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SOME REMARKS ON HARMONIC TYPE MATRICES

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Abstract

In 1915, Theisinger proved that all harmonic numbers are not integers except for the first one. In 1862, Wolstenholme proved that the numerator of the reduced form of the harmonic number H_{p-1} is divisible by p^2 and the numerator of the reduced form of the generalized harmonic number $H_{p-1}^{(2)}$ is divisible by p for all primes $p \geq 5$. In this note, we define harmonic type matrices and our goal is to extend Theisinger's and Wolstenholme's results to harmonic type matrices.

1. Introduction

The *n-th harmonic number*, denoted by H_n, is defined as the partial sum of the harmonic series

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

The integrality and divisibility properties of these numbers have been studied widely. For instance, in 1915 Theisinger [9] proved that all harmonic numbers H_n except for H_1 are not integers. In 1918, Kürschák [5] showed that the difference $H_n - H_m$ between two distinct harmonic numbers is never an integer. In 1819, Babbage [1] proved that if p is an odd prime, then the numerator of the reduced form of H_{p-1} is divisible by p. Furthermore, in 1862, Wolstenholme [10] showed that if $p \geq 5$, then p^2 divides the numerator of the reduced form of H_{p-1}. The *n-th generalized harmonic number of order* r, denoted by $H_n^{(r)}$, is defined by

$$H_n^{(r)} = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{n^r}.$$

Note that $H_n^{(1)} = H_n$ for all $n \in \mathbb{Z}^+$. For $r > 1$, the generalized harmonic number $H_n^{(r)}$ is indeed the partial sum of a convergent series. For that reason, it is not interesting whether the generalized harmonic numbers are integers or not. However, their divisibility properties are still quite striking. For instance, in 1862, Wolstenholme [10] showed that if $p \geq 5$, then the numerator of the reduced form of $H_{p-1}^{(2)}$

is divisible by p. Furthermore, if $p \geq 5$ with $p-1 \nmid r$, then the numerator of the reduced form of $H_{p-1}^{(r)}$ is divisible by p; see [4].

In this note, we define harmonic type matrices and generalize the integrality and divisibility results of harmonic numbers and generalized harmonic numbers to harmonic type matrices. First, we give the definition of harmonic type matrices. Unless otherwise stated, if A is a matrix with non-zero entries, then $\frac{1}{A}$ denotes the matrix whose entries are the multiplicative inverses of the entries of the matrix A.

Definition 1. The *n-th harmonic matrix of size m*, denoted by $h_n^{(m\times m)}$, is defined as

$$h_n^{(m\times m)} = \sum_A \frac{1}{A},$$

where A runs over all $m \times m$ matrices whose entries are integers between 1 and n.

Recall that a complex square matrix A is called *non-singular* if $\det(A)$ is non-zero.

Definition 2. The *n-th non-singular harmonic matrix of size m*, denoted by $H_n^{(m\times m)}$, is defined by

$$H_n^{(m\times m)} = \sum_A \frac{1}{A},$$

where the sum ranges over all $m\times m$ non-singular matrices whose entries are integers between 1 and n.

Observe that if $m = 1$, then we have that $h_n^{(1\times 1)} = H_n^{(1\times 1)} = H_n$.

2. Divisibility Properties of Harmonic Type Matrices

In this section, we analyze the divisibility properties of the harmonic type matrices $h_n^{(m\times m)}$ and $H_n^{(m\times m)}$. We will show that if p is a prime number greater than or equal to 5, then the numerators of all entries of the $(p-1)$-th harmonic matrix $h_{p-1}^{(n\times n)}$ of size n is divisible by p^2 for all positive integers n. This result generalizes Wolstenholme's theorem [10]. Unlike this, we will give a counterexample that (see Example 4) the numerators of all entries of the fourth non-singular harmonic matrix $H_4^{(2\times 2)}$ of size 2 are not divisible by 5. In addition, we prove an analogue of Wolstenholme's theorem for the special subsets $G_{p,n}$, $T_{p,n}$ and $G_{p,n,r}$ of the matrix ring $M_n(\mathbb{Z})$ defined as follows:

$$G_{p,n} = \{A = (a_{ij}) \in M_n(\mathbb{Z}) \mid 1 \leq a_{ij} \leq p-1,\ p \nmid \det(A)\},$$

$$T_{p,n} = \{A = (a_{ij}) \in M_n(\mathbb{Z}) \mid 1 \leq a_{ij} \leq p-1,\ p \nmid \mathrm{Tr}(A)\},$$

$$G_{p,n,r} = \{A = (a_{ij}) \in M_n(\mathbb{Z}) \mid a_{ij} \in \{1, 2^r, \ldots, (p-1)^r\},\ p \nmid \det(A)\},$$

where n and r are positive integers and p is a prime number. Since we have that

$$\sum_{A\in G_{p,1}} \frac{1}{A} = H_{p-1},$$

$$\sum_{A\in T_{p,1}} \frac{1}{A} = H_{p-1},$$

$$\sum_{A\in G_{p,1,r}} \frac{1}{A} = H_{p-1}^{(r)},$$

the numerators of all entries of the first two 1×1 matrices above are divisible by p^2 for all $p \geq 5$, while the numerators of all entries of the last 1×1 matrix are divisible by p for all $p \geq 5$ with $p - 1 \nmid r$. From now on, for any $r \in \{1, ..., p-1\}$, let $G_{p,n}^r$ denote the subset of $G_{p,n}$ such that the determinants of matrices in $G_{p,n}^r$ are congruent to r modulo p, that is,

