0\right)^2 + \left(Y - Y_0\right)^2\right] \\
& +4d\begin{bmatrix} \left(X - X_0\right)^2 \\ +\left(Y - Y_0\right)^2 \\ -a^2 + d^2 \end{bmatrix}\begin{bmatrix} \left(X - X_0\right)\sin\left(\beta\right) \\ -\left(Y - Y_0\right)\cos\left(\beta\right) \\ +h \end{bmatrix}\begin{bmatrix} \begin{bmatrix} \left(X - X_0\right)\cos\left(\beta\right) \\ +\left(Y - Y_0\right)\sin\left(\beta\right) \end{bmatrix} b\sin\left(\alpha\right) \\ +\begin{bmatrix} -\left(X - X_0\right)\sin\left(\beta\right) \\ +\left(Y - Y_0\right)\cos\left(\beta\right) \end{bmatrix}\begin{pmatrix} d \\ -b\cos\left(\alpha\right) \end{pmatrix} \end{bmatrix} \\
& -4d^2\begin{bmatrix} \left[\left(X - X_0\right)\cos\left(\beta\right) + \left(Y - Y_0\right)\sin\left(\beta\right)\right]\left(d - b\cos\left(\alpha\right)\right) \\ -\left[-\left(X - X_0\right)\sin\left(\beta\right) + \left(Y - Y_0\right)\cos\left(\beta\right)\right] b\sin\left(\alpha\right) \end{bmatrix}^2 = 0
\end{aligned} \tag{32}
$$

The equation is complicated to be handled term by term. The first term includes square of a circle-like expression and gives first two important geometric components which work as boundaries of possible area for the coupler-curve. The term can be rearranged as follows:

$$
\begin{gathered}
f^2\left[(X-X_0)^2+(Y-Y_0)^2-\left(a^2-d^2\right)\right]\left[(X-X_0)^2+(Y-Y_0)^2-\left(a^2-d^2\right)\right] \\
\Downarrow \\
f^2\left[(X-X_0)^2+(Y-Y_0)^2-(a-d)^2\right]\left[(X-X_0)^2+(Y-Y_0)^2-(a+d)^2\right] \\
+4d^2 f^2\left[(X-X_0)^2+(Y-Y_0)^2\right]
\end{gathered} \tag{33}
$$

The second term includes one of the geometric components that is expected to appear in the equation which is the line on which the slider moves. The term multiplying the square of the slider axis line expression may be transformed into one of the circles obtained from the first term. Then the second term is rearranged as follows:

$$
\begin{gathered}
4d^2\left[(X-X_0)\sin(\beta)-(Y-Y_0)\cos(\beta)+h\right]^2\left[(X-X_0)^2+(Y-Y_0)^2\right] \\
\Downarrow \\
4d^2\begin{bmatrix}(X-X_0)\sin(\beta) \\ -(Y-Y_0)\cos(\beta)+h\end{bmatrix}^2\begin{bmatrix}(X-X_0)^2+(Y-Y_0)^2 \\ -(a+d)^2\end{bmatrix} \\
+4d^2(a+d)^2\left[(X-X_0)\sin(\beta)-(Y-Y_0)\cos(\beta)+h\right]^2
\end{gathered} \tag{34}
$$

The third term in equation (32) is the most complicated one. This term is product of three expressions. First expression needs to be transformed into one of the circles from the first term. The second expression is the slider axis line that appears in the second term. The last expression can be transformed into linear combination of two lines: the slider axis of the mechanism and the slider axis of the cognate of the mechanism. The cognate mechanism is shown in Figure 7 in Section 2.2.

$$
4d\begin{bmatrix}\left(X-X_0\right)^2\\+\left(Y-Y_0\right)^2\\-\left(a+d\right)^2\\+2d\left(a+d\right)\end{bmatrix}\begin{bmatrix}\left(X-X_0\right)\sin\left(\beta\right)\\-\left(Y-Y_0\right)\cos\left(\beta\right)\\+h\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\left(X-X_0\right)\cos\left(\beta\right)\\+\left(Y-Y_0\right)\sin\left(\beta\right)\end{bmatrix}b\sin\left(\alpha\right)\\+\begin{bmatrix}-\left(X-X_0\right)\sin\left(\beta\right)\\+\left(Y-Y_0\right)\cos\left(\beta\right)\end{bmatrix}\begin{pmatrix}d\\-b\cos\left(\alpha\right)\end{pmatrix}\end{bmatrix}
$$

$$
\Downarrow
$$

$$
\begin{gathered}
4d\begin{bmatrix}\left(X-X_0\right)^2\\+\left(Y-Y_0\right)^2\\-\left(a+d\right)^2\end{bmatrix}\begin{bmatrix}\left(X-X_0\right)\sin\left(\beta\right)\\-\left(Y-Y_0\right)\cos\left(\beta\right)\\+h\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\left(X-X_0\right)b\sin\left(\alpha+\beta\right)\\-\left(Y-Y_0\right)b\cos\left(\alpha+\beta\right)+dh\end{bmatrix}\\d\begin{bmatrix}-\left(X-X_0\right)\sin\left(\beta\right)\\+\left(Y-Y_0\right)\cos\left(\beta\right)-h\end{bmatrix}\end{bmatrix}\\
+8d^2\left(a+d\right)\begin{bmatrix}\left(X-X_0\right)\sin\left(\beta\right)\\-\left(Y-Y_0\right)\cos\left(\beta\right)\\+h\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\left(X-X_0\right)b\sin\left(\alpha+\beta\right)\\-\left(Y-Y_0\right)b\cos\left(\alpha+\beta\right)+dh\end{bmatrix}\\d\begin{bmatrix}-\left(X-X_0\right)\sin\left(\beta\right)\\+\left(Y-Y_0\right)\cos\left(\beta\right)-h\end{bmatrix}\end{bmatrix}
\end{gathered}
\qquad (35)
$$

The last term in equation (32) appears to be one of the simplest terms but square of the given expression should be calculated in explicit form for upcoming steps to be taken. The expression in the last term or the explicit form of its square does not directly offer any expression expected to appear in the equation.

$$
\begin{gathered}
-4d^2\begin{bmatrix}\left[\left(X-X_0\right)\mathrm{c}_\beta+\left(Y-Y_0\right)\mathrm{s}_\beta\right]\left(d-b\mathrm{c}_\alpha\right)\\-\left[-\left(X-X_0\right)\mathrm{s}_\beta+\left(Y-Y_0\right)\mathrm{c}_\beta\right]b\mathrm{s}_\alpha\end{bmatrix}^2\Rightarrow\\
-4d^2\begin{bmatrix}\begin{bmatrix}\left(X-X_0\right)c_\beta\\+\left(Y-Y_0\right)s_\beta\end{bmatrix}^2\left(d-bc_\alpha\right)^2+\begin{bmatrix}-\left(X-X_0\right)s_\beta\\+\left(Y-Y_0\right)c_\beta\end{bmatrix}^2b^2s^2{}_\alpha\\-2\begin{bmatrix}\left(X-X_0\right)c_\beta\\+\left(Y-Y_0\right)s_\beta\end{bmatrix}\begin{bmatrix}-\left(X-X_0\right)s_\beta\\+\left(Y-Y_0\right)c_\beta\end{bmatrix}\left(d-bc_\alpha\right)b\mathrm{s}_\alpha\end{bmatrix}\Rightarrow\\
-4d^2\begin{bmatrix}\left(X-X_0\right)^2\left[c^2{}_\beta\left(d-bc_\alpha\right)^2+s^2{}_\beta b^2s^2{}_\alpha\right]\\+\left(Y-Y_0\right)^2\left[s^2{}_\beta\left(d-bc_\alpha\right)^2+c^2{}_\beta b^2s^2{}_\alpha\right]\\+2\left(X-X_0\right)\left(Y-Y_0\right)c_\beta s_\beta\left[\left(d-bc_\alpha\right)^2-b^2s^2{}_\alpha\right]\\-2\left(d-bc_\alpha\right)bs_\alpha s_\beta c_\beta\left[-\left(X-X_0\right)^2+\left(Y-Y_0\right)^2\right]\\-2\left(d-bc_\alpha\right)bs_\alpha\left(X-X_0\right)\left(Y-Y_0\right)\left(c^2{}_\beta-s^2{}_\beta\right)\end{bmatrix}
\end{gathered}
\qquad (36)
$$

where c_α, s_α, c_β, and s_β are abbreviations of cos(α), sin(α), cos(β) and sin(β).

Substituting all terms in their rearranged form back into coupler-curve equation, an equation with 11 terms is obtained. The equation already contains the useful line and circle expressions that are expected to appear. Defining them could make things easier for the rest of the derivation procedure

$$\begin{aligned} C_1 &= \left(X - X_0\right)^2 + \left(Y - Y_0\right)^2 - \left(a - d\right)^2 \\ C_2 &= \left(X - X_0\right)^2 + \left(Y - Y_0\right)^2 - \left(a + d\right)^2 \\ L_1 &= -\left(X - X_0\right)\sin\left(\beta\right) + \left(Y - Y_0\right)\cos\left(\beta\right) - h \\ L_2 &= \left(X - X_0\right)\sin\left(\alpha + \beta\right) - \left(Y - Y_0\right)\cos\left(\alpha + \beta\right) + gh \end{aligned} \tag{37}$$

where $g = d/b$. Then, the form of the coupler-curve equation in 11 terms is obtained as

$$\begin{gathered} f^2 C_1 C_2 + 4d^2 f^2 \left[\left(X - X_0\right)^2 + \left(Y - Y_0\right)^2\right] + 4d^2 L_1^2 C_2 \\ +4d^2\left(a + d\right)^2 L_1^2 + 4dC_2\left(-L_1\right)\left(bL_2 + dL_1\right) - 8d^2\left(ad + d^2\right)L_1^2 \\ +8bd^2\left(a + d\right)\left(-L_1\right)L_2 - 4d^2\left(X - X_0\right)^2\left[c^2_\beta\left(d - bc_\alpha\right)^2 + s^2_\beta b^2 s^2_\alpha\right] \\ -4d^2\left(Y - Y_0\right)^2\left[s^2_\beta\left(d - bc_\alpha\right)^2 + c^2_\beta b^2 s^2_\alpha\right] \\ -8d^2\left(X - X_0\right)\left(Y - Y_0\right)c_\beta s_\beta\left(d^2 - 2bdc_\alpha + b^2c^2_\alpha - b^2s^2_\alpha\right) \\ -8d^2\begin{bmatrix} s_\beta c_\beta\left[\left(X - X_0\right)^2 - \left(Y - Y_0\right)^2\right] \\ +\left(X - X_0\right)\left(Y - Y_0\right)\left(c^2_\beta - s^2_\beta\right) \end{bmatrix}\left(d - bc_\alpha\right)bs_\alpha = 0 \end{gathered} \tag{38}$$

The fifth term in equation (38) is partially cancelled out by the third term in the equation. The remaining part of the fifth term has C_2 as the common term with the first term. Therefore, the first, third and fifth terms result in a shorter expression fully in expected geometric components of the equation and therefore will form the first term of the final form of the coupler-curve equation

$$\left(f^2 C_1 - 4bdL_1L_2\right)C_2 \tag{39}$$

The fourth and sixth terms in equation (38) have $4d^2L_1^2$ in common and the rest of the terms partially cancel out each other. The two terms result in a simpler expression:

$$4d^2\left(a^2-d^2\right)L_1^2 \tag{40}$$

The second term in equation (38) is summation of two terms. First of them and the eighth term in equation (38) have $4d^2\left(X-X_0\right)^2$ in common. Similarly, second part of the second term in equation (38) and the ninth term in equation (38) have $4d^2\left(Y-Y_0\right)^2$ in common. The link length *f* appears in the equation as the short form of $d^2+b^2-2bd\cos\left(\alpha\right)$. Therefore, second, eighth and nineth terms in equation (38) can be added up to obtain

$$\begin{aligned}&4d^2\left(X-X_0\right)^2\left[s^2{}_{\beta}\left(d-bc_{\alpha}\right)^2+c^2{}_{\beta}b^2s^2{}_{\alpha}\right]\\&+4d^2\left(Y-Y_0\right)^2\left[c^2{}_{\beta}\left(d-bc_{\alpha}\right)^2+s^2{}_{\beta}b^2s^2{}_{\alpha}\right]\end{aligned} \tag{41}$$

Finally, the last two terms in equation (38) and expression (41) can be added and transformed into a much useful form:

$$
\begin{aligned}
&4d^2\left(X-X_0\right)^2\left[s^2{}_\beta\left(d-bc_\alpha\right)^2+c^2{}_\beta b^2 s^2{}_\alpha\right] \\
&+4d^2\left(Y-Y_0\right)^2\left[c^2{}_\beta\left(d-bc_\alpha\right)^2+s^2{}_\beta b^2 s^2{}_\alpha\right] \\
&-8d^2\left(X-X_0\right)\left(Y-Y_0\right)c_\beta s_\beta\left[\left(d-bc_\alpha\right)^2-b^2 s^2{}_\alpha\right] \\
&+8d^2\begin{bmatrix} s_\beta c_\beta\left[-\left(X-X_0\right)^2+\left(Y-Y_0\right)^2\right] \\ +\left(X-X_0\right)\left(Y-Y_0\right)\left(c^2{}_\beta-s^2{}_\beta\right)\end{bmatrix}\left(d-bc_\alpha\right)bs_\alpha
\end{aligned}
$$

$$\Downarrow$$

$$
4d^2\left\{\begin{matrix}
\left(X-X_0\right)^2\begin{bmatrix} s^2{}_\beta\left(d-bc_\alpha\right)^2 \\ +c^2{}_\beta b^2 s^2{}_\alpha\end{bmatrix}-2\left(d-bc_\alpha\right)bs_\alpha s_\beta c_\beta\left(X-X_0\right)^2 \\
+\left(Y-Y_0\right)^2\begin{bmatrix} c^2{}_\beta\left(d-bc_\alpha\right)^2 \\ +s^2{}_\beta b^2 s^2{}_\alpha\end{bmatrix}+2\left(d-bc_\alpha\right)bs_\alpha s_\beta c_\beta\left(Y-Y_0\right)^2 \\
-2\left(X-X_0\right)\left(Y-Y_0\right)c_\beta s_\beta\left[\left(d-bc_\alpha\right)^2-b^2 s^2{}_\alpha\right] \\
+2\left(X-X_0\right)\left(Y-Y_0\right)\left(d-bc_\alpha\right)bs_\alpha\left(c^2{}_\beta-s^2{}_\beta\right)
\end{matrix}\right\}
$$

$$\Downarrow$$

$$
4d^2\left\{\begin{matrix}
\left(X-X_0\right)^2\left[s_\beta\left(d-bc_\alpha\right)-c_\beta bs_\alpha\right]^2 \\
+\left(Y-Y_0\right)^2\left[c_\beta\left(d-bc_\alpha\right)+s_\beta bs_\alpha\right]^2 \\
-2\left(X-X_0\right)\left(Y-Y_0\right)c_\beta s_\beta\left[\left(d-bc_\alpha\right)^2-b^2 s^2{}_\alpha\right] \\
+2\left(X-X_0\right)\left(Y-Y_0\right)\left(d-bc_\alpha\right)bs_\alpha\left(c^2{}_\beta-s^2{}_\beta\right)
\end{matrix}\right\}
$$

$$\Downarrow$$

$$
4d^2\begin{bmatrix}\left(X-X_0\right)\left(ds_\beta-bs_\beta c_\alpha-bc_\beta s_\alpha\right) \\ -\left(Y-Y_0\right)\left(dc_\beta-bc_\beta c_\alpha+bs_\beta s_\alpha\right)\end{bmatrix}^2
$$

$$\Downarrow$$

$$
4d^2\begin{bmatrix} d\left[\left(X-X_0\right)s_\beta-\left(Y-Y_0\right)c_\beta+h\right] \\ -b\left[\left(X-X_0\right)s_{\alpha+\beta}-\left(Y-Y_0\right)c_{\alpha+\beta}+gh\right]\end{bmatrix}^2 \tag{42}
$$

Substituting all expressions (39) to (42) back into equation (38)

$$
\begin{aligned}
&\left(f^2 C_1-4bdL_1L_2\right)C_2 \\
&-8bd^2\left(a+d\right)L_1L_2+4d^2\left(a^2-d^2\right)L_1{}^2+4d^2\left(dL_1+bL_2\right)^2=0
\end{aligned} \tag{43}
$$

However, this form may be further simplified as

$$
\begin{gathered}
\left(f^2C_1-4bdL_1L_2\right)C_2-8bd^2\left(a+d\right)L_1L_2 \\
+4d^2\left(a^2-d^2\right)L_1^{\ 2}+4d^2\left(dL_1+bL_2\right)^2=0 \\
\Downarrow \\
\left(f^2C_1-4bdL_1L_2\right)C_2-8abd^2L_1L_2-8bd^3L_1L_2 \\
+4a^2d^2L_1^{\ 2}-4d^4L_1^{\ 2}+4b^2d^2L_2^{\ 2}+8bd^3L_1L_2+4d^4L_1^{\ 2}=0 \\
\Downarrow \\
\left(f^2C_1-4bdL_1L_2\right)C_2+4d^2\left(a^2L_1^{\ 2}-2abL_1L_2+b^2L_2^{\ 2}\right)=0 \\
\Downarrow \\
\left(f^2C_1-4bdL_1L_2\right)C_2+4d^2\left(aL_1-bL_2\right)^2=0
\end{gathered}
\tag{44}
$$

Final form of the equation can be expressed in open form as

$$
\begin{gathered}
\left\{\begin{matrix} f^2\left[\left(X-X_0\right)^2+\left(Y-Y_0\right)^2-\left(a-d\right)^2\right] \\ -4bd\begin{bmatrix}\left(X-X_0\right)\sin\left(\alpha+\beta\right) \\ -\left(Y-Y_0\right)\cos\left(\alpha+\beta\right)+gh\end{bmatrix}\begin{bmatrix}-\left(X-X_0\right)\sin\left(\beta\right) \\ +\left(Y-Y_0\right)\cos\left(\beta\right)-h\end{bmatrix}\end{matrix}\right\}\begin{bmatrix}\left(X-X_0\right)^2 \\ +\left(Y-Y_0\right)^2 \\ -\left(a+d\right)^2\end{bmatrix} \\
+4d^2\left\{a\begin{bmatrix}-\left(X-X_0\right)\sin\left(\beta\right) \\ +\left(Y-Y_0\right)\cos\left(\beta\right)-h\end{bmatrix}-b\begin{bmatrix}\left(X-X_0\right)b\sin\left(\alpha+\beta\right) \\ -\left(Y-Y_0\right)b\cos\left(\alpha+\beta\right)+dh\end{bmatrix}\right\}^2=0
\end{gathered}
\tag{45}
$$

Equation (45) comprises two circles with radii $a - d$ and $a + d$ and with center (X_0, Y_0) and two lines (slider axes of the two cognates) (see Figure 6). The coupler-curve must lie within the area between the two circles.

