EXACTLY SOLVABLE QUANTUM PARAMETRIC OSCILLATORS IN HIGHER DIMENSIONS

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ABSTRACT

EXACTLY SOLVABLE QUANTUM PARAMETRIC OSCILLATORS IN HIGHER DIMENSIONS

The purpose of this thesis is to study the dynamics of the generalized quantum parametric oscillators in one and higher dimensions and present exactly solvable models. First, time-evolution of the nonclassical states for a one-dimensional quantum parametric oscillator corresponding to the most general quadratic Hamiltonian is found explicitly, and the squeezing properties of the wave packets are analyzed. Then, initial boundary value problems for the generalized quantum parametric oscillator with Dirichlet and Robin boundary conditions imposed at a moving boundary are introduced. Solutions corresponding to different types of initial data and homogeneous boundary conditions are found to examine the influence of the moving boundaries. Besides, an N-dimensional generalized quantum harmonic oscillator with time-dependent parameters is considered and its solution is obtained by using the evolution operator method. Exactly solvable quantum models are introduced and for each model, the squeezing and displacement properties of the time-evolved coherent states are studied. Finally, time-dependent Schrödinger equation describing a generalized two-dimensional quantum coupled parametric oscillator in the presence of time-variable external fields is solved using the evolution operator method. The propagator and time-evolution of eigenstates and coherent states are derived explicitly in terms of solutions to the corresponding system of coupled classical equations of motion. In addition, a Cauchy-Euler type quantum oscillator with increasing mass and decreasing frequency in time-dependent magnetic and electric fields is introduced. Based on the explicit results, squeezing properties of the wave packets and their trajectories in the two-dimensional configuration space are discussed according to the influence of the time-variable parameters and external fields.

ÖZET

YÜKSEK BOYUTTA TAM ÇÖZÜLEBİLEN KUANTUM PARAMETRİK OSİLATÖRLER

Bu tezin amacı bir ve yüksek boyutlarda genelleştirilmiş kuantum parametrik osilatörlerin dinamiğini çalışmak ve tam çözülebilen modeller sunmaktır. İlk olarak, ikinci dereceden en genel Hamiltonyen'e karşılık gelen bir boyutlu bir kuantum parametrik osilatör için klasik olmayan durumların zamanla evrimi açıkça bulunmuş ve dalga paketlerinin sıkışma özellikleri analiz edilmiştir. Daha sonra, hareketli bir sınıra dayatılan Dirichlet ve Robin sınır koşullarına sahip genelleştirilmiş kuantum parametrik osilatör için başlangıç sınır değer problemleri tanıtılmıştır. Hareketli sınırların etkisini inceleyebilmek için farklı türdeki başlangıç verilerine ve homojen sınır koşullarına karşılık gelen çözümler bulunmuştur. Ayrıca, zamana bağlı parametrelere sahip N boyutlu bir genelleştirilmiş kuantum harmonik osilatör ele alınmış ve çözümü evrim operatörü yöntemini kullanarak elde edilmiştir. Tam çözülebilen kuantum modeller tanıtılmış ve her bir model için zamanla evrimleşmiş eş uyumlu durumların sıkışma ve yer değişme özellikleri çalışılmıştır. Son olarak, evrim operatörü yöntemini kullanarak, zamana bağlı olarak değişen dış alanların varlığında genelleştirilmiş iki boyutlu bir kuantum parametrik bağlaşım osilatörünü tanımlayan zamana bağlı Schrödinger denklemi çözülmüştür. Üretici ve özdurumların ve eş uyumlu durumların zamanla evrimi karşılık gelen bağlantılı klasik hareket denklemlerinin sisteminin çözümleri cinsinden açıkça türetilmiştir. Ek olarak, zamana bağlı manyetik ve elektrik alanlarda artan kütle ve azalan frekansa sahip Cauchy-Euler tipi bir kuantum osilatör tanıtılmıştır. Açık sonuçlara dayanarak, dalga paketlerinin sıkışma özellikleri ve iki boyutlu konfigürasyon uzayındaki yörüngeleri, zamana bağlı parametrelerinin ve dış alanların etkisine göre tartışılmıştır.

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CHAPTER 1

INTRODUCTION

Quantum harmonic oscillators with explicitly time-dependent Hamiltonians have attracted substantial interest in the literature since they have applications in many branches of physics, such as quantum optics, quantum fluid dynamics, ion-traps, cosmology, quantum information and quantum computation. To understand better behavior of such quantum systems, it is always important to have exactly solvable models. The best known one is the Caldirola-Kanai oscillator with an exponentially increasing mass, which is widely used to study dissipation in quantum mechanics (Caldirola, 1941), (Kanai, 1948). There are many powerful approaches for solving one-dimensional non-stationary quantum oscillator problems, such as Feynman path integral (Feynman, 1951), Husimi ansatz (Husimi, 1953), Lewis-Riesenfeld invariant (Lewis & Riesenfeld, 1969), Malkin-Man'ko-Trifonov (Malkin, Man'ko & Trifonov, 1970), (Malkin, Man'ko & Trifonov, 1971), and Wei-Norman approaches (Wei & Norman, 1963).

Coherent states and squeezed coherent states of the quantum harmonic oscillator are known since the beginning of quantum mechanics. Indeed, the 'non-spreading wave packets' of the harmonic oscillator were proposed by Schrödinger (Schrödinger, 1926), and followed by Kennard (Kennard, 1927). Then the same states were derived as eigenstates of the non-Hermitian annihilation operator $\hat{a}$ (Iwata, 1951). However, in the literature, the name 'coherent states' appeared for the first time in the paper (Glauber, 1963). For the standard quantum harmonic oscillator (SQHO) defined by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2} + \frac{\omega_0^2}{2}\hat{q}^2,$$

where $\hat{q}$ is position operator, and $\hat{p} = -i\hbar\partial/\partial q$ is the momentum operator, $\omega_0 > 0$ is the constant frequency and mass is $m = 1$, coherent states are minimum uncertainty states with equal uncertainties in both quadratures, whose dynamics most closely resemble the classical states. According to this, they are known as the most classical states among the quantum states. On the other hand, the squeezed coherent states, which can be considered as generalizations of the coherent states, are one simplest representations of the nonclassical states. A straightforward technique for describing the nonclassical states is quadrature squeezing. For states that satisfy the Heisenberg uncertainty relation $\Delta\hat{q}\Delta\hat{p} \geq \hbar/2$, a quadrature is said to be squeezed if uncertainty in that quadrature is smaller than $\hbar/2$. In the simplest case, squeezed coherent states obey the minimum uncertainty principle, but have less uncertainty in one quadrature at the expense of increased uncertainty in the other. Essential properties of squeezed coherent states were derived in (Stoler, 1970), (Stoler, 1971), (Yuen, 1976) and then extensively investigated by many authors, (Henry, 1988), (Trifonov, 1994), (Nieto, 1997), (Nieto & Truax, 1993), (Dodonov, 2002), (Dodonov & Man'ko, 2003), (Malkin & Man'ko, 1979), (Perelomov, 1986).

Coherent and squeezed states of standard quantum harmonic oscillator (SQHO) can be generated by using different but equivalent approaches. One way to obtain coherent states is applying the unitary displacement operator $\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}\right)$, where $\hat{a}^{\dagger}$ denotes Hermitian conjugate of the operator $\hat{a}$, and α^{*} denotes complex conjugate of α, to the ground state. On the other hand, squeezed states can be found by application of the unitary squeeze operator $\hat{S}(z) = \exp[(z\hat{a}^{\dagger 2} - z^{*}\hat{a}^{2})/2]$, $z \in \mathbb{C}$. Using this formalism, the displaced and squeezed number states of SQHO, and their time-evolution were derived explicitly in (Nieto & Simmons, 1979), (Nieto, 1996), (Nieto, 1997). It was shown that the time-evolved squeezed coherent states of SQHO correspond to wave packets whose width oscillates with time, the minimum uncertainty is no longer preserved during time-evolution, and their peak follows the classical trajectory.

Moreover, there are other interesting types of nonclassical states, such as even-odd coherent states and even-odd displaced squeezed states. As known, coherent states of SQHO are not orthogonal, but superposition of these states generates new ones, which are orthogonal and called even-odd coherent states. They were introduced in (Dodonov, Malkin & Man'ko, 1974). The even-odd coherent states are eigenstates of the operator $\hat{a}^2$, and considered as the simplest examples of Schrödinger's cat states. They can be obtained by applying the displacement operators $\hat{D}_{+}(\alpha) = \cosh(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a})$ and $\hat{D}_{-}(\alpha) = \sinh(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a})$, $\alpha \in \mathbb{C}$, to the ground state of SQHO. A detailed analysis of the nonclassical properties of these states is given in (Gerry, 1993), (Buzek & Knight, 1991), (Buzek, Vidiella-Barranco & Knight, 1992). Besides, in (Choi, 2004), time development of even-odd coherent states were found by using Lewis-Riesenfeld invariant approach.

The even-odd displaced squeezed states were proposed in (Fan & Zhang, 1994) and it was shown that even-displaced squeezed states exhibit stronger squeezing than squeezed coherent states of SQHO. There are two different but equivalent ways of defining the even-odd displaced squeezed states; one way is taking the superposition of squeezed coherent states, and the other one is applying the operators $\hat{D}_{+}(\alpha)$, $\hat{D}_{-}(\alpha)$, $\alpha \in \mathbb{C}$, on the squeezed ground state. A comparison of the even-odd coherent states and the even-odd displaced squeezed states was given in (Nieto, 1996), and the nonclassical properties of the even-odd displaced squeezed states were discussed in (Zhu, Wang & Li, 1993), (Xin, Wang, Hirayama & Matumoto, 1994).

There are other possibilities of generating nonclassical states. For example, by adding to $\hat{H}_0$, at some moment of time, a term of the form $\frac{1}{2}\omega_1^2 q^2 - f_0 q$, which clearly corresponds to change of frequency and displacement, one can construct displaced and squeezed number states. On the other hand, when the oscillator has time-dependent mass $\mu(t)$ and/or frequency $\omega(t)$, squeezing effects appear naturally due to the time-variable parameters. As a consequence, the evolution operator of the quadratic parametric oscillator can be considered as some kind of generalized squeezing operator (Dodonov & Man'ko, 2003). In other words, evolution itself is a displacement and squeezing process. Coherent and squeezed states of this generalized oscillator were investigated in (Choi & Kim, 2004), (Choi, 2006), (Choi & Nahm, 2007), using Lewis-Riesenfeld invariant approach (Lewis & Riesenfeld, 1969). In (Atılgan Büyükaşık, 2018), the squeezing and resonance properties of coherent states for generalized Caldirola-Kanai type models were investigated.

We consider the time-evolution problem for a quantum parametric oscillator described by a generalized quadratic Hamiltonian

$$\hat{H}_g(t) = \frac{\hat{p}^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}^2 + \frac{B(t)}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) + D(t)\hat{p} + E(t)\hat{q} + F(t)\hat{I}, \tag{1.1}$$

where $\mu(t)$, $\omega(t)$, $B(t)$, $D(t)$, $E(t)$, $F(t)$ are real-valued parameters depending on time. Recently, in (Atılgan Büyükaşık & Çayiç, 2016), by using the Wei-Norman technique and by properly choosing the ordering of the exponential operators, we found the exact evolution operator for a quantum parametric oscillator described by a Hamiltonian with $SU(1,1) \oplus h(4)$ group structure. The significance of our results is that, for a time-dependent one-dimensional Schrödinger equation with the most general quadratic in position and momentum Hamiltonian, we were able to determine the evolution operator explicitly in terms of two linearly independent homogeneous solutions and a particular solution to the corresponding classical equation of motion. This allowed us to give exact description of the quantum dynamics and its relation with the corresponding classical motion. According to this, we find the time development of the squeezed coherent states (Atılgan Büyükaşık & Çayiç, 2019), the even-odd coherent states, and the even-odd displaced squeezed states. Then, we analyze their squeezing and displacement properties in detail. We also construct time-dependent quantum dynamical invariants using the evolution operator formalism and study the relationship between the dynamical invariants and quantum states.

The Schrödinger equation subjected to time-dependent moving boundary conditions is another interesting problem. Fermi presented this type of problem related to the study of cosmic radiation in (Fermi, 1949). Then, in many works initial boundary value problems (IBVPs) with moving boundaries were studied. In general, finding solutions of such problems is not possible for an arbitrary boundary function. A well-known approach for solving a moving boundary problem is transforming it into a problem with a fixed boundary. However, exactly solvable models are rare over a fixed line segment of the real line, especially when the potential is time-dependent. In (Makowski & Dembinski, 1991), it was shown that even in the case of a free particle bouncing between two infinitely high walls, there exists an exact solution only when the moving boundary $L(t)$ satisfies $L^3\ddot{L}$ ≡const. Then, in 1992, Makowski introduced a "cut-off oscillator" with a moving infinite potential wall and a time-dependent frequency (Makowski, 1992). He was able to find particular solutions only when the boundary $s(t)$ satisfies $\ddot{s}(t) + \omega^2(t)s(t) = 0$ and realized that moving boundaries generate additional phase factors in the solution, both time-dependent and coordinate-dependent phases. Alternatively, the supersymmetry approach was used to find a class of exactly solvable potentials in (Jana & Roy, 2008).

We introduce an IBVP for a one-dimensional quantum oscillator related to Hamiltonian (1.1) defined on a domain $s(t) < q < \infty$, $0 < t < T$, with a Dirichlet boundary condition imposed at a moving boundary $q = s(t)$. Before finding solutions to problems with moving boundaries, we first consider an IBVP defined on the fixed half-line

$0 < q < \infty$. We note that, time-dependent Schrödinger equation with the most general quadratic Hamiltonian $\hat{H}_g(t)$ given by (1.1) is not symmetric with respect to space inversion and it is not easy to solve the problem on the fixed half-line with Dirichlet boundary condition imposed at $q = 0$. However, when the external fields $D(t) = E(t) = 0$, the Schrödinger equation is invariant under space inversion and we solve the Dirichlet IBVP defined on the fixed half-line analytically. Then, we consider the Dirichlet IBVP for the generalized quantum parametric oscillator described by Hamiltonian (1.1) in the presence of all terms and defined on the domain $s(t) < q < \infty$, $0 < t < T$. We prove that if the boundary can be written as a linear combination of homogenous and particular solutions of the corresponding classical equation of motion in position space, then it is possible to find exact analytical solutions to these problems. Indeed, redefining the coordinate allows us to replace the moving boundary with a fixed one, and consequently, the IBVP with a moving boundary transforms into a fixed half-line problem. In this case, the boundary $s(t)$ generates new terms in the Hamiltonian. Although the transformed Schrödinger equation with the new Hamiltonian is more complicated, by using the Wei-Norman Lie algebraic approach we solve the IBVP for a certain family of moving boundaries. Furthermore, we introduce an IBVP for the generalized quantum parametric oscillator with a Robin boundary condition imposed at a boundary $q = s(t)$ on a domain $s(t) < q < \infty$, $0 < t < T$. We show that if the time-dependent boundary is prescribed in a certain way, the Robin IBVP can be solved analytically.

We also consider the time-evolution problem for a quantum system in higher dimensions. Dynamics in higher dimensions is an extensive area of research since it always brings new questions and attracts more interest . Based on this motivation, we first consider an N-dimensional quantum harmonic oscillator described by the Hamiltonian

$$\hat{H}(t) \equiv \sum_{j=1}^{N} \left(\frac{\hat{p}_j^2}{2\mu_j(t)} + \frac{\mu_j(t)\omega_j^2(t)}{2}\hat{q}_j^2 + \frac{B_j(t)}{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j) + D_j(t)\hat{p}_j + E_j(t)\hat{q}_j + F_j(t) \right), \tag{1.2}$$

where all time-dependent parameters are real valued. The corresponding N-dimensional time-dependent Schrödinger equation is separable, and one can write formal solutions in terms of solutions to the N one-dimensional time-dependent problems. Such multidimensional quantum harmonic oscillators were studied before by different approaches (Burgan, Feix, Fijalkow & Munier, 1979), (Ray & Hartley, 1982), (Malkin & Man'ko, 1979).

To solve one-dimensional non-stationary quantum oscillator problem, we use our results obtained in (Atılgan Büyükaşık & Çayiç, 2016), which are based on Wei-Norman Lie algebraic approach. Then, we focus mainly on the time-development of coherent states of N-dimensional harmonic oscillator. We also aim to investigate explicitly the influence of squeezing parameters $B_j(t)$ and the displacement parameters $D_j(t)$, $E_j(t)$ on the wave packets.

Finally, we consider time-evolution problem for a quantum system described by a generalized two-dimensional quadratic Hamiltonian of the form

$$\hat{\mathcal{H}}_{gen}(t) = \sum_{j=1}^{2} \Big(\frac{\hat{p}_j^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}_j^2 + \frac{B(t)}{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j) + D_j(t)\hat{p}_j + E_j(t)\hat{q}_j \Big) + \lambda(t)\Big(\hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1\Big), \quad (1.3)$$

where all time-dependent parameters are real valued. This Hamiltonian is usually used to describe quantum particles in two-dimensional space and comprises many fundamental physical systems as subcases. A significant physical and mathematical distinction can be done according to the coupling parameter $\lambda(t)$. When $\lambda(t) = 0$, one has a two-dimensional quantum parametric oscillator with time-dependent mass $\mu(t) > 0$, frequency $\omega(t)$, squeezing parameter $B(t)$ and driving forces $D_j(t)$, $E_j(t)$, $j = 1, 2$. Since in that case Hamiltonian (1.3) is separable, formally one can speak about two independent one-dimensional oscillators.

Clearly, wave function solutions of the two-dimensional oscillator described by (1.3) when $\lambda(t) = 0$, can be easily written as a product of solutions to the one-dimensional problem (Malkin, Man'ko & Trifonov, 1970), (Malkin, Man'ko & Trifonov, 1973). On the other hand, when $\lambda(t) \neq 0$, that is in the presence of the angular momentum operator $\hat{L} = \hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1$, Hamiltonian (1.3) can be used to describe the motion of a charged particle in time-dependent magnetic and electric fields. In that context, parameter $\lambda(t)$ is known as the Larmor frequency, $\omega(t)$ is the modulated frequency, and $E_j(t)$, $j = 1, 2$ are parameters of the external electric field. The problem of a charged particle in magnetic and electric fields is addressed in numerous research articles and has applications in electromagnetic theory, quantum optics, plasma physics, etc. For non-stationary systems, including a charged particle in a time-dependent electromagnetic field, Lewis and

Riesenfeld derived explicitly time-dependent quadratic invariants (Lewis & Riesenfeld, 1969). Soon after, Malkin, Man'ko and Trifonov suggested the use of linear in position and momentum invariants (Malkin, Man'ko & Trifonov, 1969), (Malkin, Man'ko & Trifonov, 1970) and constructed two-dimensional coherent states of Gaussian type, that can be considered as a generalization of the Glauber coherent states of the one-dimensional harmonic oscillator. For a recent review of various families of coherent states, squeezed states and their generalizations for a charged particle in a magnetic field, including Gaussian and non-Gaussian states, one can see the work of Dodonov (Dodonov, 2018).

We solve the two-dimensional quantum parametric oscillator described by the generalized quadratic Hamiltonian (1.3) using the evolution operator approach (Atılgan Büyükaşık & Çayiç, 2022). We find the exact evolution operator by first applying a simple unitary transformation to decouple the Schrödinger equation, and then using Wei-Norman Lie algebraic technique. This gives the evolution operator of the problem as a finite product of unitary exponential operators being generators of a Lie group associated with the closed Lie algebra describing the Hamiltonian. A crucial point in the Lie algebraic techniques is to find all time-variable coefficients that completely determine the evolution operator as product of Lie group generators. Usually this requires solution of a large nonlinear system of ordinary differential equations, which is not always an easy task, and in most works it is usually solved by quadratures. The utility of our results is that all time-variable coefficients in the formulation of the evolution operator for the quantum problem are found explicitly in terms of the solutions to the corresponding system of classical equations of motion. Then, the propagator (Green's function), time-evolution of the wave functions, expectations of position and momentum and their uncertainties are also found in terms of the classical solutions. Furthermore, using the evolution operator formalism, we also construct linear and quadratic quantum invariants and compare our results by those obtained using the MMT- and the LR- approaches.

The main goal of this thesis is to provide exact and explicit results of the prescribed evolution problems that allows us to investigate the influence of the time-dependent parameters and external terms on the dynamics of the quantum particle. We focus on the squeezing properties of the wave packets and their trajectories in the presence of time-dependent driving forces. For this purpose, the thesis is organised as follows.

Chapter 2 provides some essential tools that are useful for our further studies.

The coordinate representation of the quantum states, such as coherent states, squeezed coherent states, etc., of the SQHO and their properties are given.

In Chapter 3, we present an IVP for time-dependent Schrödinger equation corresponding to the generalized Hamiltonian $\hat{H}_g(t)$ defined by (1.1). Then using the exact evolution operator of the generalized quantum parametric oscillator, we explicitly obtain the time-evolution of the squeezed coherent states, even-odd coherent states, and even-odd displaced squeezed states of the SQHO. We also find the expectation values and uncertainties of position and momentum. This allows us to investigate the nonclassical properties of quantum states according to the complex parameter α of the displacement operator $\hat{D}(\alpha)$, the complex parameter z of the squeeze operator $\hat{S}(z)$, and the time-dependent parameters of the Hamiltonian $\hat{H}_g(t)$. As an application, we construct an exactly solvable model for a generalized Caldirola-Kanai oscillator. We find the time-evolution of the quantum states and discuss their properties, and construct many illustrative figures.

In Chapter 4, we introduce an IBVP for the generalized quantum parametric oscillator described by the Hamiltonian $\hat{H}_g(t)$ given by (1.1) with a Dirichlet boundary condition imposed at a moving $q = s(t)$ in a domain $s(t) < q < \infty$, $0 < t < T$. We first solve an IBVP for a quantum parametric oscillator with Hamiltonian (1.1) when the external fields are zero, i.e., $D(t) = E(t) = 0$, defined on the fixed half-line with the homogeneous Dirichlet boundary condition imposed at $q = 0$. Then, to solve the IBVP for the generalized oscillator, we pass to a moving coordinate system and this transforms the moving IBVP to a fixed half-line problem. We prove that if the boundary can be written as a linear combination of homogenous and particular solutions of the corresponding classical equation of motion in position space, then it is possible to find exact analytical solutions to these problems. We also provide exact solutions of the IBVP for some particular initial functions and homogeneous boundary condition. As an application, we construct an exactly solvable quantum model with specific frequency modification and analyze the influence of the moving boundaries on the solution. Moreover, we introduce and solve an IBVP for the generalized quantum oscillator with a Robin boundary condition.

In Chapter 5, we provide coordinate representation of the exact time-evolution operator for the N-dimensional Schrödinger equation described by Hamiltonian (1.2). Then, we find the exact time-evolution of the eigenstates and coherent states. As known, solutions of the quantum dynamical problem are completely determined by the solutions of the corresponding classical equations, which could be exactly solvable or not, depending on the parameters of the Hamiltonian. Therefore, many properties of the time-evolved quantum states depend on that parameters. In general, parameters $B_j(t)$ modify the frequency of the classical oscillator and can change it essentially. For this reason, we discuss the corresponding classical equations and introduce all parameters $B_j(t)$ for which the structure of the standard harmonic oscillator in position space is preserved. Choosing sinusoidal parameters $D_j(t)$, we also discuss the classical trajectories of the wave packets. After that, we introduce exactly solvable models and for every model we give some examples and illustrative figures.

In Chapter 6, we introduce the classical Hamiltonian corresponding to (1.3) and find solutions to the associated system of coupled classical equations of motion. Then, we obtain explicitly the evolution operator and the propagator (Green's function or fundamental solution) for the time-dependent Schrödinger equation with Hamiltonian (1.3) in terms of the classical solutions. We also describe the exact time-evolution of harmonic oscillator eigenstates and Glauber coherent states under the influence of the generalized Hamiltonian (1.3). In addition to this, we find the dynamical invariants for the quantum problem and compare the results in the present thesis by those obtained by the MMT- and the LR- techniques. As a generalization of the one-dimensional Cauchy-Euler type dissipative oscillator in (Atılgan Büyükaşık & Çayiç, 2016), we introduce an exactly solvable Cauchy-Euler type quantum parametric oscillator in time-dependent magnetic and electric fields, discuss the dynamical properties of the quantum states and using concrete numerical values we draw some illustrative plots.

Chapter 7 includes brief discussion and concluding remarks.

CHAPTER 2

PRELIMINARIES

In this chapter, we briefly recall the main concepts used in quantum mechanics. Also, for completeness and later use of the results, we review the definition and properties of coherent states, squeezed states and non-Gaussian oscillatory states for the SQHO.

2.1. Basic Concepts of Quantum Mechanics

In this section, we present the postulates of quantum mechanics and some of their important consequences.

Postulate 1 *The state of a quantum mechanical system is completely specified by a complex valued function $\psi(x,t)$, which depends upon the coordinates of the particle(s) and on time, in a Hilbert space. This function is called the wave function or state function and the probability of finding the particle between x and $x+dx$ is proportional to $|\psi(x,t)|^2 dx$.*

The wave function must satisfy certain mathematical conditions according to its probabilistic interpretation. As known, the net probability of finding a single particle at some point in space must be unity. So this leads to the normalization condition, $\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$.

Moreover, the function $\rho(x,t) = |\psi(x,t)|^2$ is called **the probability density function.**

Postulate 2 *To every observable in quantum mechanics, there corresponds a linear Hermitian operator in a Hilbert space.*

This postulate asserts that each quantum observable, such as position, momentum, energy, etc., is mathematically represented by a linear Hermitian operator in an infinite dimensional separable Hilbert space. In quantum mechanics, such operators are called observable operators. As a consequence of this postulate, eigenvalues of the observable operators are real.

Postulate 3 *If the result of a measurement of an observable operator $\hat{A}$ is the number λ, then λ must be one of the eigenvalues satisfying the eigenvalue equation $\hat{A}\psi = \lambda\psi$, where ψ is the corresponding eigenstate.*

According to this postulate, the only real values that can be measured for an observable operator $\hat{A}$ are the eigenvalues of $\hat{A}$. Although measurements must always yield an eigenvalue, the initial state before measuring does not have to be an eigenstate of $\hat{A}$. So it is almost impossible to know which one of these eigenvalues will be obtained in any measurement unless the state of the system is not one of the eigenstates of the operator. To deal with this difficulty, the average of a large number of measurements under identical conditions is introduced as follows:

Definition 2.1 *The expectation value of an observable corresponding to the operator $\hat{A}$ in a state described by a normalized wave function $\psi(x,t)$ is defined by*

$$\langle \hat{A} \rangle_\psi \equiv \langle \psi, \hat{A}\psi \rangle = \int_{-\infty}^{\infty} \psi^* \hat{A} \psi dx.$$

A measurement of an observable $\hat{A}$ in the state ψ which leads to the eigenvalue λ_n, causes the wave function to collapse into the corresponding eigenstate ψ_n for any $n = 0, 1, 2, \ldots$. Thus, measurement affects the state of the system.

Moreover, as a consequence of this postulate, the eigenvectors corresponding to different eigenvalues of an observable operator $\hat{A}$ form a complete orthonormal set, that is, for all n, m, one has $\langle \psi_n, \psi_m \rangle = \delta_{nm}$ and any state vector $\psi(x)$ can be expanded in terms of eigenvectors $\{\psi_n\}$ as

$$\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) = \sum_{n=1}^{\infty} \langle \psi_n, \psi \rangle \psi_n(x). \tag{2.1}$$

Namely, the set of eigenvectors $\{\psi_n\}$ form an orthonormal basis for the corresponding Hilbert space. The equation (2.1) implies that for a normalized state vector ψ, $\|\psi\|^2 = \sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |\langle \psi_n, \psi \rangle|^2 = 1$. Also, one can easily find that $\langle \hat{A} \rangle_\psi = \sum_{n=1}^{\infty} |\langle \psi_n, \psi \rangle|^2 \lambda_n$. Hence, $|c_n|^2 = |\langle \psi_n, \psi \rangle|^2$ can be interpreted as the probability that the measurement will yield the eigenvalue λ_n of $\hat{A}$ in the normalized state $\psi(x)$.

Postulate 4 *The wave function of a system evolves in time according to the time-dependent Schrödinger equation*

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi(x,t),$$

where $\hat{H}$ is a linear Hermitian operator acting in the complex Hilbert space $L_2(\mathbb{R})$, called the Hamiltonian or energy operator.

If we consider an initial value problem (IVP) for a time-dependent Schrödinger equation

$$\begin{cases} i\hbar\dfrac{\partial\psi}{\partial t} = \hat{H}\psi(x,t), \\ \psi(x,t_0) = \psi_0(x), \end{cases} \tag{2.2}$$

then the solution is completely determined by the evolution operator $\hat{U}(t,t_0)$, which carries the initial state $\psi(x,t_0)$ into the state $\psi(x,t)$ at later time t, that is, $\psi(x,t) = \hat{U}(t,t_0)\psi(x,t_0)$. Substituting this into the IVP (2.2), one obtains the IVP for the operator equation

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}\hat{U}(t,t_0), \\ \hat{U}(t_0,t_0) = \hat{I}, \end{cases} \tag{2.3}$$

which can be seen as the definition of the evolution operator. We note that, the evolution operator of a quantum system with a time-dependent Hermitian Hamiltonian is unitary.

We also give other essential tools for later use.

Definition 2.2 ***The uncertainty*** $(\Delta\hat{A})_\psi$ *of an observable operator $\hat{A}$ is defined by the square root of the expectation value of $(\hat{A} - \langle\hat{A}\rangle_\psi)^2$ in the normalized state ψ in which $\langle\hat{A}\rangle_\psi$ is computed.*

Theorem 2.1 *(Debnath and Mikusiński, 2005) For any Hermitian operator $\hat{A}$ and any normalized state ψ, we have*

(i) $(\Delta\hat{A})^2 = \langle\hat{A}^2\rangle_\psi - \langle\hat{A}\rangle_\psi^2$,

(ii) $\langle\hat{A}^2\rangle_\psi = \|\hat{A}\psi\|^2$.

Theorem 2.2 *(**Uncertainty Principle**) (Debnath and Mikusiński, 2005) Let $\hat{A}$ and $\hat{B}$ be two Hermitian operators on a Hilbert space H, then for any state vector ψ*

$$(\Delta\hat{A})_\psi(\Delta\hat{B})_\psi \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|,$$

where the commutator is defined by $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

Corollary 2.1 *For any state vector ψ, **the Heisenberg uncertainty principle** states that*

$$(\Delta\hat{x})_\psi(\Delta\hat{p})_\psi \geq \frac{\hbar}{2}.$$

Definition 2.3 *States for which the Heisenberg uncertainty principle holds with equality are called **the minimum uncertainty states**.*

In what follows, we focus on the properties of dynamical invariants (integrals of the motion) of a quantum system. As known, solution of the IVP for a time-dependent Schrödinger equation is completely determined by the evolution operator $\hat{U}(t,t_0)$. Here, we give the relation between the dynamical invariants and the evolution operator. The following definitions and results can be found in the work (Man'ko, 1987).

Definition 2.4 *A **quantum dynamical invariant** (integral of the motion) is an operator $\hat{I}(t)$, acting on the space of states of the physical system, whose expectation value at these states does not change with time, that is, $d\langle\hat{I}\rangle_\psi/dt = 0$.*

Proposition 2.1 *An operator $\hat{I}(t)$ is a dynamical invariant for the Schrödinger equation (2.2) if and only if*

$$\left(\frac{\partial\hat{I}(t)}{\partial t} - [\hat{H}(t),\hat{I}(t)]\right)\psi(x,t) = 0$$

for any $\psi(x,t)$ being arbitrary solution of the Schrödinger equation.

Proposition 2.2 *An operator $\hat{I}(t)$ is a dynamical invariant for the Schrödinger equation (2.2) if and only if it has the form $\hat{I}(t) = \hat{U}(t,t_0)\hat{I}(t_0)\hat{U}^\dagger(t,t_0)$, where $\hat{U}(t,t_0)$ is the evolution* *operator for the IVP (2.2).*

Proposition 2.3 *The eigenvalues of an integral of the motion do not depend on time.*

Proposition 2.4 *An integral of the motion takes a solution of the Schrödinger equation into a solution of this same equation.*

2.2. Coherent States and Nonclassical States

This section consists of definition and properties of coherent states, which are considered as the most classical states, see (Perelomov, 1986) for further details. Then, squeezed coherent states and superposition of two coherent states are given and their nonclassical properties are analyzed.

2.2.1. Coherent States

First, we consider the SQHO defined by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2} + \frac{\omega_0^2}{2}\hat{q}^2, \tag{2.4}$$

where $\omega_0 > 0$ is the natural frequency, mass is $m = 1$, $\hat{q} = q$ is the position operator, and $\hat{p} = -i\hbar(\partial/\partial q)$ is the momentum operator such that $[\hat{q}, \hat{p}] = i\hbar$. Then, by introducing the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$, where

$$\hat{a} = \sqrt{\frac{\omega_0}{2\hbar}}q + \sqrt{\frac{\hbar}{2\omega_0}}\frac{\partial}{\partial q}, \quad \hat{a}^\dagger = \sqrt{\frac{\omega_0}{2\hbar}}q - \sqrt{\frac{\hbar}{2\omega_0}}\frac{\partial}{\partial q}, \tag{2.5}$$

are the annihilation and the creation operators, respectively, one may rewrite $\hat{H}_0 = \hbar\omega_0(\hat{N} + 1/2)$. The operators $\hat{a}$, $\hat{a}^\dagger$ and $\hat{N}$ satisfy the commutation relations $[\hat{a}, \hat{a}^\dagger] = \hat{I}$, $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$, $[\hat{N}, \hat{a}] = -\hat{a}$. So we have the spectrum generating algebra $\{\hat{I}, \hat{a}^\dagger, \hat{a}, \hat{N}\}$.

The eigenvalue problem for $\hat{H}_0$ is $\hat{H}_0\varphi_n(q) = E_n\varphi_n(q)$, which is also known as the time-independent Schrödinger equation. If $\varphi_n(q)$ is the eigenstate of $\hat{H}_0$ corresponding to eigenvalue E_n, then $\hat{a}\varphi_n(q)$ and $\hat{a}^\dagger\varphi_n(q)$ will also be eigenstates of $\hat{H}_0$ corresponding to eigenvalues $E_n - \hbar\omega_0$ and $E_n + \hbar\omega_0$, respectively. Therefore, the annihilation operator $\hat{a}$ reduces the energy of the state, while the creation operator $\hat{a}^\dagger$ raises it. The ground state of a system is the state with the lowest energy. So the ground state $\varphi_0(q)$ of the SQHO can be found by solving the equation $\hat{a}\varphi_0(q) = 0$. In normalized form, it will be $\varphi_0(q) = (\omega_0/\pi\hbar)^{1/4}e^{-\frac{\omega_0}{2\hbar}q^2}$. Then, by applying the creation operator to the ground state, one can find all other eigenstates of $\hat{H}_0$, which are given as

$$\varphi_n(q) = N_n e^{-\frac{\omega_0}{2\hbar}q^2} H_n(\sqrt{\omega_0/\hbar}q), \quad n = 0, 1, 2, \ldots, \tag{2.6}$$

where $E_n = (n + 1/2)\hbar\omega_0$ are the corresponding eigenvalues, $H_n(q)$ represents the n-th order Hermite polynomial and $N_n = (2^n n!)^{-1/2}(\omega_0/\pi\hbar)^{1/4}$ is the normalization constant. The collection of eigenstates $\{\varphi_n(q)\}$ of the Hamiltonian $\hat{H}_0$ forms an orthonormal basis for the space $L_2(\mathbb{R})$.

Now, we introduce coherent states of SQHO, which are discovered by Schrödinger as non-spreading wave packets in 1926, (Schrödinger, 1926). Then, after the work of Glauber (Glauber, 1963), the name 'coherent states' appeared for the first time. Coherent states of SQHO, which are also called Glauber coherent states, can be defined using different, but equivalent approaches:

(i) As minimum uncertainty states they satisfy $(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha = \hbar/2$ with equal uncertainties in both quadratures ($\omega_0 = 1$). Their motion follows the classical trajectory, and so they are the closest analogs to the classical states.

(ii) Displacement operator coherent states are obtained by applying the unitary **displacement operator**

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}), \quad \alpha \in \mathbb{C}, \tag{2.7}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are define by (2.5), to the ground state $\varphi_0(q)$ of the SQHO.

(iii) Coherent states are also known as the eigenstates of the annihilation operator $\hat{a}$ since they satisfy the eigenvalue equation $\hat{a}\phi_\alpha(q) = \alpha\phi_\alpha(q)$ for any $\alpha \in \mathbb{C}$.

Some important properties of coherent states are listed below:
For any $\alpha \in \mathbb{C}$, coherent states of the SQHO can be represented in terms of energy eigenstates (2.6) of $\hat{H}_0$ as

$$\phi_\alpha(q) = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \varphi_n(q). \tag{2.8}$$

Coherent states of the SQHO are not orthogonal. Actually, for any $\alpha, \beta \in \mathbb{C}$, they satisfy

$$\langle \phi_\alpha(q), \phi_\beta(q) \rangle = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^* \beta} \neq 0, \tag{2.9}$$

and the closure relation

$$\frac{1}{\pi} \int_{\mathbb{C}} |\phi_\alpha\rangle\langle\phi_\alpha| d^2\alpha = \hat{I}.$$

Consequently, the collection of coherent states $\{\phi_\alpha(q)\}_{\alpha\in\mathbb{C}}$ forms an overcomplete set.

The coordinate representation of coherent states can be found explicitly by using the displacement operator formalism.

Proposition 2.5 *The displacement operator $\hat{D}(\alpha)$ can be written as a product of exponential operators, which are group generators associated with the Heisenberg-Weyl algebra defined by*

$$\hat{E}_1 = iq, \quad \hat{E}_2 = \frac{\partial}{\partial q}, \quad \hat{E}_3 = i\hat{I}. \tag{2.10}$$

Proof Using (2.5), the coordinate representation of the displacement operator becomes

$$\hat{D}(\alpha) = \exp\left(-\sqrt{\frac{2\hbar}{\omega_0}} \alpha_1 \frac{\partial}{\partial q} + i \sqrt{\frac{2\omega_0}{\hbar}} \alpha_2 q \right) \tag{2.11}$$

for any $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$. Recall that, Baker-Campbell-Hausdorff (BCH) formula says that for any operators $\hat{X}$ and $\hat{Y}$, if $[\hat{X}, \hat{Y}]$ commutes with both $\hat{X}$ and $\hat{Y}$, then $e^{\hat{X}+\hat{Y}} = e^{-[\hat{X},\hat{Y}]/2} e^{\hat{X}} e^{\hat{Y}}$.

Indeed, the commutator of the operators $\hat{E}_1$ and $\hat{E}_2$ is $[\hat{E}_1, \hat{E}_2] = -\hat{E}_3$. So from BCH formula, the coordinate representation (2.11) of the displacement operator can be rewritten as a product of exponential operators in the form

$$\hat{D}(\alpha) = \exp(-i\alpha_1\alpha_2)\exp\left(i\sqrt{\frac{2\omega_0}{\hbar}}\alpha_2 q\right)\exp\left(-\sqrt{\frac{2\hbar}{\omega_0}}\alpha_1\frac{\partial}{\partial q}\right). \tag{2.12}$$

□

Then, application of the disentangled form (2.12) of $\hat{D}(\alpha)$ to the ground state $\varphi_0(q)$ gives the well-known coherent states of SQHO

$$\phi_\alpha(q) = \left(\frac{\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \times \exp\left[-i\alpha_1\alpha_2\right]\exp\left[i\sqrt{\frac{2\omega_0}{\hbar}}\alpha_2 q\right] \times \exp\left[-\frac{\omega_0}{2\hbar}\left(q-\sqrt{\frac{2\hbar}{\omega_0}}\alpha_1\right)^2\right] \tag{2.13}$$

for any $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$.

2.2.2. Squeezed Coherent States

The squeezed coherent states of SQHO are generalizations of the coherent states, which in the simplest case obey the minimum uncertainty principle, but have less uncertainty in one quadrature at the expense of increased uncertainty in the other. Their main properties were derived by Stoler (Stoler, 1970), (Stoler, 1971), and Yuen (Yuen, 1976). Squeezed states can be defined as a result of applying the squeeze operator.

The **squeeze operator** is a unitary operator mostly known in the form

$$\hat{S}(z) = \exp\left[\frac{1}{2}(z\hat{a}^{\dagger 2} - z^*\hat{a}^2)\right], \quad z = z_1 + iz_2, \ \ z_1, z_2 \in R, \tag{2.14}$$

where operators $\hat{a}$ and $\hat{a}^\dagger$ are given by (2.5), (Stoler, 1970), (Nieto, 1996). In coordinate representation, it becomes

$$\hat{S}(z_1, z_2) = \exp\left[-\frac{i}{\hbar\omega_0}z_2\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} - \frac{\omega_0^2}{2}q^2\right) - z_1\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right]. \tag{2.15}$$

Now, using polar representation $z = re^{i\theta}$, with $r \geq 0$, $\theta \in [0, 2\pi)$, one can write the squeeze operator as

$$\hat{S}(r,\theta) = \exp\left[r\left(i\frac{\omega_0}{2\hbar}(\sin\theta)q^2 - (\cos\theta)(q\frac{\partial}{\partial q} + \frac{1}{2}) + i\frac{\hbar}{2\omega_0}\sin\theta\frac{\partial^2}{\partial q^2}\right)\right]. \tag{2.16}$$

Then, it can be disentangled as a product of exponential operators, which are generators of the SU(1,1) group corresponding to Lie algebra defined by

$$\hat{K}_- = -\frac{i}{2}\frac{\partial^2}{\partial q^2}, \qquad \hat{K}_+ = \frac{i}{2}q^2, \qquad \hat{K}_0 = \frac{1}{2}(q\frac{\partial}{\partial q} + \frac{1}{2}), \tag{2.17}$$

that is,

$$\hat{S}(r,\theta) = \exp\left[\frac{i}{2}f_\theta(r)q^2\right]\exp\left[h_\theta(r)\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right]\exp\left[-\frac{i}{2}g_\theta(r)\frac{\partial^2}{\partial q^2}\right], \tag{2.18}$$

where $f_\theta(r), g_\theta(r)$ and $h_\theta(r)$ are real-valued functions. Indeed, taking the derivative with respect to r in (2.16) and (2.18), and comparing the results, we find that

$$\begin{aligned} f_\theta(r) &= \frac{\omega_0}{\hbar\sin\theta}\left(\frac{y'_{1,\theta}(r)}{y_{1,\theta}(r)}\right), \quad f_\theta(0) = 0, \\ g_\theta(r) &= -\frac{\hbar\sin\theta}{\omega_0}y^2_{1,\theta}(0)\left(\frac{y_{2,\theta}(r)}{y_{1,\theta}(r)}\right), \quad g_\theta(0) = 0, \\ h_\theta(r) &= -\left(r\cos\theta - \ln\left|\frac{y_{1,\theta}(r)}{y_{1,\theta}(0)}\right|\right), \quad h_\theta(0) = 0, \end{aligned}$$

where $y_{1,\theta}(r),\ y_{2,\theta}(r)$ are two independent solutions of the classical inverted oscillator

$$y''_\theta(r) + 2(\cos\theta)y'_\theta(r) - (\sin^2\theta)y_\theta(r) = 0, \quad r \geq 0, \quad 0 \leq \theta < 2\pi, \tag{2.19}$$

satisfying the initial conditions $y_{1,\theta}(0) = y_0 \neq 0$, $y'_{1,\theta}(0) = 0$; $y_{2,\theta}(0) = 0$, $y'_{2,\theta}(0) = 1/y_0$ (prime denotes derivative with respect to r). In terms of solutions of this differential equation, the squeeze operator becomes

$$\hat{S}(r,\theta) = \sqrt{\frac{y_0}{e^{r\cos\theta}y_{1,\theta}(r)}}\exp\Big[\frac{i\omega_0}{2\hbar\sin\theta}\Big(\frac{y'_{1,\theta}(r)}{y_{1,\theta}(r)}\Big)q^2\Big]$$
$$\exp\Big[-\Big(r\cos\theta - \ln\Big|\frac{y_{1,\theta}(r)}{y_0}\Big|\Big)q\frac{\partial}{\partial q}\Big]\exp\Big[-\frac{i\hbar\sin\theta}{2\omega_0}y_0^2\Big(\frac{y_{2,\theta}(r)}{y_{1,\theta}(r)}\Big)\frac{\partial^2}{\partial q^2}\Big],$$

and since $y_{1,\theta}(r) = y_0 e^{-r\cos\theta}(\cosh r + \cos\theta\sinh r)$, $y_{2,\theta}(r) = y_0^{-1}e^{-r\cos\theta}\sinh r$, we have explicitly

$$\begin{aligned}\hat{S}(r,\theta) &= \frac{1}{\sqrt{\cosh r + \cos\theta\sinh r}} \times \exp\Big[\frac{i\omega_0}{2\hbar}\Big(\frac{\sin\theta\sinh r}{\cosh r + \cos\theta\sinh r}\Big)q^2\Big] \\ &\times\exp\Big[-\ln(\cosh r + \cos\theta\sinh r)q\frac{\partial}{\partial q}\Big] \\ &\times\exp\Big[\frac{i\hbar}{2\omega_0}\Big(\frac{\sin\theta\sinh r}{\cosh r + \cos\theta\sinh r}\Big)\frac{\partial^2}{\partial q^2}\Big]. \end{aligned} \tag{2.20}$$

This form of the operator, which we derived in (Atılgan Büyükaşık & Çayiç, 2019), coincides with the squeeze operator derived by Nieto in (Nieto, 1996), but with slightly different approach.