$$G_{p,n}^r = \{A \in G_{p,n} \mid \det(A) \equiv r \pmod p\}.$$

Notice that we may identify the matrices in $G_{p,n}$ with the elements of the general linear group $GL_n(\mathbb{F}_p)$ with non-zero entries. In fact, if $r = 1$, then the matrices in $G_{p,n}^1$ can be considered as the elements of the special linear group $SL_n(\mathbb{F}_p)$ with non-zero entries. In this section, we prove the following results which generalize Wolstenholme's theorem. Also, our theorems extend Babbage's result from $\mathbb{F}_p^\times$ to the multi-dimensional structures $GL_n(\mathbb{F}_p^\times)$ and $SL_n(\mathbb{F}_p^\times)$. Here is our first result.

Theorem 1. *If $n > 1$ and $p \geq 5$, then the numerators of all entries of the matrix*

$$\sum_{A\in G_{p,n}^r} \frac{1}{A}$$

are divisible by p^2.

Proof. Note that for any $r \in \{1, ..., p-1\}$, we have $G_{p,1}^r = \{r\}$. Hence, we deduce that

$$\sum_{A\in G_{p,1}^r} \frac{1}{A} = \frac{1}{r}$$

and p^2 does not divide the numerator of $1/r$. Now, assume that $n > 1$ and $p \geq 5$. Let P_k^{ij} denote the subset of $G_{p,n}^r$ consisting of matrices whose ij-th entry is k, where $k = 1, 2, \dots p-1$, that is, $P_k^{ij} = \{(a_{ij}) \in G_{p,n}^r \mid a_{ij} = k\}$. Note that $\{P_k^{ij} \mid k = 1, 2, \dots, p-1\}$ is a partition of $G_{p,n}^r$, namely, we have that

$$G_{p,n}^r = \bigsqcup_{k=1}^{p-1} P_k^{ij}.$$

First, we show that the sets in the partition have the same cardinality. For a matrix $A = (a_{ij}) \in M_n(\mathbb{Z})$, we denote by $(A)_p$ the matrix whose entries are the remainders of the entries of A divided by p and $(a_{ij})_p$ denotes the ij-th entry of the matrix $(A)_p$. We define a function $\phi : P_1^{ij} \longrightarrow P_k^{ij}$ as $\phi(A) = (BA)_p$, where $B = (b_{ij})$ is a diagonal matrix such that if i is not 1, then $b_{ii} = k$, $b_{11} = k'$ and if $i = 1$ then $b_{ii} = k$, $b_{nn} = k'$, where k' denotes the multiplicative inverse of k modulo p, and the other entries equal 1:

$$B = \begin{pmatrix} k' & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & k & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} k & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & k' \end{pmatrix}.$$

When $A \in P_1^{ij}$, we have that the entries of $(BA)_p$ are integers between 1 and $p-1$ and $((BA)_{ij})_p = k$. Recall from linear algebra that, we have

$$\det(BA) = \det(B)\det(A),$$

$$\det((BA)_p) \equiv \det(BA) \pmod{p}.$$

From these fundamental observations, we obtain that

$$\det((BA)_p) \equiv r \pmod{p}$$

which implies that $\phi(A) \in P_k^{ij}$ when $A \in P_1^{ij}$. In order to show that ϕ is an injection, suppose that $A = (a_{ij}) \in P_1^{ij}$, $C = (c_{ij}) \in P_1^{ij}$ and $\phi(A) = \phi(C)$. We will show that $A = C$. Since the rows of the matrices A and C, except for the first and i-th rows, remain stable under the map ϕ, it suffices to show that their entries in the first and i-th rows are equal. The entries of the images $\phi(A)$ and $\phi(C)$ in the first row are of the form $(k'a_{1\beta})_p$ and $(k'c_{1\beta})_p$, where $\beta = 1, \ldots, n$. Since all the entries of the matrices A and C are integers between 1 and $p-1$, we have that $(k'a_{1\beta})_p = (k'c_{1\beta})_p$ implies $(a_{1\beta})_p = (c_{1\beta})_p$, which in turn yields $a_{1\beta} = c_{1\beta}$ where $\beta = 1, \ldots, n$. Similarly, we also have $a_{i\beta} = c_{i\beta}$, where $\beta = 1, \ldots, n$. Thus, we obtain $A = C$, that is, ϕ is an injection. Conversely, we define an injection, $\psi : P_k^{ij} \longrightarrow P_1^{ij}$, in a similar way by $\psi(M) = (NM)_p$, where $N = (n_{ij})$ is a diagonal matrix such that if i is not 1, then $n_{11} = k$, $n_{ii} = k'$ and if $i = 1$, then $n_{ii} = k'$ and $n_{nn} = k$ and the other entries equal 1:

$$N = \begin{pmatrix} k & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & k' & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \quad \text{or} \quad N = \begin{pmatrix} k' & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & k \end{pmatrix}.$$