The expressions (34) and (35) obtained during the transformation of the equation may be obtained differently. Instead of obtaining a circle with radius $a + d$ the same expressions might have been converted to include circle with radius $|a - d|$ so that some signs in the resulting equation change. Therefore, the coupler-curve equation maybe obtained with same components also as:

$$\left(f^2C_2 - 4bdL_1L_2\right)C_1 + 4d^2\left(aL_1 + bL_2\right)^2 = 0$$

or

$$\left\{ \begin{array}{c} f^2\left[\left(X-X_0\right)^2 + \left(Y-Y_0\right)^2 - \left(a+d\right)^2\right] \\ -4d\begin{bmatrix} -\left(X-X_0\right)\sin\left(\beta\right) \\ +\left(Y-Y_0\right)\cos\left(\beta\right) - h \end{bmatrix}\begin{bmatrix} \left(X-X_0\right)b\sin\left(\alpha+\beta\right) \\ -\left(Y-Y_0\right)b\cos\left(\alpha+\beta\right) + dh \end{bmatrix} \end{array} \right\}\begin{bmatrix} \left(X-X_0\right)^2 \\ +\left(Y-Y_0\right)^2 \\ -\left(a-d\right)^2 \end{bmatrix} \quad (46)$$

$$+4d^2\left\{ a\begin{bmatrix} -\left(X-X_0\right)\sin\left(\beta\right) \\ +\left(Y-Y_0\right)\cos\left(\beta\right) - h \end{bmatrix} + b\begin{bmatrix} \left(X-X_0\right)\sin\left(\alpha+\beta\right) \\ -\left(Y-Y_0\right)\cos\left(\alpha+\beta\right) + gh \end{bmatrix} \right\}^2 = 0$$

Circle expression C_2 in the first term of coupler-curve equation is multiplied with an ellipse expression $f^2C_1 - 4bdL_1L_2$ that is a combination of the other circle and the two slider axes. The two circles are concentric with center at the fixed pivot coordinates (X_0, Y_0). The β angle is the angle between the X-axis of the global coordinate frame and the local x-axis, which is along the slider axis of the slider-crank mechanism. The α angle is one of the inner angles of coupler triangle as shown in Figure 5, which is also equal to the angle between the slider axes of the two cognates. The radii of the two circles give us the crank and coupler length values a and d. The distance from the circle center to the slider axis of the original mechanism is h and to the slider axis of cognate mechanism is gh where $g = d/b$.

As a result, the coupler-curve equation is simplified into two terms. The first term is the product of an ellipse and a circle, and the second term is the square of a line which is a linear combination of two slider axes. Both ellipses and lines formed in equations (45) and (46) are shown for a sample mechanism in Figure 6. The circle expression with radius $a + d$ in equation (45) is labelled as Circle 1 and the one with radius $|a - d|$ in equation (46) is labelled as Circle 2. Ellipse 1 is the ellipse expression multiplied with Circle 1 in equation (45) and Ellipse 2 is the multiplier of Circle 2 in equation (46). The line formulated as $aL_1 - bL_2$ intersects the coupler-curve at the tangent point of Circle 1. Similarly, the other linear combination of slider axes $aL_1 + bL_2$ intersects the coupler-curve at the tangent point of Circle 2. These two special points could be useful for path generation problems.

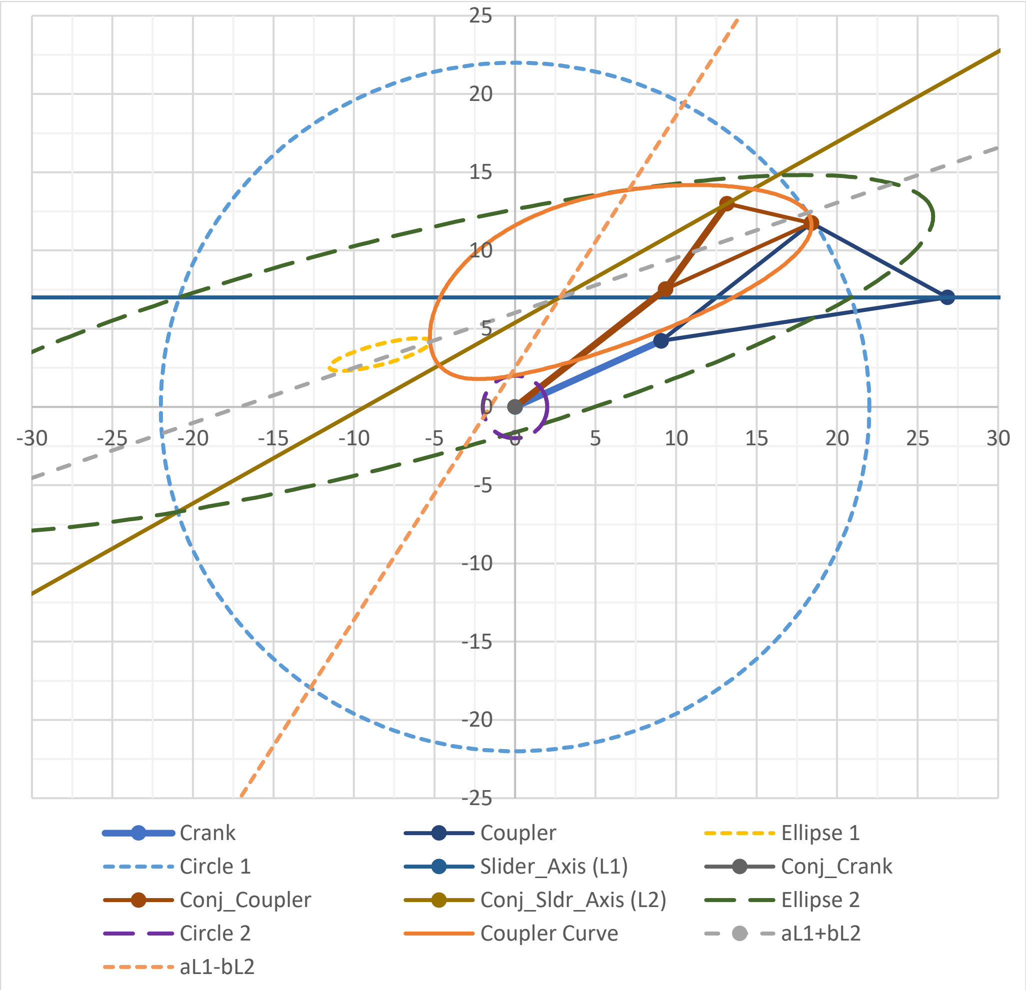

Figure 6. A slider-crank mechanism, its cognate, coupler-curve and components

The full lines represent the slider axes, L_1 and L_2. The orange full curve represents the coupler-curve. The ellipses are shown in dashed or dotted forms in the figure. The ellipses are in the same form (dotted or dashed) with the circle inside their equation. The two combination of slider axes, $aL_1 \pm bL_2$ are also drawn in the same form of the ellipse that exist in the same equation.

The geometric properties of the ellipses are such that they are similar, and both have the same center and orientation. The larger ellipse E_2 is simply a scaled version of the smaller ellipse E_1. The similarity ratio of the two ellipses is the same ratio of radii of the circles. The combination of the slider axes forms two lines which intersect the coupler-curve at two points and these two points are the tangent points of ellipse and circles that form the rest of the coupler-curve equation. $aL_1 + bL_2$ line intersect the coupler-curve at the tangent points of the curve to the E_1 and C_1 and $aL_1 - bL_2$ line at tangents of E_2 and C_2.

The radii of the circles C_1, C_2 ($a + d$ and $|a - d|$), the distances of the lines L_1, L_2 (h and gh) to the center of the circle and the angle between L_1 and L_2 (α) can be considered as the 5 geometric invariants (under coordinate transformation) of the coupler-curve equation of the slider-crank mechanism.

There are three coordinate transformation parameters in equation (45): X_0, Y_0 and β. When these three coordinate transformation parameters are set to zero, the global and local coordinate frames in Figure 5 become coincident but the rest of the geometric properties are conserved. For further investigation of the coupler curve geometric properties, simplified version of equation (45) is preferred:

$$\begin{bmatrix} f^2\left[x^2+y^2-(a-d)^2\right] \\ -4bd\,(y-h)\left(\sin(\alpha)x-\cos(\alpha)y+gh\right) \end{bmatrix}\left[x^2+y^2-(a+d)^2\right] \\ +4d^2\left[a(y-h)-b\left(\sin(\alpha)x-\cos(\alpha)y+gh\right)\right]^2=0 \tag{47}$$

There are three points of interest: the common center of circles, the common center of ellipses and the intersection of slider axes. The coordinates of these points can be calculated parametrically. The common center of circles, the location of fixed pivot, is at (X_0, Y_0) or the origin for the case in equation (47). The intersection of the slider axes can be calculated as:

$$x_i=\frac{\cos(\alpha)h-gh}{\sin(\alpha)} \quad ; \quad y_i=h \tag{48}$$

Finally, for the common center of ellipses, the equation of the ellipse in equation (47) is:

$$\left(b^2+d^2-2bd\cos(\alpha)\right)x^2-4bd\sin(\alpha)xy+\left(b^2+d^2+2bd\cos(\alpha)\right)y^2 \\ +4bdh\sin(\alpha)x-4dh\left(d+b\cos(\alpha)\right)y+4d^2h^2-f^2(a-d)^2=0 \tag{49}$$

The equation (49) is in the form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ and the center point coordinates are:

$$x_e = \frac{2CD - BE}{B^2 - 4AC} = \frac{-2bdh\sin(\alpha)}{\left(b^2 - d^2\right)}$$
$$y_e = \frac{2AE - BD}{B^2 - 4AC} = \frac{-2dh\left[d - b\cos(\alpha)\right]}{\left(b^2 - d^2\right)} \tag{50}$$

Then, if two vectors from circle center to the ellipse center and to the intersection point of slider axes are drawn, these vectors turn out to be perpendicular to each other:

$$\tan(\eta_{CI}) = \frac{y_i}{x_i} = \frac{h}{\dfrac{\cos(\alpha)h - gh}{\sin(\alpha)}} = \frac{\sin(\alpha)}{\cos(\alpha) - g}$$
$$\tan(\eta_{CE}) = \frac{y_e}{x_e} = \frac{\dfrac{-2dh\left[d - b\cos(\alpha)\right]}{\left(b^2 - d^2\right)}}{\dfrac{-2bdh\sin(\alpha)}{\left(b^2 - d^2\right)}} = \frac{g - \cos(\alpha)}{\sin(\alpha)} \tag{51}$$
$$\tan(\eta_{CI})\tan(\eta_{CE}) = \frac{\sin(\alpha)}{\cos(\alpha) - g}\frac{g - \cos(\alpha)}{\sin(\alpha)} = -1$$

2.2. An Alternative Representation of the Coupler-Curve Equation

Although the representation of coupler-curve equation in terms of simpler geometric components give an idea about possible solutions to the path generation problem, a representation in polynomial form is more likely to be used in an approximation-based path synthesis method. Therefore, the form obtained in previous section should be extended to a polynomial form. Equation (47) is free of the coordinate transformation parameters, hence is made of 5 design parameters and easier to convert into the polynomial form. . After obtaining the formulation in 5 design parameters, coordinate transformation may be applied to obtain a generalized 8 parameter equation. The equation (45) without coordinate transformation parameters becomes simplified as in equation (49).

This equation is based on local coordinate frame *M*(*x*, *y*) in Figure 5. The simplified version of the figure is given in Figure 7 which also includes the cognate mechanism.

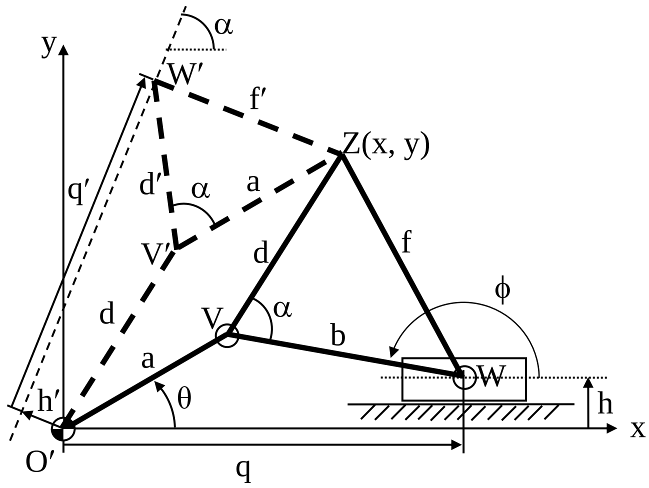

Figure 7. The slider-crank mechanism with its cognate

Then, the coupler-curve equation is expanded as

$$
\begin{gathered}
f^2\left(x^2+y^2\right)^2 \\
-4bd\sin(\alpha)\left(x^2+y^2\right)xy \\
4bd\cos(\alpha)\left(x^2+y^2\right)y^2 \\
4bdh\sin(\alpha)\left(x^2+y^2\right)x \\
-4dh\left(b\cos(\alpha)+d\right)\left(x^2+y^2\right)y \\
+\left[4d^2h^2-2f^2\left((a-d)^2+(a+d)^2\right)\right]\left(x^2+y^2\right) \\
+4b^2d^2\sin^2(\alpha)x^2 \\
+\left[4bd(a+d)^2\sin(\alpha)-8d^2b\sin(\alpha)\left(a+b\cos(\alpha)\right)\right]xy \\
+4d\left[d\left(b\cos(\alpha)+a\right)^2-(a+d)^2b\cos(\alpha)\right]y^2 \\
+4bdh(d+a)(d-a)\sin(\alpha)x \\
+4dh(a+d)(d-a)\left(d-b\cos(\alpha)\right)y \\
+f^2(a-d)^2(a+d)^2=0
\end{gathered}
\tag{52}
$$

The expanded version in equation (52) could be written in shorter form as

$$
\begin{gathered}
A\left(x^2+y^2\right)^2+4B\left(x^2+y^2\right)xy+4C\left(x^2+y^2\right)y^2+4D\left(x^2+y^2\right)x \\
+4E\left(x^2+y^2\right)y+F\left(x^2+y^2\right)+Gx^2+Hxy+Iy^2+4Jx+4Ky+L=0
\end{gathered}
\tag{53}
$$

The 12 equation coefficients are given in terms of 5 design parameters. Only three of the coefficients (*A*, *F* and *L*) include the f^2 term which can be expressed as $f^2 = b^2 + d^2 - 2bd\cos(\alpha)$. The coefficients could be simplified by dividing equation (52) by b^2 and introducing a new variable $g = d/b$ to completely replace parameter *b*. Radius of the circles mentioned in the previous section may also be defined as new variables to replace *d* and *a* as $r_1 = d + a$ and $r_2 = d - a$. Then, the new coefficient set defined in new design parameters are given as

$$
\begin{aligned}
A &= 1 + g^2 - 2gc\alpha \\
B &= -gs\alpha \\
C &= gc\alpha \\
D &= ghs\alpha \\
E &= -gh(g + c\alpha) \\
F &= 4g^2h^2 - (1 + g^2 - 2gc\alpha)(r_1^2 + r_2^2) \\
G &= \left[(r_1 + r_2)s\alpha\right]^2 \\
H &= 2s\alpha\left[(r_1 + r_2)^2(g - c\alpha) - 2gr_1r_2\right] \\
I &= (r_1 + r_2)^2(g - c\alpha)^2 - 4r_1r_2(g^2 - gc\alpha) \\
J &= ghr_1r_2s\alpha \\
K &= ghr_1r_2(g - c\alpha) \\
L &= (1 + g^2 - 2gc\alpha)r_1^2r_2^2
\end{aligned}
\tag{54}
$$

Some of the coefficients are simpler and it is easier to determine the design parameters from them. These simple coefficients can be selected as independent coefficients and the rest of the coefficients from set (54) can be defined in terms of these 5 independent coefficients. The selected coefficients are *B*, *C*, *D*, *G* and *J*. The remaining coefficients are expressed in terms of these 5 independent coefficients as

$$
\begin{aligned}
A &= B^2 + (C-1)^2 \\
E &= \frac{D\left(B^2 + C^2 + C\right)}{B} \\
F &= \frac{4D^2\left(B^2 + C^2\right) - G\left(B^2 + C^2\right)\left(B^2 + (C-1)^2\right)}{B^2} + \frac{2\left(B^2 + (C-1)^2\right)J}{D} \\
H &= \frac{4BJ}{D} - 2\left(\frac{G\left(B^2 + C^2 - C\right)}{B}\right) \\
I &= \frac{G\left(B^2 + C^2 - C\right)^2}{B^2} - \frac{4J\left(B^2 + C^2 - C\right)}{D} \\
K &= -\frac{J\left(B^2 + C^2 - C\right)}{B} = -\frac{EJ}{D} \\
L &= \frac{\left(B^2 + (C-1)^2\right)J^2}{D^2} = \frac{AJ^2}{D^2}
\end{aligned}
\tag{55}
$$

How the design parameters are defined in terms of the independent coefficients is as follows

$$
\begin{aligned}
g &= \sqrt{B^2 + C^2} \\
\alpha &= \tan^{-1}\left(-B/C\right) \\
h &= -D/B \\
r_1 r_2 &= J/D \\
d &= \frac{r_1 + r_2}{2} = \sqrt{\frac{G}{4\sin^2(\alpha)}} \\
a &= \frac{r_1 - r_2}{2} = \sqrt{\frac{G}{4\sin^2(\alpha)} - \frac{J}{D}} \\
b &= dg
\end{aligned}
\tag{56}
$$

Substituting the coefficients given as in (55) into the equation (53) result in the coupler-curve equation of the slider-crank mechanism in another form:

$$
\begin{aligned}
\Gamma = &\left[B^2+(C-1)^2\right]\left(x^2+y^2\right)^2+4B\left(x^2+y^2\right)xy+4C\left(x^2+y^2\right)y^2 \\
&+4D\left(x^2+y^2\right)x+4\frac{D\left(B^2+C^2+C\right)}{B}\left(x^2+y^2\right)y \\
&+\left(\frac{\left(B^2+C^2\right)\left\{4D^2-G\left[B^2+(C-1)^2\right]\right\}}{B^2}+\frac{2\left[B^2+(C-1)^2\right]J}{D}\right)\left(x^2+y^2\right) \\
&+Gx^2+\left(\frac{4BJ}{D}-2\frac{G\left(B^2+C^2-C\right)}{B}\right)xy \\
&+\left(\frac{G\left(B^2+C^2-C\right)^2}{B^2}-\frac{4J\left(B^2+C^2-C\right)}{D}\right)y^2 \\
&+4Jx-4\frac{J\left(B^2+C^2-C\right)}{B}y+\frac{\left[B^2+(C-1)^2\right]J^2}{D^2}=0
\end{aligned}
\tag{57}
$$

The non-linear equation has 5 independent parameters. The equation becomes much more complicated when the coordinate transformation is applied. Therefore, a general form still has the same complexity problem. However, the form of the coupler-curve equation in (57) is be useful for the method explained in Chapter 3.

CHAPTER 3

4-BAR COUPLER-CURVE EQUATION

The 4-bar coupler-curve equation is well analysed in the literature. The existing studies on the 4-bar coupler-curve is explained in the Chapter 1. The main properties of the coupler-curve equation are trinodal, tri-circular and sextic. Sextic simply means that the sum of the powers of highest order unknowns in a term is at most six. Trinodal means that this curve may at most cross itself (have real double points) three times. Tri-circular means that the highest order terms in the equation may be expressed in a single term as $(x^2 + y^2)^3$.

The coupler-curve equation can be obtained as a combination of different linear and quadratic expressions (a line, circle or ellipse). The problem is defined in 9-parameters (X_0, Y_0, β, a, b, d, a, h, m). In Figure 8, $g = d/b$. The 4-bar mechanism is shown together with its cognates and the links having the same angular velocity are drawn with same type of line.