The **squeezed coherent states** of SQHO, which we denote by $\chi^0_{\alpha,r,\theta}(q)$ are defined by

$$\chi^0_{\alpha,r,\theta}(q) = \hat{D}(\alpha)\hat{S}(r,\theta)\varphi_0(q). \tag{2.21}$$

Applying first the squeeze operator, then the displacement operator to the ground state, we explicitly get

$$\begin{aligned}\chi^0_{\alpha,r,\theta}(q) &= \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{e^{-i\alpha_1\alpha_2}}{\sqrt{S^0_{r,\theta}}} \times \exp\Big[-\frac{i}{2}\int_0^r \frac{\sin\theta}{(S^0_{r,\theta})^2}dr\Big]\exp\Big[i\alpha_2\sqrt{\frac{2\omega_0}{\hbar}}q\Big] \\ &\times\exp\Big[\frac{i\omega_0}{2\hbar}\sin\theta\sinh(2r)\Big(\frac{q-\alpha_1\sqrt{2\hbar/\omega_0}}{S^0_{r,\theta}}\Big)^2\Big] \\ &\times\exp\Big[-\frac{\omega_0}{2\hbar}\Big(\frac{q-\alpha_1\sqrt{2\hbar/\omega_0}}{S^0_{r,\theta}}\Big)^2\Big], \end{aligned} \tag{2.22}$$

where

$$S^0_{r,\theta} = e^{r\cos\theta}\sqrt{\frac{(y_{1,\theta}(r))^2 + y_0^4 \sin^2\theta (y_{2,\theta}(r))^2}{y_0^2}} = \sqrt{\cosh 2r + \cos\theta \sinh 2r}, \tag{2.23}$$

denotes the initial squeezing, that is $S^0_{r,\theta}$ is the squeezing coefficient due to the action of the squeeze operator $\hat{S}(r,\theta)$ on the ground state. We note that for $\alpha = 0$, Eq.(2.22) gives the squeezed ground state, which we denote by $\chi^0_{r,\theta}(q)$.

It is not difficult to show that for $\alpha = \alpha_1 + i\alpha_2$ with α_1, α_2 being real numbers, expectation values of coordinate and momentum at squeezed states are the same as for the coherent states and found as $\langle\hat{q}\rangle_\alpha = \sqrt{2\hbar/\omega_0}\alpha_1$ and $\langle\hat{p}\rangle_\alpha = \sqrt{2\omega_0\hbar}\alpha_2$, respectively. On the other hand, uncertainties at $\chi^0_{\alpha,r,\theta}(q)$ are

$$(\Delta\hat{q})^0_{r,\theta} = \sqrt{\frac{\hbar}{2\omega_0}}S^0_{r,\theta}, \quad (\Delta\hat{p})^0_{r,\theta} = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{S^0_{r,\theta}}\sqrt{1+\sin^2\theta\sinh^2(2r)},$$
$$(\Delta\hat{q}\Delta\hat{p})^0_{r,\theta} = \frac{\hbar}{2}\sqrt{1+\sin^2\theta\sinh^2(2r)}.$$

Clearly, the uncertainties depend on the squeezing parameters r and θ. From these results, it can be seen that squeezed coherent states are minimum uncertainty states only when z is real. Moreover, according to some special values of the phase θ in $z = r\exp(i\theta)$ uncertainties are as follows:

(1) If $\theta = 0$ and $\theta = \pi$, $(z = \pm r)$, then $S^0_{r,\theta} = e^{\pm r}$, and

$$(\Delta\hat{q})^0 = \sqrt{\frac{\hbar}{2\omega_0}}e^{\pm r}, \quad (\Delta\hat{p})^0 = \sqrt{\frac{\omega_0\hbar}{2}}e^{\mp r}, \quad (\Delta\hat{q}\Delta\hat{p})^0 = \frac{\hbar}{2}.$$

Thus, when $z = r$, one has minimum uncertainty state, which for large values of r spreads in position space, and is highly localized in momentum space. When $z = -r$, minimum uncertainty state is highly localized in position for large r, at the expense of spreading in momentum.

(2) If $\theta = \pi/2$ and $\theta = 3\pi/2$, $(z = \pm ir)$, then $S^0_{r,\theta} = \sqrt{\cosh 2r}$, and

$$(\Delta\hat{q})^0 = \sqrt{\frac{\hbar}{2\omega_0}}\sqrt{\cosh 2r}, \quad (\Delta\hat{p})^0 = \sqrt{\frac{\omega_0\hbar}{2}}\sqrt{\cosh 2r}, \quad (\Delta\hat{q}\Delta\hat{p})^0 = \frac{\hbar}{2}\cosh 2r.$$

In that case, the state is not minimum uncertainty, and uncertainties increase with increasing r.

Squeezed coherent states of SQHO may be defined in an alternative way following the approach of Yuen, (Yuen, 1976). In this definition, the state is generated by applying the displacement operator and then the squeeze operator on the ground state

$$\Upsilon^0_{\beta,r,\theta}(q) = \hat{S}(r,\theta)\hat{D}(\beta)\varphi_0(q), \quad \beta = \beta_1 + i\beta_2, \ \beta_1, \beta_2 \in \mathbb{R}, \tag{2.24}$$

where $\hat{D}(\beta)$ and $\hat{S}(r,\theta)$ are introduced by (2.7) and (2.15), respectively.

Consider the operators

$$\hat{b} = \lambda\hat{a} + \mu\hat{a}^\dagger, \quad \hat{b}^\dagger = \lambda\hat{a}^\dagger + \mu^*\hat{a}, \tag{2.25}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are defined by (2.5), $\lambda = \cosh r$, and $\mu = -e^{i\theta}\sinh r$ with $|\lambda|^2 - |\mu|^2 = 1$. So these operators obey the boson commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$. Recall that any transformation that leaves the commutator invariant is called a canonical transformation. The **Bogoliubov transformation** is a canonical transformation that maps the bosonic operators $\hat{a}$ and $\hat{a}^\dagger$ to $\hat{b}$ and $\hat{b}^\dagger$. In addition, a theorem of von Neumann says that every canonical transformation can be represented as a unitary transformation, (Von Neumann, 1931). In fact, using the definition of the unitary operator $\hat{S}(r,\theta)$ given by (2.23) one can show that the operators $\hat{b}$ and $\hat{b}^\dagger$ satisfy the following relations

$$\hat{b} = \hat{S}(r,\theta)\hat{a}\hat{S}^\dagger(r,\theta), \quad \hat{b}^\dagger = \hat{S}(r,\theta)\hat{a}^\dagger\hat{S}^\dagger(r,\theta). \tag{2.26}$$

As a consequence of this, the operators $\hat{b}$ and $\hat{b}^\dagger$ have similar properties with the operators

$\hat{a}$ and $\hat{a}^{\dagger}$, and they are called **pseudo-annihilation and creation operators**.

Proposition 2.6 *(Yuen, 1976) The squeezed states $\Upsilon^{0}_{\beta,r,\theta}(q)$ are eigenstates of the pseudo-annihilation operator $\hat{b}$ with complex eigenvalue β, that is,*

$$\hat{b}\Upsilon^{0}_{\beta,r,\theta}(q) = \beta\Upsilon^{0}_{\beta,r,\theta}(q).$$

Therefore, $\hat{b}$ and $\hat{b}^{\dagger}$ have the same eigenvalues with respect to $\chi^{0}_{\beta,r,\theta}(q)$ as do $\hat{a}$ and $\hat{a}^{\dagger}$ with respect to the coherent states $\phi_{\beta}(q)$. Note that, the pseudo-annihilation operator $\hat{b}$ is defined in terms of the operators $\hat{a}$ and $\hat{a}^{\dagger}$ and due to this the squeezed states $\Upsilon^{0}_{\beta,r,\theta}(q)$ are called **two-photon coherent states** (TPCS), (Yuen, 1976).

Remark 2.1 *The two definitions (2.21) and (2.24) of the squeezed states yield the same state if the displacement parameters $\alpha, \beta \in \mathbb{C}$ are related by*

$$\alpha = \cosh r\beta + e^{i\theta}\sinh r\beta^{*} \tag{2.27}$$

for any $r \geq 0$, $\theta \in [0, 2\pi)$.

2.2.3. Even and Odd Coherent States

Using the operator $\hat{D}(\alpha)$ given by (2.7), one can construct the following displacement operators

$$\begin{aligned} \hat{D}_{+}(\alpha) &= \cosh(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}) = \tfrac{1}{2}(\hat{D}(\alpha) + \hat{D}(-\alpha)), \\ \hat{D}_{-}(\alpha) &= \sinh(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}) = \tfrac{1}{2}(\hat{D}(\alpha) - \hat{D}(-\alpha)), \end{aligned} \tag{2.28}$$

where $\hat{a}$ and $\hat{a}^{\dagger}$ are given by (2.5). Then, the functions defined by

$$\phi^{e}_{\alpha}(q) = \hat{D}_{+}(\alpha)\varphi_{0}(q), \quad \phi^{o}_{\alpha}(q) = \hat{D}_{-}(\alpha)\varphi_{0}(q) \tag{2.29}$$

are even and odd, respectively, with respect to α and q. Therefore, they are called **even and odd coherent states** in (Dodonov,1974). Actually, these states are superpositions of coherent states

$$\phi^e_\alpha(q) = N_+(\phi(q,\alpha) + \phi(q,-\alpha)), \quad \phi^o_\alpha(q) = N_-(\phi(q,\alpha) - \phi(q,-\alpha)), \tag{2.30}$$

where we denote $\phi_\alpha(q) \equiv \phi(q,\alpha)$, and using the property (2.9), it is easy to show that they are orthogonal, i.e., $\langle\phi^e_\alpha(q), \phi^o_\beta(q)\rangle = 0$ for any complex numbers α, β. The normalization constants for even and odd coherent states are respectively

$$N_+ = \frac{e^{|\alpha|^2/2}}{2\sqrt{\cosh|\alpha|^2}}, \quad N_- = \frac{e^{|\alpha|^2/2}}{2\sqrt{\sinh|\alpha|^2}}.$$

Here, we give some important properties of even and odd coherent states, details of which can be found in the work (Gerry, 1993).

Proposition 2.7 *The annihilation operator $\hat{a}$ acts on even and odd coherent states as*

$$\hat{a}\phi^e_\alpha(q) = \alpha\sqrt{\tanh|\alpha|^2}\phi^o_\alpha(q), \quad \hat{a}\phi^o_\alpha(q) = \alpha\sqrt{\coth|\alpha|^2}\phi^e_\alpha(q),$$

where $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, and $|\alpha|^2 = \alpha_1^2 + \alpha_2^2$.

Therefore, unlike coherent cases, even and odd coherent states are not eigenstates of the annihilation operator. The eigenvalue equation for the even-odd coherent states are given in the following corollary.

Corollary 2.2 *Even and odd coherent states are eigenstates of the operator $\hat{a}^2$, that is*

$$\hat{a}^2\phi^{e,o}_\alpha(q) = \alpha^2\phi^{e,o}_\alpha(q).$$

Proposition 2.8 *Even and odd coherent states have power series expansions in terms of* *the even and odd eigenstates $\varphi_{2n}(q)$, $\varphi_{2n+1}(q)$, respectively,*

$$\phi^e_\alpha(q) = \frac{1}{\sqrt{\cosh|\alpha|^2}}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{(2n)!}}\varphi_{2n}(q),$$
$$\phi^o_\alpha(q) = \frac{1}{\sqrt{\sinh|\alpha|^2}}\sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}}\varphi_{2n+1}(q), \quad \alpha \in \mathbb{C}.$$

For any complex number α, the explicit form of normalized even and odd coherent states are

$$\phi^e_\alpha(q) = \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{e^{-\alpha^2/2}}{\sqrt{\cosh|\alpha|^2}}\cosh\left(\sqrt{\frac{2\omega_0}{\hbar}}\alpha q\right)\exp\left(-\frac{\omega_0}{2\hbar}q^2\right), \tag{2.31}$$

$$\phi^o_\alpha(q) = \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{e^{-\alpha^2/2}}{\sqrt{\sinh|\alpha|^2}}\sinh\left(\sqrt{\frac{2\omega_0}{\hbar}}\alpha q\right)\exp\left(-\frac{\omega_0}{2\hbar}q^2\right). \tag{2.32}$$

Squeezing properties:

As known, squeezing exists in position or momentum variable if the variance of one of the operators is smaller than the value $\sqrt{\hbar/2}$.

The expectation value of position and momentum operators at the even and odd coherent states are zero, that is $\langle\hat{q}\rangle^{e,o}_\alpha = 0$, and $\langle\hat{p}\rangle^{e,o}_\alpha = 0$ for any $\alpha \in \mathbb{C}$. Then, the variances in position and momentum operators at even coherent states are found as

$$(\Delta\hat{q})^e_\alpha = \sqrt{\frac{\hbar}{2\omega_0}}\sqrt{1+2|\alpha|^2\tanh|\alpha|^2+2\Re(\alpha^2)},$$
$$(\Delta\hat{p})^e_\alpha = \sqrt{\frac{\omega_0\hbar}{2}}\sqrt{1+2|\alpha|^2\tanh|\alpha|^2-2\Re(\alpha^2)},$$

where $\Re(\alpha^2)$ denotes the real part of α^2. Using polar representation, $\alpha = \upsilon e^{i\vartheta}$, (for simplicity take $\omega_0 = 1$), the variances at even coherent states are

$$(\Delta\hat{q})^e_{\upsilon,\vartheta} = \sqrt{\frac{\hbar}{2}}\sqrt{1 + 2\upsilon^2(\tanh\upsilon^2 + \cos 2\vartheta)},$$
$$(\Delta\hat{p})^e_{\upsilon,\vartheta} = \sqrt{\frac{\hbar}{2}}\sqrt{1 + 2\upsilon^2(\tanh\upsilon^2 - \cos 2\vartheta)}.$$

So they exhibit squeezing in momentum when $\vartheta \in [0, \pi/4) \bigcup (3\pi/4, \pi)$ since $\tanh\upsilon^2 < 1$ for all $\upsilon > 0$. Squeezing in position exists when $\vartheta \in (\pi/4, 3\pi/4)$. Moreover, for $\vartheta = 0$, one has the maximum squeezing in momentum while maximum squeezing in position is for $\vartheta = \pi/2$.

The variances in position and momentum operators at odd coherent states become

$$(\Delta\hat{q})^o_\alpha = \sqrt{\frac{\hbar}{2\omega_0}}\sqrt{1 + 2|\alpha|^2\coth|\alpha|^2 + 2\Re(\alpha^2)},$$
$$(\Delta\hat{p})^o_\alpha = \sqrt{\frac{\omega_0\hbar}{2}}\sqrt{1 + 2|\alpha|^2\coth|\alpha|^2 - 2\Re(\alpha^2)}.$$

In polar representation, they are

$$(\Delta\hat{q})^o_{\upsilon,\vartheta} = \sqrt{\frac{\hbar}{2}}\sqrt{1 + 2\upsilon^2(\coth\upsilon^2 + \cos 2\vartheta)},$$
$$(\Delta\hat{p})^o_{\upsilon,\vartheta} = \sqrt{\frac{\hbar}{2}}\sqrt{1 + 2\upsilon^2(\coth\upsilon^2 - \cos 2\vartheta)}.$$

Since $\coth\vartheta > 1$ for any $\upsilon > 0$, there is no squeezing in odd coherent states.

So, the uncertainty relation at even and odd coherent states follow as

$$(\Delta\hat{q})^e_\alpha(\Delta\hat{p})^e_\alpha = \frac{\hbar}{2}\sqrt{\left(1 + 2|\alpha|^2\tanh|\alpha|^2\right)^2 - 4\left(\Re(\alpha^2)\right)^2},$$
$$(\Delta\hat{q})^o_\alpha(\Delta\hat{p})^o_\alpha = \frac{\hbar}{2}\sqrt{\left(1 + 2|\alpha|^2\coth|\alpha|^2\right)^2 - 4\left(\Re(\alpha^2)\right)^2},$$

and they satisfy the Heisenberg uncertainty principle, i.e, $(\Delta\hat{q}\Delta\hat{p})^e_\alpha > \hbar/2$, and $(\Delta\hat{q}\Delta\hat{p})^o_\alpha > \hbar/2$. These relations show that, even and odd coherent states are not minimum uncertainty states and the uncertainty relation depends on the value of complex number α.

2.2.4. Even and Odd Displaced Squeezed States

Even and odd displaced squeezed states are constructed by applying the displacement operators $\hat{D}_+(\alpha)$ and $\hat{D}_-(\alpha)$ defined by equations (2.28) on the squeezed ground state

$$\chi^e_{\alpha,r,\theta}(q) = \hat{D}_+(\alpha)\hat{S}(r,\theta)\varphi_0(q), \quad \chi^o_{\alpha,r,\theta}(q) = \hat{D}_-(\alpha)\hat{S}(r,\theta)\varphi_0(q) \tag{2.33}$$

for any $\alpha \in \mathbb{C}$, $r \geq 0$, $\theta \in [0,\pi)$. Denoting the squeezed coherent states $\chi^0_{\alpha,r,\theta}(q) \equiv \chi_{r,\theta}(q,\alpha)$, we notice that even and odd displaced squeezed states are superpositions of squeezed coherent states

$$\chi^e_{\alpha,r,\theta}(q) = N_e\left(\chi_{r,\theta}(q,\alpha) + \chi_{r,\theta}(q,-\alpha)\right), \quad \chi^o_{\alpha,r,\theta}(q) = N_o\left(\chi_{r,\theta}(q,\alpha) - \chi_{r,\theta}(q,-\alpha)\right).$$

Taking, for simplicity, $\theta = 0$, which corresponds to the case $z \in \mathbb{R}$ in (2.14), normalized forms of these states become

$$\chi^e_{\alpha,r}(q) = \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4} \frac{e^{-\alpha_r^2/2}}{e^{r/2}\sqrt{\cosh\alpha_r}} \cosh\left(\sqrt{\frac{2\omega_0}{\hbar}}\alpha_r q\right) \exp\left(-\frac{\omega_0}{2\hbar}\left(\frac{q^2}{e^{2r}}\right)\right),$$
$$\chi^o_{\alpha,r}(q) = \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4} \frac{e^{-\alpha_r^2/2}}{e^{r/2}\sqrt{\sinh\alpha_r}} \sinh\left(\sqrt{\frac{2\omega_0}{\hbar}}\alpha_r q\right) \exp\left(-\frac{\omega_0}{2\hbar}\left(\frac{q^2}{e^{2r}}\right)\right),$$

where we denote $\alpha_r = e^{-r}\alpha_1 + ie^r\alpha_2$ with α_1, α_2 being real numbers.

Expectation values of position and momentum operators at the even and odd displaced squeezed states are zero, that is $\langle\hat{q}\rangle^{e,o}_{\alpha,r} = 0$ and $\langle\hat{p}\rangle^{e,o}_{\alpha,r} = 0$ for any $\alpha = \alpha_1 + i\alpha_2$,

$\alpha_1, \alpha_2 \in \mathbb{R}$. Then, the uncertainties at even displaced squeezed states become

$$(\Delta\hat{q})^e_{\alpha,r} = \sqrt{\frac{\hbar}{2\omega_0}}e^r\sqrt{1+2|\alpha_r|^2\tanh|\alpha_r|^2+2\Re(\alpha_r^2)},$$
$$(\Delta\hat{p})^e_{\alpha,r} = \sqrt{\frac{\omega_0\hbar}{2}}e^{-r}\sqrt{1+2|\alpha_r|^2\tanh|\alpha_r|^2-2\Re(\alpha_r^2)}.$$

In polar representation $\alpha_r = \upsilon_r e^{\vartheta_r}$, $\upsilon_r \geq 0$, $\vartheta_r \in [0, 2\pi)$, (when $\omega_0 = 1$,) uncertainties in position and momentum operators at even displaced squeezed states become

$$(\Delta\hat{q})^e_{\upsilon_r,\vartheta_r} = \sqrt{\frac{\hbar}{2}}e^r\sqrt{1+2\upsilon_r^2(\tanh\upsilon_r^2+\cos 2\vartheta_r)},$$
$$(\Delta\hat{p})^e_{\upsilon_r,\vartheta_r} = \sqrt{\frac{\hbar}{2}}e^{-r}\sqrt{1+2\upsilon_r^2(\tanh\upsilon_r^2-\cos 2\vartheta_r)}.$$

More precisely, for some special values of ϑ_r, squeezing properties are as follows:

(1) For small values of υ_r, since $\tanh\upsilon_r < 1$, squeezing exists in momentum if $\vartheta_r = 0$ or $\vartheta_r = \pi$, while it exists in position for $\vartheta_r = \pi/2$ or $\vartheta_r = 3\pi/2$.

(2) If $\vartheta_r = \pi/4$ or $\vartheta_r = 3\pi/4$, these states spread in position space and they are highly localized in momentum space for large values of υ_r.

The uncertainties at odd displaced squeezed states are

$$(\Delta\hat{q})^o_{\alpha,r} = \sqrt{\frac{\hbar}{2\omega_0}}e^r\sqrt{1+2|\alpha_r|^2\coth|\alpha_r|^2+2\Re(\alpha_r^2)},$$
$$(\Delta\hat{p})^o_{\alpha,r} = \sqrt{\frac{\omega_0\hbar}{2}}e^{-r}\sqrt{1+2|\alpha_r|^2\coth|\alpha_r|^2-2\Re(\alpha_r^2)}.$$

In polar representation form, they become

$$(\Delta\hat{q})^o_{\upsilon_r,\vartheta_r} = \sqrt{\frac{\hbar}{2}}e^r\sqrt{1+2\upsilon_r^2(\coth\upsilon_r^2+\cos 2\vartheta_r)},$$
$$(\Delta\hat{p})^o_{\upsilon_r,\vartheta_r} = \sqrt{\frac{\hbar}{2}}e^{-r}\sqrt{1+2\upsilon_r^2(\coth\upsilon_r^2-\cos 2\vartheta_r)}.$$

Since $\coth \upsilon_r > 1$ for all $\upsilon_r > 0$, squeezing just exists in momentum for large values of r, and there is no squeezing in position space. Finally, uncertainty relations follow as

$$(\Delta\hat{q})^e_\alpha(\Delta\hat{p})^e_\alpha = \frac{\hbar}{2}\sqrt{\left(1+2|\alpha_r|^2\tanh|\alpha_r|^2\right)^2 - 4\left(\Re(\alpha_r^2)\right)^2},$$
$$(\Delta\hat{q})^o_\alpha(\Delta\hat{p})^o_\alpha = \frac{\hbar}{2}\sqrt{\left(1+2|\alpha_r|^2\coth|\alpha_r|^2\right)^2 - 4\left(\Re(\alpha_r^2)\right)^2},$$

and they satisfy the Heisenberg uncertainty relation, that is, $(\Delta\hat{q}\Delta\hat{p})^e_\alpha > \hbar/2$, and $(\Delta\hat{q}\Delta\hat{p})^o_\alpha > \hbar/2$. Also, they show that even and odd displaced squeezed states are not minimum uncertainty states and the uncertainty relation depends on the value of complex number α and the parameter r of the squeeze operator.

Alternative approach:

By replacing the order of application of the operators in the definition of even-odd displaced squeezed states given by (2.33), one can obtain the following states,

$$\Upsilon^e_{\beta,r,\theta}(q) = \hat{S}(r,\theta)\hat{D}_+(\beta)\varphi_0(q) = \hat{S}(r,\theta)\phi^e_\beta(q),$$
$$\Upsilon^o_{\beta,r,\theta}(q) = \hat{S}(r,\theta)\hat{D}_-(\beta)\varphi_0(q) = \hat{S}(r,\theta)\phi^o_\beta(q), \tag{2.34}$$

where $\phi^e_\beta(q)$ and $\phi^o_\beta(q)$ are given by (2.31) and (2.32), respectively. These states can be considered as **squeezed even-odd coherent states**. The definition (2.33) for even-odd displaced squeezed states and the one (2.34) for squeezed even-odd coherent states yield the same state if the complex parameters α and β satisfy the equation $\beta = \cosh r\alpha - e^{i\theta}\sinh r\alpha^*$ for any $r \geq 0$ and $\theta \in [0, 2\pi)$. One can derive the following relations for the squeezed even-odd coherent states.

Proposition 2.9 *(Xin, Wang, Hirayama & Matumoto, 1994) The pseudo-annihilation operator $\hat{b}$ acts on squeezed even and odd coherent states as*

$$\hat{b}\Upsilon^e_\beta(q) = \beta\sqrt{\tanh|\beta|^2}\,\Upsilon^o_\beta(q), \quad \hat{b}\Upsilon^o_\beta(q) = \beta\sqrt{\coth|\beta|^2}\,\Upsilon^e_\beta(q), \quad \beta\in\mathbb{C}.$$

Corollary 2.3 *(Xin, Wang, Hirayama & Matumoto, 1994) Squeezed even and odd co-*

herent states are eigenstates of the operator $\hat{b}^2$, that is,

$$\hat{b}^2\Upsilon_\beta^{e,o}(q) = \beta^2\Upsilon_\beta^{e,o}(q), \quad \beta \in \mathbb{C}.$$

Therefore, the squeezed even-odd coherent states are eigenstates of the operator $\hat{b}^2$ corresponding to the eigenvalue β^2 with β being complex number.

CHAPTER 3

DYNAMICS OF THE GENERALIZED ONE-DIMENSIONAL QUANTUM PARAMETRIC OSCILLATOR

In this chapter, we consider an initial value problem for the Schrödinger equation

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\Psi(q,t) = \hat{H}_g(t)\Psi(q,t), \quad t > 0, \\ \Psi(q,t_0) = \Psi_0(q), \quad -\infty < q < \infty, \end{cases} \tag{3.1}$$

with the most general quadratic Hamiltonian

$$\hat{H}_g(t) = \frac{-\hbar^2}{2\mu(t)}\frac{\partial^2}{\partial q^2} + \frac{\mu(t)\omega^2(t)}{2}q^2 - i\hbar\frac{B(t)}{2}(q\frac{\partial}{\partial q} + \frac{\partial}{\partial q}q) - i\hbar D(t)\frac{\partial}{\partial q} + E(t)q + F(t), \tag{3.2}$$

where $\mu(t) > 0, \omega^2(t) > 0, B(t), D(t), E(t)$, and $F(t)$ are real-valued parameters depending on time. Since the Hamiltonian $\hat{H}_g(t)$ is a linear combination of generators of the $SU(1,1)$ and the Heisenberg-Weyl Lie algebras, the evolution operator for the Schrödinger equation can be obtained using the Wei-Norman algebraic approach (Wei & Norman, 1963), and for details one can see (Atılgan Büyükaşık & Çayiç, 2016). It is a product of exponential operators, corresponding to multiplication, displacement, squeeze and generalized rotation as follows

$$\begin{aligned} \hat{U}_g(t,t_0) &= \exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\zeta(s)ds\right) \times \exp\left(ip_p(t)q\right) \times \exp\left(-x_p(t)\frac{\partial}{\partial q}\right) \\ &\times \exp\left(i\frac{\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)q^2\right) \times \exp\left(\ln\left|\frac{x_1(t_0)}{x_1(t)}\right|\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right) \\ &\times \exp\left(\frac{i}{2}\hbar x_1^2(t_0)\left(\frac{x_2(t)}{x_1(t)}\right)\frac{\partial^2}{\partial q^2}\right), \end{aligned} \tag{3.3}$$

where

$$\zeta(t) = \frac{-\mu(t)}{2}\left[\Big(\dot{x}_p(t) - B(t)x_p(t)\Big)^2 - \omega^2(t)x_p^2(t) - D^2(t) + \frac{2F(t)}{\mu(t)}\right]. \tag{3.4}$$

Here, $x_1(t)$, $x_2(t)$ are linearly independent homogeneous solutions of the classical equation of motion

$$\ddot{x} + \frac{\dot{\mu}}{\mu}\dot{x} + \left(\omega^2(t) - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B\right)\right)x = \dot{D} + \left(\frac{\dot{\mu}}{\mu} + B\right)D - \frac{1}{\mu}E, \tag{3.5}$$

satisfying the initial conditions $x_1(t_0) = x_0 \neq 0$, $\dot{x}_1(t_0) = x_0B(t_0)$, $x_2(t_0) = 0$, $\dot{x}_2(t_0) = 1/\mu(t_0)x_0$, and $x_p(t)$ is a particular solution of (3.5) satisfying $x_p(t_0) = 0$, $\dot{x}_p(t_0) = E(t_0)$. The corresponding equation for momentum is

$$\ddot{p} - \frac{\dot{(\mu\omega^2)}}{\mu\omega^2}\dot{p} + \left(\omega^2 + \left(\dot{B} - B^2 - \frac{\dot{(\mu\omega^2)}}{\mu\omega^2}B\right)\right)p = -\mu\omega^2 D - \dot{E} + \left(\frac{\dot{(\mu\omega^2)}}{\mu\omega^2} + B\right)E \tag{3.6}$$

with homogeneous solutions $p_1(t) = \mu(t)\Big(\dot{x}_1(t) - B(t)x_1(t)\Big)$, $p_2(t) = \mu(t)\Big(\dot{x}_2(t) - B(t)x_2(t)\Big)$, and particular solution $p_p(t) = \mu(t)\Big(\dot{x}_p(t) - B(t)x_p(t) - D(t)\Big)$.

3.1. Time-Evolved Coherent States

First, we recall the generalized time-evolved coherent states which are found by applying the displacement and evolution operators to the ground state, that is $\Phi_\alpha(q,t) = \hat{U}_g(t,t_0)\hat{D}(\alpha)\varphi_0(q)$. To be able to compare with the generalized squeezed coherent states derived in next sections, we give their explicit representation as found in (Atılgan Büyükaşık

& Çayiç, 2016), that is,

$$\begin{aligned}\Phi_\alpha(q,t) &= \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{\epsilon(t)}}\times\exp\left[-\frac{1}{2}\left(\frac{(\epsilon^*(t))^2}{|\epsilon(t)|^2}\alpha^2+|\alpha|^2\right)\right]\\ &\times\exp\left[\frac{i}{\hbar}\int_{t_0}^t\zeta(s)ds\right]\times\exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_p(t)-B(t)x_p(t)-D(t)\Big)q\right]\\ &\times\exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon(t)|-B(t)\right)(q-x_p(t))^2\right]\\ &\times\exp\left[\sqrt{\frac{2\omega_0}{\hbar}}\frac{(q-x_p(t))\alpha}{\epsilon(t)}\right]\times\exp\left[\frac{-\omega_0}{2\hbar}\frac{(q-x_p(t))^2}{|\epsilon(t)|^2}\right],\end{aligned} \tag{3.7}$$

where

$$\epsilon(t)=\frac{x_1(t)}{x_0}+i\omega_0x_0x_2(t)=|\epsilon(t)|e^{i\nu(t)}, \tag{3.8}$$

whose modulus and phase are

$$|\epsilon(t)|=\sqrt{\frac{x_1^2(t)}{x_0^2}+(\omega_0x_0)^2x_2^2(t)},\quad \nu(t)=\int_{t_0}^t\frac{d\xi}{\mu(\xi)|\epsilon(\xi)|^2} \tag{3.9}$$

and $\zeta(t)$ is given by (3.4). The corresponding probability densities become

$$\rho_\alpha(q,t)=\sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{|\epsilon(t)|}\exp\Big[-\Big(\frac{\omega_0}{\hbar}\Big)\Big(\frac{q-\langle\hat{q}\rangle_\alpha(t)}{|\epsilon(t)|}\Big)^2\Big], \tag{3.10}$$

where the squeezing coefficient $|\epsilon(t)|$ is given by (3.9) and expectation values are

$$\langle\hat{q}\rangle_\alpha(t)=\sqrt{\frac{2\hbar}{\omega_0}}\left(\frac{\alpha_1}{x_0}x_1(t)+\alpha_2(\omega_0x_0)x_2(t)\right)+x_p(t), \tag{3.11}$$

$$\langle\hat{p}\rangle_\alpha(t)=\sqrt{\frac{2\hbar}{\omega_0}}\Big(\frac{\alpha_1}{x_0}p_1(t)+\alpha_2(\omega_0x_0)p_2(t)\Big)+p_p(t), \tag{3.12}$$

showing that the center of the wave packets follows the classical trajectory. Then, uncertainties in terms of $|\epsilon(t)|$ are of the form

$$
\begin{aligned}
(\Delta\hat{q})_\alpha(t) &= \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon(t)|, \\
(\Delta\hat{p})_\alpha(t) &= \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon(t)|}\sqrt{1+\frac{\mu^2(t)|\epsilon(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon(t)|}{dt}-B(t)\right)^2}, \\
(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha(t) &= \frac{\hbar}{2}\sqrt{1+\frac{\mu^2(t)|\epsilon(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon(t)|}{dt}-B(t)\right)^2}.
\end{aligned}
$$

We note that, if $x_1(t)$, $x_2(t)$ are solutions of the simple harmonic oscillator $\ddot{x}+\omega_0^2 x = 0$ corresponding to Hamiltonian $\hat{H}_0 = \hat{p}^2/2+\omega_0^2\hat{q}^2$, then we have $|\epsilon(t)| = 1$ so there is no squeezing of the wave packets. However, in general $|\epsilon(t)|$ depends on time, which shows that time-evolution of coherent states do not preserve the minimum uncertainty, and squeezing properties depend on parameters $\mu(t), \omega^2(t)$ and $B(t)$ of the Hamiltonian.

3.2. Quantum Dynamical Invariants

In what follows, we construct time-dependent linear invariants for the quantum system using the evolution operator formalism. It is based on the fact that, if time-development of a given quantum system is described by the unitary evolution operator $\hat{U}(t,t_0)$, then any operator of the form $\hat{A}(t) = \hat{U}(t,t_0)\hat{A}(t_0)\hat{U}^\dagger(t,t_0)$ is an integral of motion or a dynamical invariant, (Man'ko, 1987).

For the generalized quantum parametric oscillator with Hamiltonian $\hat{H}_g(t)$ given by (3.2), using the evolution operator (3.3), one can find dynamical invariants that are linear in coordinate and momentum

$$
\hat{A}(t) = \hat{U}_g(t,t_0)\,\hat{a}\,\hat{U}_g^\dagger(t,t_0), \quad \hat{A}^\dagger(t) = \hat{U}_g(t,t_0)\,\hat{a}^\dagger\,\hat{U}_g^\dagger(t,t_0), \tag{3.13}
$$

where the lowering and raising operators, $\hat{a}$ and $\hat{a}^\dagger$ are given by (2.5). Explicit calculations give us

$$\hat{A}(t) = \frac{-i}{\sqrt{2\omega_0\hbar}}\left[\mu(t)\Big(\dot{\epsilon}(t) - B(t)\epsilon(t)\Big)(\hat{q} - x_p(t)) - \epsilon(t)(\hat{p} - p_p(t))\right], \tag{3.14}$$

$$\hat{A}^\dagger(t) = \frac{i}{\sqrt{2\omega_0\hbar}}\left[\mu(t)\Big(\dot{\epsilon}^*(t) - B(t)\epsilon^*(t)\Big)(\hat{q} - x_p(t)) - \epsilon^*(t)(\hat{p} - p_p(t))\right], \tag{3.15}$$

where $\epsilon(t)$ is defined by equation (3.8), and it is a complex solution of the homogeneous part of equation (3.5), that is

$$\ddot{\epsilon}(t) + \frac{\dot{\mu}}{\mu}\dot{\epsilon}(t) + \left(\omega^2(t) - \left(\dot{B}(t) + B^2(t) + \frac{\dot{\mu}}{\mu}B(t)\right)\right)\epsilon(t) = 0 \tag{3.16}$$

satisfying the IC's $\epsilon(t_0) = 1$, $\dot{\epsilon}(t_0) = B(t_0) + i\omega_0/\mu(t_0)$. Therefore, using the Wronskian $W(t) = W(\epsilon(t), \epsilon^*(t)) = \epsilon(t)\dot{\epsilon}^*(t) - \epsilon^*(t)\dot{\epsilon}(t) = -2i\omega_0/\mu(t)$, we can show that the linear invariants $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ that are explicitly given by equations (3.14) and (3.15) satisfy commutation relations $[\hat{A}(t), \hat{A}^\dagger(t)] = 1$, and that can be seen also as **generalized lowering and raising operators**.

Moreover, coherent states $\Phi_\alpha(q,t)$ found in (3.7) by construction are eigenstates of the generalized lowering operator $\hat{A}(t)$ corresponding to complex eigenvalue α. Indeed, if $\phi_\alpha(q)$ are eigenstates of $\hat{a}$ so that $\hat{a}\phi_\alpha(q) = \alpha\phi_\alpha(q)$, then

$$\hat{A}(t)\Phi_\alpha(q,t) = \hat{U}(t,t_0)\hat{a}\hat{U}^\dagger(t,t_0)\hat{U}(t,t_0)\phi_\alpha(q) = \alpha\hat{U}(t,t_0)\phi_\alpha(q),$$

from which it follows $\hat{A}(t)\Phi_\alpha(q,t) = \alpha\Phi_\alpha(q,t)$ for any $\alpha \in \mathbb{C}$.

3.3. Time-Evolved Squeezed Coherent States

In this section, we obtain time evolution of squeezed coherent states under the generalized evolution operator $\hat{U}_g(t,t_0)$, (Atılgan Büyükaşık & Çayiç, 2019). First, we give the time evolution of the squeezed ground state, that is,

$$\chi_{r,\theta}(q,t) = \hat{U}_g(t,t_0)\hat{S}(r,\theta)\varphi_0(q) = \hat{U}_g(t,t_0)\chi^0_{r,\theta}(q), \tag{3.17}$$

which explicitly becomes

$$\begin{aligned}
\chi_{r,\theta}(q,t) \;=\;& \Big(\frac{\omega_0}{\pi\hbar}\Big)^{\frac{1}{4}} \frac{1}{\sqrt{|Q_{r,\theta}(t)|}} \times \exp\Big[-\frac{i}{2}\int_0^r \frac{\sin\theta dr}{(S^0_{r,\theta})^2}\Big] \\
&\times \exp\Big[-\frac{i}{2}\int_{t_0}^t \frac{\omega_0 dt}{\mu(s)|Q_{r,\theta}(s)|^2}\Big] \times \exp\Big[\frac{i}{\hbar}\int_{t_0}^t \zeta(s)ds\Big] \times \exp\Big[\frac{i}{\hbar}p_p(t)q\Big] \\
&\times \exp\Big[\frac{i\omega_0}{2\hbar}\Big[\sin\theta\sinh(2r) + \Big(\frac{x_0^2\omega_0(1+\sin^2\theta\sinh^2(2r))}{(S^0_{r,\theta})^2}\Big)\frac{x_2(t)}{x_1(t)}\Big]\Big(\frac{q-x_p(t)}{|Q_{r,\theta}(t)|}\Big)^2\Big] \\
&\times \exp\Big[\frac{i\mu(t)}{2\hbar}\Big(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\Big)(q-x_p(t))^2\Big] \times \exp\Big[-\frac{\omega_0}{2\hbar}\Big(\frac{q-x_p(t)}{|Q_{r,\theta}(t)|}\Big)^2\Big].
\end{aligned} \tag{3.18}$$

Next, the time evolution of the squeezed coherent states under the generalized evolution operator $\hat{U}_g(t,t_0)$ is found according to

$$\chi_{\alpha,r,\theta}(q,t) = \hat{U}_g(t,t_0)\hat{D}(\alpha)\hat{S}(r,\theta)\varphi_0(q) = \hat{U}_g(t,t_0)\chi^0_{\alpha,r,\theta}(q),$$

and this gives the generalized time-dependent squeezed coherent states in the form

$$\begin{aligned}
\chi_{\alpha,r,\theta}(q,t) =& \Big(\frac{\omega_0}{\pi\hbar}\Big)^{\frac{1}{4}} \frac{1}{\sqrt{|Q_{r,\theta}(t)|}} \exp\Big[-\frac{i}{2}\int_0^r \frac{\sin\theta dr}{(S^0_{r,\theta})^2}\Big] \exp\Big[-\frac{i}{2}\int_{t_0}^t \frac{\omega_0 dt}{\mu(s)|Q_{r,\theta}(s)|^2}\Big] \\
&\exp\Big[\frac{i}{\hbar}\int_{t_0}^t \zeta(s)ds\Big] \exp\Big[\frac{i}{\hbar}p_p(t)q\Big] \exp\Big[\frac{i\mu(t)}{2\hbar}\Big(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\Big)(q-x_p(t))^2\Big] \\
&\exp\Big\{\frac{i\omega_0}{2\hbar}\Big[\sin\theta\sinh(2r) + \Big(\frac{x_0^2\omega_0(1+\sin^2\theta\sinh^2(2r))}{(S^0_{r,\theta})^2}\Big)\frac{x_2(t)}{x_1(t)}\Big] \\
&\times\Big(\frac{(q-x_p(t)) - \sqrt{2\hbar/\omega_0}x_0^{-1}x_1(t)\lambda(\alpha,r,\theta)}{|Q_{r,\theta}(t)|}\Big)^2\Big\} \\
&\exp\Big[-\frac{\omega_0}{2\hbar}\Big(\frac{(q-x_p(t)) - \sqrt{2\hbar/\omega_0}x_0^{-1}x_1(t)\lambda(\alpha,r,\theta)}{|Q_{r,\theta}(t)|}\Big)^2\Big],
\end{aligned} \tag{3.19}$$

where $\lambda(\alpha, r, \theta) = \lambda_1(\alpha, r, \theta) + i\lambda_2(\alpha, r, \theta)$ with

$$\lambda_1(\alpha, r, \theta) = \alpha_1 - \frac{\sin\theta \sinh(2r) S^0_{r,\theta}}{1 + \sin^2\theta \sinh^2(2r)}\alpha_2, \quad \lambda_2(\alpha, r, \theta) = \frac{S^0_{r,\theta}}{1 + \sin^2\theta \sinh^2(2r)}\alpha_2.$$

For both states (3.18) and (3.19), the initial squeezing coefficient $S^0_{r,\theta}$ is given by (2.23), and $Q_{r,\theta}(t)$ is denoted as

$$Q_{r,\theta}(t) = \left(\frac{S^0_{r,\theta}}{x_0}x_1(t) + \frac{x_0\omega_0 \sin\theta \sinh(2r)}{S^0_{r,\theta}}x_2(t)\right) + i\left(\frac{x_0\omega_0}{S^0_{r,\theta}}x_2(t)\right), \tag{3.20}$$

with

$$|Q_{r,\theta}(t)| = \sqrt{\left(\frac{S^0_{r,\theta}}{x_0}x_1(t) + \frac{x_0\omega_0 \sin\theta \sinh(2r)}{S^0_{r,\theta}}x_2(t)\right)^2 + \left(\frac{x_0\omega_0}{S^0_{r,\theta}}x_2(t)\right)^2} \tag{3.21}$$

being the generalized squeezing coefficient, for which we note the following properties:

(i) It depends on the squeezing parameters $r \geq 0$ and $\theta \in [0, 2\pi)$ of the squeeze operator.