One can similarly check that ψ is an injection. Thus, we deduce that P_1^{ij} and P_k^{ij} have the same cardinality. Therefore, all sets in the partition

$$\{P_k^{ij} \mid k = 1, 2, \ldots, p-1\}$$

have the same cardinality. As we have that

$$\sum_{A \in G_{p,n}^r} \frac{1}{A} = \sum_{k=1}^{p-1} \left(\sum_{A \in P_k^{ij}} \frac{1}{A} \right),$$

we will show that the numerators of all entries on the right-hand side are divisible by p^2. Now,

$$\sum_{k=1}^{p-1} \left(\sum_{A \in P_k^{ij}} \frac{1}{A} \right) = \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & |P_1^{ij}|1 & * & \cdots & * \\ * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * \end{pmatrix} + \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & |P_2^{ij}|\frac{1}{2} & * & \cdots & * \\ * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * \end{pmatrix}$$

$$+ \cdots + \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & |P_{p-1}^{ij}|\frac{1}{p-1} & * & \cdots & * \\ * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * \end{pmatrix}.$$

However, P_k^{ij} and P_ℓ^{ij} have the same cardinality for any integers $k, \ell = 1, 2, \ldots, p-1$. Therefore, we obtain that

$$\sum_{A \in G_{p,n}^r} \frac{1}{A} = \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & |P_1^{ij}|H_{p-1} & * & \cdots & * \\ * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * \end{pmatrix}.$$

By Wolstenholme's theorem [10], we know that the numerator of the reduced form of H_{p-1} is divisible by p^2. Thus, the numerator of the i-th row and the j-th column of the matrix

$$\sum_{A \in G_{p,n}^r} \frac{1}{A}$$

is divisible by p^2. Since i and j are arbitrary, we conclude that the numerators of all entries of the matrix

$$\sum_{A \in G_{p,n}^r} \frac{1}{A}$$

are divisible by p^2. The proof is now completed. □

Corollary 1. *If $p \geq 5$, then the numerators of all entries of the matrix*

$$\sum_{A \in G_{p,n}} \frac{1}{A}$$

are divisible by p^2.

Proof. We have already observed that

$$\sum_{A \in G_{p,1}} \frac{1}{A} = H_{p-1}.$$

By Wolstenholme's theorem [10], the assertion is true for the case $n = 1$. Now, suppose that $n > 1$. Then, we have that

$$\sum_{A \in G_{p,n}} \frac{1}{A} = \sum_{r=1}^{p-1} \left(\sum_{A \in G_{p,n}^r} \frac{1}{A} \right).$$

By Theorem 1, the numerators of all entries of the inner sum are divisible by p^2. This completes the proof. □

Corollary 2. *If $p \geq 5$, then the numerators of all entries of the $(p-1)$-th harmonic matrix $h_{p-1}^{(n \times n)}$ of size n are divisible by p^2.*

Proof. Suppose that $m \geq 2$ and $n > 2$. Note that for any $k, \ell \in \{1, 2, \ldots, n\}$, the number of $m \times m$ matrices whose entries are integers between 1 and n and the ij-th entry is k equals to the number of $m \times m$ matrices whose entries are integers between 1 and n and the ij-th entry is ℓ. Thus, we have that

$$h_n^{(m \times m)} = \sum_{A} \frac{1}{A} = n^{m^2-1} \begin{pmatrix} H_n & \cdots & H_n \\ \vdots & & \vdots \\ H_n & \cdots & H_n \end{pmatrix}, \tag{1}$$

where A runs over all $m \times m$ matrices whose entries are integers between 1 and n. Now, the result easily follows. □

Theorem 2. *If $p \geq 5$, then the numerators of all entries of the matrix*

$$\sum_{A \in T_{p,n}} \frac{1}{A}$$

are divisible by p^2.

Proof. Assume that P_k^{ij} denotes the subset of $T_{p,n}$ consisting of matrices such that the ij-th entry is k, where $k = 1, 2, \dots p-1$, that is, $P_k^{ij} = \{(a_{ij}) \in T_{p,n} \mid a_{ij} = k\}$. As in Theorem 1, $\{P_k^{ij} \mid k = 1, 2, \dots, p-1\}$ is a partition of $T_{p,n}$ and the sets in the partition have the same cardinality. The function $\phi : P_1^{ij} \longrightarrow P_k^{ij}$ defined as $\phi(A) = (BA)_p$, where $B = (b_{ij})$ is a diagonal matrix such that $b_{ii} = k$ for all i, that is,

$$B = \begin{pmatrix} k & & \\ & \ddots & \\ & & k \end{pmatrix},$$

is an injection. When $A \in P_1^{ij}$, the entries of $(BA)_p$ are integers between 1 and $p-1$ and $((BA)_{ij})_p = k$. We recall that