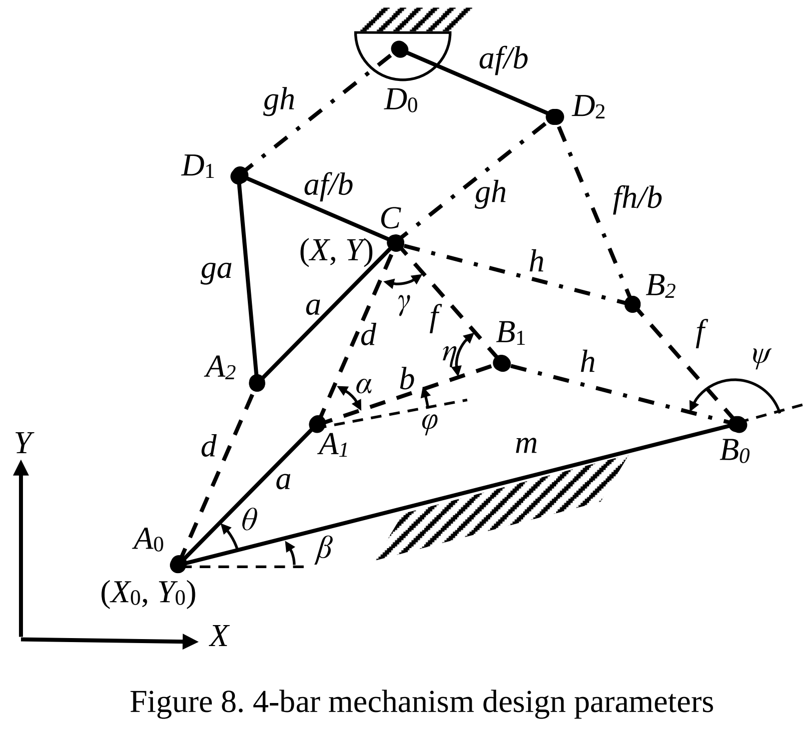

Figure 8. 4-bar mechanism design parameters

3.1. Coupler-curve Equation Derivation

The coupler point coordinates (X, Y) can be obtained in the following three different ways through the three fixed joints:

$$X + iY = ae^{i(\beta+\theta)} + de^{i(\beta+\varphi+\alpha)} \tag{58}$$

$$X + iY = me^{i\beta} + he^{i(\beta+\psi)} - be^{i(\beta+\varphi)} + de^{i(\beta+\varphi+\alpha)} \tag{59}$$

$$X + iY = gme^{i(\beta+\alpha)} + ghe^{i(\beta+\psi+\alpha)} - gae^{i(\beta+\theta+\alpha)} + ae^{i(\beta+\theta)} \tag{60}$$

where $|A_0D_0| = gm$. Equations (58)-(60) are dependent as:

$$ge^{i\alpha}\,[(59) - (58)] = [(60) - (58)] \tag{61}$$

Actually, equation (60) represents the loop closure equations. Decomposing equations (58)-(60) into their real and imaginary parts, the following 6 equations are obtained:

$$X = X_0 + ac_{\theta+\beta} + dc_{\alpha+\beta+\varphi} \tag{62}$$

$$X = X_0 + mc_{\beta} + hc_{\beta+\psi} - bc_{\beta+\varphi} + dc_{\alpha+\beta+\varphi} \tag{63}$$

$$X = X_0 + mgc_{\alpha+\beta} + hgc_{\alpha+\psi+\beta} - agc_{\alpha+\beta+\theta} + ac_{\beta+\theta} \tag{64}$$

$$Y = Y_0 + as_{\beta+\theta} + ds_{\alpha+\beta+\varphi} \tag{65}$$

$$Y = Y_0 + ms_{\beta} + hs_{\beta+\psi} - bs_{\beta+\varphi} + ds_{\alpha+\beta+\varphi} \tag{66}$$

$$Y = Y_0 + mgs_{\alpha+\beta} + hgs_{\alpha+\beta+\psi} - ags_{\alpha+\beta+\theta} + as_{\beta+\theta} \tag{67}$$

Each of the expressions for the *X*- or *Y*-coordinate of the coupler point contains two of the three motion parameters: θ, φ, ψ. Equations (62) and (65) are for dyad A_0A_1C. Similarly, equations (63) and (66) are for dyad B_0B_1C and equations (64) and (67) are for dyad D_0D_1C. Any group of two out of three dyads can be used to obtain the coupler-curve equation. Therefore, there will be three versions of the equation. The three versions of the equations are derived in the following sub-sections.

3.1.1. Coupler-Curve Equation Using Dyads A_0A_1C and B_0B_1C

Equations (58) and (59) have the angle φ in common. Each complex equation will be rearranged to eliminate the other motion parameters. Rearranging equation (62)

$$\begin{gathered}\left(ac_{\theta+\beta}\right)^2=\left(X_0+dc_{\alpha+\beta+\varphi}-X\right)^2\\ a^2c^2{}_{\theta+\beta}=X_0{}^2+d^2c^2{}_{\alpha+\beta+\varphi}+X^2+2X_0dc_{\alpha+\beta+\varphi}-2X_0X-2Xdc_{\alpha+\beta+\varphi}\end{gathered}\tag{68}$$

Similarly, rearranging equation (65):

$$\begin{gathered}\left(as_{\theta+\beta}\right)^2=\left(Y_0+ds_{\alpha+\beta+\varphi}-Y\right)^2\quad\Rightarrow\\ a^2s^2{}_{\theta+\beta}=Y_0{}^2+d^2s^2{}_{\alpha+\beta+\varphi}+Y^2+2Y_0ds_{\alpha+\beta+\varphi}-2Y_0Y-2Yds_{\alpha+\beta+\varphi}\end{gathered}\tag{69}$$

Adding equations (68) and (69) side by side eliminates θ:

$$\left(X-X_0\right)^2+\left(Y-Y_0\right)^2+d^2-a^2=2dc_{\alpha+\beta+\varphi}\left(X-X_0\right)+2ds_{\alpha+\beta+\varphi}\left(Y-Y_0\right)\tag{70}$$

Equation (70) has only φ as the motion parameter. Rearranging equation (63):

$$\begin{gathered}\left(-hc_{\beta+\psi}\right)^2=\left(X_0+mc_\beta-bc_{\beta+\varphi}+dc_{\alpha+\beta+\varphi}-X\right)^2\quad\Rightarrow\\ h^2c^2{}_{\beta+\psi}=X_0{}^2+m^2c^2{}_\beta+b^2c^2{}_{\beta+\varphi}+d^2c^2{}_{\alpha+\beta+\varphi}+X^2\\ +2X_0mc_\beta-2X_0bc_{\beta+\varphi}+2X_0dc_{\alpha+\beta+\varphi}-2X_0X-2mbc_\beta c_{\beta+\varphi}\\ +2mdc_\beta c_{\alpha+\beta+\varphi}-2mXc_\beta-2bdc_{\beta+\varphi}c_{\alpha+\beta+\varphi}+2Xbc_{\beta+\varphi}-2Xdc_{\alpha+\beta+\varphi}\end{gathered}\tag{71}$$

Rearranging equation (66):

$$\begin{gathered}\left(-hs_{\beta+\psi}\right)^2=\left(Y_0+ms_\beta-bs_{\beta+\varphi}+ds_{\alpha+\beta+\varphi}-Y\right)^2 \quad \Rightarrow \\ h^2s^2{}_{\beta+\psi}=Y_0{}^2+m^2s^2{}_\beta+b^2s^2{}_{\beta+\varphi}+d^2s^2{}_{\alpha+\beta+\varphi}+Y^2 \\ +2Y_0ms_\beta-2Y_0bs_{\beta+\varphi}+2Y_0dc_{\alpha+\beta+\varphi}-2Y_0Y-2mbs_\beta s_{\beta+\varphi} \\ +2mds_\beta s_{\alpha+\beta+\varphi}-2mYs_\beta-2bds_{\beta+\varphi}s_{\alpha+\beta+\varphi}+2Ybs_{\beta+\varphi}-2Yds_{\alpha+\beta+\varphi}\end{gathered} \tag{72}$$

Adding equations (71) and (72) side by side eliminates ψ:

$$\begin{gathered}h^2=X_0{}^2+Y_0{}^2+m^2+b^2+d^2+X^2+Y^2 \\ +2m\left[\left(X_0-X\right)c_\beta+\left(Y_0-Y\right)s_\beta\right]-2X_0X-2Y_0Y-2bdc_\alpha \\ +2\left(X-X_0\right)bc_{\beta+\varphi}+2\left(Y-Y_0\right)bs_{\beta+\varphi}-2mbc_\varphi \\ +2mdc_\beta c_{\alpha+\beta+\varphi}+2mds_\beta s_{\alpha+\beta+\varphi}+2d\left[\left(X_0-X\right)c_{\alpha+\beta+\varphi}+\left(Y_0-Y\right)s_{\alpha+\beta+\varphi}\right]\end{gathered} \tag{73}$$

The only motion parameter left in the equations (70) and (73) is φ. Eliminating common motion parameter φ from two equations, a single equation can be obtained without any motion parameter. However, the trigonometric functions with φ include other constant angles too. Therefore, some trigonometric conversions should be used to obtain both equations in the form $A\cos(\alpha+\varphi)+B\sin(\alpha+\varphi)=C$:

$$\begin{gathered}c_{\beta+\varphi}=c_{\alpha+\beta+\varphi}c_\alpha+s_{\alpha+\beta+\varphi}s_\alpha \\ s_{\beta+\varphi}=s_{\alpha+\beta+\varphi}c_\alpha-c_{\alpha+\beta+\varphi}s_\alpha \\ c_\varphi=c_{\alpha+\beta+\varphi}c_{\alpha+\beta}+s_{\alpha+\beta+\varphi}s_{\alpha+\beta}\end{gathered} \tag{74}$$

Then, equation (73) can be rewritten as:

$$\begin{gathered}\left(X-X_0-mc_\beta\right)^2+\left(Y-Y_0-ms_\beta\right)^2+f^2-h^2= \\ 2\left[\left(d-bc_\alpha\right)\left(X-X_0-mc_\beta\right)+\left(Y-Y_0-ms_\beta\right)bs_\alpha\right]c_{\alpha+\beta+\varphi} \\ +2\left[-\left(X-X_0-mc_\beta\right)bs_\alpha+\left(d-bc_\alpha\right)\left(Y-Y_0-ms_\beta\right)\right]s_{\alpha+\beta+\varphi}\end{gathered} \tag{75}$$

Equations (70) and (75) are linear in terms of $c_{\alpha+\beta+\varphi}$ and $s_{\alpha+\beta+\varphi}$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} c_{\alpha+\beta+\varphi} \\ s_{\alpha+\beta+\varphi} \end{bmatrix} = \begin{bmatrix} E \\ F \end{bmatrix} \tag{76}$$

where

$$\begin{gathered} A = 2d\left(X - X_0\right) \\ B = 2d\left(Y - Y_0\right) \\ C = 2\left[\left(d - bc_\alpha\right)\left(X - X_0 - mc_\beta\right) + bs_\alpha\left(Y - Y_0 - ms_\beta\right)\right] \\ D = 2\left[-bs_\alpha\left(X - X_0 - mc_\beta\right) + \left(d - bc_\alpha\right)\left(Y - Y_0 - ms_\beta\right)\right] \\ E = \left(X - X_0\right)^2 + \left(Y - Y_0\right)^2 + d^2 - a^2 \\ F = \left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 + f^2 - h^2 \end{gathered} \tag{77}$$

Then, solving for cosine and sine of the angle $\alpha + \beta + \varphi$

$$\Delta = AD - BC \quad c_{\alpha+\beta+\varphi} = \frac{ED - BF}{\Delta} \quad s_{\alpha+\beta+\varphi} = \frac{AF - EC}{\Delta} \tag{78}$$

Finally, substituting cosine and sine terms into $c^2_{\alpha+\beta+\varphi} + s^2_{\alpha+\beta+\varphi} = 1$:

$$\left(ED - BF\right)^2 + \left(AF - EC\right)^2 = \left(AD - BC\right)^2 \tag{79}$$

Equation (79) is the same form obtained by Roberts (1875). The *E* and *F* terms in equation (79) are similar to the circle terms obtained for the slider-crank mechanism. Therefore, their squares are easier to deal with. Rearranging the equation (79):

$$E^2\left(C^2 + D^2\right) + F^2\left(A^2 + B^2\right) - 2EF\left(AC + BD\right) = \left(AD - BC\right)^2 \tag{80}$$

Open form of equation (80):

$$
\begin{aligned}
&f^2\left[\begin{array}{c}\left(X-X_0\right)^2\\+\left(Y-Y_0\right)^2\\+d^2-a^2\end{array}\right]^2\left[\begin{array}{c}\left(X-X_0-mc_\beta\right)^2\\+\left(Y-Y_0-ms_\beta\right)^2\end{array}\right]+d^2\left[\begin{array}{c}\left(X-X_0-mc_\beta\right)^2\\+\left(Y-Y_0-ms_\beta\right)^2\\+f^2-h^2\end{array}\right]^2\left[\begin{array}{c}\left(X-X_0\right)^2\\+\left(Y-Y_0\right)^2\end{array}\right]\\
&-2d\left[\begin{array}{c}\left(X-X_0\right)^2\\+\left(Y-Y_0\right)^2\\+d^2-a^2\end{array}\right]\left[\begin{array}{c}\left(X-X_0-mc_\beta\right)^2\\+\left(Y-Y_0-ms_\beta\right)^2\\+f^2-h^2\end{array}\right]\left[\begin{array}{c}\left[\begin{array}{c}\left(X-X_0\right)\left(X-X_0-mc_\beta\right)\\+\left(Y-Y_0\right)\left(Y-Y_0-ms_\beta\right)\end{array}\right]\left(d-bc_\alpha\right)\\+\left[\begin{array}{c}\left(X-X_0\right)\left(Y-Y_0-ms_\beta\right)\\-\left(X-X_0-mc_\beta\right)\left(Y-Y_0\right)\end{array}\right]bs_\alpha\end{array}\right]\\
&=4d^2\left[\begin{array}{c}-bs_\alpha\left[\left(X-X_0\right)\left(X-X_0-mc_\beta\right)+\left(Y-Y_0\right)\left(Y-Y_0-ms_\beta\right)\right]\\+\left(d-bc_\alpha\right)\left[\left(X-X_0\right)\left(Y-Y_0-ms_\beta\right)-\left(X-X_0-mc_\beta\right)\left(Y-Y_0\right)\right]\end{array}\right]^2
\end{aligned}
\tag{81}
$$

First and second terms in the equation are rearranged to obtain circle terms:

$$
\begin{aligned}
&f^2\left[\begin{array}{c}\left(X-X_0\right)^2+\left(Y-Y_0\right)^2\\-\left(a-d\right)^2\end{array}\right]\left[\begin{array}{c}\left(X-X_0\right)^2+\left(Y-Y_0\right)^2\\-\left(a+d\right)^2\end{array}\right]\left[\begin{array}{c}\left(X-X_0-mc_\beta\right)^2\\+\left(Y-Y_0-ms_\beta\right)^2\end{array}\right]\\
&+d^2\left[\begin{array}{c}\left(X-X_0-mc_\beta\right)^2\\+\left(Y-Y_0-ms_\beta\right)^2-\left(h-f\right)^2\end{array}\right]\left[\begin{array}{c}\left(X-X_0-mc_\beta\right)^2\\+\left(Y-Y_0-ms_\beta\right)^2-\left(h+f\right)^2\end{array}\right]\left[\begin{array}{c}\left(X-X_0\right)^2\\+\left(Y-Y_0\right)^2\end{array}\right]\\
&+8d^2f^2\left[\left(X-X_0\right)^2+\left(Y-Y_0\right)^2\right]\left[\left(X-X_0-mc_\beta\right)^2+\left(Y-Y_0-ms_\beta\right)^2\right]\\
&-2d\left[\begin{array}{c}\left(X-X_0\right)^2\\+\left(Y-Y_0\right)^2\\-\left(d-a\right)^2\\+2d\left(d-a\right)\end{array}\right]\left[\begin{array}{c}\left(X-X_0-mc_\beta\right)^2\\+\left(Y-Y_0-ms_\beta\right)^2\\-\left(h-f\right)^2\\+2f\left(f-h\right)\end{array}\right]\left[\begin{array}{c}\left(d-bc_\alpha\right)\left[\begin{array}{c}\left(X-X_0\right)\left(X-X_0-mc_\beta\right)\\+\left(Y-Y_0\right)\left(Y-Y_0-ms_\beta\right)\end{array}\right]\\+bs_\alpha\left[\begin{array}{c}\left(X-X_0\right)\left(Y-Y_0-ms_\beta\right)\\-\left(Y-Y_0\right)\left(X-X_0-mc_\beta\right)\end{array}\right]\end{array}\right]\\
&=4d^2\left[\begin{array}{c}-bs_\alpha\left[\left(X-X_0\right)\left(X-X_0-mc_\beta\right)+\left(Y-Y_0\right)\left(Y-Y_0-ms_\beta\right)\right]\\+\left(d-bc_\alpha\right)\left[\left(X-X_0\right)\left(Y-Y_0-ms_\beta\right)-\left(X-X_0-mc_\beta\right)\left(Y-Y_0\right)\right]\end{array}\right]^2
\end{aligned}
\tag{82}
$$

Some of the expressions in the equation are the expected circles and lines. Defining them will help in rearranging the equation:

$$
\begin{aligned}
C_{A1} &= \left(X - X_0\right)^2 + \left(Y - Y_0\right)^2 - \left(d - a\right)^2 \\
C_{A2} &= \left(X - X_0\right)^2 + \left(Y - Y_0\right)^2 - \left(d + a\right)^2 \\
C_{B1} &= \left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 - \left(f - h\right)^2 \\
C_{B2} &= \left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 - \left(f + h\right)^2 \\
P_A &= \left(X - X_0\right)^2 + \left(Y - Y_0\right)^2 \\
P_B &= \left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 \\
C_{AB} &= \left(X - X_0\right)\left(X - X_0 - mc_\beta\right) + \left(Y - Y_0\right)\left(Y - Y_0 - ms_\beta\right) \\
L_{AB} &= \left(X - X_0\right)\left(Y - Y_0 - ms_\beta\right) - \left(X - X_0 - mc_\beta\right)\left(Y - Y_0\right)
\end{aligned}
\tag{83}
$$

C_{A1} and C_{A2} are the two circles centered at A_0 with radii $d - a$ and $d + a$. Similarly, C_{B1} and C_{B2} are the two circles centered at B_0 with radii $f - h$ and $f + h$. P_A and P_B represent points at A_0 and B_0. C_{AB} is the circle with center at the midpoint of A_0B_0 and passing through A_0 and B_0. L_{AB} is the line through A_0 and B_0. Substituting equation (83) into equation (82):

$$
\begin{aligned}
& f^2 C_{A1} C_{A2} P_B + d^2 C_{B1} C_{B2} P_A + 8d^2 f^2 P_A P_B \\
& -2d\left[C_{A1} + 2a\left(d - a\right)\right]\left[C_{B1} + 2h\left(f - h\right)\right]\left[\left(d - bc_\alpha\right)C_{AB} + bs_\alpha L_{AB}\right] \\
& = 4d^2\left[-bs_\alpha C_{AB} + \left(d - bc_\alpha\right)L_{AB}\right]^2
\end{aligned}
\tag{84}
$$

The third term in equation (84) is the result of obtaining circles in first two terms in equation (82). Half of the third term in equation (84) result from the conversion of the first term in equation (80) and other half result from the second term in equation (81). The term on the right hand side of equation (84) is the circle of singular foci, i.e. the circle that pass through the three center points A_0, B_0 and D_0.