(ii) It depends on the solutions $x_1(t)$ and $x_2(t)$ of the classical equation of motion, which in turn depend on the time-variable parameters $\mu(t)$, $\omega^2(t)$ and $B(t)$ of the Hamiltonian.

(iii) At initial time $t = t_0$, the generalized squeezing reduces to the initial squeezing, that is $|Q_{r,\theta}(t_0)| = S^0_{r,\theta}$.

(iv) For $r = 0$, we have $S^0_{r,\theta}\big|_{r=0} = 1$ and $|Q_{r,\theta}(t)|\big|_{r=0} = |\epsilon(t)|$. That is, when the squeezing parameter r is zero, the generalized squeezing coefficient $|Q_{r,\theta}(t)|$ reduces to the squeezing coefficient $|\epsilon(t)|$ defined by (3.9) for the coherent states.

Now, probability densities at time-evolved squeezed coherent states become

$$\rho_{\alpha,r,\theta}(q,t) = \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{|Q_{r,\theta}(t)|}\exp\left\{-\left[\sqrt{\frac{\omega_0}{\hbar}}\left(\frac{q - \langle\hat{q}\rangle_\alpha(t)}{|Q_{r,\theta}|}\right)\right]^2\right\}, \tag{3.22}$$

where expectation values of position and momentum are the same as for the coherent states, and given by (3.11) and (3.12), but uncertainties and uncertainty product become

$$(\Delta\hat{q})_{r,\theta}(t) = \sqrt{\frac{\hbar}{2\omega_0}}|Q_{r,\theta}(t)|,$$
$$(\Delta\hat{p})_{r,\theta}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|Q_{r,\theta}(t)|}\sqrt{1+\frac{\mu^2(t)|Q_{r,\theta}(t)|^4}{\omega_0^2}\Big(\frac{d\ln|Q_{r,\theta}(t)|}{dt}-B(t)\Big)^2},$$
$$(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t) = \frac{\hbar}{2}\sqrt{1+\frac{\mu^2(t)|Q_{r,\theta}(t)|^4}{\omega_0^2}\Big(\frac{d\ln|Q_{r,\theta}(t)|}{dt}-B(t)\Big)^2}.$$

Therefore, time-evolved squeezed coherent states satisfy the Heisenberg uncertainty relation, that is, $(\Delta\hat{q}\Delta\hat{p})_{r,\theta}(t) > \hbar/2$. However, they are not minimum uncertainty states. The squeezing properties of these states depend on the time-dependent parameters $\mu(t)$, $\omega(t)$ and $B(t)$ of the Hamiltonian $\hat{H}_g(t)$ given by (3.2) as in the time-dependent coherent states. Also, the squeezing depends on the parameters r and θ of the squeeze operator.

We also consider the case in which the phase $\theta = 0$ in order to compare the results with those to be obtained later. When $\theta = 0$, the time-evolved squeezed coherent states become

$$\begin{aligned}\chi_{\alpha,r}(q,t) &= \Big(\frac{\omega_0}{\pi\hbar}\Big)^{1/4}\frac{1}{\sqrt{\epsilon_r(t)}}\times\exp\left[-\frac{1}{2}\left(\frac{(\epsilon_r^*(t))^2}{|\epsilon_r(t)|^2}\alpha_r^2+|\alpha_r|^2\right)\right]\\ &\times\exp\left[\frac{i}{\hbar}\int_{t_0}^{t}\zeta(s)ds\right]\times\exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_p(t)-B(t)x_p(t)-D(t)\Big)q\right]\\ &\times\exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon_r(t)|-B(t)\right)(q-x_p(t))^2\right]\\ &\times\exp\left[\sqrt{\frac{2\omega_0}{\hbar}}\frac{(q-x_p(t))\alpha_r}{\epsilon_r(t)}\right]\times\exp\left[\frac{-\omega_0}{2\hbar}\frac{(q-x_p(t))^2}{|\epsilon_r(t)|^2}\right],\end{aligned} \qquad (3.23)$$

where the complex parameter $\alpha_r = e^{-r}\alpha_1 + ie^{r}\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$ depends on $r \geq 0$, and the corresponding probability densities are found as

$$\rho_{\alpha,r}(q,t) = \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{|\epsilon_r(t)|}\exp\left\{-\left[\sqrt{\frac{\omega_0}{\hbar}}\left(\frac{q-\langle\hat{q}\rangle_\alpha(t)}{|\epsilon_r(t)|}\right)\right]^2\right\}.$$

Furthermore, in this case, the generalized squeezing coefficient turns out to be

$$|\epsilon_r(t)| = |Q_{r,\theta}(t)|\Big|_{\theta=0} = \sqrt{e^{2r}x_1^2(t) + \omega_0^2 e^{-2r}x_2^2(t)}, \quad (x_0 = 1) \tag{3.24}$$

but the expectation value of position $\langle\hat{q}\rangle_\alpha(t)$ is the same as in (3.11).

Recall that the squeezed coherent states of SQHO can be defined alternatively following Yuen's approach. The states generated by this approach are called two-photon coherent states, which we denote by $\Upsilon^0_{\beta,r,\theta}(q) = \hat{S}(r,\theta)\hat{D}(\beta)\varphi_0(q)$ for any $\beta \in \mathbb{C}$, $r \geq 0$, $\theta \in [0, 2\pi)$. The squeezed coherent states $\chi^0_{\alpha,r,\theta}(q)$ and two-photon coherent states $\Upsilon^0_{\beta,r,\theta}(q)$ are equivalent when $\alpha = \cosh r\beta + e^{i\theta}\sinh r\beta^*$ for any $\alpha, \beta \in \mathbb{C}$, $r \geq 0$, $\theta \in [0, 2\pi)$. Moreover, two-photon coherent states of SQHO are eigenstates of the pseudo-annihilation operator $\hat{b}$ which is defined as a linear combination of the annihilation and creation operator $\hat{a}$, $\hat{a}^\dagger$ in the form $\hat{b} = \cosh r\hat{a} - e^{i\theta}\sinh r\hat{a}^\dagger$. It is possible to find a relation between the time-dependent dynamical invariants $\hat{A}(t)$, $\hat{A}^\dagger(t)$ given by (3.14), (3.15), respectively, and time-evolved squeezed coherent states. For this, we first find the **time-evolved two-photon coherent states** that can be obtained according to

$$\Upsilon_{\beta,r\theta}(q,t) = \hat{U}_g(t,t_0)\hat{S}(r,\theta)\hat{D}(\beta)\varphi_0(q) = \hat{U}_g(t,t_0)\Upsilon^0_{\beta,r,\theta}(q). \tag{3.25}$$

Since the squeeze operator and the displacement operator do not commute and satisfy the equation $\hat{D}(\alpha)\hat{S}(r,\theta) = \hat{S}(r,\theta)D(\cosh r\alpha e^{i\theta}\sinh r\alpha^*)$, the definition (3.25) will give us time-dependent squeezed coherent states when $\beta = \cosh r\alpha - e^{i\theta}\sinh r\alpha^*$.

Then, we construct the following time-dependent operators

$$\hat{B}(t) = \hat{U}_g(t,t_0)\,\hat{b}\,\hat{U}_g^\dagger(t,t_0), \quad \hat{B}^\dagger(t) = \hat{U}_g(t,t_0)\,\hat{b}^\dagger\,\hat{U}_g^\dagger(t,t_0), \tag{3.26}$$

where the pseudo-annihilation and creation operators $\hat{b}$ and $\hat{b}^\dagger$ are given by (2.25). The following proposition asserts that the operators $\hat{B}(t)$ and $\hat{B}^\dagger(t)$ can be written as a linear combination of the operators $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ formulated by (3.14) and (3.15), respectively.

Proposition 3.1 *There exists a Bogoliubov transformation between the dynamical invari-*

ants $\hat{A}(t)$, $\hat{A}^\dagger(t)$ and the operators $\hat{B}(t)$, $\hat{B}^\dagger(t)$.

Proof Using the equations in (2.25), we obtain

$$\begin{aligned}
\hat{B}(t) &= \hat{U}_g(t,t_0)\,\hat{b}\,\hat{U}_g^\dagger(t,t_0) \\
&= \hat{U}_g(t,t_0)\left((\cosh r)\hat{a} - (e^{i\theta}\sinh r)\hat{a}^\dagger\right)\hat{U}_g^\dagger(t,t_0) \\
&= (\cosh r)\hat{U}_g(t,t_0)\,\hat{a}\,\hat{U}_g^\dagger(t,t_0) - (e^{i\theta}\sinh r)\hat{U}_g(t,t_0)\,\hat{a}^\dagger\,\hat{U}_g^\dagger(t,t_0) \\
&= (\cosh r)\hat{A}(t) - (e^{i\theta}\sinh r)\hat{A}^\dagger(t),
\end{aligned}$$

and analogously we get $\hat{B}^\dagger(t) = (\cosh r)\hat{A}^\dagger(t) - (e^{-i\theta}\sinh r)\hat{A}(t)$. Let us denote $u = \cosh r$ and $v = e^{i\theta}\sinh r$. Then $|u|^2 - |v|^2 = 1$ and this implies that the commutator leaves invariant under this transformation, i.e, $[\hat{B}(t), \hat{B}^\dagger(t)] = 1$. So this is a canonical Bogoliubov transformation. □

The commutator $[\hat{B}(t), \hat{B}^\dagger(t)] = 1$ provides $\hat{B}(t)$ and $\hat{B}^\dagger(t)$ with properties exactly similar to those of $\hat{A}(t)$ and $\hat{A}^\dagger(t)$. In addition, there is an eigenvalue equation that time-dependent two-photon coherent satisfies, given as follows:

Proposition 3.2 *Time-dependent two photon coherent states $\Upsilon_{\beta,r,\theta}(q,t)$ are eigenstates of the operator $\hat{B}(t)$.*

Proof By using the definitions of the operator $\hat{B}(t)$ and time-dependent two photon coherent states $\Upsilon_{\beta,r,\theta}(q,t)$, we have

$$\hat{B}(t)\Upsilon_{\beta,r,\theta}(q,t) = \hat{U}_g(t,t_0)\,\hat{b}\,\hat{U}_g^\dagger(t,t_0)\hat{U}_g(t,t_0)\,\hat{S}(r,\theta)\,\hat{D}(\beta)\varphi_0(q).$$

Indeed, since $\hat{b} = \hat{S}(r,\theta)\,\hat{a}\,\hat{S}^\dagger(r,\theta)$, and $\phi_\beta(q)$ are eigenstates of $\hat{a}$, the desired result follows as

$$\begin{aligned}
\hat{B}(t)\Upsilon_{\beta,r,\theta}(q,t) &= \hat{U}_g(t,t_0)\,\hat{S}(r,\theta)\,\hat{a}\,\hat{S}^\dagger(r,\theta)\,\hat{S}(r,\theta)\,\hat{D}(\beta)\,\varphi_0(q) \\
&= \hat{U}_g(t,t_0)\,\hat{S}(r,\theta)\,\hat{a}\,\phi_\beta(q) \\
&= \beta\,\hat{U}_g(t,t_0)\,\hat{S}(r,\theta)\,\phi_\beta(q) \\
&= \beta\,\Upsilon_{\beta,r,\theta}(q,t),
\end{aligned}$$

which shows that the states $\Upsilon_{\beta,r,\theta}(q,t)$ are eigenstates of the operator $\hat{B}(t)$ corresponding to complex eigenvalue β. □

By taking the phase θ as zero, we write the time-evolved two-photon coherent states in explicit form

$$\begin{aligned}
\Upsilon_{\beta,r}(q,t) &= \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{\epsilon_r(t)}}\times\exp\left[-\frac{1}{2}\left(\frac{(\epsilon_r^*(t))^2}{|\epsilon_r(t)|^2}\beta^2+|\beta|^2\right)\right]\\
&\times\exp\left[\frac{i}{\hbar}\int_{t_0}^{t}\zeta(s)ds\right]\times\exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_p(t)-B(t)x_p(t)-D(t)\Big)q\right]\\
&\times\exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon_r(t)|-B(t)\right)(q-x_p(t))^2\right]\\
&\times\exp\left[\sqrt{\frac{2\omega_0}{\hbar}}\frac{(q-x_p(t))\beta}{\epsilon_r(t)}\right]\times\exp\left[\frac{-\omega_0}{2\hbar}\frac{(q-x_p(t))^2}{|\epsilon_r(t)|^2}\right].
\end{aligned}\tag{3.27}$$

The corresponding probability densities are

$$\rho_{\beta,r}(q,t)=\sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{|\epsilon_r(t)|}\exp\left\{-\left[\sqrt{\frac{\omega_0}{\hbar}}\left(\frac{q-\langle\hat{q}\rangle_\beta(t)}{|\epsilon_r(t)|}\right)\right]^2\right\},$$

where the generalized squeezing coefficient $|\epsilon_r(t)|$ is defined by (3.24) and expectation values are found as

$$\begin{aligned}
\langle\hat{q}\rangle_\beta(t) &= \sqrt{\frac{2\hbar}{\omega_0}}(\beta_1 e^r x_1(t)+\omega_0\beta_2 e^{-r}x_2(t))+x_p(t),\\
\langle\hat{p}\rangle_\beta(t) &= \sqrt{\frac{2\hbar}{\omega_0}}(\beta_1 e^r p_1(t)+\omega_0\beta_2 e^{-r}p_2(t))+p_p(t).
\end{aligned}$$

Comparing the states (3.27) with the ones obtained by (3.23), we realize that the squeezing coefficients are the same for both of them. Therefore, the uncertainties and the uncertainty product at these states remain the same. (Note also that, the uncertainties and the uncertainty product are independent of the complex parameters α or β.) However, the expectation values at the two-photon coherent states depend on the squeezing parameters $r\geq 0$ and $\theta\in[0,2\pi)$, while the expectation values at squeezed coherent states are independent of them.

3.4. Time-Evolved Even and Odd Coherent States

The even and odd coherent states for SQHO are defined by applying the displacement operators $\hat{D}_+(\alpha)$ and $\hat{D}_-(\alpha)$ to the ground state $\varphi_0(q)$, that is, $\phi^e_\alpha(q) = \hat{D}_+(\alpha)\varphi_0(q)$ and $\phi^o_\alpha(q) = \hat{D}_-(\alpha)\varphi_0(q)$, where $\hat{D}_+(\alpha)$, $\hat{D}_-(\alpha)$ are given by (2.28). In this section, we solve the IVP (3.1) for the generalized parametric oscillator by taking the initial states to be the even-odd coherent states for SQHO.

First, if the initial state of the IVP (3.1) is taken to be $\Psi_0(q) = \phi^e_\alpha(q)$, then time evolved even coherent states are found as $\Phi^e_\alpha(q,t) = \hat{U}_g(t,t_0)\phi^e_\alpha(q)$, and explicitly

$$\begin{aligned}
\Phi^e_\alpha(q,t) &= \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{\cosh(|\alpha|^2)\epsilon(t)}} \times \exp\left(-\frac{\epsilon^*(t)}{2\epsilon(t)}\alpha^2\right) \\
&\times \exp\left(\frac{i}{\hbar}\int_{t_0}^{t} \zeta(s)ds\right) \times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_p(t) - B(t)x_p(t) - D(t)\Big)q\right] \\
&\times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon(t)| - B(t)\right)(q - x_p(t))^2\right] \\
&\times \cosh\left[\sqrt{\frac{2\omega_0}{\hbar}}\frac{(q - x_p(t))\alpha}{\epsilon(t)}\right] \times \exp\left[\frac{-\omega_0}{2\hbar}\frac{(q - x_p(t))^2}{|\epsilon(t)|^2}\right]. \qquad (3.28)
\end{aligned}$$

The corresponding probability densities become

$$\begin{aligned}
\rho^e_\alpha(q,t) &= \sqrt{\frac{\omega_0}{\pi\hbar}} \frac{1}{2\cosh(|\alpha|^2)|\epsilon(t)|} \times \exp\left[\frac{\left(\Im(\alpha\epsilon^*(t))\right)^2 - \left(\Re(\alpha\epsilon^*(t))\right)^2}{|\epsilon(t)|^2}\right] \\
&\times \exp\left[-\frac{\omega_0}{\hbar}\left(\frac{q - x_p(t)}{|\epsilon(t)|}\right)^2\right] \times \left\{\cosh\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Re(\alpha\epsilon^*(t))\frac{q - x_p(t)}{|\epsilon(t)|}\right)\right. \\
&\left. + \cos\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Im(\alpha\epsilon^*(t))\frac{q - x_p(t)}{|\epsilon(t)|}\right)\right\}, \qquad (3.29)
\end{aligned}$$

where the squeezing coefficient $|\epsilon(t)|$ is given by (3.9), $\Re(.)$ and $\Im(.)$ represents the real and imaginary parts of the given complex functions, respectively.

Expectation values of position and momentum at time-evolved even coherent states are $\langle\hat{q}\rangle^e(t) = x_p(t)$, $\langle\hat{p}\rangle^e(t) = p_p(t)$, where $x_p(t)$ and $p_p(t)$ are the particular solutions of the classical equations of motion in position and momentum spaces, which are found as (3.5) and (3.6), respectively. So they depend on all of the time-variable parameters of the

Hamiltonian $\hat{H}_g(t)$ given by (3.2). However, the uncertainties and the uncertainty product (for details, see Appendix A)

$$(\Delta\hat{q})^e_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon(t)|\sqrt{\Pi^e_q(t)}, \quad (\Delta\hat{p})^e_\alpha(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon(t)|}\sqrt{\Pi^e_p(t)},$$
$$(\Delta\hat{q})^e_\alpha(\Delta\hat{p})^e_\alpha(t) = \frac{\hbar}{2}\sqrt{\Pi^e_q(t)\Pi^e_p(t)}, \tag{3.30}$$

where

$$\begin{aligned}
\Pi^e_q(t) &= 1 + 2|\alpha|^2\tanh|\alpha|^2 + \frac{2}{|\epsilon(t)|^2}\Re(\alpha\epsilon^*(t))^2,\\
\Pi^e_p(t) &= \left(1 + \frac{\mu^2(t)|\epsilon(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon(t)|}{dt} - B(t)\right)^2\right)\left(1 + 2|\alpha|^2\tanh|\alpha|^2\right)\\
&+ \frac{2|\epsilon(t)|^2}{\omega_0^2}\Re\left[\alpha^2(\mu(t)(\dot{\epsilon}^*(t) - B(t)\epsilon^*(t)))^2\right],
\end{aligned}$$

depend just on the values $\mu(t)$, $\omega(t)$, $B(t)$ and the complex parameter α.

Next, we take the initial state of the IVP (3.1) as $\Psi_0(q) = \phi^o_\alpha(q)$, then time evolved odd coherent states will be $\Phi^o_\alpha(q,t) = \hat{U}_g(t,t_0)\phi^o_\alpha(q)$, and explicitly we found

$$\begin{aligned}
\Phi^o_\alpha(q,t) &= \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{\cosh(|\alpha|^2)\epsilon(t)}}\times\exp\left(-\frac{\epsilon^*(t)}{2\epsilon(t)}\alpha^2\right)\\
&\times\exp\left(\frac{i}{\hbar}\int_{t_0}^t\zeta(s)ds\right)\times\exp\left[\frac{i\mu(t)}{\hbar}\left(\dot{x}_p(t) - B(t)x_p(t) - D(t)\right)q\right]\\
&\times\exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon(t)| - B(t)\right)(q - x_p(t))^2\right]\\
&\times\sinh\left[\sqrt{\frac{2\omega_0}{\hbar}}\frac{(q - x_p(t))\alpha}{\epsilon(t)}\right]\exp\left[\frac{-\omega_0}{2\hbar}\frac{(q - x_p(t))^2}{|\epsilon(t)|^2}\right].
\end{aligned} \tag{3.31}$$

The corresponding probability densities become

$$\rho_{\alpha}^{o}(q,t) = \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{2\cosh(|\alpha|^2)|\epsilon(t)|} \times \exp\left[\frac{\left(\Im(\alpha\epsilon^*(t))\right)^2 - \left(\Re(\alpha\epsilon^*(t))\right)^2}{|\epsilon(t)|^2}\right]$$
$$\exp\left[-\frac{\omega_0}{\hbar}\left(\frac{q-x_p(t)}{|\epsilon(t)|}\right)^2\right] \times \left\{\cosh\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Re(\alpha\epsilon^*(t))\frac{q-x_p(t)}{|\epsilon(t)|}\right)\right.$$
$$\left. -\cos\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Im(\alpha\epsilon^*(t))\frac{q-x_p(t)}{|\epsilon(t)|}\right)\right\}, \qquad (3.32)$$

where the squeezing coefficient $|\epsilon(t)|$ is given by (3.9).

Expectation values of position and momentum at time-evolved odd coherent states are the same as for time-evolved even coherent states, that is, $\langle\hat{q}\rangle^{o}(t) = x_p(t)$, $\langle\hat{p}\rangle^{o}(t) = p_p(t)$. On the other hand, the uncertainties and the uncertainty product are

$$(\Delta\hat{q})_{\alpha}^{o}(t) = \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon(t)|\sqrt{\Pi_q^o(t)}, \quad (\Delta\hat{p})_{\alpha}^{o}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon(t)|}\sqrt{\Pi_p^o(t)}, \qquad (3.33)$$
$$(\Delta\hat{q})_{\alpha}^{o}(\Delta\hat{p})_{\alpha}^{o}(t) = \frac{\hbar}{2}\sqrt{\Pi_q^o(t)\Pi_p^o(t)}$$

where

$$\Pi_q^o(t) = 1 + 2|\alpha|^2\coth|\alpha|^2 + \frac{2}{|\epsilon(t)|^2}\Re(\alpha\epsilon^*(t))^2,$$
$$\Pi_p^o(t) = \left(1 + \frac{\mu^2(t)|\epsilon(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon(t)|}{dt} - B(t)\right)^2\right)\left(1 + 2|\alpha|^2\coth|\alpha|^2\right)$$
$$+ \frac{2|\epsilon(t)|^2}{\omega_0^2}\Re\left[\alpha^2\left(\mu(t)(\dot{\epsilon}^*(t) - B(t)\epsilon^*(t))\right)^2\right].$$

Therefore, they depend on the time-dependent parameters $\mu(t)$, $\omega(t)$ and $B(t)$ of the Hamiltonian $\hat{H}_g(t)$ and the parameter $\alpha \in \mathbb{C}$.

Below, using the dynamical invariants defined by (3.14) and (3.15), we give some important properties of time dependent even-odd coherent states.

Proposition 3.3 *The generalized lowering operator $\hat{A}(t)$ acts on the time-dependent even-*

odd coherent states as

$$\hat{A}(t)\Phi^e_\alpha(q,t) = \alpha\sqrt{\tanh|\alpha|^2}\Phi^o_\alpha(q,t), \quad \hat{A}(t)\Phi^o_\alpha(q,t) = \alpha\sqrt{\coth|\alpha|^2}\Phi^e_\alpha(q,t) \tag{3.34}$$

for any $\alpha \in \mathbb{C}$.

Proof From the definition of the invariant $\hat{A}(t)$, we have $\hat{A}(t) = \hat{U}_g(t,t_0)\hat{a}\hat{U}^\dagger_g(t,t_0)$. Also, time-dependent even coherent states are defined as $\Phi^e_\alpha(q,t) = \hat{U}_g(t,t_0)\phi^e_\alpha(q)$. So by applying $\hat{A}(t)$ to the states $\Phi^e_\alpha(q,t)$, and using Proposition 2.7, we obtain

$$\begin{aligned}
\hat{A}(t)\Phi^e_\alpha(q,t) &= \hat{U}_g(t,t_0)\hat{a}\hat{U}^\dagger_g(t,t_0)\hat{U}_g(t,t_0)\phi^e_\alpha(q) \\
&= \hat{U}_g(t,t_0)\hat{a}\phi^e_\alpha(q) \\
&= \alpha\sqrt{\tanh|\alpha|^2}\hat{U}_g(t,t_0)\phi^o_\alpha(q) \\
&= \alpha\sqrt{\tanh|\alpha|^2}\Phi^o_\alpha(q,t)
\end{aligned}$$

for any $\alpha \in \mathbb{C}$, which proves the first part of the proposition. Similarly, we can show the other part:

$$\begin{aligned}
\hat{A}(t)\Phi^o_\alpha(q,t) &= \hat{U}_g(t,t_0)\hat{a}\hat{U}^\dagger_g(t,t_0)\hat{U}_g(t,t_0)\phi^o_\alpha(q) \\
&= \hat{U}_g(t,t_0)\hat{a}\phi^o_\alpha(q) \\
&= \alpha\sqrt{\coth|\alpha|^2}\hat{U}_g(t,t_0)\phi^e_\alpha(q) \\
&= \alpha\sqrt{\coth|\alpha|^2}\Phi^e_\alpha(q,t)
\end{aligned}$$

for any $\alpha \in \mathbb{C}$. □

Therefore, as expected, the time-evolved even-odd coherent states are not eigenstates of the generalized lowering operator $\hat{A}(t)$.

Corollary 3.1 *Time-dependent even and odd coherent states are eigenstates of the operator $\hat{A}^2(t)$.*

Proof From the previous proposition, we obtain

$$\begin{aligned}
\hat{A}^2(t)\Phi^e_\alpha(q,t) &= \hat{A}(t)\hat{A}(t)\Phi^e_\alpha(q,t) \\
&= \alpha\sqrt{\tanh|\alpha|^2}\hat{A}(t)\Phi^o_\alpha(q,t) \\
&= \left(\alpha\sqrt{\tanh|\alpha|^2}\right)\left(\alpha\sqrt{\coth|\alpha|^2}\right)\Phi^e_\alpha(q,t) \\
&= \alpha^2\Phi^e_\alpha(q,t),
\end{aligned}$$

and

$$\begin{aligned}
\hat{A}^2(t)\Phi^o_\alpha(q,t) &= \hat{A}(t)\hat{A}(t)\Phi^o_\alpha(q,t) \\
&= \alpha\sqrt{\coth|\alpha|^2}\hat{A}(t)\Phi^e_\alpha(q,t) \\
&= \left(\alpha\sqrt{\coth|\alpha|^2}\right)\left(\alpha\sqrt{\tanh|\alpha|^2}\right)\Phi^o_\alpha(q,t) \\
&= \alpha^2\Phi^o_\alpha(q,t)
\end{aligned}$$

for any $\alpha \in \mathbb{C}$. Therefore, even-odd coherent states are eigenstates of the operator $\hat{A}^2(t)$ corresponding to the eigenvalue α^2, $\alpha \in \mathbb{C}$. □

3.5. Time-Evolved Even and Odd Displaced Squeezed States

The even-odd displaced squeezed states of standard harmonic oscillator are constructed by using the displacement operators $\hat{D}_+(\alpha)$, $\hat{D}_-(\alpha)$ and the squeeze operator $\hat{S}(r,\theta)$ as $\chi^e_{\alpha,r,\theta}(q) = \hat{D}_+(\alpha)\hat{S}(r,\theta)\varphi_0(q)$, and $\chi^o_{\alpha,r,\theta}(q) = \hat{D}_-(\alpha)\hat{S}(r,\theta)\varphi_0(q)$, where $\alpha \in \mathbb{C}$, $r \geq 0$, $\theta \in [0, 2\pi)$, and $\varphi_0(q)$ is the ground state of the standard Hamiltonian $\hat{H}_0$. So they are quantum superposition of squeezed coherent states $\chi^0_{\alpha,r}(q)$ given by (2.21). In what follows, we solve the IVP (3.1) for the generalized oscillator with IC's (*i*) $\Psi_0(q) = \chi^e_{\alpha,r}(q)$, (*ii*) $\Psi_0(q) = \chi^o_{\alpha,r}(q)$, where for simplicity, we take $\theta = 0$ to find time evolution of even-odd displaced squeezed states.

If $\Psi_0(q) = \chi^e_{\alpha,r}(q)$, then time-evolved even displaced squeezed states will be $\chi^e_{\alpha,r}(q,t) = \hat{U}_g(t,t_0)\chi^e_{\alpha,r}(q)$, and explicitly

$$\begin{aligned}
\chi^e_{\alpha,r}(q,t) &= \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{\cosh(|\alpha_r|^2)\epsilon_r(t)}} \times \exp\left(-\frac{\epsilon_r^*(t)}{2\epsilon_r(t)}\alpha_r^2\right) \\
&\times \exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\zeta(s)ds\right) \times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_p(t) - B(t)x_p(t) - D(t)\Big)q\right] \\
&\times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon_r(t)| - B(t)\right)(q - x_p(t))^2\right] \\
&\times \cosh\left[\sqrt{\frac{2\omega_0}{\hbar}}\frac{(q - x_p(t))\alpha_r}{\epsilon_r(t)}\right] \times \exp\left[\frac{-\omega_0}{2\hbar}\frac{(q - x_p(t))^2}{|\epsilon_r(t)|^2}\right]. \qquad (3.35)
\end{aligned}$$

The corresponding probability density $\rho^e_{\alpha,r}(q,t) = |\chi^e_{\alpha,r}(q,t)|^2$ is then

$$\begin{aligned}
\rho^e_{\alpha,r}(q,t) &= \sqrt{\frac{\omega_0}{\pi\hbar}} \frac{1}{2\cosh(|\alpha_r|^2)|\epsilon_r(t)|} \times \exp\left[\frac{(\Im(\alpha_r\epsilon_r^*(t)))^2 - (\Re(\alpha_r\epsilon_r^*(t)))^2}{|\epsilon_r(t)|^2}\right] \\
&\times \exp\left[-\frac{\omega_0}{\hbar}\left(\frac{q - x_p(t)}{|\epsilon_r(t)|}\right)^2\right] \times \left\{\cosh\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Re(\alpha_r\epsilon_r^*(t))\frac{q - x_p(t)}{|\epsilon_r(t)|}\right)\right. \\
&\left. + \cos\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Im(\alpha_r\epsilon_r^*(t))\frac{q - x_p(t)}{|\epsilon_r(t)|}\right)\right\}, \qquad (3.36)
\end{aligned}$$

where the squeezing coefficient $|\epsilon_r(t)|$ is defined by (3.24), and $\alpha_r = e^{-r}\alpha_1 + ie^r\alpha_2$ with α_1, α_2 being real constants.

Expectation values of position and momentum are the same as for the time-evolved even coherent states, i.e., $\langle\hat{q}\rangle^e = x_p(t)$, and $\langle\hat{p}\rangle^e = p_p(t)$. So they do not depend on the squeezing parameter r and the complex number α. However, uncertainties and uncertainty product

$$\begin{aligned}
(\Delta\hat{q})^e_{\alpha,r}(t) = \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon_r(t)|\sqrt{\Pi^e_q(t)}, \quad (\Delta\hat{p})^e_{\alpha,r}(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon_r(t)|}\sqrt{\Pi^e_p(t)}, \\
(\Delta\hat{q})^e_{\alpha,r}(\Delta\hat{p})^e_{\alpha}(t) = \frac{\hbar}{2}\sqrt{\Pi^e_q(t)\Pi^e_p(t)} \qquad (3.37)
\end{aligned}$$

where

$$
\begin{aligned}
\Pi_q^e(t) &= 1 + 2|\alpha|^2 \tanh|\alpha_r|^2 + \frac{2}{|\epsilon_r(t)|^2}\Re(\alpha_r\epsilon_r^*(t))^2, \\
\Pi_p^e(t) &= \left(1 + \frac{\mu^2(t)|\epsilon_r(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon_r(t)|}{dt} - B(t)\right)^2\right)\left(1 + 2|\alpha_r|^2\tanh|\alpha_r|^2\right) \\
&+ \frac{2|\epsilon_r(t)|^2}{\omega_0^2}\Re\left[\alpha_r^2(\mu(t)(\dot{\epsilon_r}^*(t) - B(t)\epsilon_r^*(t)))^2\right],
\end{aligned}
$$

depend on $r \geq 0$, the complex number $\alpha_r = e^{-r}\alpha_1 + ie^r\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$ and also on the parameters $\mu(t)$, $\omega(t)$, and $B(t)$ of the Hamiltonian (3.2).

Now, if $\Psi_0(q) = \chi_{\alpha,r}^o(q)$ in(3.1), then we can find the time-evolved odd displaced squeezed states by using the generalized evolution operator as $\chi_{\alpha,r}^o(q,t) = \hat{U}_g(t,t_0)\chi_{\alpha,r}^o(q)$. After doing calculations, they are found explicitly

$$
\begin{aligned}
\chi_{\alpha,r}^o(q,t) &= \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{\cosh(|\alpha_r|^2)\epsilon_r(t)}} \times \exp\left(-\frac{\epsilon_r^*(t)}{2\epsilon_r(t)}\alpha_r^2\right) \\
&\times \exp\left(\frac{i}{\hbar}\int_{t_0}^t \zeta(s)ds\right) \times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_p(t) - B(t)x_p(t) - D(t)\Big)q\right] \\
&\times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon_r(t)| - B(t)\right)(q - x_p(t))^2\right] \\
&\times \sinh\left[\sqrt{\frac{2\omega_0}{\hbar}}\frac{(q - x_p(t))\alpha_r}{\epsilon_r(t)}\right] \times \exp\left[\frac{-\omega_0}{2\hbar}\frac{(q - x_p(t))^2}{|\epsilon_r(t)|^2}\right]. \qquad (3.38)
\end{aligned}
$$

The probability is then

$$
\begin{aligned}
\rho_{\alpha,r}^o(q,t) &= \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{2\cosh(|\alpha_r|^2)|\epsilon_r(t)|} \times \exp\left[\frac{(\Im(\alpha_r\epsilon_r^*(t)))^2 - (\Re(\alpha_r\epsilon_r^*(t)))^2}{|\epsilon_r(t)|^2}\right] \\
&\times \exp\left[-\frac{\omega_0}{\hbar}\left(\frac{q - x_p(t)}{|\epsilon_r(t)|}\right)^2\right] \times \left\{\cosh\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Re(\alpha_r\epsilon_r^*(t))\frac{q - x_p(t)}{|\epsilon_r(t)|}\right)\right. \\
&\left. - \cos\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Im(\alpha_r\epsilon_r^*(t))\frac{q - x_p(t)}{|\epsilon_r(t)|}\right)\right\}, \qquad (3.39)
\end{aligned}
$$

where $r \geq 0$, $\alpha_r = e^{-r}\alpha_1 + ie^r\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$ and $|\epsilon_r(t)|$ is given by (3.24).

Expectation values of position and momentum in time-evolved odd displaced squeezed states are $\langle\hat{q}\rangle^o = x_p(t)$, and $\langle\hat{p}\rangle^o = p_p(t)$. The corresponding uncertainties and uncertainty product are of the form

$$(\Delta\hat{q})^{o}_{\alpha,r}(t) = \sqrt{\tfrac{\hbar}{2\omega_0}}|\epsilon_r(t)|\sqrt{\Pi^{o}_{q}(t)}, \quad (\Delta\hat{p})^{o}_{\alpha,r}(t) = \sqrt{\tfrac{\omega_0\hbar}{2}}\tfrac{1}{|\epsilon_r(t)|}\sqrt{\Pi^{o}_{p}(t)},$$
$$(\Delta\hat{q})^{o}_{\alpha,r}(\Delta\hat{p})^{e}_{\alpha}(t) = \tfrac{\hbar}{2}\sqrt{\Pi^{o}_{q}(t)\Pi^{o}_{p}(t)} \tag{3.40}$$

where

$$\begin{aligned}
\Pi^{o}_{q}(t) &= 1 + 2|\alpha_r|^2\tanh|\alpha_r|^2 + \frac{2}{|\epsilon_r(t)|^2}\Re(\alpha_r\epsilon^*_r(t))^2, \\
\Pi^{o}_{p}(t) &= \left(1 + \frac{\mu^2(t)|\epsilon_r(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon_r(t)|}{dt} - B(t)\right)^2\right)\left(1 + 2|\alpha_r|^2\tanh|\alpha_r|^2\right) \\
&+ \frac{2|\epsilon_r(t)|^2}{\omega_0^2}\Re\left[\alpha_r^2(\mu(t)(\dot{\epsilon_r}^*(t) - B(t)\epsilon^*_r(t)))^2\right]
\end{aligned}$$

with $\alpha_r = e^{-r}\alpha_1 + ie^{r}\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$.

Furthermore, by replacing the order of application of the operators $\hat{S}(r,\theta)$ and $\hat{D}_{\pm}(\beta)$, in the definition of the even-odd displaced squeezed states $\chi^{e,o}_{\beta,r,\theta}(q)$ for any $\beta \in \mathbb{C}$, $r \geq 0$ and $\theta \in [0, 2\pi)$, we construct **time-dependent squeezed even-odd coherent states** $\Upsilon^{e,o}_{\beta,r,\theta}(q)$. Here, by taking $\theta = 0$, we apply the generalized evolution operator $\hat{U}_g(t,t_0)$ to the squeezed even-odd coherent states of SQHO and obtain their time evolution.

First, we find the exact form of the time evolved squeezed even coherent states $\Upsilon^{e}_{\beta,r}(q,t) = \hat{U}_g(t,t_0)\Upsilon^{e}_{\beta,r}(q)$ as

$$\begin{aligned}
\Upsilon^{e}_{\beta,r}(q,t) &= \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{\cosh(|\beta|^2)\epsilon_r(t)}} \times \exp\left(-\frac{\epsilon^*_r(t)}{2\epsilon_r(t)}\beta^2\right) \\
&\times \exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\zeta(s)ds\right) \times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_p(t) - B(t)x_p(t) - D(t)\Big)q\right] \\
&\times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon_r(t)| - B(t)\right)(q - x_p(t))^2\right] \\
&\times \cosh\left[\sqrt{\frac{2\omega_0}{\hbar}}\frac{(q - x_p(t))\beta}{\epsilon_r(t)}\right] \times \exp\left[\frac{-\omega_0}{2\hbar}\frac{(q - x_p(t))^2}{|\epsilon_r(t)|^2}\right].
\end{aligned} \tag{3.41}$$

Then, the probability density follows as

$$
\begin{aligned}
\rho^e_{\beta,r}(q,t) &= \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{2\cosh(|\beta|^2)|\epsilon_r(t)|}\times\exp\left[\frac{\left(\Im(\_r\epsilon_r^*(t))\right)^2-\left(\Re(\alpha_r\epsilon_r^*(t))\right)^2}{|\epsilon_r(t)|^2}\right]\\
&\times\exp\left[-\frac{\omega_0}{\hbar}\left(\frac{q-x_p(t)}{|\epsilon_r(t)|}\right)^2\right]\times\left\{\cosh\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Re(\beta\epsilon_r^*(t))\frac{q-x_p(t)}{|\epsilon_r(t)|}\right)\right.\\
&\left.+\cos\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Im(\beta\epsilon_r^*(t))\frac{q-x_p(t)}{|\epsilon_r(t)|}\right)\right\}, \qquad (3.42)
\end{aligned}
$$

where the squeezing coefficient $|\epsilon_r(t)|$ is given by (3.24).

The change in the order of application of the squeeze operator $\hat{S}(r,\theta)$ and the displacement operator $\hat{D}_+(\beta)$ does not affect expectation values. So expectation of position and momentum are the same as for time evolved even displaced squeezed states, that is, $\langle\hat{q}\rangle^e(t) = x_p(t)$, $\langle\hat{p}\rangle^e(t) = p_p(t)$, where $x_p(t)$ and $p_p(t)$ are particular solutions of the classical equations of motion. However, uncertainties and uncertainty product are

$$
\begin{gathered}
(\Delta\hat{q})^e_{\beta,r}(t) = \sqrt{\tfrac{\hbar}{2\omega_0}}|\epsilon_r(t)|\sqrt{\Pi^e_q(t)}, \quad (\Delta\hat{p})^e_{\beta,r}(t) = \sqrt{\tfrac{\omega_0\hbar}{2}}\tfrac{1}{|\epsilon_r(t)|}\sqrt{\Pi^e_p(t)},\\
(\Delta\hat{q})^e_{\beta,r}(\Delta\hat{p})^e_\alpha(t) = \tfrac{\hbar}{2}\sqrt{\Pi^e_q(t)\Pi^e_p(t)}
\end{gathered} \qquad (3.43)
$$

where

$$
\Pi^e_q(t) = 1+2|\beta|^2\tanh|\beta|^2+\frac{2}{|\epsilon_r(t)|^2}\Re(\beta\epsilon_r^*(t))^2,
$$

and

$$
\begin{aligned}
\Pi^e_p(t) &= \left(1+\frac{\mu^2(t)|\epsilon_r(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon_r(t)|}{dt}-B(t)\right)^2\right)\left(1+2|\beta|^2\tanh|\beta|^2\right)\\
&+ \frac{2|\epsilon_r(t)|^2}{\omega_0^2}\Re\left[\beta^2(\mu(t)(\dot{\epsilon}_r{}^*(t)-B(t)\epsilon_r^*(t)))^2\right],
\end{aligned}
$$

depend on the squeezing parameter $r \geq 0$, and the complex number β.

Now, we find time-evolved squeezed odd coherent states $\Upsilon^o_{\beta,r}(q,t) = \hat{U}_g(t,t_0)\Upsilon^o_{\beta,r}(q)$ explicitly as

$$
\begin{aligned}
\Upsilon^o_{\beta,r}(q,t) &= \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{\cosh(|\beta|^2)\epsilon_r(t)}} \times \exp\left(-\frac{\epsilon_r^*(t)}{2\epsilon_r(t)}\beta^2\right) \\
&\times \exp\left(\frac{i}{\hbar}\int_{t_0}^t \zeta(s)ds\right) \times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_p(t) - B(t)x_p(t) - D(t)\Big)q\right] \\
&\times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon_r(t)| - B(t)\right)(q - x_p(t))^2\right] \\
&\times \sinh\left[\sqrt{\frac{2\omega_0}{\hbar}}\frac{(q - x_p(t))\beta}{\epsilon_r(t)}\right] \times \exp\left[\frac{-\omega_0}{2\hbar}\frac{(q - x_p(t))^2}{|\epsilon_r(t)|^2}\right], \qquad (3.44)
\end{aligned}
$$

and the corresponding probability density is

$$
\begin{aligned}
\rho^o_{\beta,r}(q,t) &= \frac{\sqrt{\omega_0/(\pi\hbar)}}{2\cosh(|\beta|^2)|\epsilon_r(t)|} \times \exp\left[\frac{(\Im(\_r\epsilon_r^*(t)))^2 - (\Re(\alpha_r\epsilon_r^*(t)))^2}{|\epsilon_r(t)|^2}\right] \\
&\times \exp\left[-\frac{\omega_0}{\hbar}\left(\frac{q - x_p(t)}{|\epsilon_r(t)|}\right)^2\right] \times \left\{\cosh\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Re(\beta\epsilon_r^*(t))\frac{q - x_p(t)}{|\epsilon_r(t)|}\right)\right. \\
&\left. - \cos\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Im(\beta\epsilon_r^*(t))\frac{q - x_p(t)}{|\epsilon_r(t)|}\right)\right\}, \qquad (3.45)
\end{aligned}
$$

with the squeezing coefficient $|\epsilon_r(t)|$ defined by (3.24).