$$\mathrm{Tr}(BA) = k\mathrm{Tr}(A),$$

$$\mathrm{Tr}((BA)_p) \equiv \mathrm{Tr}(BA) \pmod{p}.$$

Thus, from these basic results, we have $p \nmid \mathrm{Tr}((BA)_p)$, which implies that $\phi(A) \in P_k^{ij}$ when $A \in P_1^{ij}$. If $A = (a_{ij}) \in P_1^{ij}$, $C = (c_{ij}) \in P_1^{ij}$ and $\phi(A) = \phi(C)$, then we have $(ka_{ij})_p = (kc_{ij})_p$. Since a_{ij} and c_{ij} are between 1 and $p-1$, we obtain $a_{ij} = c_{ij}$ and so $A = C$. Hence, ϕ is an injection. Conversely, we define an injection, $\psi : P_k^{ij} \longrightarrow P_1^{ij}$, in a similar way by $\psi(M) = (NM)_p$, where $N = (n_{ij})$ is a diagonal matrix such that $n_{ii} = k'$, where k' denotes the multiplicative inverse of k modulo p, that is,

$$N = \begin{pmatrix} k' & & \\ & \ddots & \\ & & k' \end{pmatrix}.$$

One can similarly check that ψ is an injection. Hence, we obtain that P_1^{ij} and P_k^{ij} have the same cardinality. Therefore, the sets in the partition

$$\{P_k^{ij} \mid k = 1, 2, \dots, p-1\}$$

have the same cardinality. Now,

$$\sum_{k=1}^{p-1} \left(\sum_{A \in P_k^{ij}} \frac{1}{A} \right) = \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & |P_1^{ij}|1 & * & \cdots & * \\ * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * \end{pmatrix} + \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & |P_2^{ij}|\frac{1}{2} & * & \cdots & * \\ * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * \end{pmatrix}$$

$$+ \cdots + \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & |P_{p-1}^{ij}|\frac{1}{p-1} & * & \cdots & * \\ * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * \end{pmatrix}.$$

Thus, we obtain that

$$\sum_{A\in T_{p,n}} \frac{1}{A} = \begin{pmatrix} * & \dots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & |P_1^{ij}|H_{p-1} & * & \cdots & * \\ * & \dots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \dots & * & * & \cdots & * \end{pmatrix}.$$

The rest of the proof follows immediately as in Theorem 1. □

Theorem 3. *If $p \geq 5$ and $p-1 \nmid r$, then the numerators of all entries of the matrix*

$$\sum_{A\in G_{p,n,r}} \frac{1}{A}$$

are divisible by p.

Proof. We apply the same technique as in the previous theorem. Suppose that P_k^{ij} denotes the subset of $G_{p,n,r}$ consisting of matrices such that the ij-th entry is k^r, where $k = 1, 2, \dots p-1$, that is, $P_k^{ij} = \{(a_{ij}) \in G_{p,n,r} \mid a_{ij} = k^r\}$. Then, $\{P_k^{ij} \mid k = 1, 2, \dots, p-1\}$ is a partition of $G_{p,n,r}$ and the elements of the partition have the same cardinality. Thus, we have the representation

$$\sum_{A\in G_{p,n,r}} \frac{1}{A} = \sum_{k=1}^{p-1} \left(\sum_{A\in P_k^{ij}} \frac{1}{A} \right).$$

This yields that

$$\sum_{A\in G_{p,n,r}} \frac{1}{A} = \begin{pmatrix} * & \dots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & |P_1^{ij}|H_{p-1}^{(r)} & * & \cdots & * \\ * & \dots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \dots & * & * & \cdots & * \end{pmatrix}.$$

Since $p-1 \nmid r$, by [4] the numerator of the reduced form of $H_{p-1}^{(r)}$ is divisible by p. Thus, the result follows. □

The following example indicates that the analogue of Wolstenholme's theorem does not hold for the fourth non-singular harmonic matrix $H_4^{(2\times 2)}$ of size 2.

Example 4. We have

$$H_4^{(2\times 2)} = \sum_{A} \frac{1}{A} = \begin{pmatrix} 233/2 & 233/2 \\ 233/2 & 233/2 \end{pmatrix},$$

where A runs over all $m\times m$ non-singular matrices whose entries are integers between 1 and 4. However, the numerator of 233/2 is not divisible by 5.

3. Integrality Properties of Harmonic Type Matrices

In this section, we generalize Theisinger's theorem [9] for the harmonic type matrices $h_n^{(m\times m)}$ and $H_n^{(m\times m)}$. For a given non-zero integer a, the *p-adic valuation* of a, denoted by $\nu_p(a) = m$, indicates that p^m divides a but p^{m+1} does not divide a. By convention, we set $\nu_p(0) = \infty$. For a non-zero rational number $q = m/n$ where $m, n \in \mathbb{Z}$, we define $\nu_p(q) = \nu_p(m) - \nu_p(n)$. One can immediately see that

$$\nu_p(ab) = \nu_p(a) + \nu_p(b),$$

$$\nu_p(a+b) \geq \min\{\nu_p(a), \nu_p(b)\}$$

for all rational numbers a, b.

Our first theorem in this section generalizes Theisinger's result [9] to non-singular type harmonic matrices of size 2.