Noticing that $d - bc_\alpha = fc_\gamma$ and $bs_\alpha = fs_\gamma$ in triangle A_1B_1C, the two circle expressions in equation (84) can be expressed with different parameters:

$$
\begin{aligned}
C_1 &: \quad \left(d - bc_\alpha\right)C_{AB} + bs_\alpha L_{AB} = f\left(c_\gamma C_{AB} + s_\gamma L_{AB}\right) \\
C_{SF} &: \quad -bs_\alpha C_{AB} + \left(d - bc_\alpha\right)L_{AB} = f\left(-s_\gamma C_{AB} + c_\gamma L_{AB}\right)
\end{aligned}
\tag{85}
$$

Substituting the two circles in equation (85) into equation (84), an equation without the link length b and the angle α is obtained:

$$\begin{aligned} &f^2 C_{A1} C_{A2} P_B + d^2 C_{B1} C_{B2} P_A + 8d^2 f^2 P_A P_B \\ &-2df\left[C_{A1} + 2a(d-a)\right]\left[C_{B1} + 2h(f-h)\right]C_1 = 4d^2 f^2 C_{SF}{}^2 \end{aligned} \tag{86}$$

The design parameters in equation (86) are X_0, Y_0, β, a, d, f, α, h, m. Half of the third term and square of the circle of singular foci can be united as follows:

$$\begin{aligned}
P_A P_B - C_{SF}{}^2 &= \left[(X-X_0)^2 + (Y-Y_0)^2\right]\left[(X-X_0-mc_\beta)^2 + (Y-Y_0-ms_\beta)^2\right] \\
&\quad - \begin{bmatrix} -s_\gamma\left[(X-X_0)(X-X_0-mc_\beta) + (Y-Y_0)(Y-Y_0-ms_\beta)\right] \\ +c_\gamma\left[(X-X_0)(Y-Y_0-ms_\beta) - (X-X_0-mc_\beta)(Y-Y_0)\right] \end{bmatrix}^2 \\
&= \left\{ \begin{array}{c}
(X-X_0)^2(X-X_0-mc_\beta)^2 + (X-X_0)^2(Y-Y_0-ms_\beta)^2 \\
+(Y-Y_0)^2(X-X_0-mc_\beta)^2 + (Y-Y_0)^2(Y-Y_0-ms_\beta)^2 \\
-s^2{}_\gamma(X-X_0)^2(X-X_0-mc_\beta)^2 - s^2{}_\gamma(Y-Y_0)^2(Y-Y_0-ms_\beta)^2 \\
-2s^2{}_\gamma(X-X_0)(Y-Y_0)(X-X_0-mc_\beta)(Y-Y_0-ms_\beta) \\
-c^2{}_\gamma(X-X_0)^2(Y-Y_0-ms_\beta)^2 - c^2{}_\gamma(X-X_0-mc_\beta)^2(Y-Y_0)^2 \\
+2c^2{}_\gamma(X-X_0)(Y-Y_0)(X-X_0-mc_\beta)(Y-Y_0-ms_\beta) \\
+2s_\gamma c_\gamma \begin{bmatrix} (X-X_0)(X-X_0-mc_\beta) \\ +(Y-Y_0)(Y-Y_0-ms_\beta) \end{bmatrix}\begin{bmatrix} (X-X_0)(Y-Y_0-ms_\beta) \\ -(X-X_0-mc_\beta)(Y-Y_0) \end{bmatrix}
\end{array} \right\} \\
&= \left\{ \begin{array}{c}
(X-X_0)^2(X-X_0-mc_\beta)^2 - s^2{}_\gamma(X-X_0)^2(X-X_0-mc_\beta)^2 \\
+(X-X_0)^2(Y-Y_0-ms_\beta)^2 - c^2{}_\gamma(X-X_0)^2(Y-Y_0-ms_\beta)^2 \\
+(Y-Y_0)^2(X-X_0-mc_\beta)^2 - c^2{}_\gamma(X-X_0-mc_\beta)^2(Y-Y_0)^2 \\
+(Y-Y_0)^2(Y-Y_0-ms_\beta)^2 - s^2{}_\gamma(Y-Y_0)^2(Y-Y_0-ms_\beta)^2 \\
-2s^2{}_\gamma(X-X_0)(Y-Y_0)(X-X_0-mc_\beta)(Y-Y_0-ms_\beta) \\
+2c^2{}_\gamma(X-X_0)(Y-Y_0)(X-X_0-mc_\beta)(Y-Y_0-ms_\beta) \\
+2s_\gamma c_\gamma \begin{bmatrix} (X-X_0)(X-X_0-mc_\beta) \\ +(Y-Y_0)(Y-Y_0-ms_\beta) \end{bmatrix}\begin{bmatrix} (X-X_0)(Y-Y_0-ms_\beta) \\ -(X-X_0-mc_\beta)(Y-Y_0) \end{bmatrix}
\end{array} \right\}
\end{aligned}$$

$$
= \left\{ \begin{array}{c} c^2_\gamma \left(X - X_0 \right)^2 \left(X - X_0 - mc_\beta \right)^2 + c^2_\gamma \left(Y - Y_0 \right)^2 \left(Y - Y_0 - ms_\beta \right)^2 \\ +2c^2_\gamma \left(X - X_0 \right)\left(Y - Y_0 \right)\left(X - X_0 - mc_\beta \right)\left(Y - Y_0 - ms_\beta \right) \\ +s^2_\gamma \left(X - X_0 \right)^2 \left(Y - Y_0 - ms_\beta \right)^2 + s^2_\gamma \left(Y - Y_0 \right)^2 \left(X - X_0 - mc_\beta \right)^2 \\ -2s^2_\gamma \left(X - X_0 \right)\left(Y - Y_0 \right)\left(X - X_0 - mc_\beta \right)\left(Y - Y_0 - ms_\beta \right) \\ +2s_\gamma c_\gamma \begin{bmatrix} \left(X - X_0 \right)\left(X - X_0 - mc_\beta \right) \\ +\left(Y - Y_0 \right)\left(Y - Y_0 - ms_\beta \right) \end{bmatrix} \begin{bmatrix} \left(X - X_0 \right)\left(Y - Y_0 - ms_\beta \right) \\ -\left(X - X_0 - mc_\beta \right)\left(Y - Y_0 \right) \end{bmatrix} \end{array} \right\}
$$

$$
= \left\{ \begin{array}{c} c^2_\gamma \left[\left(X - X_0 \right)\left(X - X_0 - mc_\beta \right) + \left(Y - Y_0 \right)\left(Y - Y_0 - ms_\beta \right) \right]^2 \\ +s^2_\gamma \left[\left(X - X_0 \right)\left(Y - Y_0 - ms_\beta \right) - \left(Y - Y_0 \right)\left(X - X_0 - mc_\beta \right) \right]^2 \\ +2s_\gamma c_\gamma \begin{bmatrix} \left(X - X_0 \right)\left(X - X_0 - mc_\beta \right) \\ +\left(Y - Y_0 \right)\left(Y - Y_0 - ms_\beta \right) \end{bmatrix} \begin{bmatrix} \left(X - X_0 \right)\left(Y - Y_0 - ms_\beta \right) \\ -\left(X - X_0 - mc_\beta \right)\left(Y - Y_0 \right) \end{bmatrix} \end{array} \right\} \qquad (87)
$$

$$
= \left\{ \begin{array}{c} c_\gamma \left[\left(X - X_0 \right)\left(X - X_0 - mc_\beta \right) + \left(Y - Y_0 \right)\left(Y - Y_0 - ms_\beta \right) \right] \\ +s_\gamma \left[\left(X - X_0 \right)\left(Y - Y_0 - ms_\beta \right) + \left(Y - Y_0 \right)\left(X - X_0 - mc_\beta \right) \right] \end{array} \right\}^2
$$

Therefore, the coupler-curve equation can be expressed with 5 terms which consist of the components defined in equations (83) and (85):

$$
\begin{array}{c} f^2 C_{A1} C_{A2} P_B + d^2 C_{B1} C_{B2} P_A + 4d^2 f^2 P_A P_B \\ -2df \left[C_{A1} + 2a\left(d - a \right) \right]\left[C_{B1} + 2h\left(f - h \right) \right] C_1 + 4d^2 f^2 C_1^2 = 0 \end{array} \qquad (88)
$$

A similar procedure may be applied to the two other couple of dyads. The coupler-curve equation to be obtained in the end should be at the same format with different components and coefficients. Some of the components should be common among the different versions of the coupler-curve equation.

3.1.2. Coupler-Curve Equation Using Dyads B_0B_1C and D_0D_1C

Equations (59) and (60) have the angle ψ in common. φ should be eliminated from equations (63) and (66), however the trigonometric functions are more complicated than the ones in Section 3.1.1. Equations (59) and (60) should be rearranged to obtain a set of equations in the form $A\cos(\alpha + \varphi) + B\sin(\alpha + \varphi) = C$. Rearranging equation (63):

$$X = X_0 + mc_\beta + hc_{\beta+\psi} + \left(dc_\alpha - b\right)c_{\beta+\varphi} - ds_\alpha s_{\beta+\varphi} \tag{89}$$

Rearranging equation (66):

$$Y = Y_0 + ms_\beta + hs_{\beta+\psi} + ds_\alpha c_{\beta+\varphi} + \left(dc_\alpha - b\right)s_{\beta+\varphi} \tag{90}$$

Equations (89) and (90) are linear in terms of sine and cosine of $\beta + \varphi$:

$$\begin{bmatrix} dc_\alpha - b & -ds_\alpha \\ ds_\alpha & dc_\alpha - b \end{bmatrix}\begin{bmatrix} c_{\beta+\varphi} \\ s_{\beta+\varphi} \end{bmatrix} = \begin{bmatrix} X - X_0 - mc_\beta - hc_{\beta+\psi} \\ Y - Y_0 - ms_\beta - hs_{\beta+\psi} \end{bmatrix} \tag{91}$$

Solving for $c_{\beta+\varphi}$ and $s_{\beta+\varphi}$ then substituting into $c^2_{\beta+\varphi} + s^2_{\beta+\varphi} = 1$:

$$\begin{aligned} &\left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 + h^2 - f^2 \\ &= 2hc_{\beta+\psi}\left(X - X_0 - mc_\beta\right) + 2hs_{\beta+\psi}\left(Y - Y_0 - ms_\beta\right) \end{aligned} \tag{92}$$

The same procedure is required for the scalar components of equation (60). Rearranging equation (64):

$$X - X_0 - mgc_{\alpha+\beta} - hgc_{\alpha+\beta+\psi} = a\left(1 - gc_\alpha\right)c_{\beta+\theta} + ags_\alpha s_{\beta+\theta} \tag{93}$$

Rearranging equation (67):

$$Y - Y_0 - mgs_{\alpha+\beta} - hgs_{\alpha+\beta+\psi} = -ags_\alpha c_{\beta+\theta} + a\left(1 - gc_\alpha\right)s_{\beta+\theta} \tag{94}$$

Equations (93) and (94) are linear in terms of θ:

$$\begin{bmatrix} a(1-gc_\alpha) & ags_\alpha \\ -ags_\alpha & a(1-gc_\alpha) \end{bmatrix} \begin{bmatrix} c_{\beta+\theta} \\ s_{\beta+\theta} \end{bmatrix} = \begin{bmatrix} X - X_0 - mgc_{\alpha+\beta} - hgc_{\alpha+\beta+\psi} \\ Y - Y_0 - mgs_{\alpha+\beta} - hgs_{\alpha+\beta+\psi} \end{bmatrix} \tag{95}$$

Solving for $c_{\beta+\theta}$ and $s_{\beta+\theta}$ then substituting into $c^2_{\beta+\theta} + s^2_{\beta+\theta} = 1$:

$$\begin{aligned} &\left(X - X_0 - mgc_{\alpha+\beta}\right)^2 + \left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 + h^2g^2 - a^2t^2 \\ &= 2hgc_{\alpha+\beta+\psi}\left(X - X_0 - mgc_{\alpha+\beta}\right) + 2hgs_{\alpha+\beta+\psi}\left(Y - Y_0 - mgs_{\alpha+\beta}\right) \end{aligned} \tag{96}$$

where $t = f/b$. The only motion parameter left in equations (92) and (96) is ψ. To eliminate ψ, we first use the trigonometric equations below.

$$c_{\beta+\psi} = c_{\alpha+\beta+\psi}c_\alpha + s_{\alpha+\beta+\psi}s_\alpha \qquad s_{\beta+\psi} = s_{\alpha+\beta+\psi}c_\alpha - c_{\alpha+\beta+\psi}s_\alpha \tag{97}$$

Then, equation (92) can be rewritten as:

$$\begin{aligned} &\left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 + h^2 - f^2 \\ &= \left[2h\left(X - X_0 - mc_\beta\right)c_\alpha - 2h\left(Y - Y_0 - ms_\beta\right)s_\alpha\right]c_{\alpha+\beta+\psi} \\ &+ \left[2h\left(X - X_0 - mc_\beta\right)s_\alpha + 2h\left(Y - Y_0 - ms_\beta\right)c_\alpha\right]s_{\alpha+\beta+\psi} \end{aligned} \tag{98}$$

Equations (96) and (98) are linear in terms of $c_{\alpha+\beta+\psi}$ and $s_{\alpha+\beta+\psi}$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} c_{\alpha+\beta+\psi} \\ s_{\alpha+\beta+\psi} \end{bmatrix} = \begin{bmatrix} E \\ F \end{bmatrix} \tag{99}$$

where

$$
\begin{aligned}
A &= 2hg\left(X - X_0 - mgc_{\alpha+\beta}\right) \\
B &= 2hg\left(Y - Y_0 - mgs_{\alpha+\beta}\right) \\
C &= 2h\left[\left(X - X_0 - mc_\beta\right)c_\alpha - \left(Y - Y_0 - ms_\beta\right)s_\alpha\right] \\
D &= 2h\left[\left(X - X_0 - mc_\beta\right)s_\alpha + \left(Y - Y_0 - ms_\beta\right)c_\alpha\right] \\
E &= \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 + \left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 + h^2g^2 - a^2t^2 \\
F &= \left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 + h^2 - f^2
\end{aligned}
\qquad (100)
$$

Then, solving for $c_{\alpha+\beta+\psi}$ and $s_{\alpha+\beta+\psi}$:

$$
\Delta = AD - BC \quad c_{\alpha+\beta+\psi} = \frac{ED - BF}{\Delta} \quad s_{\alpha+\beta+\psi} = \frac{AF - EC}{\Delta} \qquad (101)
$$

and substituting into $c^2_{\alpha+\beta+\psi} + s^2_{\alpha+\beta+\psi} = 1$:

$$
\left(ED - BF\right)^2 + \left(AF - EC\right)^2 = \left(AD - BC\right)^2 \qquad (102)
$$

Once again, the equation is in the same form of Roberts's (1875). The right-hand side of the equation still gives the circle of singular foci. Expanding the left-hand side:

$$
E^2\left(C^2 + D^2\right) + F^2\left(A^2 + B^2\right) - 2EF\left(AC + BD\right) = \left(AD - BC\right)^2 \qquad (103)
$$

In open form:

$$
\begin{aligned}
&\begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 + \left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 \\ +h^2 g^2 - a^2 t^2 \end{bmatrix}^2 \begin{bmatrix} \left(X - X_0 - mc_\beta\right)^2 \\ +\left(Y - Y_0 - ms_\beta\right)^2 \end{bmatrix} \\
&+g^2 \begin{bmatrix} \left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 \\ +h^2 - f^2 \end{bmatrix}^2 \begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 \\ +\left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 \end{bmatrix} \\
&-2g \begin{bmatrix} \begin{pmatrix} X - X_0 \\ -mgc_{\alpha+\beta} \end{pmatrix}^2 \\ +\begin{pmatrix} Y - Y_0 \\ -mgs_{\alpha+\beta} \end{pmatrix}^2 \\ +h^2 g^2 - a^2 t^2 \end{bmatrix} \begin{bmatrix} \begin{pmatrix} X - X_0 \\ -mc_\beta \end{pmatrix}^2 \\ +\begin{pmatrix} Y - Y_0 \\ -ms_\beta \end{pmatrix}^2 \\ +h^2 - f^2 \end{bmatrix} \begin{bmatrix} c_\alpha \begin{bmatrix} \left(X - X_0 - mc_\beta\right)\left(X - X_0 - mgc_{\alpha+\beta}\right) \\ +\left(Y - Y_0 - ms_\beta\right)\left(Y - Y_0 - mgs_{\alpha+\beta}\right) \end{bmatrix} \\ s_\alpha \begin{bmatrix} \left(X - X_0 - mc_\beta\right)\left(Y - Y_0 - mgs_{\alpha+\beta}\right) \\ -\left(Y - Y_0 - ms_\beta\right)\left(X - X_0 - mgc_{\alpha+\beta}\right) \end{bmatrix} \end{bmatrix} \\
&= 4h^2 g^2 \begin{bmatrix} s_\alpha \begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)\left(X - X_0 - mc_\beta\right) \\ +\left(Y - Y_0 - mgs_{\alpha+\beta}\right)\left(Y - Y_0 - ms_\beta\right) \end{bmatrix} \\ +c_\alpha \begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)\left(Y - Y_0 - ms_\beta\right) \\ -\left(Y - Y_0 - mgs_{\alpha+\beta}\right)\left(X - X_0 - mc_\beta\right) \end{bmatrix} \end{bmatrix}^2
\end{aligned}
\qquad (104)
$$

Rearranging the equation:

$$
\begin{aligned}
&\begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 \\ +\left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 \\ +\left(at - hg\right)^2 \end{bmatrix}
\begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 \\ +\left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 \\ +\left(at + hg\right)^2 \end{bmatrix}
\begin{bmatrix} \left(X - X_0 - mc_{\beta}\right)^2 \\ +\left(Y - Y_0 - ms_{\beta}\right)^2 \end{bmatrix} \\
&+g^2 \begin{bmatrix} \left(X - X_0 - mc_{\beta}\right)^2 \\ +\left(Y - Y_0 - ms_{\beta}\right)^2 \\ +\left(f - h\right)^2 \end{bmatrix}
\begin{bmatrix} \left(X - X_0 - mc_{\beta}\right)^2 \\ +\left(Y - Y_0 - ms_{\beta}\right)^2 \\ +\left(f + h\right)^2 \end{bmatrix}
\begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 \\ +\left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 \end{bmatrix} \\
&+8h^2 g^2 \begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 \\ +\left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 \end{bmatrix}
\begin{bmatrix} \left(X - X_0 - mc_{\beta}\right)^2 \\ +\left(Y - Y_0 - ms_{\beta}\right)^2 \end{bmatrix} \\
&-2g \begin{bmatrix} \begin{pmatrix} X - X_0 \\ -mgc_{\alpha+\beta} \end{pmatrix}^2 \\ +\begin{pmatrix} Y - Y_0 \\ -mgs_{\alpha+\beta} \end{pmatrix}^2 \\ +h^2 g^2 - a^2 t^2 \end{bmatrix}
\begin{bmatrix} \begin{pmatrix} X - X_0 \\ -mc_{\beta} \end{pmatrix}^2 \\ +\begin{pmatrix} Y - Y_0 \\ -ms_{\beta} \end{pmatrix}^2 \\ +h^2 - f^2 \end{bmatrix}
\begin{bmatrix} c_\alpha \begin{bmatrix} \left(X - X_0 - mc_\beta\right)\left(X - X_0 - mgc_{\alpha+\beta}\right) \\ +\left(Y - Y_0 - ms_\beta\right)\left(Y - Y_0 - mgs_{\alpha+\beta}\right) \end{bmatrix} \\ s_\alpha \begin{bmatrix} \left(X - X_0 - mc_\beta\right)\left(Y - Y_0 - mgs_{\alpha+\beta}\right) \\ -\left(Y - Y_0 - ms_\alpha\right)\left(X - X_0 - mgc_{\alpha+\beta}\right) \end{bmatrix} \end{bmatrix} \\
&= 4h^2 g^2 \begin{bmatrix} s_\alpha \begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)\left(X - X_0 - mc_\beta\right) \\ +\left(Y - Y_0 - mgs_{\alpha+\beta}\right)\left(Y - Y_0 - ms_\beta\right) \end{bmatrix} \\ +c_\alpha \begin{bmatrix} \left(X - X_0 - mgc_{\alpha+\beta}\right)\left(Y - Y_0 - ms_\beta\right) \\ -\left(Y - Y_0 - mgs_{\alpha+\beta}\right)\left(X - X_0 - mc_\beta\right) \end{bmatrix} \end{bmatrix}^2
\end{aligned}
\tag{105}
$$