Expectation values of position and momentum in these states are $\langle\hat{q}\rangle^o(t) = x_p(t)$, $\langle\hat{p}\rangle^o(t) = p_p(t)$, while uncertainties and uncertainty product become

$$
\begin{gathered}
(\Delta\hat{q})^o_{\beta,r}(t) = \sqrt{\tfrac{\hbar}{2\omega_0}}|\epsilon_r(t)|\sqrt{\Pi^o_q(t)}, \quad (\Delta\hat{p})^o_{\beta,r}(t) = \sqrt{\tfrac{\omega_0\hbar}{2}}\tfrac{1}{|\epsilon_r(t)|}\sqrt{\Pi^o_p(t)}, \\
(\Delta\hat{q})^o_{\beta,r}(\Delta\hat{p})^e_\alpha(t) = \tfrac{\hbar}{2}\sqrt{\Pi^o_q(t)\Pi^o_p(t)},
\end{gathered} \qquad (3.46)
$$

where

$$
\Pi^o_q(t) = 1 + 2|\beta|^2\coth|\beta|^2 + \frac{2}{|\epsilon_r(t)|^2}\Re(\beta\epsilon_r^*(t))^2,
$$

and

$$\begin{aligned}\Pi_p^o(t) &= \left(1+\frac{\mu^2(t)|\epsilon_r(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon_r(t)|}{dt}-B(t)\right)^2\right)\left(1+2|\beta|^2\coth|\beta|^2\right)\\ &+ \frac{2|\epsilon_r(t)|^2}{\omega_0^2}\Re\left[\beta^2\left(\mu(t)(\dot{\epsilon}_r{}^*(t)-B(t)\epsilon_r^*(t))\right)^2\right].\end{aligned}$$

In Section 2.3, we have defined the operators $\hat{B}(t)=\hat{U}_g(t,t_0)\hat{b}\hat{U}_g^\dagger(t,t_0)$ and $\hat{B}^\dagger(t)=\hat{U}_g(t,t_0)\hat{b}^\dagger\hat{U}_g^\dagger(t,t_0)$, where the pseudo-annihilation and creation operators $\hat{b}$ and $\hat{b}^\dagger$ are given by (2.25). Now, we give the relations between these operators and time evolved squeezed even-odd coherent states.

Proposition 3.4 *The operator $\hat{B}(t)$ acts on the time evolved squeezed even-odd coherent states as*

$$\hat{B}(t)\Upsilon^e_{\beta,r}(q,t)=\beta\sqrt{\tanh|\beta|^2}\Upsilon^o_{\beta,r}(q,t),\quad \hat{B}(t)\Upsilon^o_{\beta,r}(q,t)=\beta\sqrt{\coth|\beta|^2}\Upsilon^e_{\beta,r}(q,t) \tag{3.47}$$

for all $\beta\in\mathbb{C}$ and $r\geq 0$.

Proof By using the definition of operator $\hat{B}(t)$, and $\Upsilon^e_{\beta,r}(q,t)=\hat{U}_g(t,t_0)\Upsilon^e_{\beta,r}(q)$, for any $\beta\in\mathbb{C}$ and $r\geq 0$, we can write

$$\hat{B}(t)\Upsilon^e_{\beta,r}(q,t)=\hat{U}_g(t,t_0)\hat{b}\hat{U}_g^\dagger(t,t_0)\hat{U}_g(t,t_0)\Upsilon^e_{\beta,r}(q).$$

Since the evolution operator $\hat{U}_g(t,t_0)$ is unitary, it satisfies $\hat{U}_g(t,t_0)\hat{U}_g^\dagger(t,t_0)=\hat{I}$. So using Proposition 2.9, we obtain

$$\begin{aligned}\hat{B}(t)\Upsilon^e_{\beta,r}(q,t) &= \beta\sqrt{\tanh|\beta|^2}\,\hat{U}_g(t,t_0)\Upsilon^o_{\beta,r}(q)\\ &= \beta\sqrt{\tanh|\beta|^2}\,\Upsilon^o_{\beta,r}(q,t),\end{aligned}$$

and in a similar way we can show that $\hat{B}(t)\Upsilon^o_{\beta,r}(q,t)=\beta\sqrt{\coth|\beta|^2}\Upsilon^e_{\beta,r}(q,t)$. □

From this proposition we conclude the following property.

Corollary 3.2 *Time-evolved squeezed even-odd coherent states are eigenstates of the operator $\hat{B}^2(t)$.*

Proof The proof follows directly from the Proposition 3.47. For any $\beta \in \mathbb{C}$ and $r \geq 0$, we have

$$\hat{B}^2(t)\Upsilon^e_{\beta,r}(q,t) = \beta\sqrt{\tanh|\beta|^2}\hat{B}(t)\Upsilon^o_{\beta,r}(q,t) = \beta^2\Upsilon^e_{\beta,r}(q,t)$$

and

$$\hat{B}^2(t)\Upsilon^o_{\beta,r}(q,t) = \beta\sqrt{\coth|\beta|^2}\hat{B}(t)\Upsilon^e_{\beta,r}(q,t) = \beta^2\Upsilon^o_{\beta,r}(q,t).$$

□

Therefore, time-evolved squeezed even-odd coherent states are eigenstates of the operator $\hat{B}^2(t)$ corresponding to the eigenvalue β^2 for any $\beta \in \mathbb{C}$.

3.6. Exactly Solvable Models

In this section, we apply our results to find and analyze the behavior of the time-evolved quantum states obtained in the previous parts. For this, we introduce and discuss an exactly solvable quantum model for a generalized Caldirola-Kanai oscillator (Caldirola, 1941), (Kanai, 1948), described by the Hamiltonian

$$\hat{H}(t) = \frac{-\hbar^2}{2}e^{-\gamma t}\frac{\partial^2}{\partial q^2} + \frac{\omega_0^2}{2}e^{\gamma t}q^2 - i\hbar\frac{B(t)}{2}\left(q\frac{\partial}{\partial q} + \frac{\partial}{\partial q}q\right) - E_0 e^{\gamma t}\cos(\omega t)q, \tag{3.48}$$

where ω_0 is a constant frequency, $\mu(t) = e^{\gamma t}, \gamma > 0$, is the exponentially increasing mass, $B(t)$ is a real-valued parameter depending on time, and driving force is taken to be of sinusoidal form with E_0 and ω being arbitrary real constants. We have seen that parameter $B(t)$ can essentially modify the original frequency of the classical oscillator (3.5), and therefore it changes solutions $x_1(t)$ and $x_2(t)$, which determine the squeezing coefficient $|\epsilon(t)|$ given by (3.9). Thus, $B(t)$ influences the uncertainties and expectation values of position and momentum as well. Besides, for squeezed coherent states and even-odd displaced squeezed states, the squeezing properties depend not only on the homogenous solutions $x_1(t)$, $x_2(t)$ of the classical equation (3.5), but also on the coefficient $z = re^{i\theta}$. Our goal is to investigate the influence of these parameters on the squeezing properties of the wave packets. Since the corresponding classical equation is

$$\ddot{x} + \gamma\dot{x} + \left(\omega_0^2 - (\dot{B}(t) + B^2(t) + \gamma B(t))\right)x = E_0\cos(\omega t), \tag{3.49}$$

by requiring that

$$-(\dot{B}(t) + B^2(t) + \gamma B(t)) = \Lambda_0^2, \tag{3.50}$$

where $\Lambda_0^2 > -\omega_0^2$, we can preserve the Caldirola-Kanai type oscillator structure. According to this, we choose $B(t) = -(\gamma/2) + \Omega'_B \tanh(\Omega'_B t)$, where $\Omega'_B = \sqrt{\gamma^2/4 - \Lambda_0^2}$ and $-\omega_0^2 < \Lambda_0^2 < \gamma^2/4$. Note that, $B(t)$ takes its minimum value at $t = 0$ with $B(0) = -\gamma/2$, and as $t \to \infty$, it increases and asymptotically approaches the upper bound $-\gamma/2 + \sqrt{\gamma^2/4 - \Lambda_0^2}$. For this choice of $B(t)$, equation (3.49) takes the form

$$\ddot{x} + \gamma\dot{x} + (\omega_0^2 + \Lambda_0^2)x = E_0\cos(\omega t), \tag{3.51}$$

with constant frequency $\omega_0^2 + \Lambda_0^2 > 0$, Λ_0^2- being the frequency modification in position space, and $\Omega_d^2 = \omega_0^2 + \Lambda_0^2 - \gamma^2/4$ gives the frequency Ω_d of the modified damped oscillator. Depending on the sign of Ω_d^2, we have three cases:

(i) $\Omega_d^2 < 0$ (overdamping),

(ii) $\Omega_d^2 = 0$ (critical damping),

(iii) $\Omega_d^2 > 0$ (underdamping).

Here, we shall treat explicitly only the more interesting case of *underdamping*. When $\Omega_d^2 > 0$, homogenous solutions of the classical equation (3.51) satisfying the required initial conditions $x_1(0) = 1$, $\dot{x}_1(0) = -\gamma/2$, and $x_2(0) = 0$, $\dot{x}_2(0) = 1$ are

$$x_1(t) = e^{-\gamma t/2}\cos(\Omega_d t), \quad x_2(t) = \frac{1}{\Omega_d}e^{-\gamma t/2}\sin(\Omega_d t),$$

and particular solution is

$$x_p(t) = A_h e^{-\gamma t/2}\cos(\Omega_d t - \beta_h) + A_p\cos(\omega t - \delta_p), \tag{3.52}$$

where A_h and θ_h are constants such that $x_p(t)$ satisfies the initial conditions $x_p(0) = 0, \dot{x}_p(0) = 0$. The amplitude and phase shift of the steady-state part are

$$A_p = \frac{E_0}{\sqrt{((\omega_0^2 + \Lambda_0^2) - \omega^2)^2 + \gamma^2\omega^2}}, \quad \delta_p = \tan^{-1}\left(\frac{\gamma\omega}{(\omega_0^2 + \Lambda_0^2) - \omega^2}\right).$$

For given γ and ω_0, the driving frequency ω at which the amplitude $A_p(\omega)$ takes maximum is known as **resonance frequency**. For this model, the resonance frequency $\omega = \omega_{res}$ and the maximum amplitude are found as

$$\omega_{res} = \sqrt{(\omega_0^2 + \Lambda_0^2) - \gamma^2/2}, \quad A_p(\omega_{res}) = \frac{D_0}{\sqrt{(\omega_0^2 + \Lambda_0^2)\gamma^2 - \frac{\gamma^4}{4}}},$$

provided that $\omega_0^2 + \Lambda_0^2 - \gamma^2/2 > 0$.

A. Time-evolved squeezed coherent states

For the squeezed coherent states $\chi_{\alpha,r,\theta}(q,t)$, $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $r \geq 0$, $\theta \in [0, 2\pi)$, the probability densities are found in the form

$$\rho_{\alpha,r,\theta}(q,t) = \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{|Q_{r,\theta}(t)|}\exp\left[-\left(\frac{\omega_0}{\hbar}\right)\left(\frac{q - \langle\hat{q}\rangle_\alpha(t)}{|Q_{r,\theta}(t)|}\right)^2\right],$$

where the expectation value of position is

$$\langle\hat{q}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{\omega_0}}e^{-\gamma t/2}\left(\alpha_1\cos(\Omega_d t) + \frac{\alpha_2\omega_0}{\Omega_d}\sin(\Omega_d t)\right) + x_p(t)$$

with $x_p(t)$ is given by (3.52), and the generalized squeezing coefficient is found as

$$|Q_{r,\theta}(t)| = e^{-\gamma t/2}\sqrt{\left(S^{\,0}_{r,\theta}\cos(\Omega_d t) + \frac{\omega_0\sin\theta\sinh(2r)}{\Omega_d S^{\,0}_{r,\theta}}\sin(\Omega_d t)\right)^2 + \left(\frac{\omega_0}{\Omega_d S^{\,0}_{r,\theta}}\sin(\Omega_d t)\right)^2},$$

which depends on r, θ and the modified frequency Ω_d. In particular, when $r = 0$, one gets the squeezing coefficient for the time-evolved coherent states $\Phi_\alpha(q,t)$, that is,

$$|Q_{r,\theta}(t)|\Big|_{r=0} \equiv |\epsilon(t)| = e^{-\gamma t/2}\sqrt{\cos^2(\Omega_d t) + \frac{\omega_0^2}{\Omega_d^2}\sin^2(\Omega_d t)}.$$

For the special choices of the phase $\theta = 0$ and $\theta = \pi(z = \pm r)$, we have the squeezing coefficient

$$|\epsilon_r(t)| = e^{-\gamma t/2}\sqrt{e^{\pm 2r}\cos^2(\Omega_d t) + \frac{\omega_0^2}{\Omega_d^2}e^{\mp 2r}\sin^2(\Omega_d t)},$$

and uncertainties become

$$\begin{aligned}
(\Delta\hat{q})_r(t) &= \sqrt{\hbar/2\omega_0}|\epsilon_r(t)|, \\
(\Delta\hat{p})_r(t) &= \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon_r(t)|}\sqrt{1 + \frac{e^{2\gamma t}|\epsilon_r|^4}{\omega_0^2}\left(\frac{d}{dt}\ln|\epsilon_r(t)| + \frac{\gamma}{2} - \Omega_B\tanh(\Omega_B t)\right)^2}, \\
(\Delta\hat{q}\Delta\hat{p})_r(t) &= \frac{\hbar}{2}\left\{1 + \frac{1}{\omega_0^2}\left[\Omega_B\tanh(\Omega_B t)(e^{\pm 2r}\cos^2(\Omega_d t) + \frac{\omega_0^2}{\Omega_d^2}e^{\mp 2r}\sin^2(\Omega_d t))\right.\right. \\
&+ \left.\left.\frac{1}{2}\left(\Omega_d e^{\pm 2r} - \frac{\omega_0^2}{\Omega_d}e^{\mp 2r}\right)\sin(2\Omega_d t)\right]^2\right\}^{1/2}.
\end{aligned}$$

In Fig.3.1(a), we show that for given values γ, ω_0 and Λ_0^2, when r increases, the amplitude of oscillations of $(\Delta\hat{q})_r(t)$ increases. As an example, in Fig.3.2(a), we plot the probability density $\rho_{\alpha,r}(q,t)$ of the ground state ($\alpha = 0$) without displacement ($x_p(t) = 0$) and observe oscillatory squeezing of the width. Fig.3.2(b) exhibits the displacement of the wave packet due to the nonzero complex parameter $\alpha = i$ when $x_p(t) = 0$. Finally, in Fig.3.2(c), we plot $\rho_{\alpha,r}(q,t)$ of the ground state ($\alpha = 0$) under periodic displacement $x_p(t) = 3\cos(15t/2 - \tan^{-1}(\sqrt{15}))/2$ at resonance frequency $\omega = \sqrt{15}/2$.

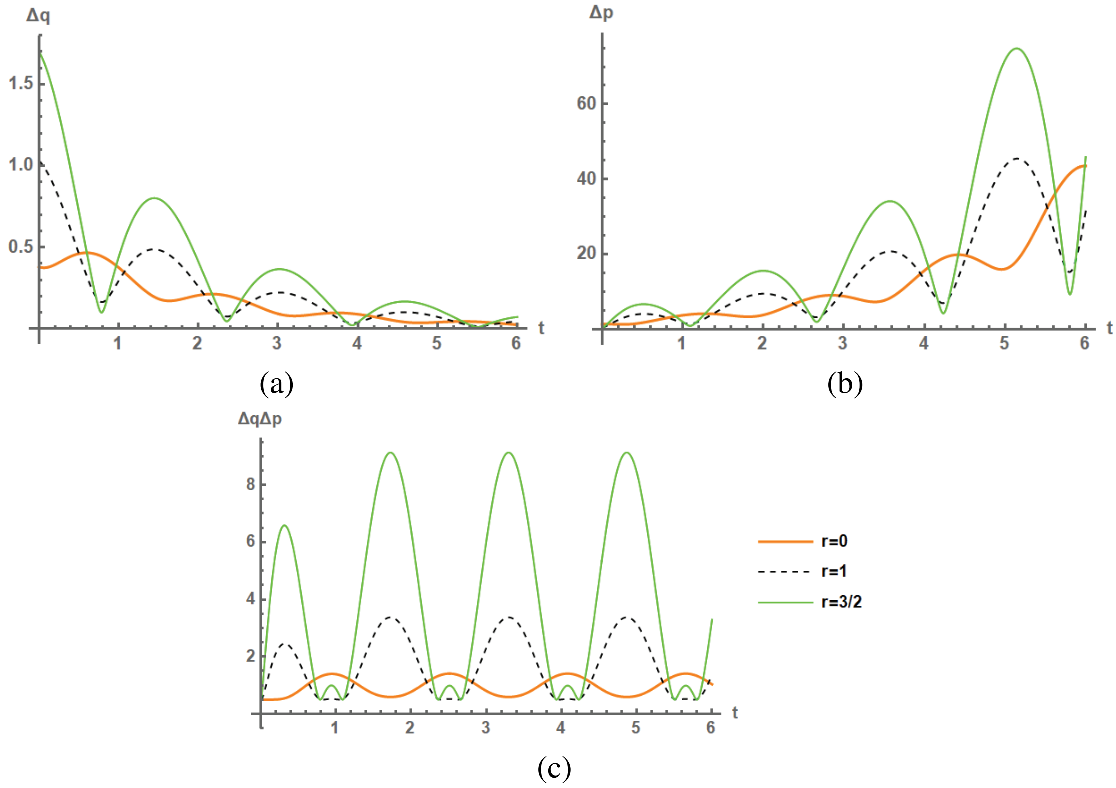


Figure 3.1. For $\omega_0 = \sqrt{12}, \gamma = 1, \Lambda_0^2 = -31/4, \Omega_d = 2, \Omega_B' = 2\sqrt{2}, r = 0, 1, 3/2, \theta = 0$. (a)Uncertainty $(\Delta\hat{q})_r(t)$, (b)Uncertainty $(\Delta\hat{p})_r(t)$, (c) Uncertainty product $(\Delta\hat{q}\Delta\hat{p})_r$.

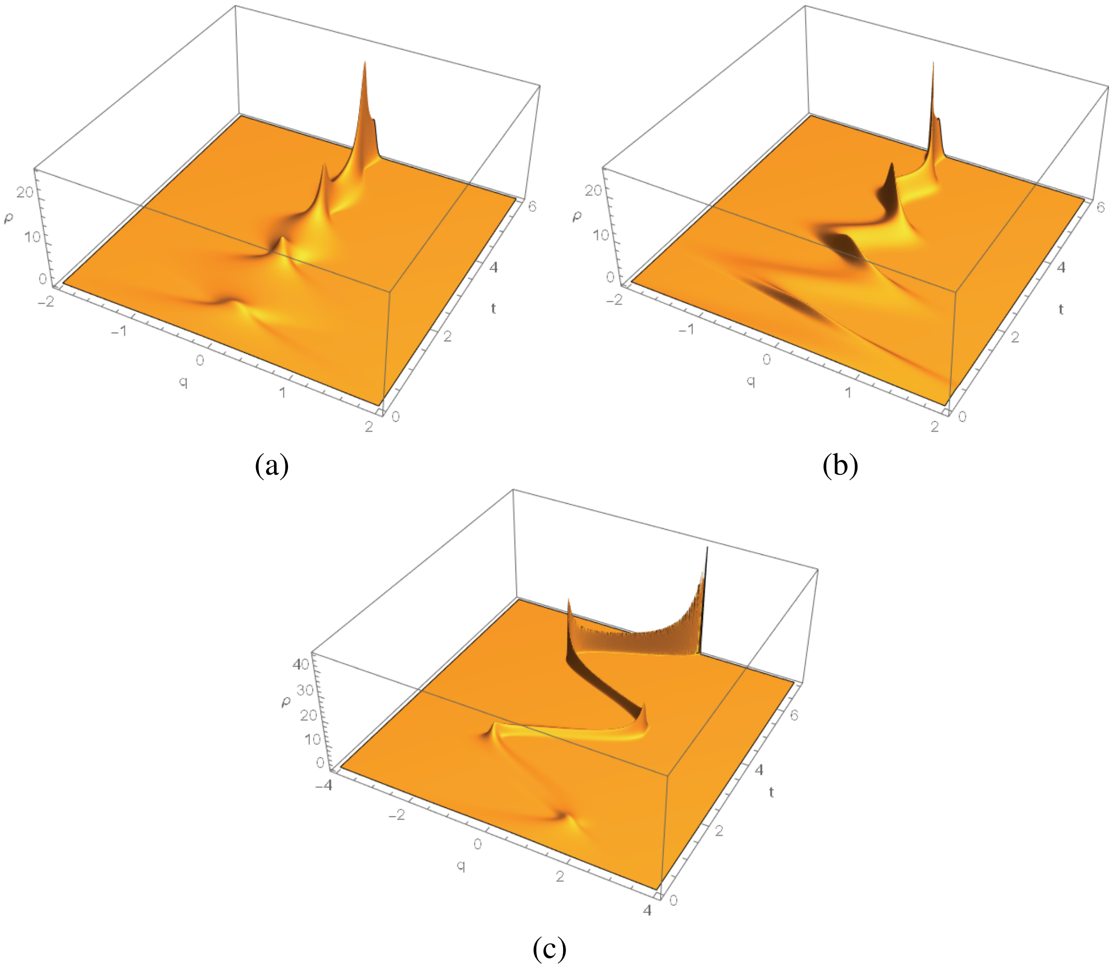


Figure 3.2. Probability density $\rho_{\alpha,r}(q,t)$ with $\gamma = 1$, $\omega_0 = \sqrt{12}$, $\hbar = 1$, $r = 1$, $\theta = 0$, $\Lambda_0^2 = -31/4$, (a) $\alpha = 0$, $x_p(t) = 0$, (b) $\alpha = i$, $x_p(t) = 0$, (c) $\alpha = 0$, $x_p(t)$ at resonance frequency $\omega_{res} = \sqrt{15}/2$, $E_0 = 3$.

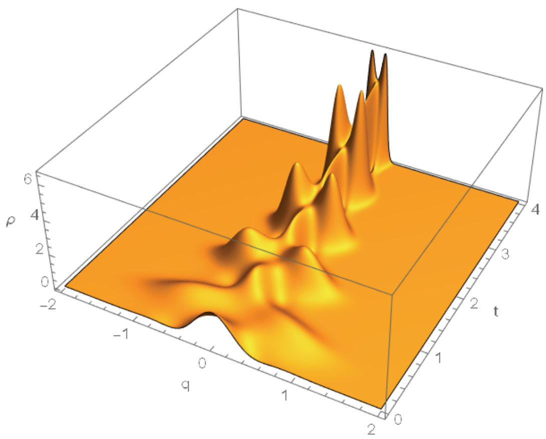
B. Time-evolved even-odd coherent states

The probability densities for the even-odd coherent states $\Phi_\alpha^{e,o}(q,t)$ are given by (3.29) and (3.32), respectively, with the squeezing coefficient

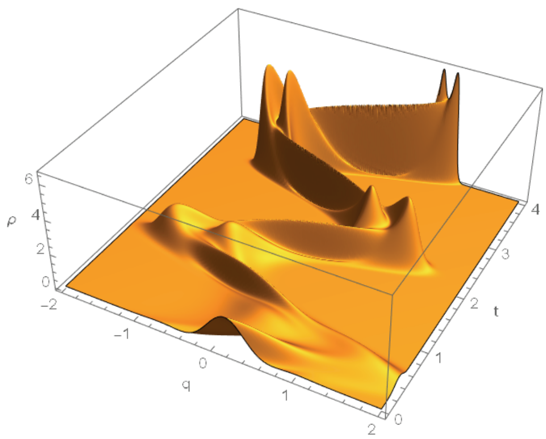
$$|\epsilon(t)| = e^{-\gamma t/2}\sqrt{\cos^2(\Omega_d t) + \frac{\omega_0^2}{\Omega_d^2}\sin^2(\Omega_d t)}, \quad \frac{\omega_0^2}{\Omega_d^2} > 1.$$

When $\Lambda_0^2 \to \gamma^2/4$, we have $\Omega_d^2 \to \omega_0^2$ and $|\epsilon(t)|$ approaches the limiting squeezing $e^{-\gamma t/2}$. In other words, for given γ and ω_0, when frequency Ω_d increases, amplitude of oscillations of $|\epsilon(t)|$ decreases. Knowing the explicit form of $|\epsilon(t)|$ allows us to find the uncertainties and uncertainty product at time-evolved even-odd coherent states using the equations in (3.30) and (3.33).

In Fig.3.3 and Fig.3.4, we give plots of the probability densities $\rho_\alpha^e(q,t)$ and $\rho_\alpha^o(q,t)$, which corresponds to $\alpha = 0$ in (3.29) and (3.32). Precisely, Fig.3.3-(a) and Fig.3.4-(a) show the evolution when $x_p(t) = 0$. Comparing these figures, we observe that while the probability density at odd coherent states vanishes at $q = 0$, the probability density at even coherent states does not. Besides, Fig.3.3-(b) and Fig.3.4-(b) exhibit how the trajectories oscillate according to the particular solution $x_p(t) = \cos(\sqrt{47}t/2 - \tan^{-1}(\sqrt{47}))$ at resonance frequency $\omega_{res} = \sqrt{23/2}$.

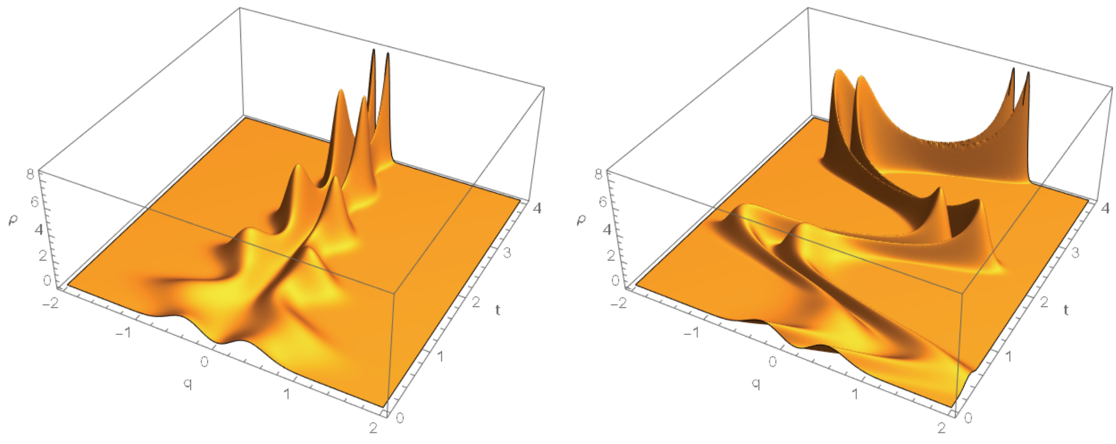


(a)



(b)

Figure 3.3. Probability density $\rho_\alpha^e(q,t)$ with $\gamma = 1$, $\omega_0 = \sqrt{12}$, $\hbar = 1$, $\Lambda_0 = 0$, $\alpha = 0$, (a) $x_p(t) = 0$, (b) $x_p(t)$ at resonance frequency $\omega_{res} = \sqrt{23/2}$, $E_0 = \sqrt{47}/2$.



(a) (b)

Figure 3.4. Probability density $\rho^{o}_{\alpha}(q,t)$ with $\gamma = 1$, $\omega_0 = \sqrt{12}$, $\hbar = 1$, $\Lambda_0 = 0$, $\alpha = 0$, (a) $x_p(t) = 0$, (b) $x_p(t)$ at resonance frequency $\omega_{res} = \sqrt{23/2}$, $E_0 = \sqrt{47}/2$.

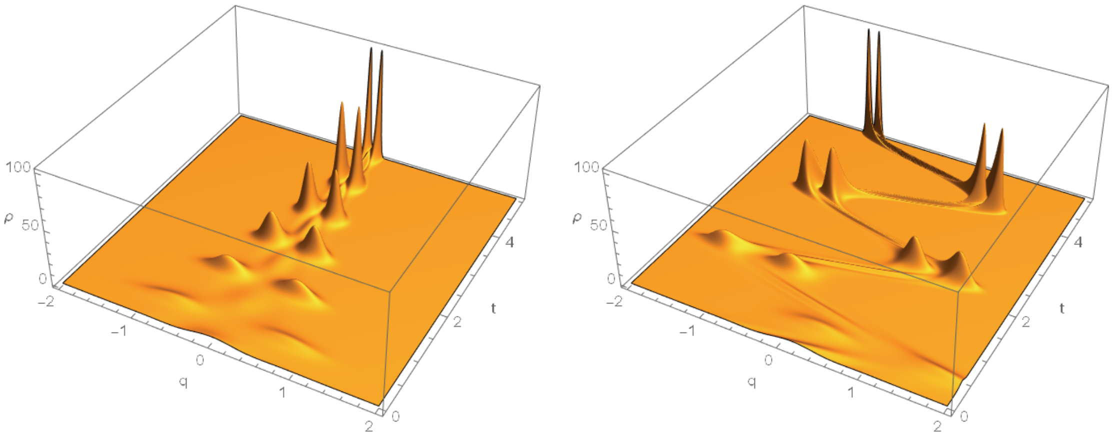
C. Time-evolved even-odd displaced squeezed states

The probability densities for the even-odd displaced squeezed states $\chi^{e,o}_{\alpha,r}(q,t)$ are given by (3.36) and (3.39), respectively, with the squeezing coefficient

$$|\epsilon_r(t)| = e^{-\gamma t/2}\sqrt{e^{2r}\cos^2(\Omega_d t) + \frac{\omega_0^2}{\Omega_d^2}e^{-2r}\sin^2(\Omega_d t)}, \quad \frac{\omega_0^2}{\Omega_d^2} > 1.$$

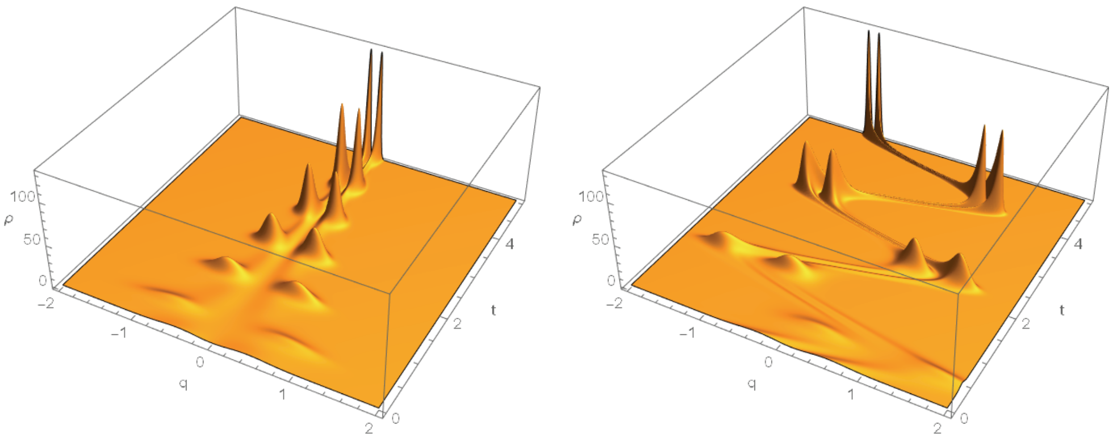
Then, we can find the uncertainties and uncertainty product at time-evolved even-odd coherent states using the equations in (3.37) and (3.40). For these states, the squeezing properties depend also on the parameter $r \geq 0$. Clearly the amplitude of oscillations of $(\Delta\hat{q})^{e,o}_r$ grows when r increases.

In the previous example, we show the time evolution of even-odd coherent states. Even-odd displaced squeezed states are obtained by squeezing them. Therefore, we plot the probability densities at time evolved even-odd displaced squeezed states and observe the squeezing according to the parameter r, see Fig.3.5 and Fig.3.6.



(a) (b)

Figure 3.5. Probability density $\rho^{e}_{\alpha,r}(q,t)$ with $\gamma = 1$, $\omega_0 = \sqrt{12}$, $\hbar = 1$, $\Lambda_0 = 0$, $r = 1/6$, $\alpha = 0$, (a) $x_p(t) = 0$, (b) $x_p(t)$ at resonance frequency $\omega_{res} = \sqrt{23/2}$, $E_0 = \sqrt{47}/2$.



(a) (b)

Figure 3.6. Probability density $\rho^{o}_{\alpha,r}(q,t)$ with $\gamma = 1$, $\omega_0 = \sqrt{12}$, $\hbar = 1$, $\Lambda_0 = 0$, $r = 1/6$, $\alpha = 0$, (a) $x_p(t) = 0$, (b) $x_p(t)$ at resonance frequency $\omega_{res} = \sqrt{23/2}$, $E_0 = \sqrt{47}/2$.

CHAPTER 4

INITIAL-BOUNDARY VALUE PROBLEMS FOR ONE-DIMENSIONAL QUANTUM PARAMETRIC OSCILLATORS WITH MOVING BOUNDARIES

In this chapter, we present an initial-boundary value problem (IBVP) for a one-dimensional generalized quantum parametric oscillator. We start with a Dirichlet boundary condition imposed at a moving boundary and show that if the boundary function is given as a linear combination of the homogenous and particular solutions of the corresponding classical equation of motion in position space, the problem can be solved analytically. As an application, we construct an exactly solvable quantum model with specific frequency modification and analyze the influence of the moving boundaries on the solution. Moreover, we introduce and solve an IBVP for the generalized quantum oscillator with a Robin boundary condition.

4.1. Dirichlet IBVP for a Quantum Parametric Oscillator on the Fixed Half-Line

In this section, we first consider an IBVP for a one-dimensional quantum parametric oscillator defined on the fixed half-line $0 < q < \infty$,

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\Psi(q,t) = \hat{H}(t)\Psi(q,t), \quad 0 < q < \infty,\ 0 < t < T, \\ \Psi(q,0) = \Psi_0(q), \quad 0 < q < \infty \\ \Psi(0,t) = 0, \quad 0 < t < T, \end{cases} \tag{4.1}$$

where $\Psi(q,t)$ is the wave function at time $0 < t < T$, $\Psi_0(q)$ is the initial state at time $t = 0$, and $\hat{H}(t)$ is a quadratic Hamiltonian given by

$$\hat{H}(t) = \frac{\hat{p}^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}^2 + \frac{B(t)}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) \tag{4.2}$$

with all time-dependent parameters being real-valued functions and $\mu(t) > 0$. We note that, the Schrödinger equation defined by (4.1) is invariant under space inversion and we can find an exact analytical solution of the Dirichlet IBVP on the half-line as given in the following proposition.

Proposition 4.1 *The Dirichlet IBVP for a quantum parametric oscillator given by (4.1) has solution of the form*

$$\Psi(q,t) = \frac{1}{\sqrt{x_1(t)}} \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)q^2\right] \times \varphi(\eta(q,t),\tau(t)), \tag{4.3}$$

where $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of the homogenous classical equation of motion

$$\ddot{x} + \frac{\dot{\mu}}{\mu}\dot{x} + \left(\omega^2 - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B\right)\right)x = 0, \quad t > 0, \tag{4.4}$$

satisfying the initial conditions $x_1(0) = 1$, $\dot{x}_1(0) = B(0)$, $x_2(0) = 0$, $\dot{x}_2(0) = 1/\mu(0)$, respectively,

$$\eta(q,t) = \frac{q}{x_1(t)}, \quad \tau(t) = -\hbar\left(\frac{x_2(t)}{x_1(t)}\right), \quad 0 < t < T, \tag{4.5}$$

and $\varphi(\eta,\tau)$ is solution of the Dirichlet IBVP for free Schrödinger equation

$$\begin{cases} i\dfrac{\partial\varphi}{\partial\tau} = \dfrac{1}{2}\dfrac{\partial^2\varphi}{\partial\eta^2}, \quad 0 < \eta < \infty, \ 0 < \tau < \tau(T), \\ \varphi(\eta,0) = \Psi_0(\eta), \quad 0 < \eta < \infty, \\ \varphi(0,\tau) = 0, \quad 0 < \tau < \tau(T). \end{cases} \tag{4.6}$$

Proof First, consider the following IVP defined on the whole real line

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\Psi(q,t) = \hat{H}(t)\Psi(q,t), & -\infty < q < \infty,\ 0 < t < T, \\ \Psi(q,0) = \Psi_0(q), & -\infty < q < \infty, \end{cases} \tag{4.7}$$

where the Hamiltonian $\hat{H}(t)$ is given by (4.2). IVP (4.7) is a special case of IVP (3.1) defined in the previous chapter. So using Eq. (3.3), we can write the explicit form of the evolution operator for IVP (4.7) in the form

$$\begin{aligned} \hat{U}(t,t_0) &= \exp\left(i\frac{\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)q^2\right)\times\exp\left(\ln\left|\frac{x_1(t_0)}{x_1(t)}\right|\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right) \\ &\quad\times\exp\left(\frac{i}{2}\hbar x_1^2(t_0)\left(\frac{x_2(t)}{x_1(t)}\right)\frac{\partial^2}{\partial q^2}\right), \end{aligned} \tag{4.8}$$

where $x_1(t)$ and $x_2(t)$ are linearly independent solutions of the classical equation (4.4) satisfying the initial conditions $x_1(0) = 1$, $\dot{x}_1(0) = B(0)$, $x_2(0) = 0$, $\dot{x}_2(0) = 1/\mu(0)$, respectively. Then, by applying $\hat{U}(t,t_0)$ to the given initial function $\Psi_0(q)$, we can find the solution of IVP (4.7) as

$$\Psi(q,t) = \frac{1}{\sqrt{x_1(t)}}\times\exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)q^2\right]\times\varphi(\eta(q,t),\tau(t)), \tag{4.9}$$

where $\varphi(\eta,\tau)$ is solution of the Dirichlet IVP for free Schrödinger equation

$$\begin{cases} i\dfrac{\partial\varphi}{\partial\tau} = \dfrac{1}{2}\dfrac{\partial^2\varphi}{\partial\eta^2}, & -\infty < \eta < \infty,\ 0 < \tau < \tau(T), \\ \varphi(\eta,0) = \Psi_0(\eta), & -\infty < \eta < \infty. \end{cases} \tag{4.10}$$

Then, $\Psi(q,t)$ found by (4.9) satisfies the initial condition $\Psi(q,0) = \Psi_0(q)$ also on the interval $0 < q < \infty$. We note that, $\tau(t)$ is positive and strictly increasing for $0 < t < T$, so that $\tau = \tau(t)$, $0 < t < T$ if and only if $t = t(\tau)$ for $0 < \tau < \tau(T)$. Therefore, solution (4.9) will satisfy the boundary condition in (4.1) only when the function $\varphi(\eta,\tau)$ satisfies $\varphi(0,\tau) = 0$. Hence, (4.9) satisfies IBVP (4.1) if $\varphi(\eta,\tau)$ satisfies IBVP (4.6) for the free Schrödinger equation. □

The Dirichlet IBVP (4.1) has solution

$$\varphi(\eta, \tau) = \int_0^{\infty} G(\eta, \xi, \tau)\varphi(\xi, 0)d\xi, \tag{4.11}$$

where $G(\eta, \xi, \tau) = K(\eta - \xi, \tau) - K(\eta + \xi, \tau)$ is the Green's function with $K(\eta, \tau)$ being the propagator of the free Schrödinger equation defined by

$$K(\eta, \tau) = \sqrt{\frac{i}{2\pi\tau}} \exp\left(\frac{-i\eta^2}{2\tau}\right). \tag{4.12}$$

As a consequence of Proposition 4.1, Dirichlet IBVP (4.1) for a quantum parametric oscillator defined on the fixed half-line has solution with integral representation of the form

$$\Psi(q, t) = \frac{1}{\sqrt{x_1(t)}} \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)q^2\right] \times \int_0^{\infty} G(\eta(q, t), \xi, \tau(t))\Psi_0(\xi)d\xi, \tag{4.13}$$

where $\eta(q, t)$ and $\tau(t)$ are given by (4.5). Thus, the exact form of the solution can be found if the integral converges for the given initial data.

4.2. Dirichlet IBVP for a Generalized Quantum Parametric Oscillator with Moving Boundary

In this section, we consider an IBVP for a time-dependent Schrödinger equation defined on the interval $s(t) < q < \infty$ and with Dirichlet boundary condition imposed at $q = s(t)$, $0 < t < T$,

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\Psi(q, t) = \hat{H}_g(t)\Psi(q, t), \quad s(t) < q < \infty,\ 0 < t < T, \\ \Psi(q, 0) = \Psi_0(q), \quad s(0) < q < \infty \\ \Psi(s(t), t) = 0, \quad 0 < t < T, \end{cases} \tag{4.14}$$

where $\Psi(q,t)$ is the wave function at time $0 < t < T$, $\Psi_0(q)$ is the initial state at time $t = 0$, and $\hat{H}_g(t)$ is the most general quadratic Hamiltonian given by

$$\hat{H}_g(t) = \frac{\hat{p}^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}^2 + \frac{B(t)}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) + D(t)\hat{p} + E(t)\hat{q} + F(t) \tag{4.15}$$

with all time-dependent parameters being real-valued functions and $\mu(t) > 0$.