Theorem 5. *All entries of the non-singular harmonic matrix $H_n^{(2\times 2)}$ of size 2 are not integers for any $n > 1$.*

Proof. Note that the set of 2×2 non-singular matrices whose entries are 1 is empty. Therefore, the first non-singular harmonic matrix $H_1^{(2\times 2)}$ of size 2 is equal to 0. Now, let $n > 1$. For any $k = 1, 2, \ldots, n$, the number m_{kn} denotes the number of 2×2 non-singular matrices whose entries are integers between 1 and n with the $(1,1)$-th entry is k. We claim that for any $i, j = 1, 2$, the number of 2×2 non-singular matrices whose entries are integers between 1 and n with the ij-th entry is k equals to m_{kn} as well. Recall the fundamental result from linear algebra that if we interchange two rows (columns) of a matrix, then the determinant will change its sign so that being non-singular will be preserved. Using this fact, our claim is observable. Hence, we obtain that

$$H_n^{(2\times 2)} = \sum_A \frac{1}{A} = \begin{pmatrix} \sum_{k=1}^n \frac{m_{kn}}{k} & \sum_{k=1}^n \frac{m_{kn}}{k} \\ \sum_{k=1}^n \frac{m_{kn}}{k} & \sum_{k=1}^n \frac{m_{kn}}{k} \end{pmatrix},$$

where A runs over all 2×2 non-singular matrices whose entries are integers between 1 and n. Now, we choose a prime number p with $n/2 < p \leq n$ and evaluate m_{pn} which equals to the number of 2×2 non-singular matrices of the form $\begin{pmatrix} p & c \\ a & d \end{pmatrix}$, where $a, c, d = 1, 2, \ldots, n$. For that reason, we need to count $\mathbb{Q}$-linearly independent vectors of the form

$$v_1 = \begin{pmatrix} p \\ a \end{pmatrix}, \qquad v_2 = \begin{pmatrix} c \\ d \end{pmatrix}.$$

We claim that the number of such linearly independent vectors is n^3-2n+1, which is independent from the choice of p. If $a=p$, then the vectors of the form $v_2=\begin{pmatrix}x\\x\end{pmatrix}$, where $x=1,2,\dots n$, are all linearly dependent with the vector $v_1=\begin{pmatrix}p\\p\end{pmatrix}$. Thus, we have n many possibilities for this case. If $a\neq p$, then since p is a prime between $n/2$ and n, the only linearly dependent vector with $v_1=\begin{pmatrix}p\\a\end{pmatrix}$ is just itself $v_2=\begin{pmatrix}p\\a\end{pmatrix}$. Since there are $n-1$ choices for a, in this case we obtain $n-1$ linearly dependent such vector tuples. Hence, we obtain totally $2n-1$ linearly dependent such vector tuples. This yields that $m_{pn}=n^3-(2n-1)=n^3-2n+1$.

Suppose the entries of the non-singular harmonic matrix $H_n^{(2\times 2)}$ of size 2 are integers. Then, we have that

$$\nu_p\left(\sum_{k=1}^{n}\frac{m_{kn}}{k}\right)\geq 0,$$

which in turn implies that $p\mid m_{pn}$. Hence, we deduce that

$$\prod_{\frac{n}{2}<p\leq n} p \mid n^3-2n+1.$$

There are at least four prime numbers between $\frac{n}{2}$ and n for all $n\geq 29$, as $R_4=29$ is the fourth Ramanujan's prime; see [6]. Therefore, we obtain the following inequality

$$\left(\frac{n}{2}\right)^4<\prod_{\frac{n}{2}<p\leq n} p\leq n^3-2n+1$$

for all $n\geq 29$. This implies that

$$n^4-16n^3+32n-16<0$$

for all $n\geq 29$. However,

$$n^4-16n^3+32n-16=n((n-16)n^2+32)-16>0$$

for all $n\geq 29$. Hence, the entries of $H_n^{(2\times 2)}$ can not be integers for all $n\geq 29$. We also computed all non-singular harmonic matrices $H_n^{(2\times 2)}$ for every $1<n\leq 28$, using SageMath [7]. Their entries are not integers, as well. Thus, the proof is done. $\square$

The next result is another generalization of [9] when the polynomial $f(X)=1$ is the constant polynomial.

Theorem 6. *For every non-zero polynomial $f(X)\in\mathbb{Z}[X]$, the product $f(n)H_n$ is not an integer for all but finitely many n.*

Proof. Indeed, what we want to prove is that there exists a constant $N = N(f)$ such that $f(n)H_n \notin \mathbb{Z}$ for all $n > N$. Let $f(X) \in \mathbb{Z}[X]$ be a non-zero polynomial of degree k and suppose that $f(X) = a_0 + a_1X + \cdots + a_kX^k$, where $a_k \neq 0$. By Bertrand's postulate we know that if $n > 2$, then $\pi(n) - \pi(\frac{n}{2}) \geq 1$. Furthermore, using the prime number theorem, one sees that

$$\pi(n) - \pi\left(\frac{n}{2}\right) \sim \frac{n/2}{\log(n/2)},$$

as $n \to \infty$. Thus, we deduce that

$$\lim_{n\to\infty} \pi(n) - \pi\left(\frac{n}{2}\right) = \infty.$$

So, we guarantee that for large N_1 there exist at least $k+1$ many prime numbers $p_1, \ldots, p_{k+1}$ with $\frac{n}{2} < p_1, \ldots, p_{k+1} \leq n$ for all $n > N_1$. There exists also N_2 such that $f(n) \neq 0$ for all $n > N_2$. Note also that $\nu_{p_i}(H_n) = -1$ for every prime number p_i from $(\frac{n}{2}, n]$. Since $\nu_{p_i}(f(n)H_n) = \nu_{p_i}(f(n)) + \nu_{p_i}(H_n)$, if $f(n)H_n \in \mathbb{Z}$, then $p_i \mid f(n)$ which in turn implies that

$$\prod_{i=1}^{k+1} p_i \mid f(n).$$

Therefore, we obtain the following inequality

$$\left(\frac{n}{2}\right)^{k+1} \leq \prod_{i=1}^{k+1} p_i \leq \sum_{j=0}^{k} |a_j| n^j.$$