In order to shorten the equation as in the previous section, let

$$
\begin{aligned}
C_{D1} &= \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 + \left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 - \left(at - hg\right)^2 \\
C_{D2} &= \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 + \left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 - \left(at + hg\right)^2 \\
C_{B1} &= \left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 - \left(f - h\right)^2 \\
C_{B2} &= \left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 - \left(f + h\right)^2 \\
P_B &= \left(X - X_0 - mc_\beta\right)^2 + \left(Y - Y_0 - ms_\beta\right)^2 \\
P_D &= \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 + \left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 \\
C_{BD} &= \left(X - X_0 - mgc_{\alpha+\beta}\right)\left(X - X_0 - mc_\beta\right) + \left(Y - Y_0 - mgs_{\alpha+\beta}\right)\left(Y - Y_0 - ms_\beta\right) \\
L_{BD} &= \left(X - X_0 - mgc_{\alpha+\beta}\right)\left(Y - Y_0 - ms_\beta\right) - \left(Y - Y_0 - mgs_{\alpha+\beta}\right)\left(X - X_0 - mc_\beta\right)
\end{aligned}
\tag{106}
$$

Short form of the equation becomes:

$$
\begin{gathered}
C_{D1}C_{D2}P_B + g^2 C_{B1}C_{B2}P_D + 8h^2 g^2 P_B P_D \\
-2g\left[C_{D1} + 2at\left(hg - at\right)\right]\left[C_{B1} + 2f\left(h - f\right)\right]\left(c_\alpha C_{BD} + s_\alpha L_{BD}\right) \\
= 4h^2 g^2\left(s_\alpha C_{BD} + c_\alpha L_{BD}\right)^2
\end{gathered} \tag{107}
$$

Half of the third term and the circle of singular foci on the right-hand side can be combined as:

$$
P_B P_D - \left(s_\alpha C_{BD} + c_\alpha L_{BD}\right)^2 = \left(c_\alpha C_{BD} + s_\alpha L_{BD}\right)^2 = C_2^{\,2} \tag{108}
$$

Final form of the coupler-curve equation becomes:

$$
\begin{gathered}
4h^2 C_{D1}C_{D2}P_B + 4h^2 g^2 C_{B1}C_{B2}P_D + 16h^4 g^2 P_B P_D \\
-8h^2 g\left[C_{D1} + 2at\left(hg - at\right)\right]\left[C_{B1} + 2f\left(h - f\right)\right]C_2 + 16h^4 g^2 C_2^{\,2} = 0
\end{gathered} \tag{109}
$$

The design parameters used in equation (109) are X_0, Y_0, β, h, f, g, at, m, α, which is a different set of parameters than the parameters in equation (88).

3.1.3. Coupler-Curve Equation Using Dyads A_0A_1C and D_0D_1C

Equations (58) and (60) have the angle θ in common. φ should be eliminated from equations (63) and (66). Rearranging equation (62):

$$
\begin{gathered}
\left(dc_{\alpha+\beta+\varphi}\right)^2 = \left(X_0 + ac_{\theta+\beta} - X\right)^2 \quad \Rightarrow \\
d^2c^2{}_{\alpha+\beta+\varphi} = X_0^{\,2} + a^2c^2{}_{\theta+\beta} + X^2 + 2X_0 ac_{\theta+\beta} - 2X_0X - 2Xac_{\theta+\beta}
\end{gathered} \tag{110}
$$

Rearranging equation (65):

$$
\begin{gathered}
\left(ds_{\alpha+\beta+\varphi}\right)^2 = \left(Y_0 + as_{\theta+\beta} - Y\right)^2 \quad \Rightarrow \\
d^2s^2{}_{\alpha+\beta+\varphi} = Y_0^{\,2} + a^2s^2{}_{\theta+\beta} + Y^2 + 2Y_0 as_{\theta+\beta} - 2Y_0Y - 2Yas_{\theta+\beta}
\end{gathered} \tag{111}
$$

Adding equations (110) and (111) side by side eliminates φ:

$$\left(X - X_0\right)^2 + \left(Y - Y_0\right)^2 + a^2 - d^2 = -2ac_{\theta+\beta}\left(X - X_0\right) - 2as_{\theta+\beta}\left(Y - Y_0\right) \qquad (112)$$

ψ should be eliminated from equations (64) and (67). Rearranging equation (64):

$$\begin{aligned} &\left(hgc_{\alpha+\beta+\psi}\right)^2 = \left(X - X_0 - mgc_{\alpha+\beta} + agc_{\alpha+\beta+\theta} - ac_{\beta+\theta}\right)^2 \quad \Rightarrow \\ &h^2g^2c^2{}_{\alpha+\beta+\psi} = \left(X - X_0 - mgc_{\alpha+\beta}\right)^2 + a^2g^2c^2{}_{\alpha+\beta+\theta} + a^2c^2{}_{\beta+\theta} \\ &-2a^2gc_{\alpha+\beta+\theta}c_{\beta+\theta} + 2a\left(X - X_0 - mgc_{\alpha+\beta}\right)\left(gc_{\alpha+\beta+\theta} - c_{\beta+\theta}\right) \end{aligned} \qquad (113)$$

Rearranging equation (67):

$$\begin{aligned} &\left(hgs_{\alpha+\beta+\psi}\right)^2 = \left(Y - Y_0 - mgs_{\alpha+\beta} + ags_{\alpha+\beta+\theta} - as_{\beta+\theta}\right)^2 \quad \Rightarrow \\ &h^2g^2s^2{}_{\alpha+\beta+\psi} = \left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 + a^2g^2s^2{}_{\alpha+\beta+\theta} + a^2s^2{}_{\beta+\theta} \\ &-2a^2gs_{\alpha+\beta+\theta}s_{\beta+\theta} + 2a\left(Y - Y_0 - mgs_{\alpha+\beta}\right)\left(gs_{\alpha+\beta+\theta} - s_{\beta+\theta}\right) \end{aligned} \qquad (114)$$

Adding equations (113) and (114) side by side eliminates ψ:

$$\begin{aligned} h^2g^2 = &\left(X - X_0 - mgc_{\alpha+\beta}\right)^2 + \left(Y - Y_0 - mgs_{\alpha+\beta}\right)^2 + a^2t^2 \\ &+2a\left(X - X_0 - mgc_{\alpha+\beta}\right)\left(gc_{\alpha+\beta+\theta} - c_{\beta+\theta}\right) \\ &+2a\left(Y - Y_0 - mgs_{\alpha+\beta}\right)\left(gs_{\alpha+\beta+\theta} - s_{\beta+\theta}\right) \end{aligned} \qquad (115)$$

The only motion parameter left in the equations (112) and (115) is θ. To eliminate this, the following trigonometric equations are used:

$$c_{\beta+\theta} = c_{\alpha+\beta+\theta}c_\alpha + s_{\alpha+\beta+\theta}s_\alpha \qquad s_{\beta+\theta} = s_{\alpha+\beta+\theta}c_\alpha - c_{\alpha+\beta+\theta}s_\alpha \qquad (116)$$

Then, equation (115) can be rewritten as:

$$
\begin{aligned}
&2a\left[\left(X-X_0-mgc_{\alpha+\beta}\right)\left(c_\alpha-g\right)-\left(Y-Y_0-mgs_{\alpha+\beta}\right)s_\alpha\right]c_{\alpha+\beta+\theta}\\
&+2a\left[\left(X-X_0-mgc_{\alpha+\beta}\right)s_\alpha+\left(Y-Y_0-mgs_{\alpha+\beta}\right)\left(c_\alpha-g\right)\right]s_{\alpha+\beta+\theta}\\
&=\left(X-X_0-mgc_{\alpha+\beta}\right)^2+\left(Y-Y_0-mgs_{\alpha+\beta}\right)^2+a^2t^2-h^2g^2
\end{aligned}
\tag{117}
$$

Equation (112) can be rearranged as

$$
\begin{aligned}
&\left(X-X_0\right)^2+\left(Y-Y_0\right)^2+a^2-d^2=\\
&-2a\left[\left(X-X_0\right)c_\alpha-\left(Y-Y_0\right)s_\alpha\right]c_{\alpha+\beta+\theta}-2a\left[\left(X-X_0\right)s_\alpha+\left(Y-Y_0\right)c_\alpha\right]s_{\alpha+\beta+\theta}
\end{aligned}
\tag{118}
$$

Equations (117) and (118) are linear in terms of $c_{\alpha+\beta+\theta}$ and $s_{\alpha+\beta+\theta}$:

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} c_{\alpha+\beta+\theta} \\ s_{\alpha+\beta+\theta} \end{bmatrix}=\begin{bmatrix} E \\ F \end{bmatrix}
\tag{119}
$$

where

$$
\begin{aligned}
A&=-2a\left[\left(X-X_0\right)c_\alpha-\left(Y-Y_0\right)s_\alpha\right]\\
B&=-2a\left[\left(X-X_0\right)s_\alpha+\left(Y-Y_0\right)c_\alpha\right]\\
C&=2a\left[\left(X-X_0-mgc_{\alpha+\beta}\right)\left(c_\alpha-g\right)-\left(Y-Y_0-mgs_{\alpha+\beta}\right)s_\alpha\right]\\
D&=2a\left[\left(X-X_0-mgc_{\alpha+\beta}\right)s_\alpha+\left(Y-Y_0-mgs_{\alpha+\beta}\right)\left(c_\alpha-g\right)\right]\\
E&=\left(X-X_0\right)^2+\left(Y-Y_0\right)^2+a^2-d^2\\
F&=\left(X-X_0-mgc_{\alpha+\beta}\right)^2+\left(Y-Y_0-mgs_{\alpha+\beta}\right)^2+a^2t^2-h^2g^2
\end{aligned}
\tag{120}
$$

Then, solving for cosine and sine of the angle $\alpha + \beta + \theta$:

$$
\Delta=AD-BC \quad c_{\alpha+\beta+\theta}=\frac{ED-BF}{\Delta} \quad s_{\alpha+\beta+\theta}=\frac{AF-EC}{\Delta}
\tag{121}
$$

and substituting into $c_{\alpha+\beta+\theta} + s_{\alpha+\beta+\theta} = 1$:

$$
\left(ED-BF\right)^2+\left(AF-EC\right)^2=\left(AD-BC\right)^2
\tag{122}
$$

Equation (122) is obtained in Roberts's (1875) form. For the further operations, the equation is rewritten as:

$$E^2\left(C^2+D^2\right)+F^2\left(A^2+B^2\right)-2EF\left(AC+BD\right)=\left(AD-BC\right)^2 \tag{123}$$

or, in open form:

$$\begin{aligned}
&4a^2t^2\begin{bmatrix}\left(X-X_0\right)^2+\left(Y-Y_0\right)^2\\ +a^2-d^2\end{bmatrix}^2\begin{bmatrix}\left(X-X_0-mgc_{\alpha+\beta}\right)^2\\ +\left(Y-Y_0-mgs_{\alpha+\beta}\right)^2\end{bmatrix}\\
&+4a^2\begin{bmatrix}\left(X-X_0-mgc_{\alpha+\beta}\right)^2+\left(Y-Y_0-mgs_{\alpha+\beta}\right)^2\\ +a^2t^2-h^2g^2\end{bmatrix}^2\begin{bmatrix}\left(X-X_0\right)^2\\ +\left(Y-Y_0\right)^2\end{bmatrix}\\
&-8a^2t\begin{bmatrix}\left(X-X_0\right)^2\\ +\left(Y-Y_0\right)^2\\ +a^2-d^2\end{bmatrix}\begin{bmatrix}\begin{pmatrix}X-X_0\\ -mgc_{\alpha+\beta}\end{pmatrix}^2\\ +\begin{pmatrix}Y-Y_0\\ -mgs_{\alpha+\beta}\end{pmatrix}^2\\ +a^2t^2-h^2g^2\end{bmatrix}\begin{bmatrix}c_\eta\begin{bmatrix}\left(X-X_0\right)\left(X-X_0-mgc_{\alpha+\beta}\right)\\ +\left(Y-Y_0\right)\left(Y-Y_0-mgs_{\alpha+\beta}\right)\end{bmatrix}\\ s_\eta\begin{bmatrix}\left(X-X_0\right)\left(Y-Y_0-mgs_{\alpha+\beta}\right)\\ -\left(Y-Y_0\right)\left(X-X_0-mgc_{\alpha+\beta}\right)\end{bmatrix}\end{bmatrix}\\
&=16a^4t^2\begin{bmatrix}-s_\eta\begin{bmatrix}\left(X-X_0\right)\left(X-X_0-mgc_{\alpha+\beta}\right)\\ +\left(Y-Y_0\right)\left(Y-Y_0-mgs_{\alpha+\beta}\right)\end{bmatrix}\\ +c_\eta\begin{bmatrix}\left(X-X_0\right)\left(Y-Y_0-mgs_{\alpha+\beta}\right)\\ -\left(Y-Y_0\right)\left(X-X_0-mgc_{\alpha+\beta}\right)\end{bmatrix}\end{bmatrix}^2
\end{aligned} \tag{124}$$

Rearranging the equation to obtain two circles for each center point:

$$
\begin{aligned}
& t^2\begin{bmatrix}(X-X_0)^2+(Y-Y_0)^2\\-(d-a)^2\end{bmatrix}\begin{bmatrix}(X-X_0)^2+(Y-Y_0)^2\\-(d+a)^2\end{bmatrix}\begin{bmatrix}(X-X_0-mgc_{\alpha+\beta})^2\\+(Y-Y_0-mgs_{\alpha+\beta})^2\end{bmatrix}\\
& +\begin{bmatrix}(X-X_0-mgc_{\alpha+\beta})^2\\+(Y-Y_0-mgs_{\alpha+\beta})^2\\-(at-hg)^2\end{bmatrix}\begin{bmatrix}(X-X_0-mgc_{\alpha+\beta})^2\\+(Y-Y_0-mgs_{\alpha+\beta})^2\\-(at+hg)^2\end{bmatrix}\begin{bmatrix}(X-X_0)^2\\+(Y-Y_0)^2\end{bmatrix}\\
& 4a^2t^2\left[(X-X_0)^2+(Y-Y_0)^2\right]\left[(X-X_0-mgc_{\alpha+\beta})^2+(Y-Y_0-mgs_{\alpha+\beta})^2\right]\\
& -2t\begin{bmatrix}(X-X_0)^2\\+(Y-Y_0)^2\\-(d-a)^2\\+2d(a-d)\end{bmatrix}\begin{bmatrix}\begin{pmatrix}X-X_0\\-mgc_{\alpha+\beta}\end{pmatrix}^2\\+\begin{pmatrix}Y-Y_0\\-mgs_{\alpha+\beta}\end{pmatrix}^2\\-(at-hg)^2\\+2hg(at-hg)\end{bmatrix}\begin{bmatrix}c_\eta\begin{bmatrix}(X-X_0)(X-X_0-mgc_{\alpha+\beta})\\+(Y-Y_0)(Y-Y_0-mgs_{\alpha+\beta})\end{bmatrix}\\s_\eta\begin{bmatrix}(X-X_0)(Y-Y_0-mgs_{\alpha+\beta})\\-(Y-Y_0)(X-X_0-mgc_{\alpha+\beta})\end{bmatrix}\end{bmatrix}\\
& =4a^2t^2\begin{bmatrix}-s_\eta\begin{bmatrix}(X-X_0)(X-X_0-mgc_{\alpha+\beta})\\+(Y-Y_0)(Y-Y_0-mgs_{\alpha+\beta})\end{bmatrix}\\+c_\eta\begin{bmatrix}(X-X_0)(Y-Y_0-mgs_{\alpha+\beta})\\-(Y-Y_0)(X-X_0-mgc_{\alpha+\beta})\end{bmatrix}\end{bmatrix}^2
\end{aligned}
\tag{125}
$$

Let

$$
\begin{aligned}
C_{A1}&=(X-X_0)^2+(Y-Y_0)^2-(d-a)^2\\
C_{B1}&=(X-X_0)^2+(Y-Y_0)^2-(d+a)^2\\
C_{D1}&=(X-X_0-mgc_{\alpha+\beta})^2+(Y-Y_0-mgs_{\alpha+\beta})^2-(at-hg)^2\\
C_{D2}&=(X-X_0-mgc_{\alpha+\beta})^2+(Y-Y_0-mgs_{\alpha+\beta})^2-(at+hg)^2\\
P_A&=(X-X_0)^2+(Y-Y_0)^2\\
P_B&=(X-X_0-mgc_{\alpha+\beta})^2+(Y-Y_0-mgs_{\alpha+\beta})^2\\
C_{AD}&=(X-X_0)(X-X_0-mgc_{\alpha+\beta})+(Y-Y_0)(Y-Y_0-mgs_{\alpha+\beta})\\
L_{AD}&=(X-X_0)(Y-Y_0-mgs_{\alpha+\beta})-(Y-Y_0)(X-X_0-mgc_{\alpha+\beta})
\end{aligned}
\tag{126}
$$

Then

$$t^2 C_{A1} C_{A2} P_D + C_{D1} C_{D2} P_A + 4a^2 t^2 P_A P_D$$
$$-2t\left[C_{A1} + 2d\left(a-d\right)\right]\left[C_{D1} + 2hg\left(at-hg\right)\right]\left(c_\eta C_{AD} + s_\eta L_{AD}\right) \tag{127}$$
$$= 4a^2 t^2 \left(-s_\eta C_{AD} + c_\eta L_{AD}\right)^2$$

Finally, the circle of singular foci is united with half of the third term in equation (127) to obtain:

$$P_A P_D - \left(s_\eta C_{AD} + c_\eta L_{AD}\right)^2 = \left(c_\eta C_{AD} - s_\eta L_{AD}\right)^2 = C_3^{\ 2} \tag{128}$$

Final form of the coupler-curve equation is obtained as:

$$t^2 C_{A1} C_{A2} P_D + C_{D1} C_{D2} P_A + 4a^2 t^2 P_A P_D$$
$$-2t\left[C_{A1} + 2d\left(a-d\right)\right]\left[C_{D1} + 2hg\left(at-hg\right)\right]\left(c_\gamma C_{AD} - s_\gamma L_{AD}\right) + 4a^2 t^2 C_3^{\ 2} = 0 \tag{129}$$

3.2. Graphical Representation of Equation Components

The components obtained in the coupler-curve equation are points, circles and lines. Six of the circles, C_{A1}, C_{A2}, C_{B1}, C_{B2}, C_{D1} and C_{D2} are centered at the fixed pivots A_0, B_0 and D_0. Each fixed pivot has two circles (Figure 9). The circle at A_0 with smaller radius is C_{A1} and the one with larger radius is C_{A2} and so on. These circles can be called boundary circles because the coupler-curve must lie within the area bounded by them. Each center point has two boundary circles and the area between the two circles is where the coupler-curve may be located. The area where the coupler-curve may exist is limited to the intersection of circular disks with centers at the fixed pivot (Figure 10).