Proposition 4.2 *Let the Dirichlet IBVP for a generalized quantum parametric oscillator with moving boundary be given by (4.14). If the boundary function $s(t)$ in the IBVP (4.14) is of the form*

$$s(t) = x_g(t) \equiv c_1 x_1(t) + c_2 x_2(t) + x_p(t), \quad c \equiv (c_1, c_2) \in \mathbb{R}^2, \tag{4.16}$$

where $x_1(t)$ and $x_2(t)$ are two linearly independent homogenous solutions and $x_p(t)$ is a particular solution of the classical equation of motion

$$\ddot{x} + \frac{\dot{\mu}}{\mu}\dot{x} + \left(\omega^2 - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B\right)\right)x = \dot{D} + \left(\frac{\dot{\mu}}{\mu} + B\right)D - \frac{1}{\mu}E, \quad t > 0, \tag{4.17}$$

satisfying the initial conditions $x_1(0) = 1$, $\dot{x}_1(0) = B(0)$, $x_2(0) = 0$, $\dot{x}_2(0) = 1/\mu(0)$, and $x_p(0) = 0$, $\dot{x}_p(0) = D(0)$, respectively, then the IBVP (4.14) has solution of the form

$$\begin{aligned}\Psi(q,t) &= \frac{1}{\sqrt{x_1(t)}} \times \exp\left[i\int_0^t L_g(\xi)d\xi\right] \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)(q - x_g(t))^2\right] \\ &\quad \times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_g(t) - B(t)x_g(t) - D(t)\Big)(q - x_g(t))\right] \times \varphi(\eta_g(q,t), \tau(t)),\end{aligned} \tag{4.18}$$

where $L_g(t)$ denotes a Lagrangian function for the moving boundary given as

$$L_g(t) = \frac{1}{2\hbar}\Big[\mu(t)\Big((\dot{x}_g(t) - B(t)x_g(t) - D(t))^2 - \omega^2(t)x_g^2(t)\Big) - 2\Big(E(t)x_g(t) + F(t)\Big)\Big], \tag{4.19}$$

the coordinate transformations are denoted as

$$\eta_g(q,t) = \frac{q - x_g(t)}{x_1(t)}, \quad \tau(t) = -\hbar\left(\frac{x_2(t)}{x_1(t)}\right), \quad 0 < t < T, \tag{4.20}$$

and $\varphi(\eta,\tau)$ is solution of the Dirichlet IBVP for free Schrödinger equation

$$\begin{cases} i\dfrac{\partial\varphi}{\partial\tau} = \dfrac{1}{2}\dfrac{\partial^2\varphi}{\partial\eta^2}, \quad 0 < \eta < \infty,\ 0 < \tau < \tau(T), \\ \varphi(\eta,0) = \exp\left(\dfrac{-ic_2}{\hbar}\eta\right)\Psi_0(\eta + c_1), \quad 0 < \eta < \infty, \\ \varphi(0,\tau) = 0, \quad 0 < \tau < \tau(T). \end{cases} \tag{4.21}$$

Proof Assume that the boundary function $s(t)$ is given by (4.16). Define a new coordinate $\tilde{q} = q - s(t)$ and denote $\widetilde{\Psi}(\tilde{q},t) = \Psi(q,t)$. After performing time and space differentiations

$$\frac{\partial\Psi}{\partial t} = -\dot{s}(t)\frac{\partial\widetilde{\Psi}}{\partial\tilde{q}} + \frac{\partial\widetilde{\Psi}}{\partial t}, \quad \frac{\partial\Psi}{\partial q} = \frac{\partial\widetilde{\Psi}}{\partial\tilde{q}}, \quad \frac{\partial^2\Psi}{\partial q^2} = \frac{\partial^2\widetilde{\Psi}}{\partial\tilde{q}^2},$$

we obtain the following IBVP defined on $0 < \tilde{q} < \infty$ for function $\widetilde{\Psi}(\tilde{q},t)$

$$\begin{cases} i\hbar\dfrac{\partial\widetilde{\Psi}}{\partial t} = \hat{H}_g^s(t)\widetilde{\Psi}, \\ \widetilde{\Psi}(\tilde{q},0) = \Psi_0(\tilde{q} + c_1), \ 0 < \tilde{q} < \infty, \\ \widetilde{\Psi}(0,t) = 0, \ 0 < t < T, \end{cases} \tag{4.22}$$

where

$$\begin{aligned} \hat{H}_g^s(t) \quad = \quad & -\frac{\hbar^2}{2\mu(t)}\frac{\partial^2}{\partial\tilde{q}^2} + \frac{\mu(t)\omega^2(t)}{2}\tilde{q}^2 - \frac{i\hbar}{2}B(t)\left(1 + 2\tilde{q}\frac{\partial}{\partial\tilde{q}}\right) \\ & + i\hbar\Big(\dot{s}(t) - B(t)s(t) - D(t)\Big)\frac{\partial}{\partial\tilde{q}} + \Big(\mu(t)\omega^2(t)s(t) + E(t)\Big)\tilde{q} \\ & + \left(\frac{\mu(t)\omega^2(t)}{2}s^2(t) + E(t)s(t) + F(t)\right). \end{aligned} \tag{4.23}$$

To solve IBVP (4.22), we first consider the following IVP defined on the whole real line

$$\begin{cases} i\hbar\dfrac{\partial\widetilde{\Psi}}{\partial t} = \hat{H}_g^s(t)\widetilde{\Psi}, \ \ 0 < t < T, \\ \widetilde{\Psi}(\tilde{q}, 0) = \Psi_0(\tilde{q} + c_1), \ \ -\infty < \tilde{q} < \infty, \end{cases} \tag{4.24}$$

where the Hamiltonian $\hat{H}_g^s(t)$ is given by (4.23). We can find the exact form of the evolution operator for IVP (4.24) defined as

$$\begin{cases} i\hbar\dfrac{d}{dt}\hat{U}_g^s(t,t_0) = \hat{H}_g^s(t)\hat{U}_g^s(t,t_0), \quad 0 < t < T, \\ \hat{U}_g^s(t_0,t_0) = \hat{I}, \end{cases} \tag{4.25}$$

by using the Wei-Norman Lie algebraic approach. In fact, Hamiltonian (4.23) can be expressed as a finite linear combination of Lie algebra generators as

$$\begin{aligned} \hat{H}_g^s(t) \;\; = \;\; & -i\Big[\frac{\hbar^2}{\mu(t)}\hat{K}_- + \mu(t)\omega^2(t)\hat{K}_+ + 2\hbar B(t)\hat{K}_0 \\ & +\hbar\Big(-\dot{s}(t) + B(t)s(t) + D(t)\Big)\hat{E}_2 + \Big(\mu(t)\omega^2(t)s(t) + E(t)\Big)\hat{E}_1 \\ & +\left(\frac{\mu(t)\omega^2(t)}{2}s^2(t) + E(t)s(t) + F(t)\right)\hat{E}_3\Big], \end{aligned} \tag{4.26}$$

where Heisenberg-Weyl algebra generators $\hat{E}_1, \hat{E}_2, \hat{E}_3$ are given by (2.10) and the generators $\hat{K}_,\hat{K}_+, \hat{K}_0$ of the $SU(1,1)$ algebra are given by (2.17). Then, the evolution operator $\hat{U}_g^s(t,t_0)$ for IVP (4.24) can be written as a product of exponential operators of the form

$$\begin{aligned} \hat{U}_g^s(t,t_0) \;\; = \;\; & \exp\Big(c(t)\hat{E}_3\Big)\times\exp\left(\frac{a(t)}{\hbar}\hat{E}_1\right)\times\exp\Big((s(t)-b(t))\hat{E}_2\Big) \\ & \times\exp\Big(f(t)\hat{K}_+\Big)\times\exp\Big(2h(t)\hat{K}_0\Big)\times\exp\Big(g(t)\hat{K}_-\Big)\times\exp\Big(-\frac{c_2}{\hbar}\hat{E}_1\Big), \end{aligned}$$

where $f(t), g(t), h(t)$ and $a(t), b(t), c(t)$ being real-valued functions to be determined. Performing time differentiation, we obtain

$$\begin{aligned}
\frac{d}{dt}\hat{U}^s_g(t,t_0) &= \left(\dot{c}\hat{E}_3\right) e^{\frac{a(t)}{\hbar}\hat{E}_1}\, e^{(s(t)-b(t))\hat{E}_2}\, e^{f(t)\hat{K}_+}\, e^{2h(t)\hat{K}_0}\, e^{g(t)\hat{K}_-}\, e^{-\frac{c_2}{\hbar}\hat{E}_1} \\
&+ e^{c(t)\hat{E}_3} \left(\frac{\dot{a}(t)}{\hbar}\hat{E}_1\right) e^{\frac{a(t)}{\hbar}\hat{E}_1}\, e^{(s(t)-b(t))\hat{E}_2}\, e^{f(t)\hat{K}_+}\, e^{2h(t)\hat{K}_0}\, e^{g(t)\hat{K}_-}\, e^{-\frac{c_2}{\hbar}\hat{E}_1} \\
&+ e^{c(t)\hat{E}_3}\, e^{\frac{a(t)}{\hbar}\hat{E}_1} \left((\dot{s}(t)-\dot{b}(t))\hat{E}_2\right) e^{(s(t)-b(t))\hat{E}_2}\, e^{f(t)\hat{K}_+}\, e^{2h(t)\hat{K}_0}\, e^{g(t)\hat{K}_-}\, e^{-\frac{c_2}{\hbar}\hat{E}_1} \\
&+ e^{c(t)\hat{E}_3}\, e^{\frac{a(t)}{\hbar}\hat{E}_1}\, e^{(s(t)-b(t))\hat{E}_2} \left(\dot{f}(t)\hat{K}_+\right) e^{f(t)\hat{K}_+}\, e^{2h(t)\hat{K}_0}\, e^{g(t)\hat{K}_-}\, e^{-\frac{c_2}{\hbar}\hat{E}_1} \\
&+ e^{c(t)\hat{E}_3}\, e^{\frac{a(t)}{\hbar}\hat{E}_1}\, e^{(s(t)-b(t))\hat{E}_2}\, e^{f(t)\hat{K}_+} \left(2\dot{h}(t)\hat{K}_0\right) e^{2h(t)\hat{K}_0}\, e^{g(t)\hat{K}_-}\, e^{-\frac{c_2}{\hbar}\hat{E}_1} \\
&+ e^{c(t)\hat{E}_3}\, e^{\frac{a(t)}{\hbar}\hat{E}_1}\, e^{(s(t)-b(t))\hat{E}_2}\, e^{f(t)\hat{K}_+}\, e^{2h(t)\hat{K}_0} \left(\dot{g}(t)\hat{K}_-\right) e^{g(t)\hat{K}_-}\, e^{-\frac{c_2}{\hbar}\hat{E}_1}. \qquad (4.27)
\end{aligned}$$

Bu using the Baker-Campbell-Hausdorff relation

$$e^{\xi\hat{A}}\,\hat{B}\,e^{-\xi\hat{A}} = \hat{B} + \xi[\hat{A},\hat{B}] + (\xi^2/2)[\hat{A},[\hat{A},\hat{B}]] + (\xi^3/3!)[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \ldots,$$

we can rewrite Eq. (4.27) in the form

$$\begin{aligned}
i\hbar\frac{d}{dt}\hat{U}_g(t,t_0) &= i\hbar\bigg[\Big(\dot{c} + \frac{1}{\hbar}a(\dot{s}-\dot{b}) + \frac{1}{2}\dot{f}(s-b)^2 + \frac{1}{\hbar}\dot{h}a(s-b) - f\dot{h}(s-b)^2 \\
&+ \dot{g}e^{-2h}\Big(\frac{1}{2\hbar^2}a^2 - \frac{1}{\hbar}fa(s-b) + \frac{1}{2}f^2(s-b)^2\Big)\Big)\hat{E}_3 \\
&+ \Big(\dot{b} - \dot{s} - \dot{h}(s-b) + \dot{g}e^{-2h}\Big(-\frac{1}{\hbar}a + f(s-b)\Big)\Big)\hat{E}_2 \\
&+ \Big(\frac{1}{\hbar}\dot{a} - \dot{f}(s-b) - \frac{1}{\hbar}\dot{h}a + 2f\dot{h}(s-b) + \dot{g}e^{-2h}\Big(\frac{1}{\hbar}fa - f^2(s-b)\Big)\Big)\hat{E}_1 \\
&+ \Big(\dot{f} - 2f\dot{h} + f^2\dot{g}e^{-2h}\Big)\hat{K}_+ \\
&+ 2\Big(\dot{h} - f\dot{g}e^{-2h}\Big)\hat{K}_0 + \Big(\dot{g}e^{-2h}\Big)\hat{K}_-\bigg]\hat{U}_g(t,t_0). \qquad (4.28)
\end{aligned}$$

Using equations (4.26) and (4.28), we compare both sides of the operator equation (4.25) and obtain that $\hat{U}^s_g(t,t_0)$ is solution of the problem if the unknown functions $f(t), g(t), h(t)$

satisfy the nonlinear system

$$\begin{aligned} \dot{f}(t)+\frac{\hbar}{\mu(t)}f^2(t)+2B(t)f(t)+\frac{\mu(t)\omega^2(t)}{\hbar}&=0, \quad f(0)=0,\\ \dot{g}(t)+\frac{\hbar}{\mu(t)}e^{2h(t)}&=0, \quad g(0)=0,\\ \dot{h}(t)+\frac{\hbar}{\mu(t)}f(t)+B(t)&=0, \quad h(0)=0, \end{aligned} \tag{4.29}$$

and $a(t), b(t), c(t)$ satisfy the nonlinear system

$$\begin{aligned} \dot{a}(t)+B(t)a(t)+\mu(t)\omega^2(t)b(t)+E(t)&=0, \quad a(0)=c_2,\\ \dot{b}(t)-B(t)b(t)-\frac{a(t)}{\mu(t)}-D(t)&=0, \quad b(0)=c_1,\\ \dot{c}(t)-\frac{a^2(t)}{2\hbar\mu(t)}+\frac{\mu(t)\omega^2(t)}{2\hbar}s^2(t)+\frac{1}{\hbar}(E(t)s(t)+F(t))&=0, \quad c(0)=0. \end{aligned} \tag{4.30}$$

Indeed, (4.29) and (4.30) are two independent systems, one for f, g, h and second for a, b, c. We realize that the first equation in the system (4.29) is an IVP for the non-linear Riccati equation, and using substitution $f(t)=\mu(t)(\dot{x}/x-B)/\hbar$, it transforms to the linear second-order homogeneous differential equation

$$\ddot{x}+\frac{\dot{\mu}}{\mu}\dot{x}+\left(\omega^2-\left(\dot{B}+B^2+\frac{\dot{\mu}}{\mu}B\right)\right)x=0, \tag{4.31}$$

with initial conditions $x(0)=1$, $\dot{x}(0)=B(0)$. We denote the solution of the IVP as $x_1(t)$ and assuming that all coefficients in Eq. (4.31) are continuous on time interval containing $t=0$, we denote a second solution by $x_2(t)$ satisfying the initial conditions $x_2(0)=0$, $\dot{x}_2(0)=1/\mu(0)$. Abel's differential equation identity gives us

$$x_2(t)=x_1(t)\int_0^t \frac{1}{\mu(s)x_1^2(s)}ds.$$

Then, the solution of system (4.29) is found in terms of two linearly independent solutions $x_1(t)$ and $x_2(t)$ of classical equation of motion (4.17) as

$$\begin{aligned} f(t) &= \frac{\mu(t)}{\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right), \\ g(t) &= -\hbar\left(\frac{x_2(t)}{x_1(t)}\right), \\ h(t) &= -\ln|x_1(t)|. \end{aligned} \tag{4.32}$$

On the other hand, to solve system (4.30), we take derivative of the first line and use the equation in the second line, which gives

$$\ddot{b} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{b} + \left(\omega^2 - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B\right)\right)b = \dot{D} + \left(\dot{B} + \frac{\dot{\mu}}{\mu}\right)D - \frac{1}{\mu}E, \tag{4.33}$$

which is the same as the classical equation (4.17) with initial conditions $b(0) = c_1$, $\dot{b}(0) = c_1 B(0) + c_2/\mu(0) + D(0)$. It implies that $b(t)$ and the boundary function $s(t)$ satisfy the same differential equation with the same initial conditions, that means $b(t) = s(t)$. Then, the solution of the system (4.30) is found as

$$\begin{aligned} a(t) &= \mu(t)\Big(\dot{x}_g(t) - B(t)x_g(t) - D(t)\Big), \\ b(t) &= s(t) = x_g(t), \\ c(t) &= \frac{1}{2\hbar}\int_0^t \mu(\xi)\Big(\left(\dot{x}_g(\xi) - B(\xi)x_g(\xi)\right)^2 - \omega^2(\xi)x_g^2(\xi)\Big) - 2\Big(E(\xi)x_g(\xi) + F(\xi)\Big)d\xi. \end{aligned} \tag{4.34}$$

After finding expressions for all unknown functions, the explicit form of the evolution operator becomes

$$\begin{aligned} \hat{U}_g^s(t,t_0) \;=\; & \exp\left(i\int_0^t L_g(\xi)d\xi\right) \times \exp\left(i\mu(t)\Big(\dot{x}_g(t) - B(t)x_g(t) - D(t)\Big)\tilde{q}\right) \\ & \times \exp\left(i\frac{\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)\tilde{q}^2\right) \times \exp\left(-\ln|x_1(t)|\left(\tilde{q}\frac{\partial}{\partial\tilde{q}} + \frac{1}{2}\right)\right) \\ & \times \exp\left(\frac{i\hbar}{2}\left(\frac{x_2(t)}{x_1(t)}\right)\frac{\partial^2}{\partial\tilde{q}^2}\right). \end{aligned} \tag{4.35}$$

By applying this form of the evolution operator to the initial function $\Psi_0(\tilde{q}+c_1)$ in (4.24), we can find the solution of IVP (4.24) as

$$\begin{aligned}\widetilde{\Psi}(\widetilde{q},t) &= \frac{1}{\sqrt{x_1(t)}}\times\exp\left[i\int_0^t L_g(\xi)d\xi\right]\times\exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)}-B(t)\right)\tilde{q}^2\right]\\ &\quad\times\exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_g(t)-B(t)x_g(t)-D(t)\Big)\tilde{q}\right]\times\varphi\Big(\frac{\tilde{q}}{x_1(t)},-\hbar\frac{x_2(t)}{x_1(t)}\Big),\end{aligned} \tag{4.36}$$

where $\varphi(\eta,\tau)$ satisfies the IVP for free Schödinger equation

$$\begin{cases} i\dfrac{\partial\varphi}{\partial\tau}=\dfrac{1}{2}\dfrac{\partial^2\varphi}{\partial\eta^2}, & -\infty<\eta<\infty,\ 0<\tau<\tau(T),\\ \varphi(\eta,0)=\exp\left(\dfrac{-ic_2}{\hbar}\eta\right)\Psi_0(\eta+c_1), & -\infty<\eta<\infty.\end{cases} \tag{4.37}$$

Now, $\widetilde{\Psi}(\widetilde{q},t)$ satisfies the homogeneous boundary condition $\widetilde{\Psi}(0,t)=0$ whenever $\varphi(\eta,\tau)$ satisfies $\varphi(0,\tau)=0$. So the function $\widetilde{\Psi}(\widetilde{q},t)$ found by (4.36) will be the solution of IBVP (4.22) on the fixed half-line if the function $\varphi(\eta,\tau)$ solves IBVP (4.21) for the free Schrödinger equation. By back substitution $\tilde{q}=q-x_g(t)$, we obtain solution (4.18) of the IBVP, satisfying the IC in (4.14). Therefore, solution (4.18) will satisfy the Dirichlet BC in (4.14) if $\varphi(\eta,\tau)$ satisfies the Dirichlet BC given in (4.21), completing the proof. □

As a result of Proposition 4.2, the Dirichlet IBVP (4.14) with moving boundary for a quantum parametric oscillator has solution with integral representation given as

$$\begin{aligned}\Psi(q,t) &= \frac{1}{\sqrt{x_1(t)}}\times\exp\left[i\int_0^t L_g(\xi)d\xi\right]\times\exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)}-B(t)\right)(q-x_g(t))^2\right]\\ &\quad\times\exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_g(t)-B(t)x_g(t)-D(t)\Big)(q-x_g(t))\right]\\ &\quad\times\int_0^\infty G(\eta_g(q,t),\xi,\tau(t))\Psi_0(\xi+c_1)e^{-ic_2\xi/\hbar}d\xi,\end{aligned} \tag{4.38}$$

provided the integral converges for the given initial data.

We notice that solution properties depend on the initial data, the time-dependent parameters of Hamiltonian (4.15), and the moving boundary $s(t)$. Here, in the case $s(t)=0$ and $D(t)=E(t)=F(t)=0$, the problem reduces to the one defined on the fixed half-line which we consider in Proposition 4.1. Therefore, the effect of the moving boundary can be seen by comparing solutions given by (4.38) and (4.13). We observe that the boundary $s(t)$ causes a shift in the position coordinate. Also, there are extra time-dependent exponentials generated by the effect of the moving boundary in the solution. One of them includes the Lagrangian $L_g(t) \equiv L(x(t), \dot{x}(t), t)$ and from the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)(x(t), \dot{x}(t), t) - \frac{\partial L}{\partial x}(x(t), \dot{x}(t), t) = 0, \tag{4.39}$$

one can obtain the classical equation of motion (4.17). So the Lagrangian $L_g(t)$ describes the motion of the boundary point. The other exponent depends linearly on the position variable. Since the arguments of both of these exponentials are pure imaginary, they contribute to the phase factor in the solution.

Finally, we note that if the parameters $D(t)$ and $E(t)$ of Hamiltonian (4.15) are nonzero, then the Schrödinger equation for the generalized quantum parametric oscillator in (4.14) is not invariant under space inversion. In this case, it is not easy to solve the corresponding IBVP on the fixed half-line $0 < q < \infty$ with Dirichlet boundary condition imposed at $q = 0$. However, solutions of some particular half-line IBVP's can be found as a consequence of Proposition 4.2 when the particular solution of the corresponding classical equation of motion is zero, that is $x_p(t) = 0$. Here, we write some particular cases which could be of interest:

i) $D(t) = E(t) = 0$.

ii) $B(t) = -\frac{\dot{\mu}(t)}{\mu(t)}$, $D(t)$−constant and $E(t) = 0$.

iii) $B(t) = -\frac{\dot{\mu}(t)}{\mu(t)}$ and $\dot{D}(t) = \frac{E(t)}{\mu(t)}$.

For each case, by letting $c_1 = c_2 = 0$ in solution (4.38), one can obtain solutions of the corresponding IBVP on the fixed half-line.

To get better insight into the problem, in what follows, we give solutions of the IBVP (4.14) corresponding to a certain initial data $\Psi_0(q)$ and homogenous Dirichlet boundary condition $\Psi_{bc}(t) = 0$.

IBVP 1- Eigenstates type initial condition: Here, we consider IBVP (4.14) with a family of initial functions that is parametrized by $\xi \equiv (\xi_1, \xi_2) \in \mathbb{R}^2$

$$\Psi^0_{\xi,n}(q) = N_n \exp\left(i\frac{\xi_2}{\hbar}(q-\xi_1)\right)\exp\left(-\frac{\omega_0}{2\hbar}(q-\xi_1)^2\right)H_n\left(\sqrt{\frac{\omega_0}{\hbar}}(q-\xi_1)\right), \quad s(0) < q < \infty, \tag{4.40}$$

where ξ_1 controls shifting, ξ_2 controls the phase factor, $\omega_0 > 0$, $N_n = (\omega_0/\pi\hbar)^{1/4}(2^n n!)^{-1/2}$, and $H_n(q)$ are Hermite polynomials for all $n = 0, 1, 2, \ldots$. For the given boundary $s(t) = x_g(t) = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$, $c = (c_1, c_2) \in \mathbb{R}^2$ and parameter $c = \xi$, the IBVP has solution of the form (4.18), where $\eta_g(q,t)$, $\tau(t)$ are given by (4.20), and $\varphi(\eta,\tau)$ is solution of the IBVP (4.21) with initial condition

$$\varphi(\eta, 0) = N_n \exp\left(-\frac{\omega_0}{2\hbar}\eta^2\right)H_n\left(\sqrt{\frac{\omega_0}{\hbar}}\eta\right), \quad \eta > 0, \tag{4.41}$$

being the normalized eigenstates of the SQHO whose Hamiltonian is $\hat{H}_0 = (\hat{p}^2 + \omega_0^2\hat{q}^2)/2$, and $\varphi(0,\tau) = 0$. So, IBVP (4.21) has solution with integral representation of the form

$$\varphi_n(\eta,\tau) = N_n \int_0^\infty \sqrt{\frac{i}{2\pi\tau}}\left(e^{-\frac{i}{2\tau}(\eta-\xi)^2} - e^{-\frac{i}{2\tau}(\eta+\xi)^2}\right)\exp\left(-\frac{\omega_0}{2\hbar}\xi^2\right)H_n\left(\sqrt{\frac{\omega_0}{\hbar}}\xi\right)d\xi \tag{4.42}$$

for all $n = 0, 1, 2, \ldots$. We notice that for odd $n = 2k+1$, $k = 0, 1, 2, \ldots$, one has

$$\varphi_{2k+1}(\eta,\tau) = N_{2k+1}\int_{-\infty}^{\infty}\sqrt{\frac{i}{2\pi\tau}}e^{-\frac{i}{2\tau}(\eta-\xi)^2}\exp\left(-\frac{\omega_0}{2\hbar}\xi^2\right)H_{2k+1}\left(\sqrt{\frac{\omega_0}{\hbar}}\xi\right)d\xi, \tag{4.43}$$

and so we can find the exact form of the solution. Then,

$$\begin{aligned}\Psi_{2k+1}(q,t) \quad = \quad & \sqrt{\frac{2}{|\epsilon(t)|}}N_{2k+1} \times \exp\left[-i\left(2k+\frac{3}{2}\right)\nu(t)\right] \times \exp\left(i\int_0^t L_g(\xi)d\xi\right) \\ & \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon(t)| - B(t)\right)(q - x_g(t))^2\right] \\ & \times \exp\left[\frac{i\mu(t)}{\hbar}\left(\dot{x}_g(t) - B(t)x_g(t) - D(t)\right)(q - x_g(t))\right] \\ & \times \exp\left[-\frac{\omega_0}{2\hbar}\frac{(q-x_g(t))^2}{|\epsilon(t)|^2}\right] \times H_{2k+1}\left(\sqrt{\frac{\omega_0}{\hbar}}\left(\frac{q-x_g(t)}{|\epsilon(t)|}\right)\right),\end{aligned} \tag{4.44}$$

and the corresponding probability density becomes

$$\rho_{2k+1}(q,t) = \frac{2N_{2k+1}^2}{|\epsilon(t)|} \exp\Big[-\frac{\omega_0}{\hbar}\frac{(q-x_g(t))^2}{|\epsilon(t)|^2}\Big] H_{2k+1}^2\left(\sqrt{\frac{\omega_0}{\hbar}}\left(\frac{q-x_g(t)}{|\epsilon(t)|}\right)\right), \tag{4.45}$$

where $|\epsilon(t)|$ is the squeezing coefficient defined by (3.9). Thus, for $k = 0, 1, 2, \ldots$, $\Psi_{2k+1}(q,t)$ found by (4.44) is an exact solution to IBVP (4.14).

IBVP 2- Coherent states type initial condition: Now, for IBVP (4.14) we take the following family of initial data

$$\Psi_\alpha^0(q) = Ae^{-i\alpha_1\alpha_2}\exp\left(i\sqrt{\frac{2\omega_0}{\hbar}}\alpha_2 q\right)\exp\left(-\frac{\omega_0}{2\hbar}\left(q-\sqrt{\frac{2\hbar}{\omega_0}}\alpha_1\right)^2\right), \; s(0) < q < \infty, \tag{4.46}$$

where A is a real constant, $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$. If $s(t) = x_g(t)$ is the moving boundary, then solution is of the form (4.18), where $\eta(q,t)$, $\tau(t)$ are given by (4.20) and $\varphi(\eta,\tau)$ is solution of IBVP (4.21) with

$$\varphi_{\tilde\alpha}(\eta,0) = A_{\tilde\alpha}\exp\left(\sqrt{\frac{2\omega_0}{\hbar}}\tilde\alpha\eta\right)\exp\left(\frac{-\omega_0}{2\hbar}\eta^2\right), \;\; \eta > 0, \tag{4.47}$$

where $\tilde\alpha = \alpha - \sqrt{1/(2\omega_0\hbar)}(\omega_0 c_1 + ic_2)$, $c_1, c_2 \in \mathbb{R}$ and $\varphi(0,\tau) = 0$. Actually, one may write $\varphi_\alpha(\eta, 0)$ as a linear combination of even and odd coherent states of SQHO

$$\varphi_{\tilde\alpha}(\eta,0) = A_1\phi_{\tilde\alpha}^e(\eta) + A_2\phi_{\tilde\alpha}^o(\eta), \tag{4.48}$$

where $A_1, A_2 \in \mathbb{R}$ and $\phi_{\tilde\alpha}^e(\eta), \phi_{\tilde\alpha}^o(\eta)$ are given by (2.31) and (2.32), respectively. Then, we can write solution of IBVP (4.21) in the form

$$\varphi_{\tilde\alpha}(\eta,\tau) = \int_0^\infty \sqrt{\frac{i}{2\pi\tau}}\left(e^{-\frac{i}{2\tau}(\eta-\xi)^2} - e^{-\frac{i}{2\tau}(\eta+\xi)^2}\right)(A_1\phi_{\tilde\alpha}^e(\xi) + A_2\phi_{\tilde\alpha}^o(\xi))d\xi. \tag{4.49}$$

Taking $A_1 = 0$, we can find the exact form of the solution by taking the following integral

$$\varphi_{\tilde{\alpha}}(\eta,\tau) = A_2 \int_{-\infty}^{\infty} \sqrt{\frac{i}{2\pi\tau}} e^{-\frac{i}{2\tau}(\eta-\xi)^2} \phi^o_{\tilde{\alpha}}(\xi) d\xi. \tag{4.50}$$

Thus, one family of normalized solutions of Dirichlet IBVP (4.14) is

$$\begin{aligned} \Psi^o_{\tilde{\alpha}}(q,t) \;=\; & \Big(\frac{\omega_0}{\pi\hbar}\Big)^{1/4} \frac{1}{\sqrt{\epsilon(t)\sinh|\tilde{\alpha}|^2}} \times \exp\Big(-\frac{(\epsilon^*(t))^2}{2|\epsilon(t)|^2}\tilde{\alpha}^2\Big) \\ & \times \exp\Big(i\int_0^t L_g(\xi)d\xi\Big) \times \exp\Big[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_g(t) - B(t)x_g(t) - D(t)\Big)(q - x_g(t))\Big] \\ & \times \exp\Big[\frac{i\mu(t)}{2\hbar}\Big(\frac{d}{dt}\ln|\epsilon(t)| - B(t)\Big)(q - x_g(t))^2\Big] \\ & \times \sinh\Big(\sqrt{\frac{2\omega_0}{\hbar}}\frac{1}{\epsilon(t)}(q - x_g(t))\tilde{\alpha}\Big) \times \exp\Big(-\frac{\omega_0}{2\hbar}\frac{(q - x_g(t))^2}{|\epsilon(t)|^2}\Big), \end{aligned} \tag{4.51}$$

and the corresponding probability density function becomes

$$\begin{aligned} \rho^o_{\alpha}(q,t) \;=\; & \sqrt{\frac{\omega_0}{\pi\hbar}} \frac{1}{|\epsilon(t)|\sinh|\alpha|^2} \times \exp\Big(\frac{\big(\Im(\alpha\epsilon^*(t))\big)^2 - \big(\Re(\alpha\epsilon^*(t))\big)^2}{|\epsilon(t)|^2}\Big) \\ & \times \exp\Big(-\frac{\omega_0}{\hbar}\frac{(q - x_g(t))^2}{|\epsilon(t)|^2}\Big)\Big\{\cosh\Big(2\sqrt{\frac{2\omega_0}{\hbar}}\Re(\alpha\epsilon^*(t))\frac{q - x_g(t)}{|\epsilon(t)|^2}\Big) \\ & - \cos\Big(2\sqrt{\frac{2\omega_0}{\hbar}}\Im(\alpha\epsilon^*(t))\frac{q - x_g(t)}{|\epsilon(t)|^2}\Big)\Big\}, \end{aligned} \tag{4.52}$$

where $|\epsilon(t)|$ is given by Eq.(3.9).

The probability density function (4.52) is in the same form as the probability density at time-evolved odd coherent states (3.32), obtained in the previous chapter. However, the boundary $s(t)$ causes a difference in the displacement of the position. Then, we also realize differences in the phase factors by comparing solution (4.51) and time-evolved odd coherent states given by (3.31).

4.3. Exacatly Solvable Caldirola-Kanai Model for the Dirichlet IBVP

In this section, we consider an exactly solvable model

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\Psi(q,t) = \hat{H}(t)\Psi(q,t), \quad s(t) < q < \infty, \quad t > 0, \\ \Psi(q,0) = \Psi_0(q), \quad s(0) < q < \infty, \\ \Psi(s(t),t) = 0, \quad t > 0, \end{cases} \tag{4.53}$$

with moving boundary $s(t)$, and the Hamiltonian

$$\hat{H}(t) = \frac{e^{-\gamma t}}{2}\hat{p}^2 + \frac{\omega_0^2 e^{\gamma t}}{2}\hat{q}^2 + e^{-\gamma t}D(t)\hat{p} - e^{\gamma t}E(t)\hat{q}, \tag{4.54}$$

where $\mu(t) = e^{\gamma t}, \gamma > 0$, is the exponentially increasing mass, $D(t), E(t)$ are arbitrary real-valued parameters depending on time. The corresponding classical equation of motion

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = e^{-\gamma t}\dot{D} + E, \quad t > 0 \tag{4.55}$$

where $\gamma > 0$ is the damping coefficient and $\omega_0 > 0$ is the natural frequency, that is the frequency of the undamped oscillator ($\gamma = 0$), has homogenous solutions

$$x_1(t) = \frac{\omega_0}{\Omega_0}e^{-\gamma t/2}\cos(\Omega_0 t - \delta_0), \quad x_2(t) = \frac{1}{\Omega_0}e^{-\gamma t/2}\sin(\Omega_0 t),$$

satisfying the initial conditions $x_1(0) = 1$, $\dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1$, and $\Omega_0 = \sqrt{\omega_0^2 - \gamma^2/4}$ is the frequency of the damped oscillator and $\delta_0 = \tan^{-1}(\gamma/(2\Omega_0))$ is the phase shift.

Periodic forces ($B(t) = 0$): If the driving forces are taken to be $D(t) = 0$, $E(t) = E_0\cos(\omega t)$, where ω is the driving frequency and E_0 is a real constant, then the particular solution of (4.55) will be

$$x_p(t) = A_h e^{-\gamma t/2}\cos(\Omega_0 t - \theta_h) + A_p\cos(\omega t - \delta_p), \tag{4.56}$$

where A_h and θ_h are constants such that $x_p(t)$ satisfies the initial conditions $x_p(0) = 0, \dot{x}_p(0) = 0$. The amplitude and phase shift of the steady-state part are

$$A_p = \frac{E_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \delta_p = \tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right), \tag{4.57}$$

and resonance frequency and maximum amplitude are found as

$$\omega_{res} = \sqrt{\omega_0^2 - \gamma^2/2}, \quad A_p(\omega_{res}) = \frac{E_0}{\sqrt{\gamma^2(\omega_0^2 - \gamma^2/4)}}, \quad \omega_0^2 - \gamma^2/2 > 0. \tag{4.58}$$

Example 1

First, we consider the IBVP (4.53) with an initial data of harmonic oscillator eigenstate type given by (4.40) for odd $n = 2k+1, k = 0, 1, 2, \ldots$. If $s(t) = x_g(t) = c_1x_1(t) + c_2x_2(t) + x_p(t)$, $c = (c_1, c_2) \in \mathbb{R}^2$ and the parameter $\xi = c$ in (4.40), then the probability densities will be found by (4.45) with the squeezing coefficient

$$|\epsilon(t)| = \frac{\omega_0}{\Omega_0} e^{-\gamma t/2} \sqrt{\cos^2(\Omega_0 t - \delta_0) + \sin^2(\Omega_0 t)}, \tag{4.59}$$

which is smooth and oscillatory for $t > 0$. We note that, the amplitude of oscillations can be increased by increasing the value of the frequency ω_0. When $\gamma \to 0$, one has $\omega_0^2/\Omega_0^2 \to 1, \delta_0 \to 0$ so that $|\epsilon(t)| \to 1$. However, when $\gamma > 1$, the amplitude of oscillations in $|\epsilon(t)|$ decreases and approaches zero as time goes to infinity.

In Fig.4.1-(i), we plot time evolution of the probability density $\rho_n(q,t)$, $n = 3$ on the fixed domain $0 < q < \infty$ for $s(t) = 0$ and $D(t) = E(t) = 0$. We observe that the probability density function is equal to zero on the fixed boundary $q = 0$. This figure shows the evolution of the Gaussian wave packet on its domain and without displacement in position. Since $n = 3$, we observe also the trajectory of a moving zero for $q > 0$. In Fig.4.1, we also give plots of time evolution of the probability density $\rho_n(q,t)$ for $n = 3$, when $c_1 = 0$, $c_2 = 1$ (*ii*) without external parameters, that is $x_p(t) = 0$, (*iii*) with periodic $x_p(t) = (4/\sqrt{47})\cos(\sqrt{23/2}t - \tan^{-1}(\sqrt{46}))$ at resonance frequency (transient parts are neglected in the figures). Fig.4.1-(ii) exhibits that the boundary point oscillates in time with decreasing amplitude. However, in Fig.4.1-(iii), the boundary point will continue to oscillate with a constant amplitude after a certain time due to the moving boundary $s(t) = x_2(t) + x_p(t)$. We observe also how the trajectories of moving zeros oscillate according to the displacement (*ii*) $s(t) = x_2(t)$, (*iii*) $s(t) = x_g(t) = x_2(t) + x_p(t)$.

Example 2

Now, in the IBVP (4.53) we take the initial function as of the form (4.46), then the probability densities are given by Eq. (4.52) with the squeezing coefficient $|\epsilon(t)|$ given by (4.59). In Fig.4.2-(i), we show the probability density $\rho^o_\alpha(q,t)$ for $\alpha = i$, when the boundary $s(t) = 0$, and we observe that the function is zero on the fixed boundary $q = 0$. In Fig.4.2-(ii), we plot $\rho^o_{\tilde{\alpha}}(q,t)$ for $\tilde{\alpha} = i$, when $c_1 = 0$, $c_2 = 2$ without external parameters, that is $x_p(t) = 0$. Due to the moving boundary $s(t) = 2x_2(t)$, one can see that the boundary point oscillates and as time increases the amplitude of oscillations decreases and approaches to zero. Fig.4.2-(iii) shows $\rho^o_{\tilde{\alpha}}(q,t)$ for $\tilde{\alpha} = i$, when $c_1 = 0$, $c_2 = 2$ with periodic $x_p(t) = (4/\sqrt{47})\cos(\sqrt{23/2}t - \tan^{-1}(\sqrt{46}))$ at resonance frequency. According to this, the amplitude of oscillations of the boundary point will not approach to zero as time increases.

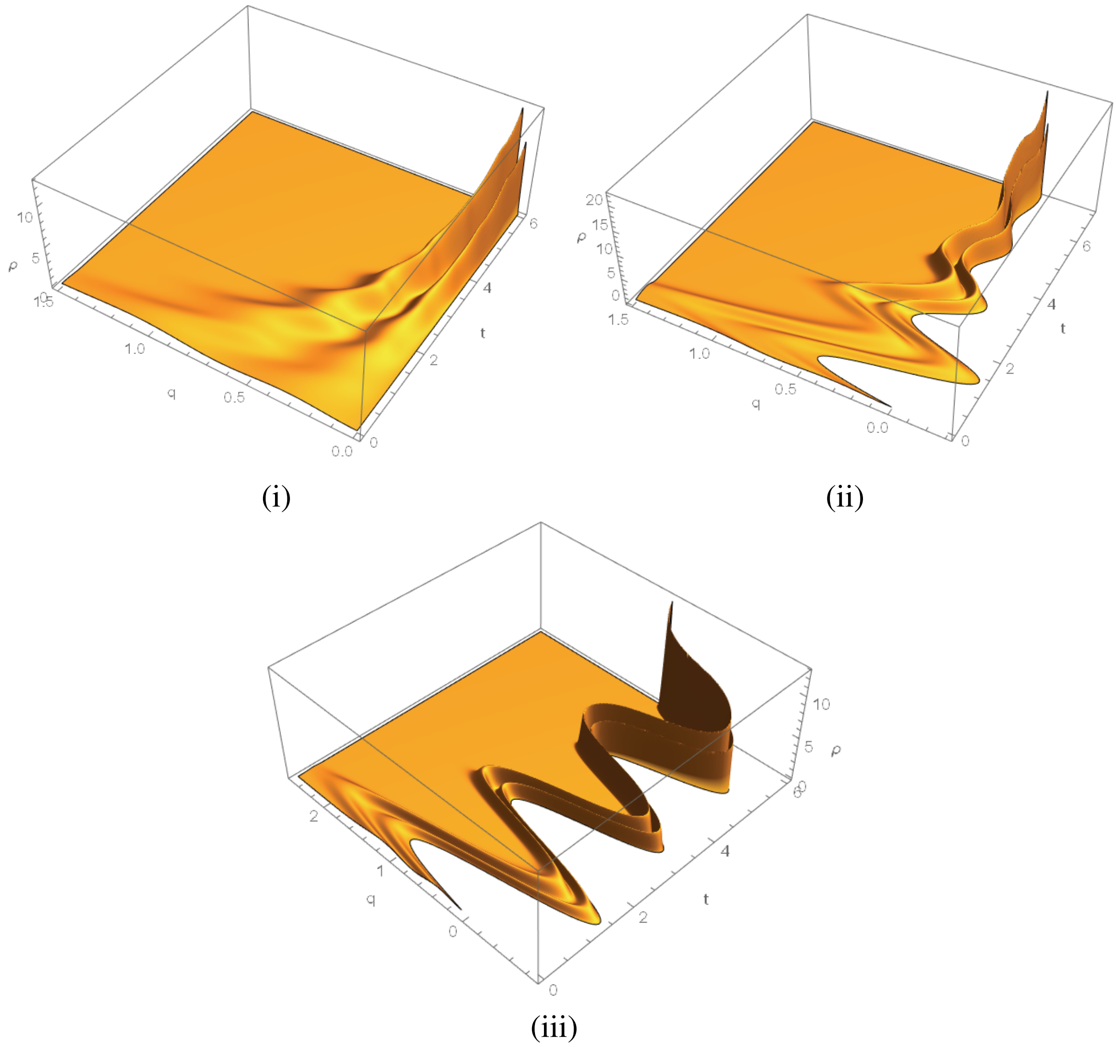


Figure 4.1. Probability density $\rho_n(q,t)$ given by (4.45) for $n = 3, \gamma = 1, \omega_0 = \sqrt{12}, \hbar = 1$, when (i) $s(t) = 0$ and $x_p(t) = 0$, (ii) $s(t) = 2x_2(t)$ and $x_p(t) = 0$, (iii) $s(t) = 2x_2(t) + x_p(t)$ with periodic $x_p(t)$ at resonance frequency $\omega_{res} = \sqrt{23/2}, E_0 = 2$.

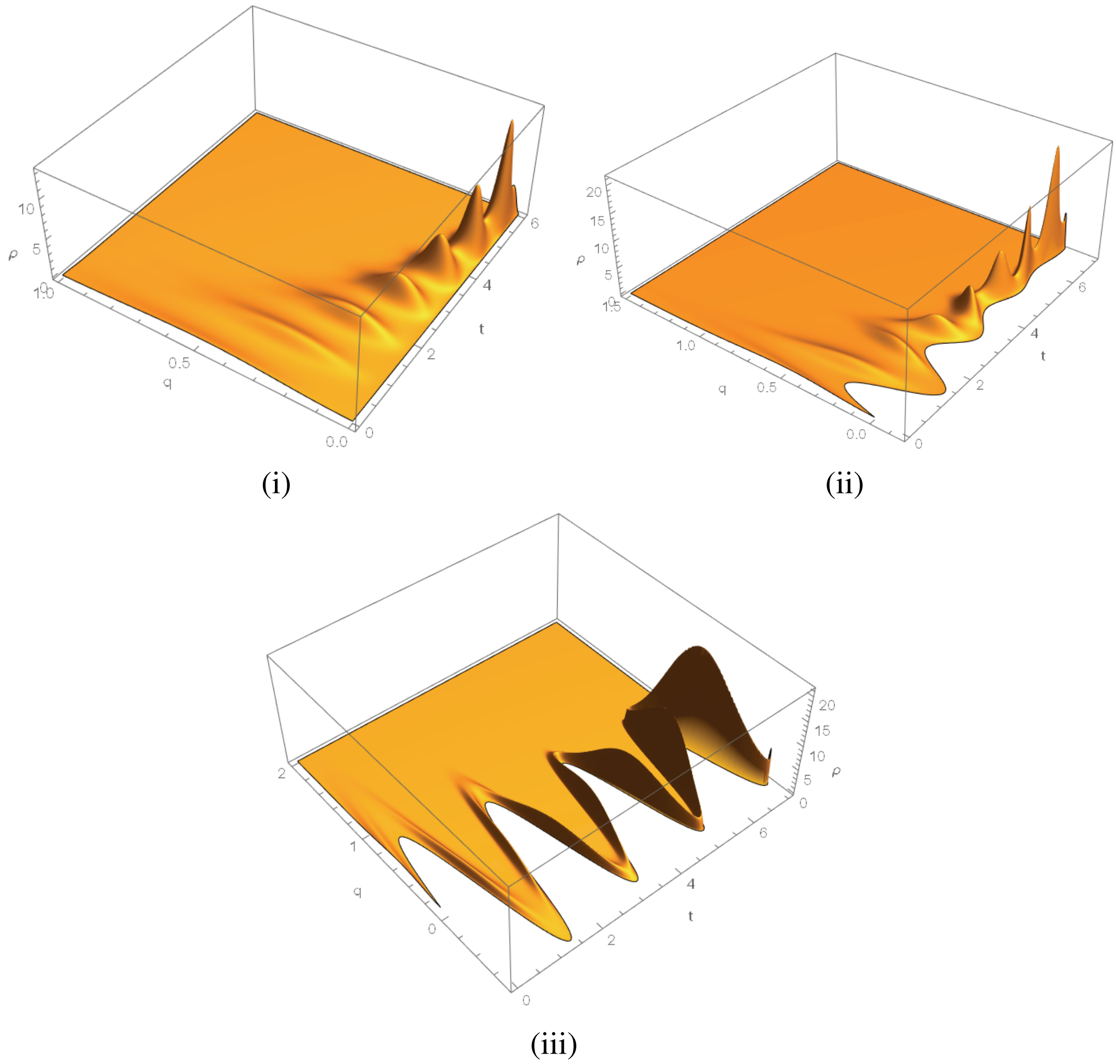


Figure 4.2. Probability density $\rho^{o}_{\tilde{\alpha}}(q,t)$ given by (4.52) for $\tilde{\alpha} = i, \gamma = 1, \omega_0 = \sqrt{12}, \hbar = 1$ when (i) $s(t) = 0$ and $x_p(t) = 0$, (ii) $s(t) = 2x_2(t)$ and $x_p(t) = 0$, (iii) $s(t) = 2x_2(t) + x_p(t)$ with periodic $x_p(t)$ at resonance frequency $\omega_{res} = \sqrt{23/2}, E_0 = 2$.