The left-hand side is a polynomial of degree $k+1$, whereas the right-hand side is a polynomial of degree k. Hence, there exists a number N_3 such that this inequality does not hold for every $n > N_3$. If $N = \max\{N_1, N_2, N_3\}$, then $f(n)H_n$ is not an integer for every $n > N$. $\square$

Remark 1. For a polynomial $p(X)$ of degree n given by

$$p(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n,$$

the height of $p(X)$, denoted as $H(p)$, is defined to be the maximum of the absolute values of its coefficients: $H(p) = \max_i |a_i|$.

Now, for given positive n and k, consider the set of polynomials with integers coefficients whose degrees and heights are up to n and k, respectively:

$$A_{n,k} = \{p \in \mathbb{Z}[X] : \deg(p) \leq n, H(p) \leq k\}.$$

This is a finite set, and furthermore we have that $|A_{n,k}| = (2k+1)^{n+1}$. By Theorem 6, we have a constant $N = N(p)$ for every non-zero polynomial $p \in A_{n,k}$ such that

$p(n)H_n \notin \mathbb{Z}$ for every $n > N$. Since $A_{n,k}$ is finite, we choose a constant N to be the largest among the numbers $N(p)$, where $p \in A_{n,k} \setminus \{0\}$. Therefore, there exists N such that $p(n)H_n \notin \mathbb{Z}$ for any $p \in A_{n,k} \setminus \{0\}$ and $n > N$.

Remark 2. Note that the number $N(f)$ in Theorem 6 can be very large with respect to the choice of $f(X)$. For instance, if $f(X) = k!$ for some positive integer k, then $f(n)H_n \in \mathbb{Z}$ for all $1 \leq n \leq k$.

The next result is also a generalization of the corresponding result of [9].

Proposition 1. *Let $m \geq 2$. All entries of the harmonic matrix $h_n^{(m\times m)}$ are not integers if and only if $n \neq 1, 2$.*

Proof. First, suppose that $n > 2$. Recall from Equation (1) that

$$h_n^{(m\times m)} = \sum_A \frac{1}{A} = n^{m^2-1} \begin{pmatrix} H_n & \cdots & H_n \\ \vdots & & \vdots \\ H_n & \cdots & H_n \end{pmatrix}.$$

By Bertrand's postulate we know that if $n > 2$, then there always exists at least one prime number p between $n/2$ and n. Choose a prime number p with $\frac{n}{2} < p < n$ and consider $\nu_p(n^{m^2-1}H_n)$. Since $\nu_p(H_n) = -1$, we have that

$$\nu_p(n^{m^2-1}H_n) = (m^2-1)\nu_p(n) + \nu_p(H_n) = 0 - 1 = -1.$$

Hence, $n^{m^2-1}H_n$ is not an integer which means that all entries of the harmonic matrix $h_n^{(m\times m)}$ are not integers.

Conversely, suppose that $n = 1$. Then, it is easy to see that

$$h_1^{(m\times m)} = 1^{m^2-1} \begin{pmatrix} H_1 & \cdots & H_1 \\ \vdots & & \vdots \\ H_1 & \cdots & H_1 \end{pmatrix} \in M_m(\mathbb{Z}).$$

Now, suppose that $n = 2$. Since $m \geq 2$, we have $m^2 - 1 \geq 3$ which implies that

$$2^{m^2-1} \cdot H_2 = 2^{m^2-1} \cdot \frac{3}{2} \in \mathbb{Z}.$$

As a consequence, this yields that

$$h_2^{(m\times m)} = 2^{m^2-1} \begin{pmatrix} H_2 & \cdots & H_2 \\ \vdots & & \vdots \\ H_2 & \cdots & H_2 \end{pmatrix} \in M_m(\mathbb{Z}),$$

and we are done. □

4. Further Results on the p-Adic Valuation of Harmonic Matrices

For each prime number p, define

$$J_p = \{n \geq 1 \mid H_n \equiv 0 \pmod{p}\}.$$

Theisinger [9] showed that $J_2 = \emptyset$ and Eswarathasan and Levine [3] conjectured that J_p is finite for all primes p. Recently, Sanna [8] proved that for $x \geq 1$ the number of integers in $J_p \cap [1, x]$ is less than $129p^{2/3}x^{0.765}$ and this was extended in [11]. Later on, the denominators of harmonic numbers were also studied, and we direct the reader to [2, 12, 13] for this.