Each of the circle and line expressions that appear in the coupler-curve equation, and the distances between these objects can be considered as the geometric invariants (under coordinate transformation) for the coupler-curve equation. Three different versions of the coupler-curve equation are derived for each cognate, and an independent set of geometric invariants can be chosen for each version of the equation. For instance, for the equation derived using fixed pivots A_0 and B_0, the radii $a + d$ and $|a - d|$ of the pair of circles concentric at A_0, the radii $f + h$ and $|f - h|$ of the pair of circles concentric at B_0, the distance $m = |A_0B_0|$ between the circle centers and the angle γ that appears in the linear combination of the C_{AB} and L_{AB} constitute a set of 6 independent geometric invariants.

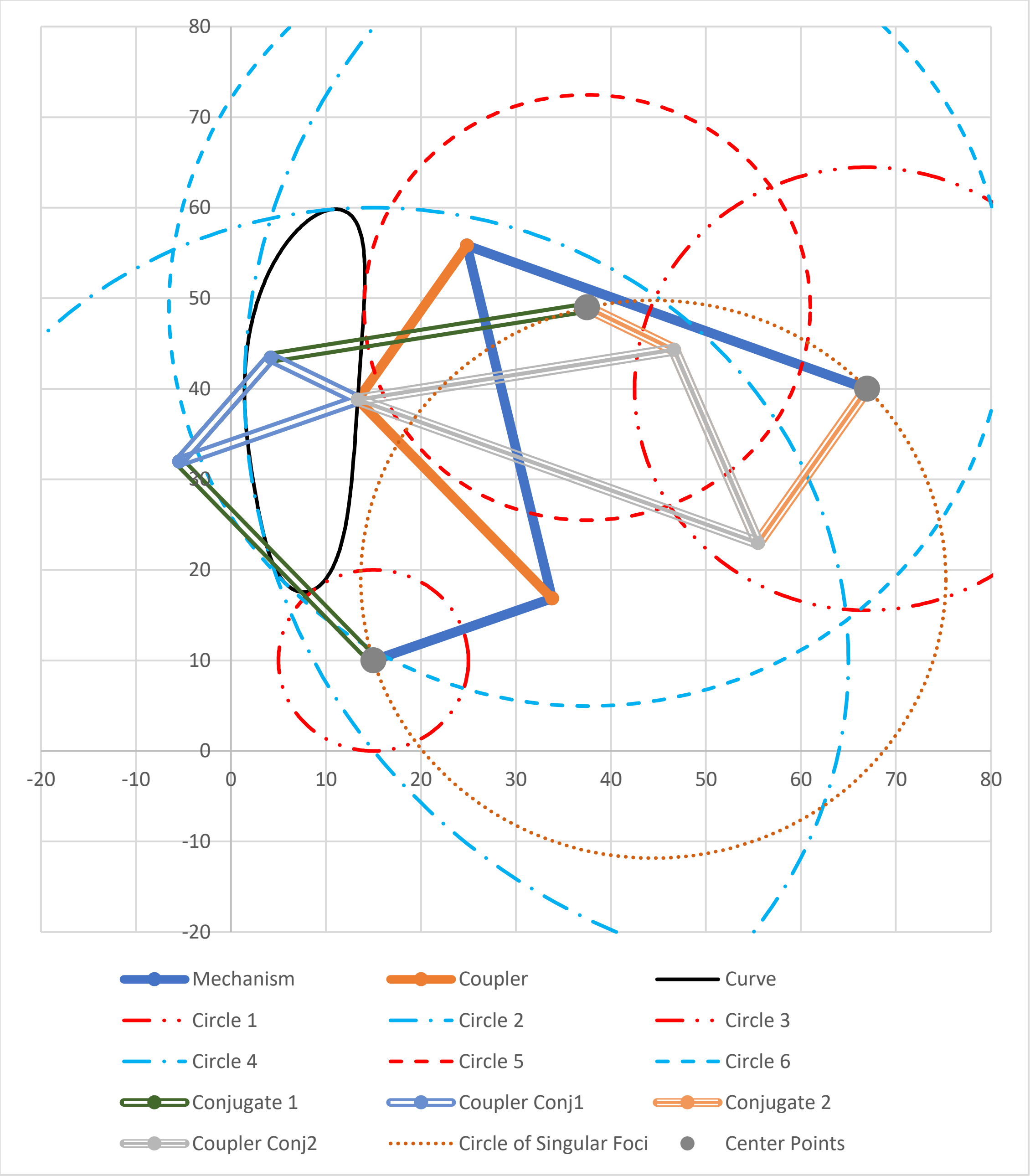

Figure 9. Graphical representation of components of the coupler-curve of a 4-bar mechanism

The boundary circles may be tangent to the coupler-curve. The tangency is guaranteed under certain conditions. Recall that Figure 8 shows the 4-bar mechanism and its cognates. When built as in Figure 8, the mechanism is over-constrained but still moves due to special link length ratios. Figure 8 shows 2 cranks. The mechanism is called crank-rocker, double-crank or double-rocker, depending on the motion of the rotating links. Each of the two concentric boundary circles with center at a fixed pivot must have a tangent point with the coupler-curve if one of the two rotating links (as in Figure 8) is a crank. A crank must match the rotation of another rotating link at another fixed pivot. The link with the same rotational speed is drawn with the same type of line in Figure 8. Therefore, one crank guarantees 4 tangent points to the coupler-curve. A crank-rocker type mechanism is guaranteed to have 4 tangent points when a double-crank mechanism is guaranteed to have each of the 6 circles to have a tangent point to the coupler-curve. A double-rocker type mechanism is not guaranteed to have tangent points.

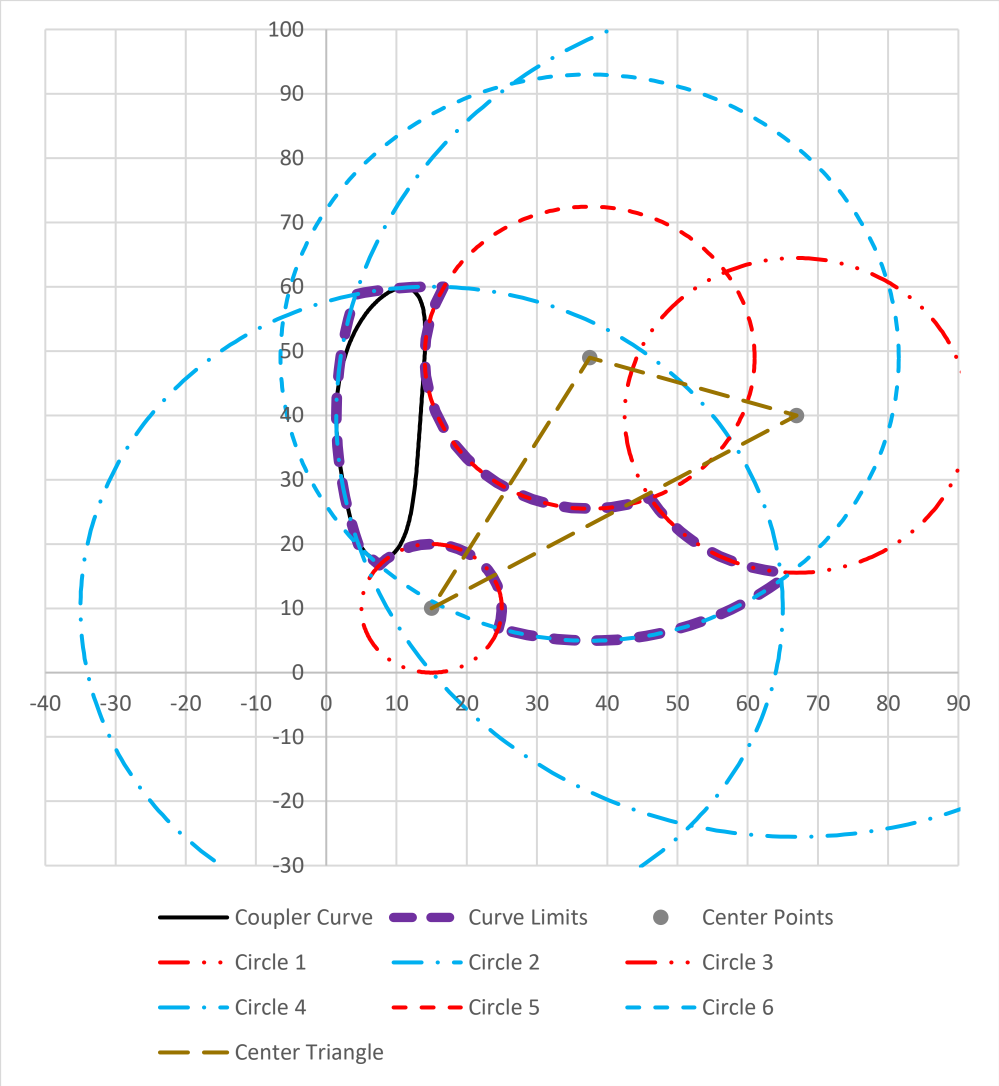

Figure 10. A 4-bar mechanism's center points, boundary circles and coupler-curve

There are three circles that pass through a pair of fixed pivots with centers at midpoints of the pivots: C_{AB}, C_{BD} and C_{AD}. Each circle is named with subscripts of pivots it passes through, such as C_{AB} passes through A_0 and B_0. There are three lines that pass through a pair of fixed pivots: L_{AB}, L_{BD} and L_{AD}. The linear combination of these three circles and lines appears within alternative forms of the coupler-curve equation. These linear combinations have special forms. The coupler triangle is similar to the triangle of the three fixed pivots. The linear combination of C_{AB} and L_{AB} connecting A_0 and B_0 have the coefficients as sine and cosine of the inner angle γ at D_0. $C_{SF} = -\mathrm{s}\gamma C_{AB} + \mathrm{c}\gamma L_{AB}$ is the circle of singular foci. Such a circle is the geometric locus of points P, such that $\angle A_0PB_0 = \gamma$. The other linear combination that appears in the coupler-curve equation is $C_1 = \mathrm{c}\gamma C_{AB} + \mathrm{s}\gamma L_{AB}$. C_1 is the geometric locus of points Q, such that $\angle A_0QB_0 = \pi/2 - \gamma$. Similar discussion is valid for pair of pivots B_0-D_0 and angle α for $C_{SF} = -\mathrm{s}\alpha C_{BD} + \mathrm{c}\alpha L_{BD}$ and $C_2 = \mathrm{c}\alpha C_{BD} + \mathrm{s}\alpha L_{BD}$. Similarly, a third alternative form of C_{SF} and a circle C_3 are associated with fixed pivots A_0-D_0. The circle of singular foci, C_{SF}, is of course an important circle, because it passes through all three fixed pivots and nodes of the coupler-curve, if there are any (Roberts, 1875). Whether the circles C_1, C_2 and C_3 have any significance is yet to be investigated. A general formulation about these circles is presented in Section 4.3.

3.3. On the Circle of Singular Foci of 4-Bar Linkages

Let $\vec{X} = x\vec{i} + y\vec{j}$ be the position vector of a point X on an xy-plane and $\vec{A} = x_A\vec{i} + y_A\vec{j}$, $\vec{B} = x_B\vec{i} + y_B\vec{j}$ be position vectors two specific points A and B.

$F_1(x,y) = \left|\overrightarrow{AX} \times \overrightarrow{BX}\right| = 0$ represents a curve, where the points on it are such that the angle between $\overrightarrow{AX}$ and $\overrightarrow{BX}$ is zero, i.e. $\vec{X}$, $\vec{A}$ and $\vec{B}$ are collinear. In other words, $F_1(x, y) = 0$ is the geometric locus of points lying on the AB line.

$F_2(x,y) = \overrightarrow{AX} \cdot \overrightarrow{BX} = 0$ represents a curve, where $\overrightarrow{AX}$ and $\overrightarrow{BX}$ are perpendicular to each other, i.e. $\angle AXB = 90°$. The geometric locus of such points is a circle with center at the midpoint M of AB and passes through A and B. The radius r_2 of the circle is the half of $|AB|$:

$$r_2 = |AM| = \frac{|AB|}{2} = \sqrt{\left(\frac{x_B - x_A}{2}\right)^2 + \left(\frac{y_B - y_A}{2}\right)^2} \tag{130}$$

Let $F_3(x,y) = \sin(\gamma) F_2(x,y) - \cos(\gamma) F_1(x,y) = 0$ such that

$$\frac{F_1(x,y)}{F_2(x,y)} = \frac{\sin(\gamma)}{\cos(\gamma)} = \frac{\left|\overrightarrow{AX} \times \overrightarrow{BX}\right|}{\overrightarrow{AX} \cdot \overrightarrow{BX}} \tag{131}$$

This represents a curve, for which the angle between $\overrightarrow{AX}$ and $\overrightarrow{BX}$ is γ. The geometric locus of such points is a circle passing through *A* and *B*, and for any point *C* on the circle, $\angle ACB = \gamma$. *C* are on the same side of line *AB* as the center O_3 of the circle for $\gamma < 90°$, but for $\gamma > 90°$, *C* and O_3 are on opposite sides of *AB* (Figure 11). For $\alpha = 90°$, $F_3(x, y) = 0$ coincides with $F_2(x, y) = 0$. Also, $\angle AO_3B = 2\gamma$ and $\angle AO_3M = \gamma$. Triangle AO_3M is a right triangle. Radius of $F_3(x, y) = 0$ is $r_3 = |AO_3| = r_2/\sin\gamma$.

Let $F_4(x,y) = \cos(\gamma) F_2(x,y) + \sin(\gamma) F_1(x,y) = 0$ such that

$$\frac{F_1(x,y)}{F_2(x,y)} = -\frac{\cos(\gamma)}{\sin(\gamma)} = \frac{\sin(\gamma + \pi/2)}{\cos(\gamma + \pi/2)} = \frac{\left|\overrightarrow{AX} \times \overrightarrow{BX}\right|}{\overrightarrow{AX} \cdot \overrightarrow{BX}} \tag{132}$$

$F_4(x, y) = 0$ is a circle passing through *A* and *B*, and for any point *E* on the circle, $\angle AEB = \delta = \pi - \gamma - \pi/2 = \pi/2 - \gamma$. For the center O_4 of the circle, $\angle AO_4B = 2\delta = \pi - 2\gamma$ and $\angle AO_4M = \pi/2 - \alpha$. Radius of $F_4(x, y) = 0$ is $r_4 = |AO_4| = r_2/\cos\gamma$. Also, $\angle O_3AO_4 = \pi/2$.

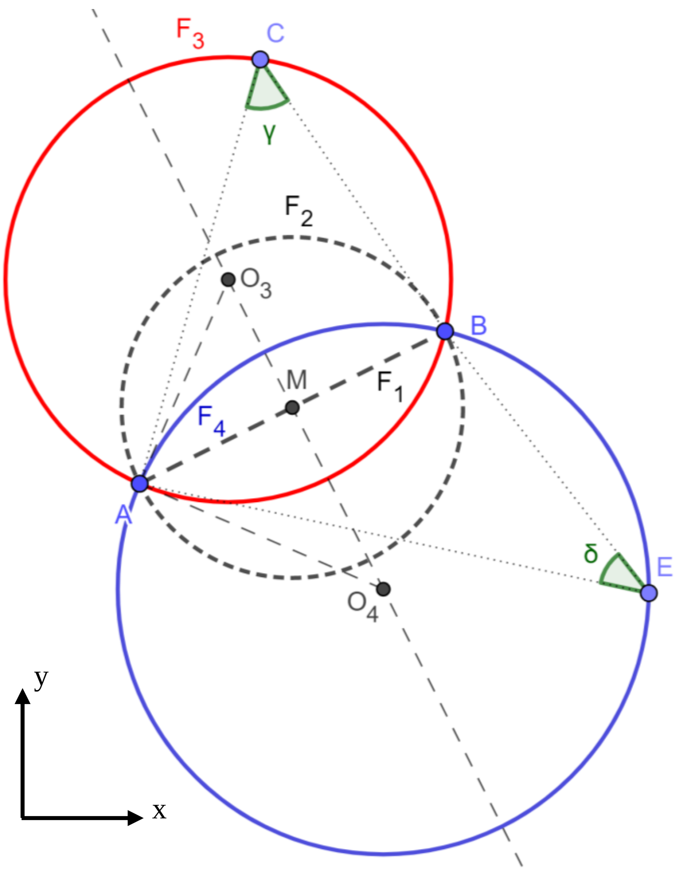

Figure 11. Circles subtending a line segment *AB* with angles γ and $\delta = \pi/2 - \gamma$

If the points *A* and *B* are the fixed pivots of a 4-bar mechanism, $F_3(x, y) = 0$ is the circle of singular foci, whereas $F_4(x, y) = 0$ is a circle, which is somehow related to the circle of singular foci.

CHAPTER 4

PATH GENERATION ALGORITHMS FOR THE SLIDER-CRANK MECHANISM

This Chapter presents two set of formulations for design. The first one is based on a design methodology for given 3 points and either the fixed pivot location, or velocity direction at two of the points. The slider translation direction is assumed to be known or selected in this method. The second method is a numerical method which makes use of the first method as initial guess. In this method, also the fixed pivot location and slider translation direction are assumed to be known or selected. The second method can be employed for 5-point precision synthesis or a least square approximation problem for more than 5 points.

4.1. Synthesis Method Based on Geometric Invariants of the Coupler-Curve

Coupler-curve equation has two tangent points to the inner and outer tangent circles C_1 and C_2. These two points, say (x_1, y_1) and (x_2, y_2), are where the linear combination of slider axes, $aL_1 \pm bL_2$ is also coincident. Therefore, the closest and farthest points to the origin may be taken as two special points. This method makes use of these two special points.

The design problem can be defined by three design points, given fixed pivot location and slider translation direction. The rest of the parameters can be computed. If the closest and furthest of three points are assumed to be the two special points that are the closest and the furthest of the curve to the fixed pivot, then the distances of the two special points to the fixed pivot give smaller and larger radii, r_1 and r_2. Hence the design parameters a and d can be found. This type design method is Blechschmidt and Uicker's (1986) method. Instead of selecting some of the points to be nodes as in Blechschmidt and Uicker (1986), here some points are selected as closest and furthest points to a fixed pivot.

Another approach to define the problem is defining the coupler-point path itself. Given the desired curve, one may select three design points, two of which are assumed to be the closest and the furthest points to the fixed pivot. The fixed pivot location is determined as the intersection of two lines perpendicular to the tangents to the curve at the two special points, because the two circles in Figure 6 are tangent to the coupler-curve at these points. The tangents to the curve define the velocity vector orientations at special points. Still, the orientation of the slider axes is to be chosen.