4.4. Robin IBVP for a Generalized Quantum Parametric Oscillator with Moving Boundary

Now, we consider an IBVP for the generalized quantum parametric oscillator defined on a time-dependent domain $s(t) < q < \infty$ $0 < t < T$, with Robin BC imposed at a moving boundary $q = s(t)$

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\Psi(q,t) = \hat{H}_g(t)\Psi(q,t), \quad s(t) < q < \infty, \quad 0 < t < T, \\ \Psi(q,0) = \Psi_0(q), \quad s(0) < q < \infty, \\ \dfrac{\partial\Psi}{\partial q}(s(t),t) - \dfrac{i}{\hbar}\beta(t)\Psi(s(t),t) = 0, \quad 0 < t < T, \end{cases} \tag{4.60}$$

where $\hat{H}_g(t)$ is the generalized Hamiltonian given by (4.15) and $\beta(t)$ is a real valued function of time.

Proposition 4.3 *Consider the Robin IBVP with the moving boundary for the generalized quantum parametric oscillator given by (4.60). If the boundary function $s(t)$ is of the form (4.16), and the function $\beta(t)$ is given by*

$$\beta(t) = p_g(t) \equiv c_1 p_1(t) + c_2 p_2(t) + p_p(t), \quad c_1, c_2 \in \mathbb{R}, \tag{4.61}$$

where $p_1(t) = \mu(t)(\dot{x}_1(t) - B(t)x_1(t))$ and $p_2(t) = \mu(t)(\dot{x}_2(t) - B(t)x_2(t))$ are two linearly independent homogeneous solutions and $p_p(t) = \mu(t)(\dot{x}_p(t) - B(t)x_p(t) - D(t))$ is a particular solution of the classical equation of motion in momentum space

$$\ddot{p} - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}\dot{p} + \left(\omega^2 + \left(\dot{B} - B^2 - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}B\right)\right)p = -\mu\omega^2 D - \dot{E} + \left(\frac{(\dot{\mu\omega^2})}{\mu\omega^2} + B\right)E, \tag{4.62}$$

then IBVP (4.60) has solution of the form

$$\Psi(q,t) = \frac{1}{\sqrt{x_1(t)}} \times \exp\left[i\int_0^t L_g(\xi)d\xi\right] \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)(q - x_g(t))^2\right]$$
$$\times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_g(t) - B(t)x_g(t) - D(t)\Big)(q - x_g(t))\right] \times \varphi(\eta_g(q,t), \tau(t)), \tag{4.63}$$

where the Lagrangian $L_g(t)$ and coordinate transformations $\eta_g(t)$, $\tau(t)$ are defined by (4.19) and (4.20), respectively. Here, $\varphi(\eta,\tau)$ is solution of the following IBVP for the free Schrödinger equation with a homogeneous Neumann boundary condition

$$\begin{cases} i\dfrac{\partial\varphi}{\partial\tau} = \dfrac{1}{2}\dfrac{\partial^2\varphi}{\partial\eta^2}, \quad 0 < \eta < \infty, \ 0 < \tau < \tau(T), \\ \varphi(\eta,0) = \exp\left(\dfrac{-ic_2}{\hbar}\eta\right)\Psi_0(\eta + c_1), \quad 0 < \eta < \infty, \\ \varphi_\eta(0,\tau) = 0, \quad 0 < \tau < \tau(T). \end{cases} \tag{4.64}$$

Proof Suppose the boundary function $s(t)$ and the function $\beta(t)$ are of the form (4.16) and (4.61), respectively. Let $\tilde{q} = q - s(t)$ and denote $\widetilde{\Psi}(\tilde{q},t) = \Psi(q,t)$. Then, we obtain an IBVP for $\widetilde{\Psi}(\tilde{q},t) = \Psi(q,t)$ defined on the fixed half-line $0 < \tilde{q} < \infty$ $\widetilde{\Psi}(\tilde{q},t)$

$$\begin{cases} i\hbar\dfrac{\partial\widetilde{\Psi}}{\partial t} = \hat{H}_g^s(t)\widetilde{\Psi}, \\ \widetilde{\Psi}(\tilde{q},0) = \Psi_0(\tilde{q} + c_1), \ 0 < \tilde{q} < \infty, \\ \dfrac{\partial\widetilde{\Psi}}{\partial\tilde{q}}(0,t) - \dfrac{i}{\hbar}\beta(t)\widetilde{\Psi}(0,t) = 0, \ 0 < t < T, \end{cases} \tag{4.65}$$

where $\hat{H}_g^s(t)$ is given by (4.23). So following the same steps in the proof of Proposition 4.2, we obtain the function $\widetilde{\Psi}(\tilde{q},t)$ as

$$\widetilde{\Psi}(\tilde{q},t) = \frac{1}{\sqrt{x_1(t)}} \times \exp\left[i\int_0^t L_g(\xi)d\xi\right] \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)\tilde{q}^2\right]$$
$$\times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_g(t) - B(t)x_g(t) - D(t)\Big)\tilde{q}\right] \times \varphi\left(\frac{\tilde{q}}{x_1(t)}, -\hbar\frac{x_2(t)}{x_1(t)}\right), \tag{4.66}$$

where $\varphi(\eta, \tau)$ satisfies (4.37) with the prescribed initial condition. Now, $\widetilde{\Psi}(\widetilde{q}, t)$ satisfies the homogeneous boundary condition in (4.65) whenever $\varphi(\eta, \tau)$ satisfies the Neumann boundary condition $\varphi_\eta(0, \tau) = 0$. So the function $\widetilde{\Psi}(\widetilde{q}, t)$ found by (4.66) will be the solution of IBVP (4.65) on the fixed half-line if the function $\varphi(\eta, \tau)$ solves the Neumann IBVP (4.64) for the free Schrödinger equation. By doing back substitution $\tilde{q} = q - x_g(t)$, we get solution (4.63) to IBVP (4.60) satisfying the prescribed IC there. Thus, solution (4.63) will satisfy the Robin BC in (4.60) if $\varphi(\eta, \tau)$ satisfies the Neumann BC given in (4.64). □

To be able to find the exact solution of the IBVP (4.60), one has to solve IBVP (4.64) with Neumann boundary condition. Solution of IBVP (4.64) is found as

$$\varphi(\eta, \tau) = \int_0^\infty N(\eta, \xi, \tau)\varphi(\xi, 0)d\xi, \tag{4.67}$$

where $N(\eta, \xi, \tau) = K(\eta - \xi, \tau) + K(\eta + \xi, \tau)$ with $K(\eta, \tau)$ being the propagator of the system given by (4.12). Then, as a consequence of Proposition (4.3), the Robin IBVP (4.60) has solution with integral representation of the form

$$\begin{aligned} \Psi(q, t) \quad = \quad & \frac{1}{\sqrt{x_1(t)}} \times \exp\left[i \int_0^t L_g(\xi)d\xi\right] \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)(q - x_g(t))^2\right] \\ & \times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_g(t) - B(t)x_g(t) - D(t)\Big)(q - x_g(t))\right] \\ & \times \int_0^\infty N(\eta_g(q, t), \xi, \tau(t))\Psi_0(\xi + c_1)e^{-ic_2\xi/\hbar}d\xi. \end{aligned} \tag{4.68}$$

Therefore, exact and explicit solutions to the IBVP (4.60) can be found only when the integral in (4.68) converges for the given initial data.

In what follows, we give solutions of the IBVP (4.60) corresponding to some particular initial functions $\Psi_0(q)$ and homogenous boundary condition $\Psi_{bc}(t) = 0$.

IBVP 1**- Eigenstates type initial condition:** First, we consider IBVP (4.60) with the family of initial functions $\Psi^0_{\xi,n}(q)$ given by (4.40). For the given boundary function $s(t) = x_g(t)$, IBVP (4.60) has the solution of the form (4.63), and $\varphi(\eta, \tau)$ is the solution to IBVP (4.64), with initial condition $\varphi(\eta, 0)$ given by (4.41). Then, solution of IBVP (4.64) will be of the form (4.42), and for $n = 2k$, $k = 0, 1, 2, \dots$, one has

$$\varphi_{2k}(\eta,\tau) = N_{2k}\int_{-\infty}^{\infty}\sqrt{\frac{i}{2\pi\tau}}e^{-\frac{i}{2\tau}(\eta-\xi)^2}\exp\Big(-\frac{\omega_0}{2\hbar}\xi^2\Big)H_{2k}\Big(\sqrt{\frac{\omega_0}{\hbar}}\xi\Big)d\xi, \tag{4.69}$$

and normalized solutions of the IBVP with moving boundary for the generalized oscillator are found as

$$\begin{aligned}\Psi_{2k}(q,t) \;\; = \;\; & \sqrt{\frac{2}{|\epsilon(t)|}}N_{2k}\exp\Big[-i\Big(2k+\frac{3}{2}\Big)\nu(t)\Big]\times\exp\Big(i\int_0^t L_g(\xi)d\xi\Big)\\ & \times\exp\Big[\frac{i\mu(t)}{2\hbar}\Big(\frac{d}{dt}\ln|\epsilon(t)| - B(t)\Big)(q-x_g(t))^2\Big]\\ & \times\exp\Big[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_g(t) - B(t)x_g(t) - D(t)\Big)(q-x_g(t))\Big]\\ & \times\exp\Big[-\frac{\omega_0}{2\hbar}\frac{(q-x_g(t))^2}{|\epsilon(t)|^2}\Big]\times H_{2k}\Big(\sqrt{\frac{\omega_0}{\hbar}}\Big(\frac{q-x_g(t)}{|\epsilon(t)|}\Big)\Big),\end{aligned} \tag{4.70}$$

and the corresponding probability densities become

$$\rho_{2k}(q,t) = \frac{2N_{2k}^2}{|\epsilon(t)|}\exp\Big[-\frac{\omega_0}{\hbar}\frac{(q-x_g(t))^2}{|\epsilon(t)|^2}\Big]\times H_{2k}^2\Big(\sqrt{\frac{\omega_0}{\hbar}}\Big(\frac{q-x_g(t)}{|\epsilon(t)|}\Big)\Big), \tag{4.71}$$

where $|\epsilon(t)|$ is the squeezing coefficient defined by (3.9).

IBVP 2- Coherent states type initial condition: Now, for IBVP (4.60), we take the initial functions $\Psi^0_\alpha(q)$ of coherent states type given by (4.46). If $s(t) = x_g(t)$ is the moving boundary, then solution will be of the form (4.63). Here, $\varphi(\eta,\tau)$ is the solution of IBVP (4.64) with initial condition $\varphi_{\tilde{\alpha}}(\eta, 0)$, given by (4.47). We can write $\varphi_{\tilde{\alpha}}(\eta, 0)$ as a linear combination of even and odd coherent states of SQHO as $\varphi_{\tilde{\alpha}}(\eta,0) = A_1\phi^e_{\tilde{\alpha}}(\eta) + A_2\phi^o_{\tilde{\alpha}}(\eta)$, where $A_1, A_2 \in \mathbb{R}$ and $\phi^e_{\tilde{\alpha}}(\eta), \phi^o_{\tilde{\alpha}}(\eta)$ are given by (2.31) and (2.32), respectively. Then, solution of IBVP (4.64) in integral representation will be

$$\varphi_{\tilde{\alpha}}(\eta,\tau) = \int_0^{\infty}\sqrt{\frac{i}{2\pi\tau}}\Big(e^{-\frac{i}{2\tau}(\eta-\xi)^2} - e^{-\frac{i}{2\tau}(\eta+\xi)^2}\Big)(A_1\phi^e_{\tilde{\alpha}}(\xi) + A_2\phi^o_{\tilde{\alpha}}(\xi))d\xi. \tag{4.72}$$

Here, taking $A_2 = 0$, we can find the exact form of the solution by taking the following integral

$$\varphi_{\tilde{\alpha}}(\eta,\tau) = A_1 \int_{-\infty}^{\infty} \sqrt{\frac{i}{2\pi\tau}} e^{-\frac{i}{2\tau}(\eta-\xi)^2} \phi^e_{\tilde{\alpha}}(\xi) d\xi. \tag{4.73}$$

So, we obtain

$$\begin{aligned}
\Psi^e_{\tilde{\alpha}}(q,t) \;=\; & \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{\epsilon(t)\sinh|\tilde{\alpha}|^2}} \exp\left(-\frac{(\epsilon^*(t))^2}{2|\epsilon(t)|^2}\tilde{\alpha}^2\right) \times \exp\left(i\int_0^t L_g(\xi)d\xi\right) \\
& \times \exp\left[\frac{i\mu(t)}{\hbar}\Big(\dot{x}_g(t) - B(t)x_g(t) - D(t)\Big)(q - x_g(t))\right] \\
& \times \exp\left[\frac{i\mu(t)}{2\hbar}\left(\frac{d}{dt}\ln|\epsilon(t)| - B(t)\right)(q - x_g(t))^2\right] \\
& \times \cosh\left(\sqrt{\frac{2\omega_0}{\hbar}}\frac{1}{\epsilon(t)}(q - x_g(t))\tilde{\alpha}\right) \times \exp\left(-\frac{\omega_0}{2\hbar}\frac{(q - x_g(t))^2}{|\epsilon(t)|^2}\right),
\end{aligned} \tag{4.74}$$

and the corresponding probability densities

$$\begin{aligned}
\rho^e_{\tilde{\alpha}}(q,t) \;=\; & \sqrt{\frac{\omega_0}{\pi\hbar}} \frac{1}{|\epsilon(t)|\sinh|\tilde{\alpha}|^2} \exp\left(\frac{\left(\Im(\tilde{\alpha}\epsilon^*(t))\right)^2 - \left(\Re(\tilde{\alpha}\epsilon^*(t))\right)^2}{|\epsilon(t)|^2}\right) \\
& \times \exp\left(-\frac{\omega_0}{\hbar}\frac{(q - x_g(t))^2}{|\epsilon(t)|^2}\right) \times \left\{\cosh\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Re(\tilde{\alpha}\epsilon^*(t))\frac{q - x_g(t)}{|\epsilon(t)|^2}\right)\right. \\
& \left. + \cos\left(2\sqrt{\frac{2\omega_0}{\hbar}}\Im(\tilde{\alpha}\epsilon^*(t))\frac{q - x_g(t)}{|\epsilon(t)|^2}\right)\right\},
\end{aligned} \tag{4.75}$$

where $|\epsilon(t)|$ is given by Eq.(3.9).

Therefore, one family of normalized solutions of Robin IBVP (4.60) for the generalized oscillator with moving boundary is of the form (4.74). We realize that the probability density function (4.75) and the probability density at time-evolved even coherent states found by (3.29) are similar. The only difference is in the displacement of the position, which is a consequence of the boundary $s(t)$. However, solution (4.74) and time-evolved even coherent states (3.28) also differ in their phase factors.

CHAPTER 5

TIME-EVOLVED COHERENT STATES OF *N*-DIMENSIONAL GENERALIZED QUANTUM HARMONIC OSCILLATORS

In this chapter, we consider N-dimensional generalized quantum harmonic oscillator with time-dependent parameters and obtain its solution using the evolution operator approach. Time-evolution of eigenstates and coherent states under the generalized evolution operator is found explicitly. Then, we introduce exactly solvable quantum models with special time-variable parameters for which the structure of the corresponding classical harmonic oscillator in position space is preserved. For each model, we study squeezing properties of the time-evolved coherent states according to the frequency modification and describe their displacement under the influence of external sinusoidal forces.

5.1. The Classical Problem

In this section, we consider a generalized N-dimensional oscillator described by the Hamiltonian $H_{cl}(t) = \sum_{j=1}^{N} H_j(x_j, p_j, t)$, where

$$H_j(x_j, p_j, t) = \frac{p_j^2}{2\mu_j(t)} + \frac{\mu_j(t)\omega_j^2(t)}{2} x_j^2 + B_j(t)x_j p_j + D_j(t)p_j + E_j(t)x_j + F_j(t), \qquad (5.1)$$

and $\mu_j(t) > 0$, $\omega_j^2(t), B_j(t), D_j(t), E_j(t)$ and $F_j(t)$, $j = 1, 2, \ldots, N$, are real-valued parameters depending on time. The corresponding Hamilton's equations of motion are

$$\dot{x}_j = \frac{\partial H_j}{\partial p_j} = B_j(t) + \frac{p_j}{\mu_j(t)} + D_j(t),$$
$$\dot{p}_j = -\frac{\partial H_j}{\partial x_j} = -(\mu_j(t)\omega_j^2(t)x_j + B_j(t)p_j + E_j(t)), \quad j = 1, 2, \ldots, N.$$

Then, for each $j = 1, 2, \ldots, N$, we have the classical equations of motion in position space

$$\ddot{x}_j + \frac{\dot{\mu}_j}{\mu_j}\dot{x}_j + \left(\omega_j^2 - \left(\dot{B}_j + B_j^2 + \frac{\dot{\mu}_j}{\mu_j}B_j\right)\right)x_j = -\frac{E_j}{\mu_j} + \dot{D}_j + \left(\frac{\dot{\mu}_j}{\mu_j} + B_j\right)D_j, \tag{5.2}$$

and oscillator equation in momentum space

$$\ddot{p}_j - \frac{\dot{(\mu_j\omega_j^2)}}{\mu_j\omega_j^2}\dot{p}_j + \left(\omega_j^2 + \dot{B}_j - B_j^2 - \frac{\dot{(\mu_j\omega_j^2)}}{\mu_j\omega_j^2}B_j\right)p_j = -\mu_j\omega_j^2 D_j - \dot{E}_j + \left(\frac{\dot{(\mu_j\omega_j^2)}}{\mu_j\omega_j^2} + B_j\right)E_j. \tag{5.3}$$

We notice that the parameter $B_j(t)$ of the mixed term in Hamiltonian (5.1) leads to modification of the original frequency $\omega_j^2(t)$, and the external parameters $B_j(t), D_j(t), E_j(t)$ all contribute to the forcing term of the oscillator for $j = 1, 2, \ldots, N$.

We denote $x_j^{(1)}(t)$, $x_j^{(2)}(t)$ to be two linearly independent homogenous solutions and $x_j^{(p)}(t)$ to be a particular solution of the corresponding classical equation of motion in position space given by (5.2), satisfying the initial conditions $x_j^{(1)}(t_0) = x_0 \neq 0$, $\dot{x}_j^{(1)}(t_0) = x_0 B_j(t_0)$, $x_j^{(2)}(t_0) = 0$, $\dot{x}_j^{(2)}(t_0) = 1/\left(x_0\mu_j(t_0)\right)$, and $x_j^{(p)}(t_0) = 0$, $\dot{x}_j^{(p)}(t_0) = E_j(t_0)$, respectively, for $j = 1, 2, \ldots, N$. Furthermore, we let $p_j^{(1)}(t)$, $p_j^{(2)}(t)$ denote two homogeneous solutions of the oscillator equation in momentum space given by (5.3), then they can be found in terms of the solutions of the classical equations in position space as

$$p_j^{(1)}(t) = \mu_j(t)(\dot{x}_j^{(1)}(t) - B_j(t)x_j^{(1)}(t) - D_j(t)),$$
$$p_j^{(2)}(t) = \mu_j(t)(\dot{x}_j^{(2)}(t) - B_j(t)x_j^{(2)}(t) - D_j(t)),$$

and particular solution will be

$$p_j^{(p)}(t) = \mu_j(t)(\dot{x}_j^{(p)}(t) - B_j(t)x_j^{(p)}(t) - D_j(t)). \tag{5.4}$$

This establishes solutions to the classical problem, whose quantization using the usual replacement $x_j \to \hat{q}_j$, $p_j \to \hat{p}_j$, $x_j p_j \to (\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j)/2$, $j = 1, 2, \ldots, N$ is discussed in next sections.

5.2. Generalized Quantum Parametric Oscillator

We consider the evolution problem describing a generalized N-dimensional quantum parametric oscillator in the presence of time-variable external fields given by

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{q},t) = \hat{H}(t)\Psi(\mathbf{q},t), \quad \mathbf{q}\in\mathbb{R}^N,\ t>t_0, \\ \Psi(\mathbf{q},t_0) = \Psi^0(\mathbf{q}), \quad \mathbf{q}\in\mathbb{R}^N, \end{cases} \tag{5.5}$$

where $\Psi(\mathbf{q},t) := \Psi(q_1,q_2,\ldots,q_N,t)$ is the wave function at time $t>t_0$; at $t=t_0$, the initial state is $\Psi^0(\mathbf{q}) := \Psi^0(q_1,q_2,\ldots,q_N)$ and time-dependent Hamiltonian $\hat{H}(t)$ is defined by

$$\hat{H}(t) \equiv \sum_{j=1}^{N}\hat{H}_j(t), \tag{5.6}$$

where

$$\hat{H}_j(t) = \frac{\hat{p}_j^2}{2\mu_j(t)} + \frac{\mu_j(t)\omega_j^2(t)}{2}\hat{q}_j^2 + \frac{B_j(t)}{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j) + D_j(t)\hat{p}_j + E_j(t)\hat{q}_j + F_j(t), \tag{5.7}$$

with $\hat{q}_j = q_j$, is the position operator, $\hat{p}_j = -i\hbar\partial/\partial q_j$ is the momentum operator for $j=1,2,\ldots,N$.

We note that, $[\hat{H}_i,\hat{H}_j] = 0$ for all $i,j = 1,2,\ldots,N$, and due to this, we have the following proposition.

Proposition 5.1 *The IVP for an N-dimensional generalized quantum parametric oscillator given by (5.5) has solution of the form*

$$\Psi(\mathbf{q},t) = \prod_{j=1}^{N}\Psi_j(q_j,t), \tag{5.8}$$

where

$$\Psi_j(q_j,t) = \sqrt{\frac{x_0}{x_j^{(1)}(t)}}\exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\zeta_j(s)ds\right)\exp\left[i\mu_j(t)\left(\dot{x}_j^{(p)}(t)-B_j(t)x_j^{(p)}(t)D_j(t)\right)q_j\right]$$
$$\exp\left[\frac{i\mu_j(t)}{2\hbar}\Big(\frac{\dot{x}_j^{(1)}(t)}{x_j^{(1)}(t)}-B_j(t)\Big)(q_j-x_j^{(p)}(t))^2\right]\varphi_j(\eta_j(q_j,t),\tau_j(t)). \quad (5.9)$$

Here, $x_j^{(1)}(t)$, $x_j^{(2)}(t)$ *are two linearly independent homogenous solutions and* $x_j^{(p)}(t)$ *is a particular solution of the corresponding classical equation of motion in position space given by (5.2) satisfying the prescribed initial conditions, and* $\varphi_j(q_j,t)$ *is solution of N-dimensional Schrödinger equation*

$$\begin{cases} i\frac{\partial}{\partial t}\varphi_j(q_j,t) = -\frac{\partial^2}{\partial q_j^2}\varphi_j(q_j,t), & q_j\in\mathbb{R},\ t>0, \\ \varphi(q_j,0)=\Psi^0(q_j), & q_j\in\mathbb{R}. \end{cases} \quad (5.10)$$

Also, we denote

$$\zeta_j(t) = \frac{-\mu_j(t)}{2}\left[\left(\dot{x}_j^{(p)}(t)-B_j(t)x_j^{(p)}(t)\right)^2-\omega_j^2(t)\left(x_j^{(p)}(t)\right)^2-D_j^2(t)+\frac{2F_j(t)}{\mu_j(t)}\right], \quad (5.11)$$

and the coordinate transformations

$$\eta_j(q_j,t) = \left(\frac{x_0}{x_j^{(1)}(t)}\right)\left(q_j-x_j^{(p)}(t)\right), \quad \tau_j(t)=\hbar x_0^2\left(\frac{x_j^{(2)}(t)}{x_j^{(1)}(t)}\right), \quad j=1,2,\dots,N. \quad (5.12)$$

Proof The dynamics of the quantum system described by Schrödinger equation (5.5) is contained in the evolution operator defined as

$$i\hbar\frac{d}{dt}\hat{U}(t,t_0)=\hat{H}(t)\hat{U}(t,t_0), \quad \hat{U}(t_0,t_0)=\hat{I}. \quad (5.13)$$

Exact form of $\hat{U}(t,t_0)$ can be found by using Wei-Norman Lie algebraic process. We can write the Hamiltonian $\hat{H}(t)$ given by (5.6) as a linear combination of Lie algebra generators as

$$\hat{H}(t) = -i\sum_{j=1}^{N}\left(\frac{\hbar^2}{\mu_j(t)}\hat{\mathcal{K}}_j^{(-)} + \mu_j(t)\omega_j^2(t)\hat{\mathcal{K}}_j^{(+)} + 2\hbar B_j(t)\hat{\mathcal{K}}_j^{(0)} + \hbar D_j(t)\hat{\mathcal{E}}_j^{(2)} + E_j(t)\hat{\mathcal{E}}_j^{(1)} + F_j(t)\hat{\mathcal{E}}_j^{(3)}\right),$$

where operators

$$\hat{\mathcal{E}}_j^{(1)} = iq_j, \quad \hat{\mathcal{E}}_j^{(2)} = \frac{\partial}{\partial q_j}, \quad \hat{\mathcal{E}}_j^{(3)} = i\hat{I},$$

are generators of Heisenberg-Weyl algebra and

$$\hat{\mathcal{K}}_j^{(-)} = -\frac{i}{2}\frac{\partial^2}{\partial q_j^2}, \quad \hat{\mathcal{K}}_j^{(+)} = \frac{i}{2}q_j^2, \quad \hat{\mathcal{K}}_j^{(0)} = \frac{1}{2}\left(q_j\frac{\partial}{\partial q_j} + \frac{1}{2}\right),$$

are generators of the su(1,1) algebra for each fixed $j = 1, 2, \ldots, N,$. Then, the evolution operator is

$$\hat{U}(t, t_0) = \prod_{j=1}^{N}\hat{U}_j(t, t_0), \tag{5.14}$$

where for each $j = 1, 2, \ldots, N$ the operator $\hat{U}_j(t, t_0)$ can be expressed as product of exponential operators

$$\hat{U}_j(t, t_0) = e^{c_j(t)\hat{\mathcal{E}}_j^{(3)}}\, e^{\frac{a_j(t)}{\hbar}\hat{\mathcal{E}}_j^{(1)}}\, e^{-b_j(t)\hat{\mathcal{E}}_j^{(2)}}\, e^{f_j(t)\hat{\mathcal{K}}_j^{(+)}}\, e^{2h_j(t)\hat{\mathcal{K}}_j^{(0)}}\, e^{g_j(t)\hat{\mathcal{K}}_j^{(-)}},$$

with $f_j(t), g_j(t), h_j(t)$ and $a_j(t), b_j(t), c_j(t)$ being real-valued functions to be determined. Substituting (5.14) and (5.6) into (5.13) and performing necessary calculations, we find that $\hat{U}(t, t_0)$ is a solution of (5.13) if the unknown functions $f_j(t), g_j(t), h_j(t)$ satisfy the nonlinear system

$$\begin{aligned}\dot{f}_j + \tfrac{\hbar}{\mu_j(t)} f_j^2 + 2B_j(t) f_j + \tfrac{\mu_j(t)\omega_j^2(t)}{\hbar} &= 0, \quad f_j(t_0) = 0,\\ \dot{g}_j + \tfrac{\hbar}{\mu_j(t)} e^{2h_j} &= 0, \quad g_j(t_0) = 0,\\ \dot{h}_j + \tfrac{\hbar}{\mu_j(t)} f_j + B_j(t) &= 0, \quad h_j(t_0) = 0,\end{aligned} \tag{5.15}$$

and $a_j(t), b_j(t), c_j(t)$ satisfy the nonlinear system

$$\begin{aligned}\dot{a}_j + B_j(t)a_j + \mu_j(t)\omega_j^2(t) b_j + E_j(t) &= 0, \quad a_j(t_0) = 0,\\ \dot{b}_j - B_j(t) b_j - \tfrac{1}{\mu_j(t)} a_j - D_j(t) &= 0, \quad b_j(t_0) = 0,\\ \dot{c}_j + \tfrac{1}{2\hbar\mu_j(t)} a_j^2 + \tfrac{D_j(t)}{\hbar} a_j - \tfrac{\mu_j(t)\omega_j^2(t)}{2\hbar} b_j^2 + \tfrac{F_j(t)}{\hbar} &= 0, \quad c_j(t_0) = 0, \quad j = 1, 2, \ldots, N.\end{aligned} \tag{5.16}$$

Then, for each $j = 1, 2, \ldots, N$, the solution of system (5.15) is found in terms of two linearly independent solutions $x_j^{(1)}(t)$ and $x_j^{(2)}(t)$ of classical system (5.2) as

$$f_j(t) = \frac{\mu_j(t)}{\hbar}\left(\frac{\dot{x}_j^{(1)}(t)}{x_j^{(1)}(t)} - B_j(t)\right), \quad g_j(t) = -\hbar x_0^2 \left(\frac{x_j^{(2)}(t)}{x_j^{(1)}(t)}\right), \quad h_j(t) = -\ln\left|\frac{x_j^{(1)}(t)}{x_0}\right|.$$

On the other hand, for each $j = 1, 2 \ldots N$, the solution of system (5.16) is obtained in terms of particular solutions of systems (5.2) and (5.3) as

$$a_j(t) = p_j^{(p)}(t), \quad b_j(t) = x_j^{(p)}(t), \quad c_j(t) = \frac{1}{\hbar}\int_{t_0}^{t} \zeta_j(s)ds.$$

Therefore, for each $j = 1, 2, \ldots, N$, we find

$$\begin{aligned}\hat{U}_j(t, t_0) = {} & \exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\zeta_j(s)ds\right)\exp\left(ip_j^{(p)}(t)q_j\right)\exp\left(-x_j^{(p)}(t)\frac{\partial}{\partial q_j}\right)\\ & \exp\left(i\frac{\mu_j(t)}{2\hbar}\left(\frac{\dot{x}_j^{(1)}(t)}{x_j^{(1)}(t)} - B_j(t)\right)q_j^2\right)\exp\left(\ln\left|\frac{x_0}{x_j^{(1)}(t)}\right|\left(q_j\frac{\partial}{\partial q_j} + \frac{1}{2}\right)\right)\exp\left(\frac{i}{2}\hbar x_0^2\left(\frac{x_j^{(2)}(t)}{x_j^{(1)}(t)}\right)\frac{\partial^2}{\partial q_j^2}\right),\end{aligned}$$

which determines $\hat{U}(t, t_0)$ explicitly. We note that,

$$\exp\left(-\frac{i\xi_j}{2}\frac{\partial^2}{\partial q_j^2}\right)\Psi_0(q_j) = \varphi(q_j;\xi_j), \tag{5.17}$$

where for $j = 1, 2, \ldots, N$, the function $\varphi(q_j; z_j)$ satisfies the Schrödinger equation (5.10). Using (5.17) and the expressions for the shift and dilatation operators respectively,

$$\exp\left(\xi_j\frac{\partial}{\partial q_j}\right)\phi(q_j) = \phi(q_j + \xi_j), \quad \exp\left(\xi_j q_j\frac{\partial}{\partial q_j}\right)\phi(q_j) = \phi(e^{\xi_j}q_j) \tag{5.18}$$

for any function $\phi(q_j)$, $j = 1, 2, \ldots, N$, the evolution operator (5.17) is applied to the initial function $\Psi^0(q_j)$, that is $\Psi_j(q_j, t) = \hat{U}_j(t, t_0)\Psi^0(q_j)$. Then, we obtain solution (5.9) of the IVP for N-dimensional generalized quantum parametric oscillator given by (5.5). □

Therefore, knowing explicitly the evolution operator allows us to obtain solution of the IVP (5.5) for any given initial function. In what follows we show the exact time-development of harmonic oscillator eigenstates and Glauber coherent states.

5.3. Time-Evolution of Harmonic Oscillator Eigenstates

First, we solve IVP (5.5) by taking the initial function to be the eigenstates $\varphi_n(\mathbf{q})$ of the N-dimensional simple harmonic oscillator, whose Hamiltonian is $\hat{H}_0 = \sum_{j=1}^{N}(\hat{p}_j^2 + \omega_0^2\hat{q}_j^2)/2$. As known, these eigenstates correspond to eigenvalues $E_n = E_{n_1} + E_{n_2} + \ldots + E_{n_N} = \hbar\omega_0(n_1 + n_2 + \ldots + n_N + N/2)$, and for $n = (n_1, n_2, \ldots, n_N)$ we have

$$\varphi_n(\mathbf{q}) = \varphi_{n_1}(q_1)\varphi_{n_2}(q_2)\ldots\varphi_{n_N}(q_N), \quad n_1, n_2, \ldots, n_N = 0, 1, 2, ...,$$

with

$$\varphi_{n_j}(q_j) = N_{n_j}e^{-\frac{\omega_0}{2\hbar}q_j^2}H_{n_j}\left(\sqrt{\frac{\omega_0}{\hbar}}q_j\right), \quad j = 1, 2, \ldots, N,$$

where $H_{n_j}(\sqrt{\omega_0/\hbar}q_j)$ are Hermite polynomials and $N_{n_j} = (\omega_0/\pi\hbar)^{1/4}(2^{n_j}n_j!)^{-1/2}$ are the normalization constants. According to this, time-evolved eigenstates of the N-dimensional oscillator (5.5) with Hamiltonian (5.6) are of the form

$$\Psi(\mathbf{q},t) = \hat{U}(t,t_0)\varphi_n(\mathbf{q}) = \prod_{j=1}^{N} \hat{U}_j(t,t_0)\varphi_{n_j}(q_j) = \prod_{j=1}^{N} \Psi_{n_j}(q_j,t), \tag{5.19}$$

and using the equations (5.18) and (5.17), $\Psi_{n_j}(q_j,t)$ are explicitly found as

$$\begin{aligned}
&\Psi_{n_j}(q_j,t) = N_{n_j}\frac{1}{\sqrt{|\epsilon_j(t)|}}\exp\Big(-\frac{iE_{n_j}}{\hbar\omega_0}\nu_j(t)\Big)\\
&\exp\left\{\frac{i}{\hbar}\Big[\int_{t_0}^{t}\zeta_j(s)ds - \frac{\mu_j(t)}{2}\Big(B_j(t) - \frac{d}{dt}\ln|\epsilon_j(t)|\Big)(q_j - x_j^{(p)}(t))^2 + p_j^{(p)}(t)q_j\Big]\right\}\\
&\exp\Big[-\frac{\omega_0}{2\hbar}\frac{(q_j - x_j^{(p)}(t))^2}{|\epsilon_j(t)|^2}\Big]H_{n_j}\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{q_j - x_j^{(p)}(t)}{|\epsilon_j(t)|}\right),
\end{aligned} \tag{5.20}$$

and the corresponding probability densities

$$\rho_{n_j}(q_j,t) = N_{n_j}^2\frac{1}{|\epsilon_j(t)|}\exp\left[-\frac{\omega_0}{\hbar}\frac{(q_j - x_j^{(p)}(t))^2}{|\epsilon_j(t)|^2}\right]H_j^2\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{q_j - x_j^{(p)}(t)}{|\epsilon_j(t)|}\right), \tag{5.21}$$

where

$$\epsilon_j(t) = \frac{x_j^{(1)}(t)}{x_0} + i(\omega_0 x_0)x_j^{(2)}(t) = |\epsilon_j(t)|e^{i\nu_j(t)}, \tag{5.22}$$

with modulus and phase

$$|\epsilon_j(t)| = \sqrt{\frac{(x_j^{(1)}(t))^2}{x_0^2} + (\omega_0 x_0)^2(x_j^{(2)}(t))^2}, \quad \nu_j(t) = \int_{t_0}^{t}\frac{\omega_0}{\mu_j(s)|\epsilon_j(s)|^2}ds \tag{5.23}$$

for each $j = 1, 2, \ldots, N$. Here, the expectation values of position and momentum at states $\Psi_n(\mathbf{q}, t)$ are

$$\langle \hat{q}_j \rangle_n(t) = \langle \Psi_n(\mathbf{q}, t) | \hat{q}_j | \Psi_n(\mathbf{q}, t) \rangle = \langle \Psi_{n_j}(q_j, t) | \hat{q}_j | \Psi_{n_j}(q_j, t) \rangle = x_j^{(p)}(t),$$
$$\langle \hat{p}_j \rangle_n(t) = \langle \Psi_n(\mathbf{q}, t) | \hat{p}_j | \Psi_n(\mathbf{q}, t) \rangle = \langle \Psi_{n_j}(q_j, t) | \hat{p}_j | \Psi_{n_j}(q_j, t) \rangle = p_j^{(p)}(t),$$

showing that they do not depend on the wave number $n = (n_1, n_2, \ldots, n_N)$ and are completely determined by the external forces. Then, the uncertainties in position and momentum are found as

$$(\Delta \hat{q}_j)_n(t) = \sqrt{\frac{\hbar}{\omega_0}\left(n_j + \frac{1}{2}\right)} |\epsilon_j(t)|,$$
$$(\Delta \hat{p}_j)_n(t) = \sqrt{\omega_0 \hbar \left(n_j + \frac{1}{2}\right) \frac{1}{|\epsilon_j(t)|^2}} \sqrt{1 + \frac{|\epsilon_j(t)|^4}{\omega_0^2}\left(\frac{d \ln |\epsilon_j(t)|}{dt} - B_j(t)\right)^2},$$

and the uncertainty relation in state $\Psi_n(\mathbf{q}, t)$ becomes

$$(\Delta \hat{q}_j)_n (\Delta \hat{p}_j)_n(t) = \hbar \left(n_j + \frac{1}{2}\right) \sqrt{1 + \frac{|\epsilon_j(t)|^4}{\omega_0^2}\left(\frac{d \ln |\epsilon_j(t)|}{dt} - B_j(t)\right)^2}. \qquad (5.24)$$

5.4. Time-Evolution of Glauber Coherent States

The coherent states of the simple N-dimensional harmonic oscillator with Hamiltonian $\hat{H}_0$ are usually defined using the unitary displacement operator

$$\hat{D}(\alpha) = \prod_{j=1}^{N} \hat{D}_j(\alpha_j) = \prod_{j=1}^{N} \exp(\alpha_j \hat{a}_j^\dagger - \alpha_j^* \hat{a}_j), \qquad (5.25)$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ and $\alpha_j = \alpha_j^{(1)} + i\alpha_j^{(2)}$, with $\alpha_j^{(1)}$, $\alpha_j^{(2)}$ being real constants and $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the annihilation and creation operators given as

$$\hat{a}_j = \sqrt{\frac{\omega_0}{2\hbar}}q_j + \sqrt{\frac{\hbar}{2\omega_0}}\frac{\partial}{\partial q_j}, \quad \hat{a}_j^\dagger = \sqrt{\frac{\omega_0}{2\hbar}}q_j - \sqrt{\frac{\hbar}{2\omega_0}}\frac{\partial}{\partial q_j}, \quad j = 1, 2, \ldots, N. \quad (5.26)$$

By applying the displacement operator $\hat{D}(\alpha)$ to the ground states $\varphi_0(\mathbf{q})$ one can find the well-known coherent states of the simple N-dimensional harmonic oscillator

$$\phi_\alpha(\mathbf{q}) = \prod_{j=1}^{N} \phi_{\alpha_j}(q_j),$$

where

$$\phi_{\alpha_j}(q_j) = \left(\frac{\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-i\alpha_j^{(1)}\alpha_j^{(2)}\right] \exp\left[i\alpha_j^{(2)}\sqrt{\frac{2\omega_0}{\hbar}}q_j\right] \exp\left[-\frac{\omega_0}{2\hbar}\Big(q_j - \sqrt{\frac{2\hbar}{\omega_0}}\alpha_j^{(1)}\Big)^2\right].$$

Then, time-evolved coherent states become

$$\Phi_\alpha(\mathbf{q}, t) = \hat{U}(t, t_0)\phi_\alpha(\mathbf{q}) = \prod_{j=1}^{N} \hat{U}_j(t, t_0)\phi_{\alpha_j}(q_j) = \prod_{j=1}^{N} \Phi_{\alpha_j}(q_j, t). \quad (5.27)$$

For $j = 1, 2, \ldots, N$, $\Phi_{\alpha_j}(q_j, t)$ are found as

$$\begin{aligned}
\Phi_{\alpha_j}(q_j, t) &= \left(\frac{\omega_0}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{\epsilon_j(t)}} \exp\left\{-\frac{1}{2}\Big(\frac{(\epsilon_j^*(t))^2}{|\epsilon_j(t)|^2}\alpha_j^2 + |\alpha_j|^2\Big)\right\} \\
&\exp\left\{\frac{i}{\hbar}\left[\int_{t_0}^{t} \zeta_j(s)ds - \frac{\mu_j(t)}{2}\Big(B_j(t) - \frac{d}{dt}\ln|\epsilon_j(t)|\Big)(q_j - x_j^{(p)}(t))^2 + p_j^{(p)}(t)q_j\right]\right\} \\
&\exp\left[-\frac{\omega_0}{2\hbar}\frac{(q_j - x_j^{(p)}(t))^2}{|\epsilon_j(t)|^2}\right] \exp\left\{\sqrt{\frac{2\omega_0}{\hbar}}\frac{1}{\epsilon_j(t)}(q_j - x_j^{(p)}(t))\alpha_j\right\},
\end{aligned}$$

and we have

$$\rho_{\alpha_j}(q_j,t) = |\Phi_{\alpha_j}(q_j,t)|^2 = \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{|\epsilon_j(t)|^2}\exp\left\{-\frac{\omega_0}{\hbar}\frac{\left(q_j - \langle\hat{q}_j\rangle_{\alpha_j}(t)\right)^2}{|\epsilon_j(t)|^2}\right\},$$

where the squeezing coefficient $|\epsilon_j(t)|$ is found by (5.23) and displacements in j-direction determined by the expectation values at coherent states $\Phi_{\alpha_j}(q_j,t)$ as

$$\langle\hat{q}_j\rangle_{\alpha_j}(t) = \sqrt{\frac{2\hbar}{\omega_0}}\left(\frac{\alpha_j^{(1)}}{x_0}x_j^{(1)}(t) + \alpha_j^{(2)}(\omega_0 x_0)x_j^{(2)}(t)\right) + x_j^{(p)}(t), \tag{5.28}$$

(5.29)

$$\langle\hat{p}_j\rangle_{\alpha_j}(t) = \sqrt{\frac{2\hbar}{\omega_0}}\left(\frac{\alpha_j^{(1)}}{x_0}p_j^{(1)}(t) + \alpha_j^{(2)}(\omega_0 x_0)p_j^{(2)}(t)\right) + p_j^{(p)}(t). \tag{5.30}$$

The uncertainties for $\hat{q}_j$ and $\hat{p}_j$ at coherent states $\Phi_\alpha(\mathbf{q},t)$ are precisely

$$(\Delta\hat{q}_j)_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon_j(t)|, \tag{5.31}$$

(5.32)

$$(\Delta\hat{p}_j)_\alpha(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon_j(t)|}\sqrt{1 + \frac{|\epsilon_j(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon_j(t)|}{dt} - B_j(t)\right)^2}, \tag{5.33}$$

and the uncertainty product becomes

$$(\Delta\hat{q}_j)_\alpha(\Delta\hat{p}_j)_\alpha(t) = \frac{\hbar}{2}\sqrt{1 + \frac{|\epsilon_j(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon_j(t)|}{dt} - B_j(t)\right)^2}. \tag{5.34}$$

As a result, we can say that time-evolved coherent states of the generalized harmonic oscillator are N-dimensional shifted and squeezed Gaussian wave packets, which in general do not preserve the minimum uncertainty. Their squeezing properties in different directions are controlled by the squeezing coefficients $|\epsilon_j(t)|$, which depends on the choice of the parameters $\mu_j(t), \omega_j^2(t)$ and $B_j(t),\ j = 1, 2, \ldots, N$. On the other hand, the displacement properties of coherent states depend also on parameters $D_j(t), E_j(t),\ j = 1, 2, \ldots, N$. Therefore, for time-evolved coherent states the center of the wave packets in position space will follow the general classical trajectory, that is the parametric curve in $\mathbb{R}^N$, given by

$$\bar{Q}_g(t) := \left(\langle \hat{q}_1 \rangle_{\alpha_1}(t), \langle \hat{q}_2 \rangle_{\alpha_2}(t), ..., \langle \hat{q}_N \rangle_{\alpha_N}(t)\right), \quad t \geq 0, \tag{5.35}$$

and in momentum space the general trajectory will be

$$\bar{P}_g(t) := \left(\langle \hat{p}_1 \rangle_{\alpha_1}(t), \langle \hat{p}_2 \rangle_{\alpha_2}(t), ..., \langle \hat{p}_N \rangle_{\alpha_N}(t)\right), \quad t \geq 0, \tag{5.36}$$

where $\langle \hat{q}_j \rangle_{\alpha_j}(t)$ and $\langle \hat{p}_j \rangle_{\alpha_j}(t)$ are defined by (5.28) and (5.30), respectively. In particular, when there are no external forces, that is $D_j(t) = 0,\ E_j(t) = 0$ for all $j = 1, 2, ..., N$, coherent state packets in position space will follow the classical trajectory

$$\bar{Q}_\alpha(t) := \left(x_{\alpha_1}(t), x_{\alpha_2}(t), ..., x_{\alpha_N}(t)\right), \quad t \geq 0, \tag{5.37}$$

determined by the homogeneous solutions $x_{\alpha_j}(t)$ and values of $\alpha_j,\ j = 1, 2, ..., N$. Similarly, in momentum space trajectory will be

$$\bar{P}_\alpha(t) := \left(p_{\alpha_1}(t), p_{\alpha_2}(t), ..., p_{\alpha_N}(t)\right), \quad t \geq 0. \tag{5.38}$$

On the other hand, when $\alpha_j = 0$ for all $j = 1, 2, ..., N$, which corresponds to the ground state, and there are external forces due to nonzero parameters $D_j(t)$ or $E_j(t)$, the classical trajectories of the coherent states will be determined only by the particular solutions, i.e.,

$$\bar{Q}_p(t) := \left(x_{p,1}(t), x_{p,2}(t), ..., x_{p,N}(t)\right), \quad t \geq 0,$$
$$\bar{P}_p(t) := \left(p_{p,1}(t), p_{p,2}(t), ..., p_{p,N}(t)\right), \quad t \geq 0.$$

We note that, classical trajectories of the time-evolved eigenstate packets (5.21) are also given by $\bar{Q}_p(t)$, $\bar{P}_p(t)$.