Now, we define $J_p^{(m\times m)}$ to be the set containing positive integers n such that the numerators of all entries of the harmonic matrix $h_n^{(m\times m)}$ of size m are divisible by p. Observe that $J_p^{(1\times 1)} = J_p$. We have shown that

$$h_n^{(m\times m)} = \sum_A \frac{1}{A} = n^{m^2-1}\begin{pmatrix} H_n & \cdots & H_n \\ \vdots & & \vdots \\ H_n & \cdots & H_n \end{pmatrix},$$

where A runs over all $m \times m$ matrices whose entries are between 1 and n. So, $n \in J_p^{(m\times m)}$ means that the numerator of $n^{m^2-1}H_n$ is divisible by p, that is,

$$J_p^{(m\times m)} = \{n \geq 1 \mid n^{m^2-1}H_n \equiv 0 \pmod{p}\}.$$

Recall that the numerator of the reduced form of H_{p-1} is divisible by p for all $p \geq 3$. Thus, $p-1 \in J_p$ for all $p \geq 3$. Note that $J_p \subseteq J_p^{(m\times m)}$ for any $m \geq 1$, and hence $p - 1 \in J_p^{(m\times m)}$. In general, we can show that $J_p^{(m\times m)} \subseteq J_p^{(\ell\times\ell)}$ whenever $m \leq \ell$. If $m \leq \ell$ and $n \in J_p^{(m\times m)}$, then $1 \leq \nu_p(n^{m^2-1}H_n) \leq \nu_p(n^{\ell^2-1}H_n)$ which yields that $n \in J_p^{(\ell\times\ell)}$. Thus, we conclude that $J_p^{(m\times m)} \subseteq J_p^{(\ell\times\ell)}$.

Proposition 2. *If $p \nmid n$, then $n \in J_p$ if and only if $n \in J_p^{(m\times m)}$.*

Proof. Suppose that $p \nmid n$ and $n \in J_p^{(m\times m)}$. Then, we have $n^{m^2-1}H_n \equiv 0 \pmod{p}$. Since n is not divisible by p, one sees that p must divide H_n. This indicates that $n \in J_p$. The converse is obvious. □

For all prime numbers p, we have that $p \notin J_p$, because $\nu_p(H_p) = -1$ which means that p does not divide the numerator of the harmonic number H_p. However, $p \in J_p^{(m\times m)}$ for all $m \geq 2$, as we have

$$\nu_p(p^{m^2-1}H_p) = \nu_p(p^{m^2-1}) + \nu_p(H_p) \geq 2^2 - 1 + (-1) = 2.$$

Furthermore, we claim that $p^k \in J_p^{(m\times m)}$ for all $k \in \mathbb{N}$. Since $\nu_p(H_{p^k}) = -k$ and $\nu_p(p^{km^2-k}) \geq 3k$, we have $\nu_p(p^{km^2-k}H_{p^k}) \geq 2k$ for all $k \in \mathbb{N}$. This implies that

$p^k \in J_p^{(m\times m)}$. Hence, for $m \geq 2$ the set $J_p^{(m\times m)}$ is infinite contrary to the conjecture of the finiteness of J_p. Now, we will discuss when the multiples of a prime number p are also in $J_p^{(m\times m)}$ for sufficiently large m.

Theorem 7. *Let p be a prime and n be a positive integer divisible by p. If $\alpha = [\log_p n]$ and $m \geq \sqrt{\frac{\alpha+1}{\nu_p(n)}+1}$, then $n \in J_p^{(m\times m)}$. Moreover, if $[\frac{n}{p^\alpha}] \notin J_p$, then $n \in J_p^{(m\times m)}$ if and only if $m \geq \sqrt{\frac{\alpha+1}{\nu_p(n)}+1}$.*

Proof. Suppose that $p \mid n$, $\alpha = [\log_p n]$ and $m \geq \sqrt{\frac{\alpha+1}{\nu_p(n)}+1}$. Then, $\nu_p(n) \geq 1$, $m^2 - 1 \geq \frac{\alpha+1}{\nu_p(n)}$ and $p^\alpha \leq n < p^{\alpha+1}$. We want to show $n \in J_p^{(m\times m)}$, that is $\nu_p(n^{m^2-1}H_n) \geq 1$. One can immediately see that $\nu_p(H_n) \geq -\alpha$, because $p^\alpha \leq n < p^{\alpha+1}$. Then, we see the desired inequality by

$$\nu_p(n^{m^2-1}H_n) = (m^2-1)\nu_p(n) + \nu_p(H_n) \geq (m^2-1)\nu_p(n) - \alpha \geq 1.$$

Now, suppose that $[\frac{n}{p^\alpha}] \notin J_p$ and $n \in J_p^{(m\times m)}$. Our aim is to obtain the inequality $m \geq \sqrt{\frac{\alpha+1}{\nu_p(n)}+1}$. By the division algorithm, write $n = qp^\alpha + r$, where $1 \leq q < p$ and $0 \leq r < p^\alpha$. Then, $[\frac{n}{p^\alpha}] = q$. We also have that

$$\begin{aligned}
&\nu_p\left(1 + \frac{1}{2} + \cdots + \frac{1}{p^\alpha - 1}\right) \geq 1 - \alpha,\\
&\nu_p\left(\frac{1}{p^\alpha + 1} + \cdots + \frac{1}{2p^\alpha - 1}\right) \geq 1 - \alpha,\\
&\qquad\qquad\vdots\\
&\nu_p\left(\frac{1}{(q-1)p^\alpha + 1} + \cdots + \frac{1}{qp^\alpha - 1}\right) \geq 1 - \alpha,\\
&\nu_p\left(\frac{1}{qp^\alpha + 1} + \cdots + \frac{1}{qp^\alpha + r}\right) \geq 1 - \alpha.
\end{aligned}$$