Then, these two tangent points must also satisfy the associated combination of the slider axes, $aL_1 \pm bL_2$ (see Figure 6). Having determined a and d, two more equations for the two special points can be formed in three unknowns.

$$\begin{aligned} a\left(y_1 - h\right) + bs_\alpha x_1 - bc_\alpha y_1 + dh = 0 \rightarrow \left(s_\alpha x_1 - c_\alpha y_1\right) b + r_1 h = -ay_1 \\ a\left(y_2 - h\right) - bs_\alpha x_2 + bc_\alpha y_2 - dh = 0 \rightarrow \left(s_\alpha x_2 - c_\alpha y_2\right) b + r_2 h = ay_2 \end{aligned} \tag{133}$$

b and h can be solved linearly in terms of the determined tangent point coordinates (x_1, y_1) and (x_2, y_2), the circle radii, (r_1 and r_2 determined from these coordinates) and design parameters a and d:

$$\begin{bmatrix} s_\alpha x_1 - c_\alpha y_1 & r_1 \\ s_\alpha x_2 - c_\alpha y_2 & r_2 \end{bmatrix} \begin{bmatrix} b \\ h \end{bmatrix} = \begin{bmatrix} -ay_1 \\ ay_2 \end{bmatrix} \rightarrow \Delta = \begin{vmatrix} s_\alpha x_1 - c_\alpha y_1 & r_1 \\ s_\alpha x_2 - c_\alpha y_2 & r_2 \end{vmatrix} \tag{134}$$

b and h can be solved using Cramer's rule:

$$\begin{aligned} b &= \frac{\begin{vmatrix} ay_2 & r_2 \\ -ay_1 & r_1 \end{vmatrix}}{\Delta} = \frac{a\left(r_1 y_2 + r_2 y_1\right)}{s_\alpha V_s - c_\alpha V_c} \\ h &= \frac{\begin{vmatrix} s_\alpha x_2 - c_\alpha y_2 & ay_2 \\ s_\alpha x_1 - c_\alpha y_1 & -ay_1 \end{vmatrix}}{\Delta} = \frac{a\left(-s_\alpha V_{xy} + 2c_\alpha y_1 y_2\right)}{s_\alpha V_s - c_\alpha V_c} \end{aligned} \tag{135}$$

where $V_c = y_2 r_1 - y_1 r_2$, $V_s = x_2 r_1 - x_1 r_2$ and $V_{xy} = x_1 y_2 + x_2 y_1$.

Rewriting slider axes in terms of known parameters and the only unknown α:

$$L_1 = s_\alpha x - c_\alpha y + \frac{d\left(-s_\alpha V_{xy} + 2c_\alpha y_1 y_2\right)}{r_1 y_2 + r_2 y_1}$$
$$L_2 = y + \frac{a\left(s_\alpha V_{xy} - 2c_\alpha y_1 y_2\right)}{s_\alpha V_s - c_\alpha V_c} \quad (136)$$

Finally, the coupler-curve equation is obtained with *b* and *h* eliminated, *α* is the only unknown and all other parameters are known. This form is a little complicated and too long to be expressed here. The only unknown *α* can be determined by picking an arbitrary point on the given curve. When tangent of the half angle substitution is used, the equation has degree 4, which can be analytically solved. Hence, there are 4 solutions for *α*, but two of them are separated by 180° to the other two. Having determined the angle *α*, and the mechanism is obtained. Figure 12 presents the flow diagram of this method.

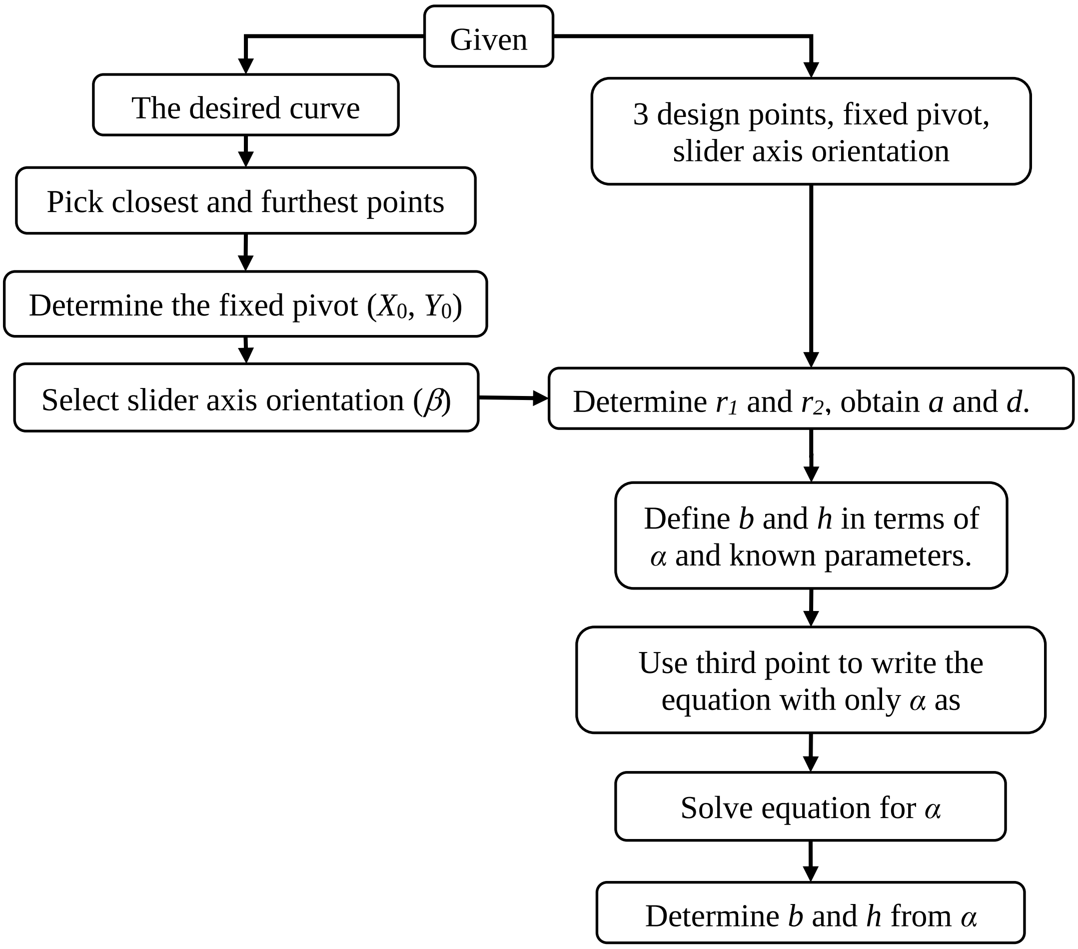

Figure 12. Flow diagram for geometric invariant-based method for slider crank mechanism

The method is numerically applied to an existing mechanism with link lengths a = 10, b = 18, d = 12, h = 7 units and α = 30°. The closest and furthest points on the coupler-curve from origin (where the fixed pivot of the mechanism is originally located) are measured approximately as (–0.3, 2.0) and (18.2, 12.4) respectively. The distance of the two points to the origin are 2.022 and 22.023, respectively. The link lengths a and d are calculated as 12.023 and 10 units, but which one is smaller or larger cannot be determined at this point. Intermediate variables are calculated as V_s = 58.85, V_c = –7.98 and V_{xy} = 32.68. Depending on whether a or d is the larger one, there are two different set of roots for α. In total, four set of parameters are calculated for four α values. The third point on the coupler-curve is required for further calculations. A random point is measured as (12.62, 6.51). One set of result for $d > a$ case is b = 18.3, h = 4.84 and α = 35.95° (Approximation #1). The other set is b = 12.7, h = –3.25 and α = 76.59° (Approximation #2). One set of results for the $a > d$ case is b = 16.55, h = –4.25 and α = 76.65° (Approximation #3). The other set is b = 19.77, h = –12.7 and α = 115.88° (Approximation #4). Resultant coupler-curves are given in Figure 13 together with the coupler-curve from which the measurements are taken. Approximation #1 appears to be the closest solution.

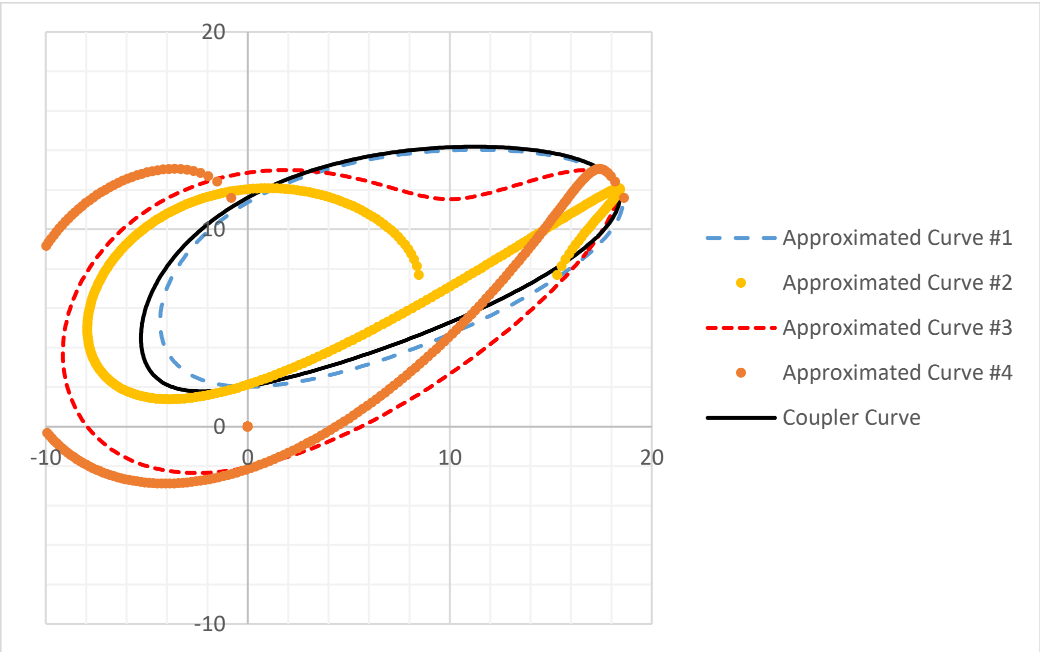

Figure 13. Graph based approximation method example

As r_1 may be $d - a$ or $a - d$ depending on which link is longer, there are two possible expressions for final step where the angle α is solved. As the equation of the coupler-curve is non-linear in terms of α, multiple number of solutions are expected. Numerical examples typically result in 4 roots, 2 of which give the same mechanism with the other two as an angle α increased by π and related link lengths are multiplied with –1. The final step proceeds using bisection method as the upper and lower limits of angle α is 0 to 2π. However, during application the α value is scanned for the whole range to determine the approximate location of the roots and bisection search is focused on these limited zones.

The generated curves must pass through two special points and the third point. The generated curves satisfy the three design (or precision in this case) points either in open configuration or the cross-configuration of the mechanism. The cross-configuration curves of the numerical example solution are not drawn in Figure 13 to keep the graph clearer. The coupler curve equation defines curves of both configurations but the given design points being satisfied by one configuration only is never guaranteed. This is called branching problem.

4.2. 5-Parameter Design Problem

The 5-parameter coupler-curve equation of slider-crank linkage is expressed in equation (57) in Chapter 2. This section explains the complexity of the problem. Solution of an equation with 5 independent parameters/coefficients means looking for solutions in a 5-dimensional space.

4.2.1. Five Coefficients of the Polynomial Form

Equation (57) is problematic because of the denominators. The function becomes undefined if the denominator of any term in the equation approaches zero. In addition, partial derivatives of rational expressions in the coupler-curve equation are required in root finding, which is harder to obtain and hence prone to errors with terms in the denominator. Multiplying equation (57) by B^2D^2 to get rid of the denominators:

$$
\begin{aligned}
\mathit{\Gamma} = {} & B^2D^2\left(B^2+(C-1)^2\right)\left(x^2+y^2\right)^2 \\
& +4B^3D^2\left(x^2+y^2\right)xy+4B^2CD^2\left(x^2+y^2\right)y^2 \\
& +4B^2D^3\left(x^2+y^2\right)x+4BD^3\left(B^2+C^2+C\right)\left(x^2+y^2\right)y \\
& +D\left\{\begin{matrix} D\left(B^2+C^2\right)\left[4D^2-G\left(B^2+(C-1)^2\right)\right] \\ +2B^2\left(B^2+(C-1)^2\right)J \end{matrix}\right\}\left(x^2+y^2\right) \\
& +B^2D^2Gx^2+BD\left(4B^2J-2DG\left(B^2+C^2-C\right)\right)xy \\
& +D\left(DG\left(B^2+C^2-C\right)^2-4B^2J\left(B^2+C^2-C\right)\right)y^2 \\
& +4B^2D^2Jx-4BD^2J\left(B^2+C^2-C\right)y+B^2\left(B^2+(C-1)^2\right)J^2=0
\end{aligned}
\tag{137}
$$

This equation is expected to be useful in root finding formulation, as none of the independent variables are in any trigonometric function. The required formulas for five precision points are given below:

$$
\begin{aligned}
S &= \sum_{i=1}^{5}\mathit{\Gamma}^2\left(x_i,y_i\right) \\
\frac{\partial S}{\partial B} &= 2\sum_{i=1}^{5}\left[\frac{\partial \mathit{\Gamma}}{\partial B}\mathit{\Gamma}\left(x_i,y_i\right)\right]=0 \\
\frac{\partial S}{\partial C} &= 2\sum_{i=1}^{5}\left[\frac{\partial \mathit{\Gamma}}{\partial C}\mathit{\Gamma}\left(x_i,y_i\right)\right]=0 \\
\frac{\partial S}{\partial D} &= 2\sum_{i=1}^{5}\left[\frac{\partial \mathit{\Gamma}}{\partial D}\mathit{\Gamma}\left(x_i,y_i\right)\right]=0 \\
\frac{\partial S}{\partial G} &= 2\sum_{i=1}^{5}\left[\frac{\partial \mathit{\Gamma}}{\partial G}\mathit{\Gamma}\left(x_i,y_i\right)\right]=0 \\
\frac{\partial S}{\partial J} &= 2\sum_{i=1}^{5}\left[\frac{\partial \mathit{\Gamma}}{\partial J}\mathit{\Gamma}\left(x_i,y_i\right)\right]=0
\end{aligned}
\tag{138}
$$

Numerical tests with this approach always resulted in trivial solutions as all terms in the equation has either B or D as a multiplier, therefore a modified approach is generated and presented in the forthcoming sub-sections.

4.2.2. Design for 5 Precision Points

This section establishes a method for path generation problems where the task is given to match 5 precision points. The maximum number of precision points is limited to the number of design parameters. This number is maximum 8 for the slider-crank linkage. As mentioned earlier, three of these parameters may be considered as coordinate transformation parameters and can be considered separately. This separation divides the design methodology study into two sections: design for 5 parameters with all coordinate transformation parameters set to zero and for 8 parameters as the most general case. The most general case is left as a future study.

The method presented in Section 4.2.1 has divergence/trivial solution problems. In this section, instead of selecting the free design parameters among the polynomial coefficients, some Euclidean invariants are selected as independent parameters. The coefficients of the equation (53) are composed of 5 design parameters: g, h, r_1, r_2 and α. By taking the partial derivatives of the equations with respect to these 5 parameters:

$$\begin{gathered}
S=\sum_{i=1}^{5}\Gamma^2\left(x_i,y_i\right)\\
\frac{\partial S}{\partial g}=2\sum_{i=1}^{5}\left[\frac{\partial \Gamma}{\partial g}\Gamma\left(x_i,y_i\right)\right]=0\\
\frac{\partial S}{\partial h}=2\sum_{i=1}^{5}\left[\frac{\partial \Gamma}{\partial h}\Gamma\left(x_i,y_i\right)\right]=0\\
\frac{\partial S}{\partial r_1}=2\sum_{i=1}^{5}\left[\frac{\partial \Gamma}{\partial r_1}\Gamma\left(x_i,y_i\right)\right]=0\\
\frac{\partial S}{\partial r_2}=2\sum_{i=1}^{5}\left[\frac{\partial \Gamma}{\partial r_2}\Gamma\left(x_i,y_i\right)\right]=0\\
\frac{\partial S}{\partial \alpha}=2\sum_{i=1}^{5}\left[\frac{\partial \Gamma}{\partial \alpha}\Gamma\left(x_i,y_i\right)\right]=0
\end{gathered} \tag{139}$$

Equation (53) is rewritten in terms of design parameters as

$$
\begin{aligned}
\Gamma &= \left(1+g^2-2g\mathrm{c}\alpha\right)\left(x^2+y^2\right)^2 - 4g\mathrm{s}\alpha\left(x^2+y^2\right)xy \\
&+4g\mathrm{c}\alpha\left(x^2+y^2\right)y^2 + 4gh\mathrm{s}\alpha\left(x^2+y^2\right)x \\
&-4gh\left(g+\mathrm{c}\alpha\right)\left(x^2+y^2\right)y + \left[4g^2h^2 - \left(1+g^2-2g\mathrm{c}\alpha\right)\left(r_1^2+r_2^2\right)\right]\left(x^2+y^2\right) \\
&+\left[\left(r_1+r_2\right)\mathrm{s}\alpha\right]^2 x^2 + 2\mathrm{s}\alpha\left[\left(r_1+r_2\right)^2\left(g-\mathrm{c}\alpha\right) - 2gr_1r_2\right]xy \\
&+\left[\left(r_1+r_2\right)^2\left(g-\mathrm{c}\alpha\right)^2 - 4r_1r_2\left(g^2-g\mathrm{c}\alpha\right)\right]y^2 \\
&+4ghr_1r_2\mathrm{s}\alpha x + 4ghr_1r_2\left(g-\mathrm{c}\alpha\right)y + \left(1+g^2-2g\mathrm{c}\alpha\right)r_1^2r_2^2 = 0
\end{aligned}
\tag{140}
$$

Taking derivative with respect to *g*:

$$
\begin{aligned}
\frac{\partial\Gamma}{\partial g} &= 2\left(g-\mathrm{c}\alpha\right)\left(x^2+y^2\right)^2 - 4\mathrm{s}\alpha\left(x^2+y^2\right)xy + 4\mathrm{c}\alpha\left(x^2+y^2\right)y^2 \\
&+4h\mathrm{s}\alpha\left(x^2+y^2\right)x - 4h\left(2g+\mathrm{c}\alpha\right)\left(x^2+y^2\right)y \\
&+2\left[4gh^2 - \left(g-\mathrm{c}\alpha\right)\left(r_1^2+r_2^2\right)\right]\left(x^2+y^2\right) + 2\mathrm{s}\alpha\left[\left(r_1+r_2\right)^2 - 2r_1r_2\right]xy \\
&+\left[2\left(r_1+r_2\right)^2\left(g-\mathrm{c}\alpha\right) - 4r_1r_2\left(2g-\mathrm{c}\alpha\right)\right]y^2 \\
&+4hr_1r_2\mathrm{s}\alpha x + 4hr_1r_2\left(2g-\mathrm{c}\alpha\right)y + \left(2g-\mathrm{c}\alpha\right)r_1^2r_2^2
\end{aligned}
\tag{141}
$$

Derivative with respect to *h*:

$$
\begin{aligned}
\frac{\partial\Gamma}{\partial h} &= 4g\mathrm{s}\alpha\left(x^2+y^2\right)x - 4g\left(g+\mathrm{c}\alpha\right)\left(x^2+y^2\right)y + 8g^2h\left(x^2+y^2\right) \\
&+4gr_1r_2\mathrm{s}\alpha x + 4gr_1r_2\left(g-\mathrm{c}\alpha\right)y
\end{aligned}
\tag{142}
$$

Derivative with respect to r_1:

$$
\begin{aligned}
\frac{\partial\Gamma}{\partial r_1} &= -2r_1\left(1+g^2-2g\mathrm{c}\alpha\right)\left(x^2+y^2\right) + 2\left(r_1+r_2\right)\mathrm{s}^2\alpha x^2 \\
&+4\mathrm{s}\alpha\left[r_1\left(g-\mathrm{c}\alpha\right) - r_2\mathrm{c}\alpha\right]xy + 2\left[\left(r_1+r_2\right)\left(g-\mathrm{c}\alpha\right)^2 - 2r_2g\left(g-\mathrm{c}\alpha\right)\right]y^2 \\
&+4ghr_2\mathrm{s}\alpha x + 4gh_1r_2\left(g-\mathrm{c}\alpha\right)y + 2\left(1+g^2-2g\mathrm{c}\alpha\right)r_1r_2^2
\end{aligned}
\tag{143}
$$