5.5. On the Classical Harmonic Oscillator

In this section, we consider a generalized N-dimensional oscillator related with the classical Hamiltonian

$$H_{cl}(t) = \sum_{j=1}^{N}\left(\frac{p_j^2}{2} + \frac{\omega_0^2}{2}x_j^2 + B_j(t)x_jp_j + E_j(t)x_j\right), \tag{5.39}$$

where mass is $m_j = 1$ for each $j = 1, 2, \ldots, N$. For this classical oscillator, we first interpret the free motion and introduce all possible cases of frequency modification, which preserve the structure. Then, we discuss the influence of the external forces.

5.5.1. Free Motion and Frequency Modification

The homogeneous classical equations of motion in position space corresponding to the Hamiltonian (5.39) are

$$\ddot{x}_j + \left(\omega_0^2 - \left(\dot{B}_j(t) + B_j^2(t)\right)\right)x_j = 0, \quad j = 1, 2, \ldots, N, \tag{5.40}$$

and to preserve the original harmonic oscillator structure, we shall choose $B_j(t)$ to satisfy the equation $(\dot{B}_j(t) + B_j^2(t)) = -\Lambda_j^2$, where $\Lambda_j^2 > -\omega_0^2$ is the frequency modification in position space for $j = 1, 2, \cdots, N$. Then, (5.40) takes the form

$$\ddot{x}_j + \Omega_j^2 x_j = 0 \tag{5.41}$$

with modified natural frequency $\Omega_j^2 = \omega_0^2 + \Lambda_j^2 > 0$ for each $j = 1, 2, \ldots, N$, On the other hand, the homogeneous equation for the corresponding momentum becomes

$$\ddot{p}_j + \left(\omega_0^2 + \Upsilon_j^2(t)\right) p_j = 0, \quad j = 1, 2, \ldots, N, \tag{5.42}$$

where $\Upsilon_j^2(t) \equiv \dot{B}_j(t) - B_j^2(t)$ is the modification of the frequency in momentum space. Now, according to above assumptions, all possibilities for $B_j(t)$ are as follows:

- $B_j(t) = 0$, (standard harmonic oscillator).
- $B_j(t) = B_j^0 -$ constant such that $0 < (B_j^0)^2 < \omega_0^2$.
- $B_j(t) = \Lambda_j' \tanh(\Lambda_j' t + \beta_j')$, $\Lambda_j' = \sqrt{|\Lambda_j^2|}$, where $-\omega_0^2 < \Lambda_j^2 < 0, \beta_j'-$arbitrary phase.
- $B_j(t) = (t + b_j)^{-1}$, b_j-arbitrary.
- $B_j(t) = -\Lambda_j \tan(\Lambda_j t + \beta_j)$, $\Lambda_j = \sqrt{\Lambda_j^2}$, where $\Lambda_j^2 > 0, \beta_j-$arbitrary.

We note that, in the last three cases, the choice of time-dependent $B_j(t)$, leads to constant frequency modification in position space by construction, but in momentum space it will depend on time.

Lissajous orbits: For the above special choices of $B_j(t)$, the position equation is given by (5.41) for each $j = 1, 2, ..., N$. Then, when there are no external forces, the coherent wave packets (3.10) in position space will be localized along the classical trajectory

$$\bar{Q}_\alpha(t) = \left(A_{\alpha_1} \cos(\Omega_1 t - \gamma_{\alpha_1}), A_{\alpha_2} \cos(\Omega_2 t - \gamma_{\alpha_2}), ..., A_{\alpha_N} \cos(\Omega_N t - \gamma_{\alpha_N})\right), \tag{5.43}$$

where amplitudes A_{α_j}, frequencies Ω_j and phases γ_{α_j} will change according to $B_j(t)$. These trajectories, especially for $N = 2$ and $N = 3$, are well-known as the Lissajous orbits, (Goldstain, Safko & Poole). If $\Omega_j = r_j\Omega$, for some $\Omega > 0$, and r_j are rational numbers for all $j = 1, 2, ..., N$, then Lissajous orbits will be periodic. Otherwise, they are not periodic, and when time increases and tends to infinity, they will pass through every point of a box in space. On the other hand, classical trajectories in momentum space in some cases will be more complicated, as we will see in next sections.

5.5.2. Forced Motion

In this part, we shall consider the response of the quantum oscillator to sinusoidal driving forces, that is $E_j(t) = -F_j \cos(\Omega'_j t)$, where Ω'_j is the driving frequency, and F_j- real constant for each $j = 1, 2, ..., N$. The corresponding classical equations with modified frequencies and forcing terms are

$$\ddot{x}_j(t) + \Omega_j^2 x_j(t) = F_j \cos(\Omega'_j t), \quad \Omega_j = \sqrt{\omega_0^2 + \Lambda_j^2}, \quad j = 1, 2, ..., N, \tag{5.44}$$

with particular solutions $x_{p,j}(t)$, $j = 1, 2, ..., N$, which depend on the frequencies Ω_j and Ω'_j, as follows:

Beats: If $\Omega'_j \neq \Omega_j$, then particular solution satisfying the IC's $x_{p,j}(0) = 0$ and $\dot{x}_{p,j}(0) = 0$ is of the form

$$x_{p,j}(t) = F_{p,j}\Big[\cos(\Omega'_j t) - \cos(\Omega_j t)\Big] = 2F_{p,j} \sin\left(\frac{(\Omega_j - \Omega'_j)t}{2}\right) \sin\left(\frac{(\Omega_j + \Omega'_j)t}{2}\right), \tag{5.45}$$

where $F_{p,j} = F_j/(\Omega_j^2 - \Omega_j'^2)$ gives the maximum amplitude of the bounded oscillations. Special case of interest occurs, when driving frequency Ω'_j is relatively close to Ω_j, so that $|\Omega_j - \Omega'_j|$ is very small compared with $(\Omega_j + \Omega'_j)$. Then one can observe formation of beats in q_j- direction. That is, $x_{p,j}(t)$ oscillates rapidly with frequency $(\Omega_j + \Omega'_j)/2$, and has slowly varying sinusoidal amplitude $2F_{p,j} \sin((\Omega_j - \Omega'_j)t/2)$, known as the envelope or modulation of the oscillations.

Resonance: If $\Omega'_j = \Omega_j$, that is the modified natural frequency is equal to the driving frequency, then particular solution satisfying prescribed IC's becomes

$$x_{p,j}(t) = \frac{F_j}{2\Omega_j} t \sin(\Omega_j t), \tag{5.46}$$

which describes oscillations whose amplitude grows linearly with time t, and leads to resonance phenomena in q_j- direction.

Since we consider multidimensional oscillators, in general one can consider models with different type of behavior in different directions according to parameters $B_j(t)$ and $D_j(t)$. For example, it is possible that beats occur in all directions, so that trajectory of the center of the wave packets will be confined in a bounded domain. But, it can happen that we have beats in one-direction, and resonance in another direction, so that motion in one direction is bounded, but in another direction it is unbounded. Clearly dynamics in multi-dimensional problems contain many possibilities according to various parameters. In what follows, to understand better some typical properties, we shall discuss special models, which behavior in different directions is of the same type according to $B_j(t)$.

5.6. N-Dimensional Standard Quantum Harmonic Oscillator

To be able to compare different models, in this section we recall some basic results for the N-dimensional SQHO with $B_j(t) = 0$ for all $j = 1, 2, ..., N$, under the influence of linear external forces, whose Hamiltonian is

$$\hat{H}_0(t) = \sum_{j=1}^{N} \frac{\hat{p}_j^2}{2} + \frac{\omega_0^2}{2} \hat{q}_j^2 - F_j \cos(\Omega_j' t) \hat{q}_j.$$

Corresponding classical equations of motion for position are

$$\ddot{x}_j(t) + \omega_0^2 x_j(t) = F_j \cos(\Omega_j' t), \quad j = 1, 2, \ldots, N, \tag{5.47}$$

with homogeneous solutions $x_{1,j}(t) = x_{0,j}\cos(\omega_0 t)$ and $x_{2,j}(t) = (1/\omega_0 x_{0,j})\sin(\omega_0 t)$, satisfying the prescribed initial conditions. Then, momentum equations become

$$\ddot{p}_j(t) + \omega_0^2 p_j(t) = -F_j\Omega'_j \sin(\Omega'_j t), \quad j = 1, 2, \ldots, N, \tag{5.48}$$

with solutions $p_{1,j}(t) = -x_{0,j}\omega_0 \sin(\omega_0 t)$ and $p_{2,j}(t) = (1/x_{0,j})\cos(\omega_0 t)$.

It follows that the probability density for time-evolved coherent states is of the form

$$\rho_\alpha(q,t) = \left(\sqrt{\frac{\omega_0}{\pi\hbar}}\right)^N \times \exp\left\{-\sum_{j=1}^{N}\left(\sqrt{\frac{\omega_0}{\hbar}}\Big(q_j - \langle\hat{q}_j\rangle_{\alpha_j}(t)\Big)\right)^2\right\}, \tag{5.49}$$

with $|\epsilon_j(t)| = 1$, for all $j = 1, 2, \ldots, N$, and uncertainties are

$$(\Delta\hat{q}_j)_\alpha = (\Delta\hat{q}_j)_{\alpha_j} = \sqrt{\frac{\hbar}{2\omega_0}}, \quad (\Delta\hat{p}_j)_\alpha = (\Delta\hat{p}_j)_{\alpha_j} = \sqrt{\frac{\omega_0\hbar}{2}}, \quad (\Delta\hat{q}_j)_\alpha(\Delta\hat{p}_j)_\alpha = \frac{\hbar}{2},$$

showing that there is no squeezing of the wave packets both in coordinate and momentum, and uncertainty product is minimum in each direction.

Expectations of position and momentum in j-direction are found using (5.28) and (5.30). Therefore, when there are no external forces, i.e. $E_j(t) = 0$ for all $j = 1, 2, \ldots, N$, coherent wave packets (5.49) will follow the trajectory $\bar{Q}_\alpha(t)$ defined by (5.37), which for this model becomes

$$\bar{Q}_\alpha(t) = \left(A_{\alpha_1}\cos(\omega_0 t - \theta_{\alpha_1}), A_{\alpha_2}\cos(\omega_0 t - \theta_{\alpha_2}), \ldots, A_{\alpha_N}\cos(\omega_0 t - \theta_{\alpha_N})\right),$$

and the trajectory in momentum space will be

$$\bar{P}_\alpha(t) = \left(-A_{\alpha_1}\omega_0\sin(\omega_0 t - \theta_{\alpha_1}), -A_{\alpha_2}\omega_0\sin(\omega_0 t - \theta_{\alpha_2}), \ldots, -A_{\alpha_N}\omega_0\sin(\omega_0 t - \theta_{\alpha_N})\right),$$

with amplitude $A_{\alpha_j} = \sqrt{2\hbar/\omega_0}\sqrt{\alpha_{1,j}^2 + \alpha_{2,j}^2} = \sqrt{2\hbar/\omega_0}\,|\alpha_j|$ and phase $\theta_{\alpha_j} = \arctan(\alpha_{2,j}/\alpha_{1,j})$.

Since frequency is the same in every direction, for SQHO both $\bar{Q}_\alpha(t)$ and $\bar{P}_\alpha(t)$ are simple curves, that is N-dimensional generalizations of lines, circles or ellipses.

Meanwhile, when $E_j(t) \neq 0$, particular solutions contributing to expectations, can be found according to the driving frequencies Ω'_j, as follows:

a) If $\Omega'_j \neq \omega_0$, then $x_{p,j}(t)$ is found by (5.45), with $\Omega_j = \omega_0$ for all $j = 1, 2, ..., N$, and

$$p_{p,j}(t) = \frac{F_j}{(\omega_0^2 - \Omega_j'^2)}\Big(- \Omega'_j \sin(\Omega'_j t) + \omega_0 \sin(\omega_0 t)\Big). \tag{5.50}$$

b) If $\Omega'_j = \omega_0$, then $x_{p,j}(t)$ is found by (5.46) and

$$p_{p,j}(t) = \frac{F_j}{2\omega_0}\Big(\sin(\omega_0 t) + \omega_0 t \cos(\omega_0 t)\Big). \tag{5.51}$$

These well-known results for the standard harmonic oscillator, show that coherent states

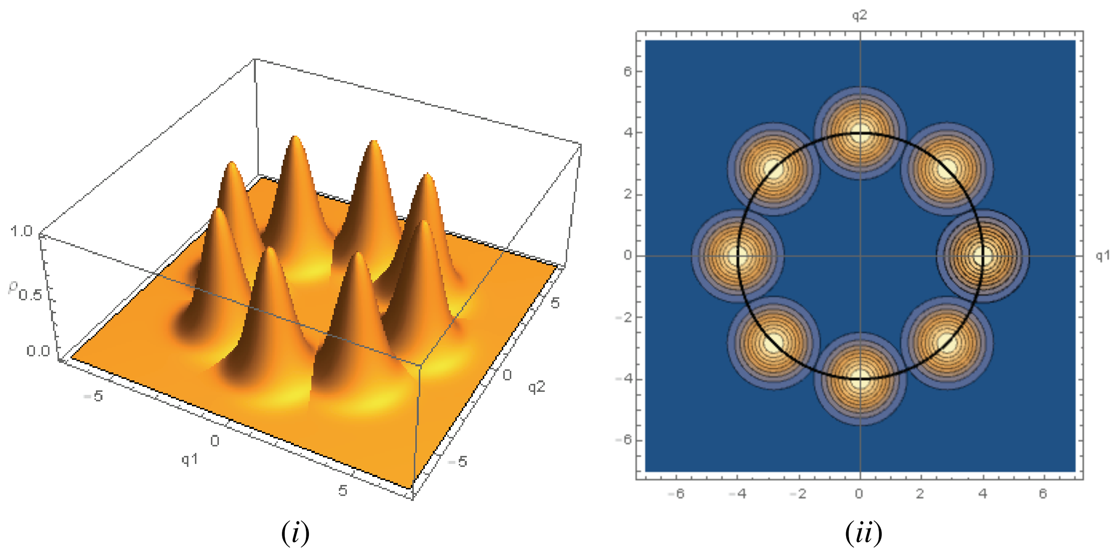


Figure 5.1. (i) Probability density $\rho_\alpha(q,t), \alpha_1 = 2\sqrt{2}, \alpha_2 = i2\sqrt{2}$ at times $t = n\pi/4$ for $n = 0, 1, \cdots, 8$, $\hbar = \omega_0 = 1$, $x_{p,j}(t) = 0$ for $j = 1, 2$. (ii) Contour plot of density and trajectory $\bar{Q}_\alpha(t) = (4\cos(t), 4\cos(t))$, $t \in [0, 2\pi]$.

are displaced Gaussian wave packets, they are minimum uncertainty states, and there is no squeezing of the wave packets. As an example for N=2, in Fig.5.1 we show time-development of probability density $\rho_\alpha(q,t) = \rho_{\alpha_1}(q_1,t)\rho_{\alpha_2}(q_2,t)$, $\alpha_1 = 2\sqrt{2}, \alpha_2 = i2\sqrt{2}$, at times $t = n\pi/4$, where $n = 0, 1, \cdots, 8$, and which center follows the circular orbit $\bar{Q}_\alpha(t)$, in case when there are no external forces.

5.7. Exactly Solvable Models

In this section, we introduce some exactly solvable models for the evolution problem (5.5) related with the Hamiltonian

$$\hat{H}(t) = \sum_{j=1}^{N} \frac{\hat{p}_j^2}{2} + \frac{\omega_0^2}{2}\hat{q}_j^2 + \frac{B_j(t)}{2}\left(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j\right) - F_j\cos(\Omega'_j t)\hat{q}_j, \tag{5.52}$$

where $B_j(t)$ are real valued time-dependent parameters and F_j, Ω'_j are real constants for each $j = 1, 2, \ldots N$.

5.7.1. Model with B_j^0– constants

First, we consider quantum oscillator with Hamiltonian (5.52) by taking the squeezing parameters as $B_j(t) \equiv B_j^0$– constants such that $0 < (B_j^0)^2 < \omega_0^2$ for all $j = 1, 2, \ldots, N$. The corresponding classical equation in position space is of the form (5.44), with modified frequency $\Omega_j = \sqrt{\omega_0^2 - (B_j^0)^2}$ smaller than the natural frequency, that is $0 < \Omega_j < \omega_0$, and solutions

$$x_{1,j}(t) = \frac{x_{0,j}\omega_0}{\Omega_j}\cos(\Omega_j t - \beta_j), \quad x_{2,j}(t) = \frac{1}{x_{0,j}\Omega_j}\sin(\Omega_j t), \quad \beta_j = \arctan(B_j^0/\Omega_j). \tag{5.53}$$

Then, the classical equation in momentum space becomes

$$\ddot{p}_j(t) + \Omega_j^2 p_j(t) = -F_j(\Omega'_j\sin(\Omega'_j t) + B_j^0\cos(\Omega'_j t)), \tag{5.54}$$

with the same modified frequency Ω_j, and its homogeneous solutions are

$$p_{1,j}(t) = -\frac{x_{0,j}\omega_0^2}{\Omega_j}\sin(\Omega_j t), \quad p_{2,j}(t) = \frac{1}{x_{0,j}}\Big(\cos(\Omega_j t) - \frac{B_j^0}{\Omega_j}\sin(\Omega_j t)\Big). \tag{5.55}$$

Thus, both position and momentum solutions oscillate with the same frequencies.

Probability density for time-evolved coherent states is of the form (5.28), with squeezing coefficient

$$|\epsilon_j(t)| = \sqrt{\frac{\omega_0^2}{\Omega_j^2}\Big(\cos^2(\Omega_j t - \beta_j) + \sin^2(\Omega_j t)\Big)}, \quad \frac{\omega_0^2}{\Omega_j^2} > 1, \tag{5.56}$$

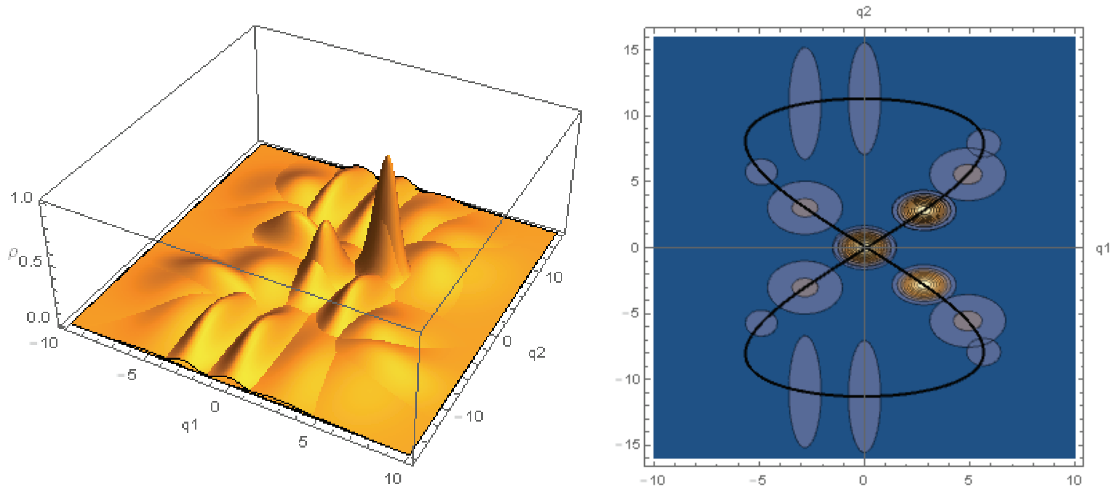
which is also periodic and oscillatory. In terms of $|\epsilon_j(t)|$, uncertainties at time-evolved coherent states are found by (5.31-5.33) and uncertainty product becomes

$$(\Delta\hat{q}_j)_{\alpha_j}(\Delta\hat{p}_j)_{\alpha_j}(t) = \frac{\hbar}{2}\Big(1 + \frac{\omega_0^2}{4\Omega_j^2}\Big[\sin(2(\Omega_j t - \beta_j)) - \sin(2\Omega_j t) + \frac{2B_j^0}{\Omega_j}\Big(\cos^2(\Omega_j t - \beta_j) + \sin^2(\Omega_j t)\Big)\Big]^2\Big)^{1/2}.$$

From (5.56), we see that when $|B_j^0|$ approaches zero, one has $(\omega_0/\Omega_j) \to 1, \beta_j \to 0$, so that $(\Delta\hat{q}_j)_{\alpha_j}(t) \to \sqrt{\hbar/2\omega_0}$, $(\Delta\hat{p}_j)_{\alpha_j}(t) \to \sqrt{\hbar\omega_0/2}$ and the uncertainty product approaches the minimum $\hbar/2$, as for the standard harmonic oscillator. For given ω_0, when $|B_j^0|$ increases and approaches ω_0, then Ω_j tends to zero, and amplitude of oscillations increases.

Now, when $E_j(t) = 0$ for all $j = 1, 2, ..., N$, then coherent wave packets in position space will follow the trajectory

$$\bar{Q}_\alpha(t) = (A_{\alpha_1}\cos(\Omega_1 t - \gamma_{\alpha_1}), A_{\alpha_2}\cos(\Omega_2 t - \gamma_{\alpha_2}), ..., A_{\alpha_N}\cos(\Omega_N t - \gamma_{\alpha_N})), \tag{5.57}$$



(*i*) (*ii*)

Figure 5.2. Model B_j^0-const: (*i*) Probability density $\rho_\alpha(q,t)$ for $B_1^0 = \frac{\sqrt{3}}{2}$, $B_2^0 = \frac{\sqrt{15}}{4}$, $\Omega_1 = \frac{1}{2}$, $\Omega_2 = \frac{1}{4}$, $\alpha_1 = \alpha_2 = 2$. (*ii*) Contour plot of probability density and Lissajous orbit $\bar{Q}_\alpha(t) = (x_{\alpha_1}(t), x_{\alpha_2}(t)) = (4\sqrt{2}\cos(\frac{t}{2} - \arctan(\sqrt{3})), 8\sqrt{2}\cos(\frac{t}{4} - \arctan(\sqrt{15}))$.

where $0 < \Omega_j < \omega_0$, for each j, with amplitudes and phases

$$A_{\alpha_j} = \sqrt{\frac{2\hbar}{\omega_0}} \frac{\sqrt{(\alpha_{1,j}B_j^0 + \alpha_{2,j}\omega_0)^2 + (\alpha_{1,j}\Omega_j)^2}}{\Omega_j}, \qquad \gamma_{\alpha_j} = \arctan\left(\frac{B_{j,0}}{\Omega_j} + \frac{\alpha_{2,j}\omega_0}{\alpha_{1,j}\Omega_j}\right), \quad (5.58)$$

and in momentum space, the corresponding trajectory $\bar{P}_\alpha(t)$ will be of the form (5.38), with

$$p_{\alpha_j}(t) = \sqrt{\frac{2\hbar}{\omega_0}}\left[-\left(\alpha_{1,j}\left(\frac{\omega_0^2}{\Omega_j}\right) + \alpha_{2,j}\omega_0\frac{B_j^0}{\Omega_j}\right)\sin(\Omega_j t) + \alpha_{2,j}\omega_0\cos(\Omega_j t)\right], \; j = 1, 2, ..., N. \quad (5.59)$$

Thus, when there are no external forces classical trajectories both in position and momentum space will be of Lissajous type.

However, if there are external forces, expectation values and general trajectories $\bar{Q}_g(t)$ and $\bar{P}_g(t)$ of the wave packets are determined according to the values of the driving frequencies. If $\Omega'_j \neq \Omega_j$, then $x_{p,j}(t)$ is given by (5.45) and $p_{p,j}(t) = \dot{x}_{p,j}(t) - B_j^0 x_{p,j}(t)$.

If $\Omega'_j = \Omega_j$, then $x_{p,j}(t)$ is given by (5.46), where $p_{p,j}(t) = \frac{F_j}{2\Omega_j}\Big((1 - B_j^0)\sin(\Omega_j t) + \Omega_j t\cos(\Omega_j t)\Big)$, for each $j = 1, 2, ..., N$.

As an example, in Fig.5.2 we plot time-evolution of probability density at times $t = n\pi/3$ for $n = 0, 1, \cdots, 24$, when there are no external forces. It explicitly shows the changes of the width and amplitude of the wave packets due to the non-zero squeezing parameters, and its contour plot confirms that the center of the wave packet follows the classical Lissajous orbit $\bar{Q}_\alpha(t)$.

5.7.2. Model 1

Now, we introduce quantum oscillator with Hamiltonian (5.52), where squeezing parameters for all $j = 1, 2, \cdots, N$ are of the form $B_j(t) = \Lambda'_j\tanh(\Lambda'_j t)$, with $0 < (\Lambda'_j)^2 < \omega_0^2$. In that case the modified frequency $\Omega_j = \sqrt{\omega_0^2 - (\Lambda'_j)^2}$ is less than the natural frequency, that is $0 < \Omega_j < \omega_0$, like in previous model with B_j^0- constants. The corresponding classical equation in position space is of the form (5.44), with homogeneous solutions $x_{1,j}(t) = x_{0,j}\cos(\Omega_j t)$, $x_{j,2}(t) = (1/\Omega_j x_{0,j})\sin(\Omega_j t)$, but the classical equation in momentum space becomes

$$\ddot{p}_j(t) + (\omega_0^2 + \Upsilon_j^2(t))p_j = F_j(\Omega'_j\sin(\Omega'_j t) - \Lambda'_j\tanh(\Lambda'_j t)\cos(\Omega'_j t)), \quad (5.60)$$

where $\Upsilon_j^2(t) = \Lambda_j'^2(1 - 2\tanh^2(\Lambda'_j(t))$ is the time-dependent frequency modification in momentum space. When $t \to \infty$, $\Upsilon_j^2(t) \to -\Lambda_j'^2$, which means as time increases frequency modification in momentum space approaches the frequency modification in position space.

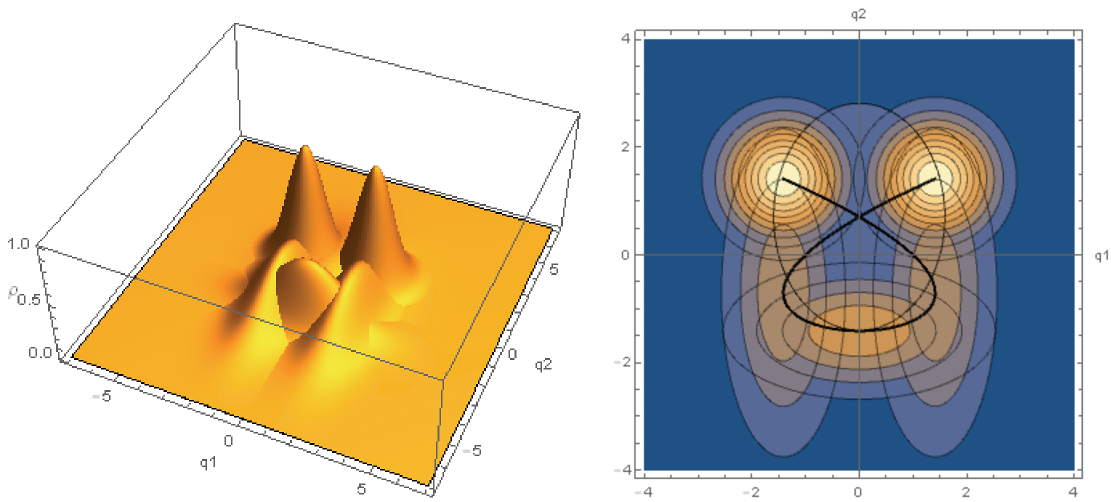
Probability density for time-evolved coherent states is of the form (3.10), with squeezing coefficients

$$|\epsilon_j(t)| = \sqrt{\cos^2(\Omega_j t) + \frac{\omega_0^2}{\Omega_j^2}\sin^2(\Omega_j t)}, \quad \frac{\omega_0^2}{\Omega_j^2} > 1, \quad (5.61)$$

where if $\Omega_j \to \omega_0$, then $|\epsilon_j(t)| \to 1$. Meanwhile, when $\Omega_j \to 0$, that is when the modified frequency decreases, then amplitude of oscillations of $|\epsilon_j(t)|$ increases. The uncertainties for $\hat{q}_j$ and $\hat{p}_j$ in terms of $|\epsilon_j(t)|$ and $B_j(t)$ will be of the form (5.31-5.33), and the uncertainty relation becomes

$$(\Delta\hat{q}_j\Delta\hat{p}_j)_\alpha = \frac{\hbar}{2}\sqrt{1+\frac{1}{\omega_0^2}\left[\Lambda_j'\tanh(\Lambda_j' t)\left(\cos^2(\Omega_j t)+\frac{\omega_0^2}{\Omega_j^2}\sin^2(\Omega_j t)\right)+\left(\frac{\Lambda_j^2}{2\Omega_j}\right)\sin(2\Omega_j t)\right]^2}. \tag{5.62}$$

Therefore, there is a periodic oscillatory squeezing in position coordinates for each $j = 1, 2, ..., N$, and period in each direction will depend on the values of Ω_j. However, the squeezing in momentum, and the uncertainty products are not periodic and become more complicated due to influence of $B_j(t)$.

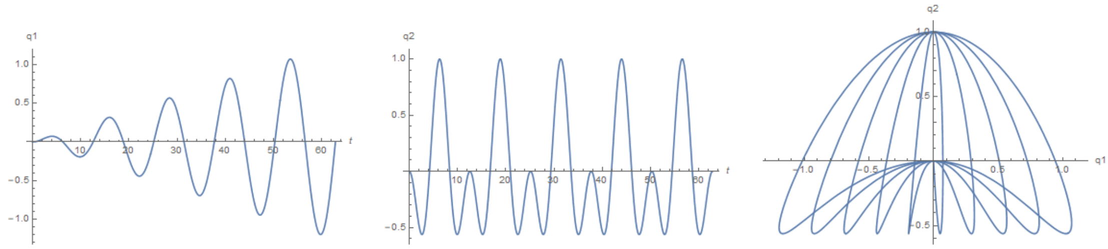


(i) (ii)

Figure 5.3. Model 1: (i) Probability density $\rho_\alpha(q,t)$ for $\Omega_1 = 1/2, \Omega_2 = 1/3, \alpha_1 = \alpha_2 = 1, \hbar = \omega_0 = 1$ and $x_{p,j}(t) = 0$ for $j = 1,2$. (ii) Contour plot and Lissajous orbit $\bar{Q}_\alpha(t) = (\sqrt{2}\cos(t/2), \sqrt{2}\cos(t/3)),\ t \in [0, 6\pi]$.

When there are no external forces wave packets in position space will follow the Lissajous trajectory

$$\bar{Q}_\alpha(t) = \left(A_{\alpha_1}\cos(\Omega_1 t-\theta_{\alpha_1}), A_{\alpha_2}\cos(\Omega_2 t-\theta_{\alpha_2}), ..., A_{\alpha_N}\cos(\Omega_N t-\theta_{\alpha_N})\right),$$



(*i*) (*ii*) (*iii*)

Figure 5.4. Model 1: (*i*) Resonance in q_1 direction: $x_{p,1}(t) = \frac{1}{50} t \sin(\frac{t}{2})$.
(*ii*) In q_2 direction: $x_{p,2}(t) = -\sin(\frac{t}{4})\sin(\frac{3}{4}t)$.
(*iii*) Trajectory $\bar{Q}_p = (x_{p,1}(t), x_{p,2}(t))$ for $t \in [0, 20\pi]$.

and in momentum space trajectory will be $\bar{P}_\alpha(t)$ defined by (5.38), with

$$p_{\alpha_j}(t) = A_{\alpha_j}\Big(-\Omega_j \sin(\Omega_j t - \theta_{\alpha_j}) - \Lambda'_j \tanh(\Lambda'_j t)\cos(\Omega_j t - \theta_{\alpha_j})\Big), \; j = 1, 2, \ldots, N, \quad (5.63)$$

where amplitudes and phases are

$$A_{\alpha_j} = \sqrt{2\hbar/\omega_0 \Omega_j^2}\sqrt{(\alpha_{1,j}\Omega_j)^2 + (\alpha_{2,j}\omega_0)^2}, \quad \theta_{\alpha_j} = \arctan(\alpha_{2,j}\omega_0/\alpha_{1,j}\Omega_j). \quad (5.64)$$

When there are external forces, expectations and trajectories will be determined according to the following cases. If $\Omega'_j \neq \Omega_j$, then $x_{p,j}(t)$ is given by (5.45) and $p_{p,j}(t) = \dot{x}_{p,j}(t) - \Lambda'_j \tanh(\Lambda'_j t) x_{p,j}(t)$. If $\Omega'_j = \Omega_j$, then $x_{p,j}(t)$ is given by (5.46) and $p_{p,j}(t) = (F_j/2\Omega_j)((1 - \Lambda'_j t \tanh(\Lambda'_j t))\sin(\Omega_j t) + \Omega_j t \cos(\Omega_j t))$.

As an example, in Fig.(5.3-*i*) we plot probability density $\rho_\alpha(q,t)$ at different times $t = n\pi$ for $n = 0, 1, \ldots, 6$, and observe their width and amplitude changes due to the squeezing parameters. Fig.5.3-(*ii*) shows the corresponding contour plot and the Lissajous orbit of the center of the coherent wave packet. In Fig.5.4, we show possible trajectory of ground state ($\alpha_1 = \alpha_2 = 0$) wave packet $\rho_\alpha(q,t)$, under the influence of forces $D_1(t) = -F_1\cos(\Omega'_1 t)$, $D_2(t) = -F_2\cos(\Omega'_2 t)$, where $\Omega_1 = \Omega'_1 = 1/2$, $\Omega_2 = 1/2, \Omega'_2 = 1$, $F_1 = 1/50, F_2 = -3/8$. It is a perpendicular superposition of resonance in q_1-direction and periodic oscillations in q_2-direction.

5.7.3. Model 2

In this section, we introduce quantum oscillator with Hamiltonian (5.52), where squeezing parameters for all $j = 1, 2, ..., N$, are of the form $B_j(t) = 1/(t+b_j)$, b_j- arbitrary nonzero constants. We note that, if $b_j < 0$, then one will have singularity at positive time $t = b_j$. Substituting $B_j(t)$ in (5.40), we see that it does not modify the natural frequency ω_0, and the classical equation in position space becomes $\ddot{x}_j(t) + \omega_0^2 x_j(t) = F_j \cos(\Omega'_j t)$, $j = 1, 2, ..., N$, which is same as for the SQHO ($B_j(t) = 0$), see eq.(5.47). But here, initial conditions change according to b_j as $x_{1,j}(0) = x_{0,j} \neq 0$, $\dot{x}_{1,j}(0) = x_{0,j}/b_j$; $x_{2,j}(0) = 0$, $\dot{x}_{2,j}(0) = 1/x_{0,j}$, and this is reflected in the amplitude and phase of the solutions as follows

$$x_{1,j}(t) = x_{0,j}\sqrt{1 + \frac{1}{\omega_0^2 b_j^2}}\cos(\omega_0 t - \delta_j), \quad x_{2,j}(t) = \frac{1}{\omega_0 x_{0,j}}\sin(\omega_0 t), \tag{5.65}$$

where $\delta_j = \arctan(1/\omega_0 b_j)$. On the other hand, classical equation in momentum space is

$$\ddot{p}_j(t) + \left(\omega_0^2 - \frac{2}{(t+b_j)^2}\right)p_j(t) = F_j(\Omega'_j \sin(\Omega'_j t) - \left(\frac{1}{t+b_j}\right)\cos(\Omega'_j t)), \tag{5.66}$$

with frequency modification $\Upsilon_j^2(t) = -2/(t+b_j)^2$ depending on time. Its solutions become

$$p_{1,j}(t) = -x_{0j}\sqrt{1 + \frac{1}{\omega_0^2 b_j^2}}\left(\omega_0 \sin(\omega_0 t - \delta_j) + \frac{1}{t+b_j}\cos(\omega_0 t - \delta_j)\right), \tag{5.67}$$

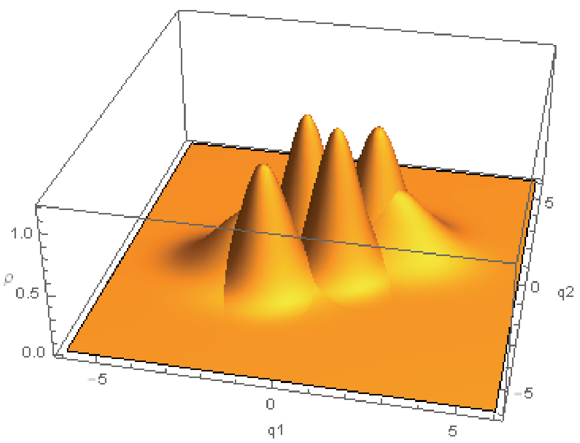
$$p_{2,j}(t) = \frac{1}{\omega_0 x_{0j}}\left(\omega_0 \cos(\omega_0 t) - \frac{1}{t+b_j}\sin(\omega_0 t)\right). \tag{5.68}$$

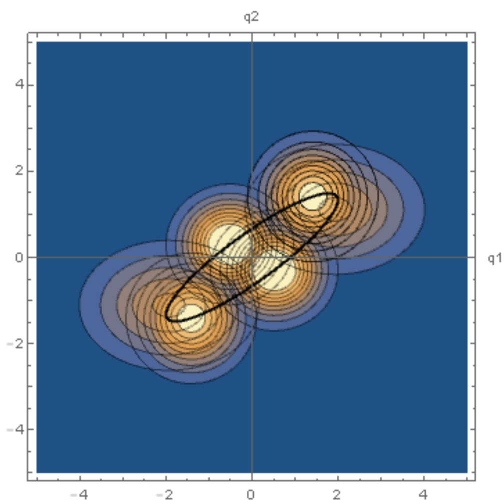
Probability density for time-evolved coherent states is of the form (3.10), with periodic and oscillatory squeezing coefficient

$$|\epsilon_j(t)| = \sqrt{\left(1 + \frac{1}{\omega_0^2 b_j^2}\right)\cos^2(\omega_0 t - \delta_j) + \sin^2(\omega_0 t)}, \tag{5.69}$$

whose period is same for all $j = 1, 2, ..., N$. Here, when $|b_j| \to \infty$, one has $\sigma_j(t) \to 1$ and amplitude of oscillations decreases, approaching the constant values as for the SQHO. On the other hand, by letting $|b_j| \to 0$, one can increase amplitude of oscillations, without changing their frequencies, which are independent of b_j. In terms of $|\epsilon_j(t)|$ uncertainties are as found in (5.31-5.33) and the uncertainty relation at coherent states is found explicitly as

$$(\Delta\hat{q}_j)_{\alpha_j}(\Delta\hat{p}_j)_{\alpha_j}(t) = \frac{\hbar}{2}\Bigg\{1 + \frac{1}{4\omega_0^2}\Bigg[\Bigg(\frac{2}{t+b_j}\Bigg)\Bigg(\frac{1+(\omega_0 b_j)^2}{(\omega_0 b_j)^2}\cos^2(\omega_0 t - \delta_j) + \sin^2(\omega_0 t)\Bigg) + \frac{1+(\omega_0 b_j)^2}{\omega_0 b_j^2}\sin(2(\omega_0 t - \delta_j)) - \omega_0 \sin(2\omega_0 t)\Bigg]^2\Bigg\}^{1/2}.$$



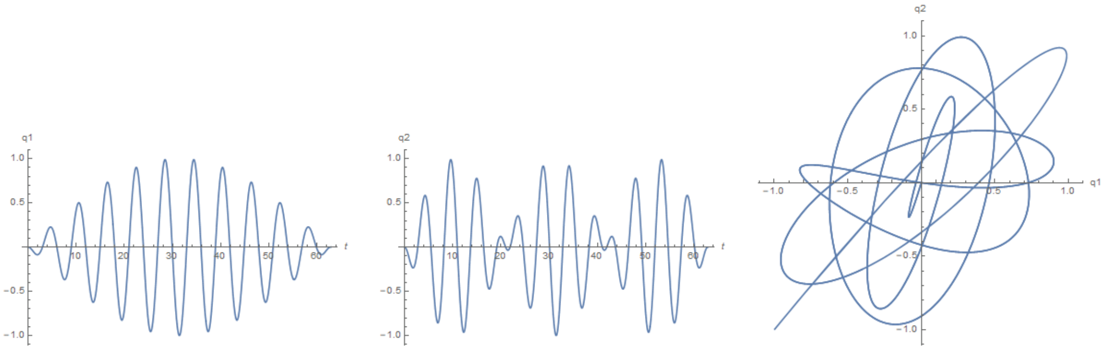


(*i*) (*ii*)

Figure 5.5. Model 2: (*i*) Probability density $\rho_\alpha(q,t), \alpha_1 = \alpha_2 = 1$, $b_1 = 1$, $b_2 = 3$, $\hbar = \omega_0 = 1$, $x_{p,j}(t) = 0$ for $j = 1, 2$ and $t = n\pi/3$, where $n = 0, 1, 2, ...6$. (*ii*) Contour plot and elliptic trajectory $\bar{Q}_\alpha(t) = (2\cos(t-\pi/4), (2\sqrt{5}/3)\cos(t-\arctan(1/3)))$, $t \in [0, 2\pi]$.

In this model, when there are no external forces the expectations and trajectory of the wave packets are given by

$$\bar{Q}_\alpha(t) = \Big(A_{\alpha_j}\cos(\omega_0 t - \gamma_{\alpha_j}), A_{\alpha_j}\cos(\omega_0 t - \gamma_{\alpha_j}), ..., A_{\alpha_j}\cos(\omega_0 t - \gamma_{\alpha_j})\Big),$$



(i) (ii) (iii)

Figure 5.6. Model 2: (i) $x_{p,1}(t) = \sin(-0.05t)\sin(1.05t)$, $\omega_0 = 1, \Omega'_1 = 1.1$. (ii) $x_{p,2}(t) = \sin(-0.15t)\sin(1.15t)$, $\omega_0 = 1, \Omega'_2 = 1.3$. (iii) Trajectory $\bar{Q}_p = (x_{p,1}(t), x_{p,2}(t))$ for $t \in [0, 20\pi]$, which is a perpendicular superposition of two beats.

which are either lines or ellipses, like for the SQHO model, but in momentum space trajectory $\bar{P}_\alpha(t)$ is found using $p_{\alpha_j}(t) = -A_{\alpha_j}\Big(\omega_0 \sin(\omega_0 t - \gamma_{\alpha_j}) + \frac{1}{t+b_j}\cos(\omega_0 t - \gamma_{\alpha_j})\Big)$ with

$$A_{\alpha_j} = \sqrt{\frac{2\hbar}{\omega_0}}\sqrt{\left(\frac{\alpha_{1,j}}{\omega_0 b_j} + \alpha_{2,j}\right)^2 + \alpha_{1,j}^2}, \quad \gamma_{\alpha_j} = \arctan\left(\frac{1}{\omega_0 b_j} + \frac{\alpha_{2,j}}{\alpha_{1,j}}\right). \tag{5.70}$$

When there are external forces, expectations and trajectories are as follows. If $\Omega'_j \neq \omega_0$, then $x_{p,j}(t)$ is given by (5.45) with $\Omega_j = \omega_0$ for all $j = 1, 2, ..., N$, and $p_{p,j}(t) = \dot{x}_{p,j}(t) - (t + b_j)^{-1}x_{p,j}(t)$. If $\Omega'_j = \omega_0$, then $x_{p,j}(t)$ is given by (5.46) and $p_{p,j}(t) = \dot{x}_{p,j}(t) - (t + b_j)^{-1}x_{p,j}(t)$.