Write the harmonic number H_n as follows:

$$\begin{aligned}
H_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{p^\alpha} + \cdots + \frac{1}{2p^\alpha} + \cdots + \frac{1}{qp^\alpha} + \frac{1}{qp^\alpha + 1} + \cdots + \frac{1}{qp^\alpha + r}\\
&= \frac{1}{p^\alpha}H_q + \left(1 + \frac{1}{2} + \cdots + \frac{1}{p^\alpha - 1}\right) + \left(\frac{1}{p^\alpha + 1} + \cdots + \frac{1}{2p^\alpha - 1}\right) + \cdots\\
&\quad + \left(\frac{1}{qp^\alpha + 1} + \cdots + \frac{1}{qp^\alpha + r}\right).
\end{aligned}$$

Since $q < p$ and $q \notin J_p$, we obtain $\nu_p(H_q) \geq 0$ and $\nu_p(H_q) \leq 0$, and these imply that $\nu_p(H_q) = 0$. Therefore, $\nu_p(H_n) = -\alpha$. Since $n \in J_p^{(m\times m)}$, we have

$$1 \leq \nu_p(n^{m^2-1}H_n) = (m^2-1)\nu_p(n) + \nu_p(H_n) = (m^2-1)\nu_p(n) - \alpha.$$

Thus, we deduce that $m \geq \sqrt{\frac{\alpha+1}{\nu_p(n)}+1}$, as desired. $\square$

Using Proposition 2 and Theorem 7, we can easily see the following result: for a given positive integer n and sufficiently large m, one has $pn \in J_p^{(m\times m)}$. So, we have $p\mathbb{Z}^+ \subset \bigcup_{m=1}^{\infty} J_p^{(m\times m)}$. If $n \notin p\mathbb{Z}^+$, then $n \in J_p$ if and only if $n \in J_p^{(m\times m)}$. Thus, we conclude that

$$\bigcup_{m=1}^{\infty} J_p^{(m\times m)} = p\mathbb{Z}^+ \cup J_p.$$

Proposition 3. *Let $m \geq 2$. If $n \in J_p^{(m\times m)}$, then $pn \in J_p^{(m\times m)}$.*

Proof. Suppose that $m \geq 2$ and $n \in J_p^{(m\times m)}$. Then, $\nu_p(n^{m^2-1}H_n) \geq 1$. That is, $(m^2-1)\nu_p(n)+\nu_p(H_n) \geq 1$. We want to show the inequality $\nu_p((pn)^{m^2-1}H_{pn}) \geq 1$. Write H_{pn} as follows:

$$H_{pn} = 1 + \frac{1}{2} + \cdots + \frac{1}{p-1} + \frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{2p-1} + \frac{1}{2p} + \cdots + \frac{1}{pn}.$$

From this, we see that $H_{pn} = \frac{H_n}{p} + A$, where $\nu_p(A) \geq 0$. Then, we have

$$\nu_p((pn)^{m^2-1}H_{pn}) = (m^2-1) + (m^2-1)\nu_p(n) + \nu_p\left(\frac{H_n}{p} + A\right).$$

Now, we have two possible cases: first, if $n \in J_p$, then $\nu_p\left(\frac{H_n}{p} + A\right) \geq 0$. It is clear that

$$(m^2-1) + (m^2-1)\nu_p(n) + \nu_p\left(\frac{H_n}{p} + A\right) \geq 3.$$

Second, if $n \notin J_p$, then $\nu_p\left(\frac{H_n}{p} + A\right) < 0$. Then, $\nu_p\left(\frac{H_n}{p} + A\right) = \nu_p(H_n) - 1$ and since $(m^2-1)\nu_p(n) + \nu_p(H_n) \geq 1$ and $m \geq 2$, we obtain that

$$(m^2-1) + (m^2-1)\nu_p(n) + \nu_p\left(\frac{H_n}{p} + A\right) \geq 3.$$

Thus, in either case, we see that $\nu_p((pn)^{m^2-1}H_{pn}) \geq 1$, which means $pn \in J_p^{(m\times m)}$. $\square$

We have observed that $p \in J_p^{(m\times m)}$ for all $m \geq 2$. Then, Theorem 3 also immediately implies that all powers p^k of p are contained in $J_p^{(m\times m)}$.

Remark 3. By Theorem 7, if $p \mid n$ and $[\frac{n}{p^\alpha}] \notin J_p$, then $n \in J_p^{(m\times m)}$ if and only if $m \geq \sqrt{\frac{\alpha+1}{\nu_p(n)}+1}$. Suppose that $[\frac{n}{p^\alpha}] = \frac{n}{p^\alpha} \in J_p$. Then, $\frac{n}{p^\alpha} \in J_p^{(2\times 2)}$. By Proposition 3, we obtain that $n \in J_p^{(2\times 2)}$. Hence, if $\frac{n}{p^\alpha} \in J_p$, then $n \in J_p^{(m\times m)}$ for all $m \geq 2$ and we do not need the bound given in Theorem 7.

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