Derivative with respect to r_2:

$$\frac{\partial \Gamma}{\partial r_2} = -2r_2\left(1+g^2-2gc\alpha\right)\left(x^2+y^2\right)+2\left(r_1+r_2\right)s^2\alpha x^2$$
$$+4s\alpha\left[r_2\left(g-c\alpha\right)-r_1c\alpha\right]xy+2\left[\left(r_1+r_2\right)\left(g-c\alpha\right)^2-2r_1\left(g^2-gc\alpha\right)\right]y^2 \quad (144)$$
$$+4ghr_1s\alpha x+4ghr_1\left(g-c\alpha\right)y+2\left(1+g^2-2gc\alpha\right)r_1^2r_2$$

Derivative with respect to α:

$$\frac{\partial \Gamma}{\partial \alpha} = 2gs\alpha\left(x^2+y^2\right)^2-4gc\alpha\left(x^2+y^2\right)xy-4gs\alpha\left(x^2+y^2\right)y^2$$
$$+4ghc\alpha\left(x^2+y^2\right)x+4ghs\alpha\left(x^2+y^2\right)y+-2gs\alpha\left(r_1^2+r_2^2\right)\left(x^2+y^2\right)$$
$$+2s\alpha c\alpha\left(r_1+r_2\right)^2x^2+2\left[\left(r_1^2+r_2^2\right)gc\alpha+\left(s^2\alpha-c^2\alpha\right)\left(r_1+r_2\right)^2\right]xy \quad (145)$$
$$+2s\alpha\left[\left(r_1+r_2\right)^2\left(g-c\alpha\right)-2r_1r_2g\right]y^2+4ghr_1r_2c\alpha x+4ghr_1r_2s\alpha y+2r_1^2r_2^2gs\alpha$$

The derivative expressions form a Jacobian matrix J when applied to five precision points. Formulations for the Newton-Raphson root finding method is derived as:

$$J\left(\bar{x}_{k+1}-\bar{x}_k\right)=-\Gamma\left(\bar{x}_k\right) \quad \Rightarrow \quad \bar{x}_{k+1}=\bar{x}_k-J^{-1}\Gamma\left(\bar{x}_k\right) \qquad k=0,\ 1,\ \ldots \qquad (146)$$

where $\bar{x}_k=\left[g_k \quad h_k \quad r_{1,k} \quad r_{2,k} \quad \alpha_k\right]^T$ is the array of design parameters at the k^{th} iteration step. Now, a proper set of initial values is needed. The root is sought in a 5-dimensional space. The geometric invariants-based method presented in Section 4.1 is helpful at this point. By selecting the closest and furthest points to the fixed pivot among the given 5 points, and assuming that they will be the closest and furthest points on the coupler-path, an initial set of values for the tangent circle radii, r_1 and r_2, can be determined. Then, b and h are derived in terms of angle α. Finally, having three more candidates (a total of five points) for third point, each point gives four solutions for angle α hence four sets of solution for five design parameters is obtained for each candidate of third point. For five precision points, 12 sets of initial values are tested, and some converge to a solution after applying the iterations in equation (146). Below is a numerical example to demonstrate the method. Five points are selected randomly at the beginning.

Table 1. 5 Precision points for numerical problem

i	x_i	y_i	$\sqrt{x_i^2 + y_i^2}$
1	5	3	5.830952
2	0	7	7
3	3	12	12.36932
4	10	12	15.6205
5	13	6.5	14.53444

The closest point to the origin is point #1 and furthest is point #4. The other three precision points are used to generate 12 sets of initial values for the problem (Table 2).

Table 2. Set of initial values for 5 precision point example

Initial value set	Design point #	Initial Values				
		g_0	h_0	a_0	d_0	α_0
1	2	-0.03867	119.3731	10.725726	4.894774	111.592
2	2	-0.12671	81.99904	4.8947737	10.72573	117.7511
3	2	-0.03867	119.3731	10.725726	4.894774	111.592
4	2	-0.12671	81.99904	4.8947737	10.72573	117.7511
5	3	-0.09414	-29.9345	10.725726	4.894774	2.957009
6	3	0.08032	-129.73	4.8947737	10.72573	138.6699
7	3	-0.09414	-29.9345	10.725726	4.894774	2.957009
8	3	0.08032	-129.73	4.8947737	10.72573	138.6699
9	5	-0.11845	4.694095	10.725726	4.894774	45.27215
10	5	0.089177	-116.507	4.8947737	10.72573	139.5692
11	5	-0.11845	4.694095	10.725726	4.894774	45.27215
12	5	0.089177	-116.507	4.8947737	10.72573	139.5692

The iteration results are listed in Table 3 in the same order with Table 2.

Table 3. Convergence results of 5 precision point example

Initial value set	Design point #	Converge?	Final Values				
			g	*h*	r_1	r_2	α
1	2	×	#NUMERR	-126.738	1.3E+10	1.06E+10	20178.41
2	2	3.691E-27	-2.04706	0.644698	15.66279	3.328551	-24900.3
3	2	×	#NUMERR	-126.738	1.3E+10	1.06E+10	20178.41
4	2	1.721E-27	-2.04706	0.644698	15.66279	3.328551	-24900.3
5	3	1.589E-22	2.047057	0.644698	-3.32855	-15.6628	12690840
6	3	1.106E-22	0.058094	-180.745	5.756558	15.65056	142.9712
7	3	1.589E-22	2.047057	0.644698	-3.32855	-15.6628	12690840
8	3	1.106E-22	0.058094	-180.745	5.756558	15.65056	142.9712
9	5	×	28.97718	210.412	-6.5E+09	4.41E+08	-1320.2
10	5	1.251E-25	0.058094	-180.745	5.756558	15.65056	142.9712
11	5	×	28.97718	210.412	-6.5E+09	4.41E+08	-1320.2
12	5	1.251E-25	0.058094	-180.745	5.756558	15.65056	142.9712

Table 4. Link Length Parameter Results for 5 precision points example

Initial value set	Design point #	Convergence?	*a*	*b*	*d*	*h*	α
1	2	#NUMERR	1.2E+09	#NUMERR	1.18E+10	-126.738	18.4096
2	2	3.691E-27	6.16712	19.43819	9.495672	0.644698	119.6769
3	2	#NUMERR	1.2E+09	#NUMERR	1.18E+10	-126.738	18.4096
4	2	1.721E-27	6.16712	19.43819	9.495672	0.644698	119.6769
5	3	1.589E-22	6.16712	-19.4382	-9.49567	0.644698	119.6769
6	3	1.106E-22	4.946999	0.621816	10.70356	-180.745	142.9712
7	3	1.589E-22	6.16712	-19.4382	-9.49567	0.644698	119.6769
8	3	1.106E-22	4.946999	0.621816	10.70356	-180.745	142.9712
9	5	#NUMERR	3.48E+09	-8.8E+10	-3E+09	210.412	119.7985
10	5	1.251E-25	4.946999	0.621816	10.70356	-180.745	142.9712
11	5	#NUMERR	3.48E+09	-8.8E+10	-3E+09	210.412	119.7985
12	5	1.251E-25	4.946999	0.621816	10.70356	-180.745	142.9712

The converge column in Table 3 is the result of the sum of the squares of coupler-curve equation at precision points with final values of design parameters. Although a total of 8 convergence results appears in Table 3, some set of parameters are practically the same. Initial value set #2, #4, #5 and #7 resulted in the same mechanism. Likewise, #6, #8, #10 and #12 are the same. Therefore, only two distinct mechanisms are obtained for solution.

4.2.3. Least Squares Approximation for 5 Parameter Design

The root finding formulation in Section 4.2.2 may also be used for least squares approximation for n many design points, n being larger than the number of design parameters, 5.

The methodology of least squares approximation is formulated as:

$$J\left(\bar{x}_{k+1}-\bar{x}_k\right)=-\Gamma\left(\bar{x}_k\right) \quad \Rightarrow \quad \bar{x}_{k+1}=\bar{x}_k-\left[\left(J^T J\right)^{-1} J^T\right]\Gamma\left(\bar{x}_k\right) \quad k=0,\,1,\,\ldots \qquad (147)$$

The sample mechanism in Section 4.1 is used once more to demonstrate the method. Recall that the mechanism link lengths are a = 10, b = 18, d = 12, h = 7 units and α = 30°. The design points are picked on the coupler-curve based on crank angle (θ in Figure 7). 8 design points are listed in Table 5. The design points are also shown graphically in Figure 14.

Table 5. List of design points

θ	x_i	y_i	$\sqrt{x_i^2+y_i^2}$
45	17.48698	13.02999	21.80769
90	11.24695	14.18403	18.10195
135	3.344845	13.02999	13.45246
180	-2.75906	9.56916	9.958978
225	-5.28055	4.794599	7.132491
270	-2.25102	1.786981	2.874087
315	8.861582	4.794599	10.07551
360	17.24094	9.56916	19.71849

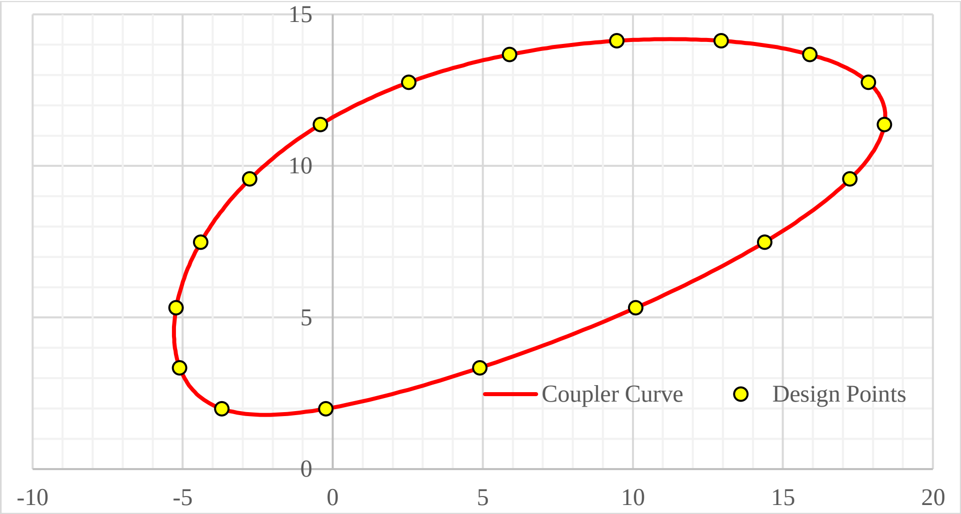

Figure 14. Coupler-curve of sample mechanism and design points

Next step is to determine the points with minimum and maximum distances to the origin, where pivot point of the crank is assumed to be located. The last column is for the distances of points to the origin, called *r*. Minimum and maximum distance points correspond to 270° and 45° crank angles, respectively. The distances of these points are approximate values for r_1 and r_2 parameters. Therefore, approximate values for *a* and *d* are calculated as in graph-based method in Section 3.2. Intermediate variables are calculated as V_s = 99.34856, V_c = -1.5206 and V_{xy} = 1.918184. Both $a > d$ and $a < d$ cases offer two roots for the coupler-curve equation; hence 4 set of parameters are to be calculated. The calculation may be done by picking any of the remaining 6 design points. Each design point results in a different set of variables. Results are presented in Table 6. The table shows *a* and *d* parameters separately. The two parameters are intentionally left in Table 6 to show how other parameters are obtained. Iterations use r_1 and r_2 parameters.

Least square approximation is applied 24 times with initial set of values given in Table 6. Convergence of the trials are listed in the last column. 4 set of initial values are set for each design point and at least one set of values from each design points converged to the original mechanism. A total of 10 set of initial values led to convergence in this case study. The link lengths/angles are listed in Table 7.

Note that the solution always approximates to one of the cognates not the other. The reason behind this is that the coupler curve equation is formed of the design parameters of that mechanism. When the coupler-curve equation is rewritten in terms of link lengths of the other cognate, the solution will always converge to that cognate.

Table 6. Set of parameters to start with the iterations

#	θ	g_0	h_0	a_0	d_0	α_0	Convergence
1	90	0.80091	8.16473	9.46680	12.34089	27.32186	Yes
2	90	1.58376	-1.80711	9.46680	12.34089	109.99036	No
3	90	0.76051	4.76070	12.34089	9.46680	48.80758	No
4	90	0.58043	-8.23646	12.34089	9.46680	143.53524	No
5	135	1.20049	4.30968	9.46680	12.34089	44.21821	Yes
6	135	0.52663	-13.69507	9.46680	12.34089	161.02112	No
7	135	0.88492	2.85928	12.34089	9.46680	61.65058	No
8	135	0.03310	-174.41606	12.34089	9.46680	177.22116	No
9	180	0.22003	33.79133	9.46680	12.34089	6.58225	No
10	180	0.60815	11.43361	9.46680	12.34089	20.14991	Yes
11	180	0.31914	16.98514	12.34089	9.46680	17.78453	Yes
12	180	0.40144	-13.31160	12.34089	9.46680	155.38920	No
13	225	0.59504	11.72515	9.46680	12.34089	19.67592	Yes
14	225	0.63097	10.95295	9.46680	12.34089	20.97858	Yes
15	225	0.09055	63.33089	12.34089	9.46680	4.33187	No
16	225	0.59421	7.65187	12.34089	9.46680	35.69000	Yes
17	315	0.50265	14.18097	9.46680	12.34089	16.37408	Yes
18	315	1.60582	1.38450	9.46680	12.34089	70.46085	No
19	315	0.90634	2.50864	12.34089	9.46680	64.45064	No
20	315	0.93011	2.09086	12.34089	9.46680	67.95801	No
21	360	0.47516	15.08544	9.46680	12.34089	15.40353	Yes
22	360	1.63382	1.11066	9.46680	12.34089	73.69044	No
23	360	0.58979	7.74229	12.34089	9.46680	35.37460	Yes
24	360	0.99647	0.09931	12.34089	9.46680	86.65923	No

Table 7. Link lengths/angles for least squares approximation solution for 5 parameters.

#	g	h	r_1	r_2	α
1	0.666667	7	-22	-2	30°
2	0.666667	7	-22	-2	30°
3	0.666667	7	2	22	30°
4	0.666667	7	2	22	30°
5	Diverged				
6	0.666667	7	2	22	30°
7	0.666667	7	-22	-2	30°
8	0.666667	7	2	22	30°
9	0.666667	7	2	22	30°
10	0.666667	7	2	22	30°
11	Diverged				
12	0.666667	7	2	22	30°
13	Diverged				
14	0.666667	7	2	22	30°
15	0.666667	7	2	22	30°
16	Diverged				
17	0.666667	7	22	2	30°
18	0.666667	7	2	22	30°
19	Diverged				
20	0.666667	7	2	22	30°
21	-0.66667	7	2	22	210°
22	Diverged				
23	Diverged				
24	0.666667	7	2	22	30°

CHAPTER 5

CONCLUSIONS

This thesis investigates the geometric properties of planar slider-crank and 4-bar mechanisms. Also, methodologies for path generation problem using the coupler-curve geometric properties are introduced.

The slider-crank coupler-curve equation is quartic and is shown to be composed of two circle and two-line components. The two circles are limiting the area where the coupler-curve may be located. The two lines are slider axes of the slider-crank mechanism and the cognate mechanism. The coupler-curve equation is expressed with two terms. The first term is one of the circles multiplied with an ellipse that is a linear combination of the other circle and multiplication of two slider axes. The second term of the coupler-curve equation is square of linear combination of slider axes. This provides a novel representation of the slider-crank mechanism coupler curve in the form of Ellipse × Circle + Line^2 which results in a set of independent Euclidean invariants of the coupler-curve.

The two circles in the slider-crank coupler-curve equation are guaranteed to have a tangent point each with the coupler-curve when the crank link can make a full rotation. That specific point on the coupler-curve satisfies both the coupler-curve equation and the circle equation. This leads to the second term in the equation to be equal to zero. The closest and furthest points on the coupler-curve meeting certain conditions inspired the path generation method introduced in Chapter 4. The method can be used to design a slider-crank mechanism for 5 precision points on the coupler-curve or more design points for an approximate design.

The slider-crank synthesis methods are demonstrated with numerical examples. For the first case study, a desired coupler curve is given, a fixed pivot location and slider direction is chosen, and closest and furthest points of the curve to the fixed pivot are determined. A third point on the curve is also used to determine all link lengths, hence a solution for three precision points is obtained. Then the method for 5 precision points is demonstrated with randomly chosen 5 points. This example results in two distinct solutions. Finally, least square approximation is applied for 8 design points as a case study. The design points are not random, but taken from an existing slider-crank mechanism, the solution converged to the original mechanism from several sets of initial values.

A future study for this approach might be keeping the fixed pivot location and slider orientation as unknown and design for all 8 constructional parameters of a slider-crank mechanism. In practice, the fixed pivot may be located in a limited area. The slider orientation can be any angle between 0° and 180°. Therefore, an algorithm may be formed that the area for possible location of fixed pivot is divided into a mesh and for each point on the mesh, the slider orientation parameter might have a limited number of values and the rest of the parameters are determined through the method introduced. This results in several mechanisms and hence coupler curves, and the one that gives best solution depending on design criteria can be chosen.

The 4-bar mechanism coupler-curve equation is a sextic and shown to be composed of circle and line expressions. The forms that already exist in the literature for more than a century are taken further and geometric components are obtained. These geometric components can be used to define some Euclidean invariant of the curve. A 4-bar mechanism has three cognates, and three coupler-curve equations can be written for each cognate. The coupler-curve equation is obtained from all three cognates in terms of limiting circles, similar to the slider-crank case and additional special circle and lines formed by fixed pivot locations.

The discovered properties of the coupler curve of the 4-bar mechanism may be helpful in a synthesis method similar to the ones introduced for the slider-crank mechanism. If the fixed pivot locations are selected, the radii of the tangent circles, hence 4 link lengths might be determined. Finally, an additional point would give the inner angle of the coupler link. The detail of this method is left as a future study.

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VITA

Talha Eraz has graduated from the bachelor program of Middle East Technical University Mechanical Engineering Department in September 2012. He started master's in mechanical engineering in İzmir Institute of Technology in 2013. During his master's study, he started in the position of research assistant in İzmir Institute of Technology Mechanical Engineering Department by January 2015. Then he attended RWTH-Aachen University Institute of Mechanism Theory and Dynamics of Machines for summer semester of 2015 under ERASMUS program and continued his master's thesis study under supervision of Burkhard Corves. He completed his master's thesis titled "Modelling and Experimental Setup of a Cable Driven System" under supervision of Dr. Gökhan Kiper in 2016. He presented the outcome of his studies for the master's thesis in EUCOMES 2016 with a conference paper titled "Design of a Cable-Driven Four-Bar Mechanism for Arm Rehabilitation". He started Ph.D. program in the same department in September 2016. He completed his Ph.D. study in December 2022 with a thesis titled "Geometric Properties of Coupler-Curve Equation of Planar Slider-Crank and Four-Bar Linkages". The outcome of his thesis study is partially presented in a journal article titled "Novel Representations for Coupler-Curve Equations of Planar Slider-Crank and 4-Bar Linkages". The article is not yet accepted for publication by the time this thesis is submitted to the İzmir Institute of Technology.