In Fig.5.5-(i), we plot probability density $\rho_\alpha(q, t)$ for $N = 2$, $b_1 = 1$ and $b_2 = 3$, $\alpha_1 = \alpha_2 = 1$, showing the width and amplitude changes at different times $t = n\pi/3$, where $n = 0, 1, 2, ...6$. Contour plot of the density in Fig.5.5-(ii), shows that the wave packet is localized along the classical orbit $\bar{Q}_\alpha(t)$, which in that example is an ellipse. In Fig.5.6, we plot a trajectory of the wave packets under the influence of special external forces. It is a perpendicular superposition of two beats.

5.7.4. Model 3

The last model, which we consider is quantum oscillator with Hamiltonian (5.52) by taking the squeezing parameters for all $j = 1, 2, ..., N$, as $B_j(t) = -\Lambda_j \tan(\Lambda_j t)$, $\Lambda_j^2 > 0$. In this model, $B_j(t)$ are periodic with singularities at times $t = ((n - 1/2)\pi)/\Lambda_j$, $n = 1, 2, ...$ The corresponding classical equations in position space are of the form (5.44), with homogeneous solutions $x_{1,j}(t) = x_{0,j} \cos(\Omega_j t)$, and $x_{2,j}(t) = (1/\Omega_j x_{0,j}) \sin(\Omega_j t)$, and modified frequency greater than the natural frequency, that is $\Omega_j = \sqrt{\omega_0^2 + \Lambda_j^2} > \omega_0$, for all $j = 1, 2, ..., N$. Then, classical equations in momentum space become

$$\ddot{p}_j + \left(\omega_0^2 + \Upsilon_j^2(t)\right) p_j = F_j(\Omega_j' \sin(\Omega_j' t) + \Lambda_j \tan(\Lambda_j t) \cos(\Omega_j' t)), \quad j = 1, 2, ..., N, \quad (5.71)$$

where we have time-dependent frequency modification $\Upsilon_j^2(t) = -\Lambda_j^2(1+2\tan^2(\Lambda_j t))$, with singularities at times $t = ((n-1/2)\pi)/\Lambda_j$, $n = 1, 2, ...$ Probability density for time-evolved coherent states is found according to (3.10), with squeezing coefficient

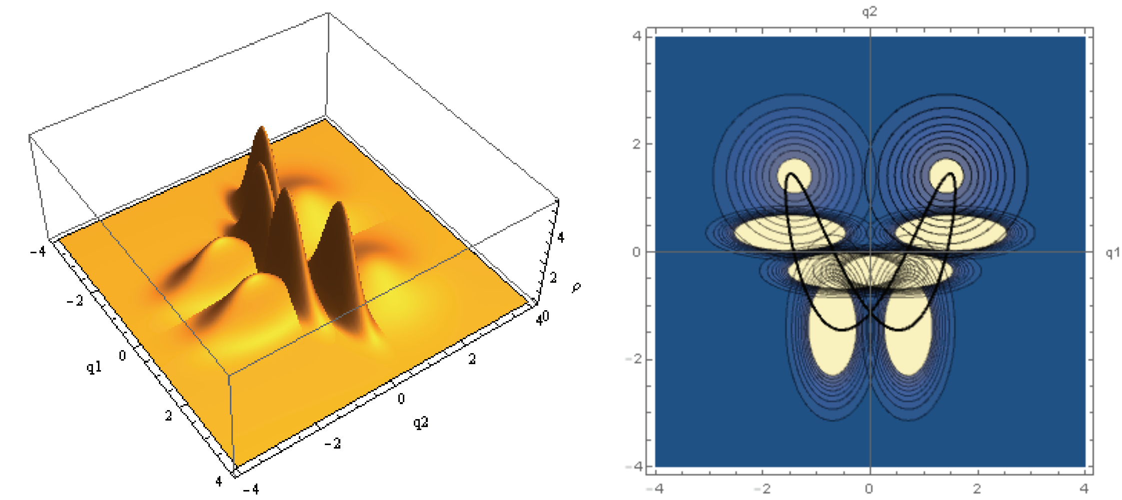
$$|\epsilon_j(t)| = \sqrt{\cos^2(\Omega_j t) + \frac{\omega_0^2}{\Omega_j^2} \sin^2(\Omega_j t)}, \quad \Omega_j > \omega_0, \quad (5.72)$$

and the uncertainties for $\hat{q}_j$ and $\hat{p}_j$ at time-evolved coherent states are given by (5.31-5.33). Then, the uncertainty product is

$$(\Delta\hat{q}_j)(\Delta\hat{p}_j)_{\alpha_j} = \frac{\hbar}{2} \sqrt{1 + \frac{1}{\omega_0^2} \left[\Lambda_j \tan(\Lambda_j t)\left(\cos^2(\Omega_j t) + \frac{\omega_0^2}{\Omega_j^2} \sin^2(\Omega_j t)\right) - \frac{\Lambda_j^2}{2\Omega_j} \sin(2\Omega_j t)\right]^2}. \quad (5.73)$$

It follows that, uncertainty in position is smooth and periodic, but uncertainty in momentum and the uncertainty product have singularities due to singularities in $B_j(t)$. Moreover, $(\Delta\hat{q}_j)_{\alpha_j}(t)$ oscillates bellow the value $(\Delta\hat{q}_j)_{\alpha_j} = 1/\sqrt{2}$, and $(\Delta\hat{p}_j)_{\alpha_j}(t)$ oscillates above it. Also, for given ω_0 both frequency Ω_j and amplitude of oscillations can be increased by increasing the value of $\Lambda_j^2 > 0$.

For this model, when $E_j(t) = 0$ for all $j = 1, 2, ..., N$, the center of the wave packets



(*i*) (*ii*)

Figure 5.7. Model 3: (*i*) Probability densities $\rho_\alpha(q,t)$ with $\alpha_1 = \alpha_2 = 1+i, \hbar = \omega_0 = 1$, $x_{p,j}(t) = 0$ for $j = 1,2$ and $t = n\pi/6, n = 0,1,2,...,6$. (*ii*) Contour plot and the trajectory $\bar{Q}_\alpha(t)$, $t \in [0,\pi]$.

will follow the Lissajous trajectories

$$\bar{Q}_\alpha(t) = (A_{\alpha_j}\cos(\Omega_j t - \theta_{\alpha_j}), A_{\alpha_j}\cos(\Omega_j t - \theta_{\alpha_j}), ..., A_{\alpha_j}\cos(\Omega_j t - \theta_{\alpha_j})),$$

but in momentum space trajectories $\bar{P}_\alpha(t)$ are more complicated, having singularities, and are found using

$$p_{\alpha_j}(t) = A_{\alpha_j}\Big(- \Omega_j \sin(\Omega_j t - \theta_{\alpha_j}) + \Lambda_j \tan(\Lambda_j t)\cos(\Omega_j t - \theta_{\alpha_j})\Big). \tag{5.74}$$

with amplitudes A_{α_j} and phases θ_{α_j} given by (5.64).

The influence of the external force $D_j(t) = -F_j\cos(\Omega'_j t)$ on the expectation values will be as follows. If $\Omega'_j \neq \Omega_j$, then $x_{p,j}(t)$ is given by (5.45) and $p_{p,j}(t) = \dot{x}_{p,j}(t) + \Lambda_j \tan(\Lambda_j t) x_{p,j}(t)$. If $\Omega'_j = \Omega_j$, then $x_{p,j}(t)$ is given by (5.46) and $p_{p,j}(t) = (F_j/2\Omega_j)((1 + \Lambda_j t\tan(\Lambda_j t))\sin(\Omega_j t) + \Omega_j t\cos(\Omega_j t))$.

As an example, for $N = 2$, when there are no external force, in Fig.5.7, we show the probability density $\rho_\alpha(q,t)$, with $\Omega_1 = 2$, $\Omega_2 = 4$, $\alpha_1 = \alpha_2 = 1+i$ at different times $t = n\pi/8$ for $n = 0, 1, \ldots, 8$ and the contour plot showing that the center of the wave packets follow the classical Lissajous orbit $\bar{Q}_\alpha(t) = (\sqrt{5/2}\cos(2t - \arctan(1/2)), \sqrt{17/8}\cos(4t - \arctan(1/4))$. In Fig.5.8-($i$), we show a trajectory of the wave packets $\rho_\alpha(q,t)$ under the influence of external forces. And it is a perpendicular superposition of two beats. In Fig.5.8-(ii), we show the trajectory $\bar{Q}_\alpha(t) = (\cos(2t), \cos(\pi t))$, which is a Lissajous orbit, of the wave packets with no external force.

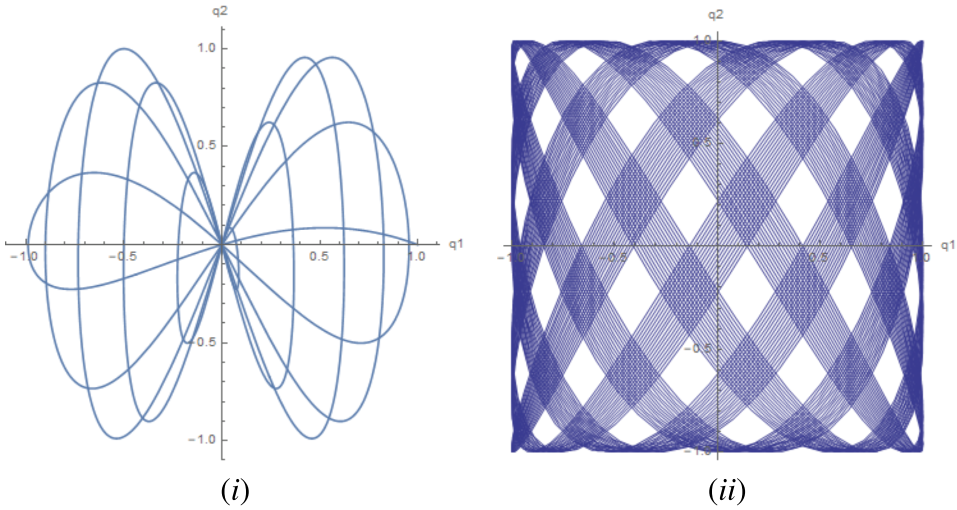


Figure 5.8. Model 3 with external force $D_j(t) = -F_j\cos(\Omega'_j t)$, $j = 1, 2$, (i) The trajectory $\bar{Q}_p(t) = (\sin((0.05)t)\sin((1.05)t), \sin((0.1)t)\sin((2.1)t))$ with $\Omega_1 = 1.1, \Omega'_1 = 1, \Omega_2 = 2.2, \Omega'_2 = 2, \alpha_1 = \alpha_2 = 0$ and $\hbar = \omega_0 = 1$, $t \in [0, 10\pi]$. (ii) The trajectory $\bar{Q}_\alpha(t) = (\cos(2t), \cos(\pi t))$ with $\Omega_1 = 2, \Omega_2 = \pi, \alpha_1 = \alpha_2 = 1/\sqrt{2}$ and $\hbar = \omega_0 = 1$, $t \in [0, 60\pi]$.

CHAPTER 6

A GENERALIZED TWO-DIMENSIONAL QUANTUM PARAMETRIC OSCILLATOR IN THE PRESENCE OF VARIABLE MAGNETIC AND ELECTRIC FIELDS

In this chapter, we introduce time-dependent Schrödinger equation describing a generalized two-dimensional quantum parametric oscillator in the presence of time-variable external fields. We solve the corresponding evolution problem by using Wei-Norman Lie algebraic approach, (Atılgan Büyükaşık & Çayiç, 2022). Then, we derive the propagator and time-evolution of the eigenstates and coherent states explicitly in terms of solutions to the corresponding system of coupled classical equations of motion. In addition, using the evolution operator formalism, we construct linear and quadratic quantum dynamical invariants that provide connection of the present results by those obtained in (Malkin, Man'ko & Trifonov, 1970) and (Lewis & Riesenfeld, 1969). Lastly, as an exactly solvable model, we introduce a Cauchy-Euler type quantum oscillator with increasing mass and decreasing frequency and in time-dependent magnetic and electric fields. Based on the explicit results for the uncertainties and expectations, squeezing properties of the wave packets and their trajectories in the two-dimensional configuration space are discussed according to the influence of the time-variable parameters and external fields.

6.1. The Classical Problem

First, we consider a classical two-dimensional oscillator described by the Hamiltonian

$$\mathcal{H}_{cl}(t) = \frac{\mathbf{P}^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\mathbf{X}^2 + B(t)\mathbf{X}\cdot\mathbf{P} + \mathbf{D}(t)\cdot\mathbf{P} + \mathbf{E}(t)\cdot\mathbf{X} + \lambda(t)L(\mathbf{X},\mathbf{P}),$$

where $\mathbf{X} = (X_1, X_2)^T$ is the position vector, $\mathbf{P} = (P_1, P_2)^T$ is the momentum vector and $L(\mathbf{X}, \mathbf{P}) = X_1 P_2 - X_2 P_1$. Also, $\mu(t) > 0, \omega^2(t), B(t)$ are real-valued parameters depending on time, $\mathbf{D}(t) = (D_1(t), D_2(t))^T$, $\mathbf{E}(t) = (E_1(t), E_2(t))^T$ are vectors of real-valued time-dependent functions, and $\lambda(t)$ is a coupling parameter. Here, we use the dot product notation $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$ for any two vectors $\mathbf{u} = (u_1, u_2)^T$, $\mathbf{v} = (v_1, v_2)^T$, and $\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u}$.

The corresponding Hamilton's equations of motion are

$$\dot{\mathbf{X}} = \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{H}_{cl}}{\partial P_1} \\ \frac{\partial \mathcal{H}_{cl}}{\partial P_2} \end{pmatrix} \equiv \frac{\mathbf{P}}{\mu(t)} + \begin{pmatrix} B(t) & -\lambda(t) \\ \lambda(t) & B(t) \end{pmatrix} \mathbf{X} + \mathbf{D}(t),$$

$$\dot{\mathbf{P}} = \begin{pmatrix} \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial \mathcal{H}_{cl}}{\partial X_1} \\ -\frac{\partial \mathcal{H}_{cl}}{\partial X_2} \end{pmatrix} \equiv -\left\{ \mu(t)\omega^2(t)\mathbf{X} + \begin{pmatrix} B(t) & \lambda(t) \\ -\lambda(t) & B(t) \end{pmatrix} \mathbf{P} + \mathbf{E}(t) \right\},$$

where 'dot' denotes derivative with respect to time. Then, the system of classical equations of motion in position space becomes

$$\ddot{\mathbf{X}} + \begin{pmatrix} \frac{\dot{\mu}}{\mu} & 2\lambda \\ -2\lambda & \frac{\dot{\mu}}{\mu} \end{pmatrix} \dot{\mathbf{X}} + \begin{pmatrix} \Omega_X(t) & \frac{\dot{\mu}}{\mu}\lambda + \dot{\lambda} \\ -\frac{\dot{\mu}}{\mu}\lambda - \dot{\lambda} & \Omega_X(t) \end{pmatrix} \mathbf{X} = \mathbf{F}_X(t), \tag{6.1}$$

and for $\lambda(t) \neq 0$ it is a system of coupled second-order differential equations. In (6.1), we have

$$\Omega_X(t) = \omega^2(t) - \left(\dot{B}(t) + B^2(t) + \frac{\dot{\mu}}{\mu} B(t) + \lambda^2(t) \right),$$

and the forcing vector term

$$\mathbf{F}_X(t) = \begin{pmatrix} \frac{\dot{\mu}}{\mu} + B(t) & \lambda(t) \\ \lambda(t) & \frac{\dot{\mu}}{\mu} + B(t) \end{pmatrix} \mathbf{D}(t) + \dot{\mathbf{D}}(t) - \frac{1}{\mu(t)} \mathbf{E}(t).$$

We note that, if $\mathbf{D} = \mathbf{0}$, then $\lambda(t)$ does not influence the forcing vector $\mathbf{F}_X(t)$. Also, in momentum space, the system of oscillator equations becomes

$$\ddot{\mathbf{P}} + \begin{pmatrix} -\frac{(\dot{\mu\omega^2})}{\mu\omega^2} & 2\lambda \\ -2\lambda & -\frac{(\dot{\mu\omega^2})}{\mu\omega^2} \end{pmatrix} \dot{\mathbf{P}} + \begin{pmatrix} \Omega_P(t) & \dot{\lambda} - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}\lambda \\ -\dot{\lambda} + \frac{(\dot{\mu\omega^2})}{\mu\omega^2}\lambda & \Omega_P(t) \end{pmatrix} \mathbf{P} = \mathbf{F}_{\mathbf{P}}(t), \tag{6.2}$$

where

$$\Omega_P(t) = \omega^2(t) + \left(\dot{B}(t) - B^2(t) - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}B(t) - \lambda^2(t)\right),$$

and the forcing term

$$\mathbf{F_P}(t) = -\mu(t)\omega^2(t)\mathbf{D}(t) - \dot{\mathbf{E}}(t) + \begin{pmatrix} \frac{(\dot{\mu\omega^2})}{\mu\omega^2} + B(t) & -\lambda(t) \\ -\lambda(t) & \frac{(\dot{\mu\omega^2})}{\mu\omega^2} + B(t) \end{pmatrix}\mathbf{E}(t).$$

Here, in momentum space, the forcing $\mathbf{F_P}(t)$ will be affected by $\lambda(t)$ in the presence of electric fields. To solve the systems (6.1) and (6.2), it is convenient to introduce the transformation of variables

$$\mathbf{x} = R_\theta(t)\mathbf{X}, \quad \mathbf{p} = R_\theta(t)\mathbf{P}, \tag{6.3}$$

where

$$R_\theta(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix}, \tag{6.4}$$

is a rotation matrix, and the rotation angle is defined as

$$\theta(t) = \int_{t_0}^{t} \lambda(s)ds. \tag{6.5}$$

Under transformation in (6.3), the coupled system (6.1) reduces to the decoupled system of two non-interacting damped oscillators

$$\ddot{\mathbf{x}} + \begin{pmatrix} \frac{\dot{\mu}}{\mu} & 0 \\ 0 & \frac{\dot{\mu}}{\mu} \end{pmatrix}\dot{\mathbf{x}} + \begin{pmatrix} \Omega_x(t) & 0 \\ 0 & \Omega_x(t) \end{pmatrix}\mathbf{x} = \widetilde{\mathbf{F}}_x(t), \tag{6.6}$$

with the same damping parameter $\Gamma(t) = \dot{\mu}(t)/\mu(t)$, and the same frequency

$$\Omega_x(t) = \omega^2(t) - \left(\dot{B}(t) + B^2(t) + \frac{\dot{\mu}}{\mu}B(t)\right),$$

which is independent on $\lambda(t)$. On the other hand, the new forcing term becomes

$$\widetilde{\mathbf{F}}_x(t) = -\frac{\widetilde{\mathbf{E}}}{\mu} + \dot{\widetilde{\mathbf{D}}} + \left(\frac{\dot{\mu}}{\mu} + B\right)\widetilde{\mathbf{D}},$$

and the relations between parameters $\mathbf{D}(t), \mathbf{E}(t)$ and $\widetilde{\mathbf{D}}(t), \widetilde{\mathbf{E}}(t)$ are found as

$$\widetilde{\mathbf{D}}(t) = R_\theta(t)\mathbf{D}(t), \quad \widetilde{\mathbf{E}}(t) = R_\theta(t)\mathbf{E}(t). \tag{6.7}$$

Similarly, in momentum space we have

$$\ddot{\mathbf{p}} + \begin{pmatrix} \frac{(\dot{\mu\omega^2})}{\mu\omega^2} & 0 \\ 0 & \frac{(\dot{\mu\omega^2})}{\mu\omega^2} \end{pmatrix} \dot{\mathbf{p}} + \begin{pmatrix} \Omega_p(t) & 0 \\ 0 & \Omega_p(t) \end{pmatrix} \mathbf{p} = \widetilde{\mathbf{F}}_p(t), \tag{6.8}$$

where the frequency and the forcing term are

$$\begin{gathered} \Omega_p(t) = \omega^2(t) + \left(\dot{B}(t) - B^2(t) - \tfrac{(\dot{\mu\omega^2})}{\mu\omega^2}B(t)\right), \\ \widetilde{\mathbf{F}}_p(t) = -\dot{\widetilde{\mathbf{E}}} + \left(\tfrac{(\dot{\mu\omega^2})}{\mu\omega^2} + B\right)\widetilde{\mathbf{E}} - \mu\omega^2\widetilde{\mathbf{D}}. \end{gathered}$$

Since the unforced part of each equation in the decoupled system (6.6) is same and it is of the form

$$\ddot{x}(t) + \frac{\dot{\mu}}{\mu}\dot{x}(t) + \Omega_x(t)x(t) = 0, \tag{6.9}$$

let $x_1^{(h)}(t)$ and $x_2^{(h)}(t)$ denote two linearly independent solutions of the homogeneous Eq.(6.9), satisfying the initial conditions, respectively

$$\begin{aligned} x_1^{(h)}(t_0) &= x_0 \neq 0, \quad \dot{x}_1^h(t_0) = x_0 B(t_0), \\ x_2^{(h)}(t_0) &= 0, \quad \dot{x}_2^{(h)}(t_0) = \frac{1}{x_0 \mu(t_0)}. \end{aligned} \tag{6.10}$$

Then, $\mathbf{x}^{(h)}(t) = (x_1^{(h)}(t), x_2^{(h)}(t))^T$ will become solution of the homogeneous part of system (6.6) with IC's (6.10). For system (6.6) in the presence of forcing terms, we let $\mathbf{x}^{(p)}(t) = (x_1^{(p)}(t), x_2^{(p)}(t))^T$ denote particular solution satisfying the initial conditions

$$\mathbf{x}^{(p)}(t_0) = \mathbf{0}, \quad \dot{\mathbf{x}}^{(p)}(t_0) = \widetilde{\mathbf{D}}(t_0).$$

Furthermore, if $p_1^{(h)}(t)$ and $p_2^{(h)}(t)$ are two homogeneous solutions of the system of oscillator equations in momentum space given by (6.8), then they can be found in terms of the solutions of the classical equation in position space as

$$\mathbf{p}^{(h)}(t) = \begin{pmatrix} p_1^{(h)}(t) \\ p_2^{(h)}(t) \end{pmatrix} = \mu(t) \left(\dot{\mathbf{x}}^{(h)}(t) - B(t)\mathbf{x}^{(h)}(t) - \widetilde{\mathbf{D}}(t) \right),$$

and particular solution will be

$$\mathbf{p}^{(p)}(t) = \begin{pmatrix} p_1^{(p)}(t) \\ p_2^{(p)}(t) \end{pmatrix} = \mu(t) \left(\dot{\mathbf{x}}^{(p)}(t) - B(t)\mathbf{x}^{(p)}(t) - \widetilde{\mathbf{D}}(t) \right).$$

As a result, it follows that

$$\mathbf{X}^{(h)}(t) \equiv \begin{pmatrix} X_1^{(h)}(t) \\ X_2^{(h)}(t) \end{pmatrix} = R_\theta^T(t)\mathbf{x}^{(h)}(t)$$

is a homogeneous solution to the coupled system (6.1) satisfying IC's

$$\mathbf{X}^{(h)}(t_0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \quad \dot{\mathbf{X}}^{(h)}(t_0) = \begin{pmatrix} x_0 B(t_0) \\ x_0\lambda(t_0) + \frac{1}{x_0\mu(t_0)} \end{pmatrix}, \tag{6.11}$$

and

$$\mathbf{X}^{(p)}(t) \equiv \begin{pmatrix} X_1^{(p)}(t) \\ X_2^{(p)}(t) \end{pmatrix} = R_\theta^T(t)\mathbf{x}^{(p)}(t)$$

is a particular solution to the forced coupled system (6.1), satisfying IC's

$$\mathbf{X}^{(p)}(t_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \dot{\mathbf{X}}^{(p)}(t_0) = \begin{pmatrix} D_1(t_0) \\ D_2(t_0) \end{pmatrix}. \tag{6.12}$$

This establishes solutions to the classical problem, whose quantization using the usual replacement $\mathbf{X} \to \hat{\mathbf{q}}$, $\mathbf{P} \to \hat{\mathbf{p}}$, $\mathbf{X}.\mathbf{P} \to (\hat{\mathbf{q}}.\hat{\mathbf{p}})/2$ is discussed in the next section.

6.2. Solution to the Generalized Quantum Parametric Oscillator

In this section, we consider time-dependent Schrödinger equation describing a generalized two-dimensional quantum parametric oscillator in the presence of time-variable external fields given by

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\Psi(\mathbf{q},t) = \hat{\mathcal{H}}_{gen}(t)\Psi(\mathbf{q},t), \quad \mathbf{q} \in \mathbf{R}^2, \ t > t_0, \\ \Psi(\mathbf{q},t_0) = \Psi^0(\mathbf{q}), \quad \mathbf{q} \in \mathbf{R}^2, \end{cases} \tag{6.13}$$

where $\Psi(\mathbf{q},t) := \Psi(q_1,q_2,t)$ is the wave function at time $t > t_0$, $\Psi^0(\mathbf{q}) := \Psi^0(q_1,q_2)$ is the initial state at time t_0, and the explicitly time-dependent Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ is

$$\hat{\mathcal{H}}_{gen}(t) = \frac{-\hbar^2}{2\mu(t)}\nabla^2 + \frac{\mu(t)\omega^2(t)}{2}\hat{\mathbf{q}}^2 - i\hbar B(t)(1 + \hat{\mathbf{q}}\cdot\nabla) - i\hbar\mathbf{D}(t)\cdot\nabla + \mathbf{E}(t)\cdot\hat{\mathbf{q}} + \lambda(t)\hat{L}. \tag{6.14}$$

Here, $\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2)^T$ is the position vector operator with $\hat{q}_j = q_j$, $j = 1, 2$, $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2)^T = -i\hbar\nabla$ is the momentum vector operator with $\nabla = (\partial/\partial q_1, \partial/\partial q_2)^T$, and $\hat{L} = \hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1$ is the angular momentum operator, which satisfies the following commutation relations

$$\left[\hat{L}, \sum_{j=1}^{2} \hat{p}_j^2\right] = 0, \quad \left[\hat{L}, \sum_{j=1}^{2} \hat{q}_j^2\right] = 0, \quad \left[\hat{L}, \sum_{j=1}^{2} (\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j)\right] = 0,$$

and

$$[\hat{L}, \hat{q}_1] = i\hat{q}_2, \quad [\hat{L}, \hat{q}_2] = -i\hat{q}_1, \quad [\hat{L}, \hat{p}_1] = i\hat{p}_2, \quad [\hat{L}, \hat{p}_2] = -i\hat{p}_1,$$

showing that $\hat{L}$ does not commute with the position and momentum operators.

Proposition 6.1 *The IVP for a two-dimensional generalized quantum parametric oscillator given by (6.13) has solution of the form*

$$\begin{aligned} \Psi(\mathbf{q}, t) &= \varphi\left(\eta(\mathbf{q}, t), \tau(t)\right) \sqrt{x_0/x_1^{(h)}(t)} \exp\left(\frac{i}{\hbar} \int_{t_0}^{t} \zeta_c(s) ds\right) \\ &\quad \exp\left[\frac{i}{\hbar}\left(\frac{\mu(t)}{2}\left(\frac{\dot{x}_1^{(h)}(t)}{x_1^{(h)}(t)} - B(t)\right)\left|\mathbf{q} - \mathbf{X}^{(p)}(t)\right|^2 + \hbar\mathbf{P}^{(p)}(t).\mathbf{q}\right)\right] \end{aligned} \tag{6.15}$$

where

$$\eta(\mathbf{q}, t) = \left|\frac{x_0}{x_1^{(h)}(t)}\right| R_\theta(t)(\mathbf{q} - \mathbf{X}^{(p)}(t)), \quad \tau(t) = \hbar x_0^2 \left(\frac{x_2^{(h)}(t)}{x_1^{(h)}(t)}\right),$$

$$\zeta_c(t) = \frac{-\left|\mathbf{P}^{(p)}(t)\right|^2}{2\mu(t)} - \mathbf{D}(t).\mathbf{P}^{(p)}(t) + \frac{\mu(t)\omega^2(t)}{2}\left|\mathbf{X}^{(p)}(t)\right|^2, \tag{6.16}$$

and $\varphi(\mathbf{q}, t)$ is solution of two-dimensional free Schrödinger equation

$$\begin{cases} i\dfrac{\partial}{\partial t}\varphi(\mathbf{q}, t) = -\dfrac{1}{2}\nabla^2\varphi(\mathbf{q}, t), & \mathbf{q} \in \mathbb{R}^2,\ t > 0, \\ \varphi(\mathbf{q}, 0) = \Psi^0(\mathbf{q}), & \mathbf{q} \in \mathbb{R}^2. \end{cases} \tag{6.17}$$

Proof Clearly, in the presence of angular momentum operator, the Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ is coupled, but one can overcome this difficulty by introducing a unitary transformation

$$\hat{U}_\theta(t, t_0) = \exp\left(\frac{i}{\hbar}\theta(t)\hat{L}\right),$$

where $\theta(t)$ is given by (6.5). Indeed, if we introduce new wave function as

$$\psi(\mathbf{q}, t) = \hat{U}_\theta(t, t_0)\Psi(\mathbf{q}, t),$$

then IVP (6.13) transforms to the IVP

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\psi(\mathbf{q}, t) = \hat{H}_{dec}(t)\psi(\mathbf{q}, t), & \mathbf{q} \in \mathbb{R}^2,\ t > t_0, \\ \psi(\mathbf{q}, t_0) = \Psi^0(\mathbf{q}), & \mathbf{q} \in \mathbb{R}^2 \end{cases} \tag{6.18}$$

with decoupled Hamiltonian

$$\hat{H}_{dec}(t) = -\frac{\hbar^2}{2\mu(t)}\nabla^2 + \frac{\mu(t)\omega^2(t)}{2}\hat{\mathbf{q}}^2 - i\hbar B(t)(1 + \hat{\mathbf{q}}\cdot\nabla) - i\hbar\widetilde{\mathbf{D}}(t)\cdot\nabla + \widetilde{\mathbf{E}}(t)\cdot\hat{\mathbf{q}}, \tag{6.19}$$

where parameters $\widetilde{\mathbf{D}}(t)$, $\widetilde{\mathbf{E}}(t)$ are defined in terms of $\mathbf{D}(t)$, $\mathbf{E}(t)$ by the relations in (6.7). Therefore, the original IVP (6.13) is reduced to solving the IVP (6.18).

The dynamics of the quantum system described by Schrödinger equation (6.18) is contained in the evolution operator defined as

$$i\hbar\frac{d}{dt}\hat{U}_{dec}(t, t_0) = \hat{H}_{dec}(t)\hat{U}_{dec}(t, t_0), \quad \hat{U}_{dec}(t_0, t_0) = \hat{I}. \tag{6.20}$$

Exact form of $\hat{U}_{dec}(t,t_0)$ can be found by using Wei-Norman Lie algebraic process. Indeed, the Hamiltonian $\hat{H}_{dec}(t)$ given by (6.19) for the decoupled oscillator can be written as time-dependent linear combination of Lie algebra generators as

$$\hat{H}_{dec}(t) = -i\left(\frac{\hbar^2}{\mu(t)}\hat{\mathcal{K}}^{(-)} + \mu(t)\omega^2(t)\hat{\mathcal{K}}^{(+)} + 2\hbar B(t)\hat{\mathcal{K}}^{(0)} + \hbar\widetilde{\mathbf{D}}(t)\cdot\hat{\mathcal{E}}^{(2)} + \widetilde{\mathbf{E}}(t)\cdot\hat{\mathcal{E}}^{(1)}\right),$$

where we denote the vector operators

$$\begin{aligned}
\hat{\mathcal{E}}^{(1)} &= \left(\hat{\mathcal{E}}_1^{(1)},\hat{\mathcal{E}}_2^{(1)}\right)^T = i\,(q_1,q_2)^T = i\mathbf{q},\\
\hat{\mathcal{E}}^{(2)} &= \left(\hat{\mathcal{E}}_1^{(2)},\hat{\mathcal{E}}_2^{(2)}\right)^T = (\partial/\partial q_1,\partial/\partial q_2)^T = \nabla,\\
\hat{\mathcal{E}}^{(3)} &= \left(\hat{\mathcal{E}}_1^{(3)},\hat{\mathcal{E}}_2^{(3)}\right)^T = (i,i)^T = i\hat{\mathbf{I}},
\end{aligned}$$

with $\hat{\mathcal{E}}_j^{(1)}$, $\hat{\mathcal{E}}_j^{(2)}$, $\hat{\mathcal{E}}_j^{(3)}$ being generators of Heisenberg-Weyl algebra for $j = 1,2$, and the operators

$$\begin{aligned}
\hat{\mathcal{K}}^{(-)} &= \hat{\mathcal{K}}_1^{(-)} + \hat{\mathcal{K}}_2^{(-)} = -\frac{i}{2}\left(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2}\right) = -\frac{i}{2}\nabla^2,\\
\hat{\mathcal{K}}^{(+)} &= \hat{\mathcal{K}}_1^{(+)} + \hat{\mathcal{K}}_2^{(+)} = \frac{i}{2}\left(q_1^2 + q_2^2\right) = \frac{i}{2}\mathbf{q}^2,\\
\hat{\mathcal{K}}^{(0)} &= \hat{\mathcal{K}}_1^{(0)} + \hat{\mathcal{K}}_2^{(0)} = \frac{1}{2}\left(1 + q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}\right) = \frac{1}{2}\left(\frac{1}{2} + \mathbf{q}.\nabla\right)
\end{aligned}$$

with $\hat{\mathcal{K}}_j^{(-)}$, $\hat{\mathcal{K}}_j^{(+)}$, $\hat{\mathcal{K}}_j^{(0)}$ being generators of the SU(1,1) algebra. Then, the evolution can be expressed as product of exponential operators

$$\begin{aligned}
\hat{U}_{dec}(t,t_0) \;&=\; \exp\left(\mathbf{c}(t)\cdot\hat{\mathcal{E}}^{(3)}\right)\times\exp\left(\frac{\mathbf{a}(t)}{\hbar}\cdot\hat{\mathcal{E}}^{(1)}\right)\times\exp\left(-\mathbf{b}(t)\cdot\hat{\mathcal{E}}^{(2)}\right)\\
&\quad\times\exp\left(f(t)\hat{\mathcal{K}}^{(+)}\right)\times\exp\left(2h(t)\hat{\mathcal{K}}^{(0)}\right)\times\exp\left(g(t)\hat{\mathcal{K}}^{(-)}\right), \qquad (6.21)
\end{aligned}$$

with $\mathbf{a}(t),\mathbf{b}(t),\mathbf{c}(t)$ being vectors of real-valued functions and $f(t),g(t),h(t)$ being real-valued functions to be determined. Substituting (6.21) and (6.19) into (6.20) and performing necessary calculations, we find that $\hat{U}_{dec}(t,t_0)$ is a solution of (6.20) if the unknown functions $f(t), g(t), h(t)$ satisfy the nonlinear system

$$\begin{aligned} \dot{f} + \tfrac{\hbar}{\mu(t)} f^2 + 2B(t) f + \tfrac{\mu(t)\omega^2(t)}{\hbar} &= 0, \quad f(t_0) = 0, \\ \dot{g} + \tfrac{\hbar}{\mu(t)} e^{2h} &= 0, \quad g(t_0) = 0, \\ \dot{h} + \tfrac{\hbar}{\mu(t)} f + B(t) &= 0, \quad h(t_0) = 0, \end{aligned} \tag{6.22}$$

and $\mathbf{a}(t), \mathbf{b}(t), \mathbf{c}(t)$ satisfy the nonlinear system

$$\begin{aligned} \dot{\mathbf{a}} + B(t)\mathbf{a} + \mu(t)\omega^2(t)\mathbf{b} + \widetilde{\mathbf{E}}(t) &= 0, \quad \mathbf{a}(t_0) = \mathbf{0}, \\ \dot{\mathbf{b}} - B(t)\mathbf{b} - \tfrac{1}{\mu(t)}\mathbf{a} - \widetilde{\mathbf{D}}(t) &= 0, \quad \mathbf{b}(t_0) = \mathbf{0}, \\ \dot{\mathbf{c}} + \tfrac{1}{2\hbar\mu(t)}\mathbf{a}^{\circ 2} + \tfrac{1}{\hbar}\left(\widetilde{\mathbf{D}}(t) \circ \mathbf{a}\right) - \tfrac{\mu(t)\omega^2(t)}{2\hbar}\mathbf{b}^{\circ 2} &= 0, \quad \mathbf{c}(t_0) = \mathbf{0}, \end{aligned} \tag{6.23}$$

where we use Hadamard product notation $\mathbf{u} \circ \mathbf{v} = (u_1 v_1, u_2 v_2)^T$ for any two vectors $\mathbf{u} = (u_1, u_2)^T$, $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{u}^{\circ 2} = (u_1^2, u_2^2)^T$. Then, the solution of system (6.22) is found in terms of two linearly independent solutions $x_1^{(h)}(t)$ and $x_2^{(h)}(t)$ of the decoupled classical system (6.6) as

$$\begin{aligned} f(t) &= \frac{\mu(t)}{\hbar}\left(\frac{\dot{x}_1^{(h)}(t)}{x_1^{(h)}(t)} - B(t)\right), \\ g(t) &= -\hbar x_0^2 \left(\frac{x_2^{(h)}(t)}{x_1^{(h)}(t)}\right), \\ h(t) &= -\ln\left|\frac{x_1^{(h)}(t)}{x_0}\right|. \end{aligned}$$

On the other hand, for each $j = 1, 2$ the solution of system (6.23) is obtained in terms of particular solutions of systems (6.6) and (6.8) as

$$\begin{aligned} \mathbf{a}(t) &= \mathbf{p}^{(p)}(t), \\ \mathbf{b}(t) &= \mathbf{x}^{(p)}(t), \\ \mathbf{c}(t) &= \frac{1}{\hbar}\int_{t_0}^{t}\left(\frac{-(\mathbf{p}^{(p)}(s))^{\circ 2}}{2\mu(s)} - \left(\widetilde{\mathbf{D}}(s) \circ \mathbf{p}^{(p)}(s)\right) + \frac{\mu(s)\omega^2(s)}{2}(\mathbf{x}^{(p)}(s))^{\circ 2}\right) ds. \end{aligned}$$

Therefore, we find $\hat{U}_{dec}(t,t_0)$ explicitly as

$$\begin{aligned}\hat{U}_{dec}(t,t_0) &= \exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\zeta_{dec}(s)ds\right)\times\exp\left(i\mathbf{p}^{(p)}(t)\cdot\mathbf{q}\right)\times\exp\left(-\mathbf{x}^{(p)}(t)\cdot\nabla\right)\\ &\quad\times\exp\left(i\frac{\mu(t)}{2\hbar}\left(\frac{\dot{x}_1^{(h)}(t)}{x_1^{(h)}(t)}-B(t)\right)|\mathbf{q}|^2\right)\times\exp\left(\ln\left|\frac{x_0}{x_1^{(h)}(t)}\right|(1+\mathbf{q}\cdot\nabla)\right)\\ &\quad\times\exp\left(\frac{i}{2}\hbar x_0^2\left(\frac{x_2^{(h)}(t)}{x_1^{(h)}(t)}\right)\nabla^2\right),\end{aligned}$$

where

$$\zeta_{dec}(t)=\frac{-\left|\mathbf{p}^{(p)}(t)\right|^2}{2\mu(t)}-\tilde{\mathbf{D}}(t)\cdot\mathbf{p}^{(p)}(t)+\frac{\mu(t)\omega^2(t)}{2}\left|\mathbf{x}^{(p)}(t)\right|^2. \tag{6.24}$$

According to the decoupling procedure discussed before, the evolution operator for IVP (6.13) will be of the form

$$\hat{\mathcal{U}}_{gen}(t,t_0)=\hat{U}_\theta^\dagger(t,t_0)\hat{U}_{dec}(t,t_0), \tag{6.25}$$

satisfying the operator equation

$$i\hbar\frac{d}{dt}\hat{\mathcal{U}}_{gen}(t,t_0)=\hat{\mathcal{H}}_{gen}(t)\hat{\mathcal{U}}_{gen}(t,t_0),\quad \hat{\mathcal{U}}_{gen}(t_0,t_0)=\hat{I}.$$

We note the action of the angular momentum operator on given initial function as

$$\exp\left(-\frac{i}{\hbar}\theta(t)\hat{L}\right)\phi_0(\mathbf{q})=\phi_0(R_\theta(t)\mathbf{q}). \tag{6.26}$$

The action of shifting and dilatation operators, respectively

$$\exp\left(\mathbf{u}(t)\cdot\nabla\right)\phi_0(\mathbf{q})=\phi_0(\mathbf{q}+\mathbf{u}(t)),\quad \exp\left(\xi\mathbf{q}\cdot\nabla\right)\phi_0(\mathbf{q})=\phi_0(e^{\xi}\mathbf{q}), \tag{6.27}$$

for any arbitrary vector of function $\mathbf{u}(t)$ and ξ constant. And, we have also

$$\exp\left(\frac{i\xi}{2}\nabla^2\right)\phi_0(\mathbf{q}) = \phi(\mathbf{q};\xi), \tag{6.28}$$

where the function $\phi(\mathbf{q};z)$ satisfies the free Schrödinger equation

$$\begin{cases} \frac{-1}{2}\nabla^2\phi(\mathbf{q};z) = i\frac{\partial}{\partial z}\phi(\mathbf{q};z) \\ \phi(\mathbf{q};z)|_{z=0} = \phi_0(\mathbf{q}). \end{cases}$$

Then, the solution of IVP (6.13) is determined as $\Psi(\mathbf{q},t) = \hat{\mathcal{U}}_{gen}(t,t_0)\Psi^0(\mathbf{q})$. □

6.2.1. The Propagator

Solution of the IVP (6.13) can be written also in the form

$$\Psi(\mathbf{q},t) = \int_{\mathbb{R}^2}\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0)\Psi^0(\mathbf{q}')d\mathbf{q}',$$

where $\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0)$ denotes the propagator of the system. The propagator is the kernel of the integral transform that converts a given initial function to a wave function solution at later times. Using the evolution operator and relation

$$\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0) = \hat{\mathcal{U}}_{gen}(t,t_0)\delta(\mathbf{q}-\mathbf{q}'), \quad \hat{\mathcal{U}}_{gen}(t_0,t_0) = \hat{I},$$

where $\delta(\mathbf{q})$ denotes the Dirac-delta distribution, one can determine the propagator explicitly. For this, first we find the propagator for the two-dimensional decoupled oscillator by using the evolution operator for the decoupled oscillator as $\mathcal{K}_{dec}(\mathbf{q},t;\mathbf{q}',t_0) =$

$\hat{U}_{dec}(t,t_0)\delta(\mathbf{q}-\mathbf{q}')$, and obtain

$$\begin{aligned}\mathcal{K}_{dec}(\mathbf{q},t;\mathbf{q}',t_0) &= \frac{-i\omega_0}{2\pi\hbar}\frac{1}{|\epsilon(t)|\sin\eta(t)}\exp\left(\frac{-i}{2\hbar}\int_{t_0}^{t}\zeta_{dec}(s)ds\right)\\ &\quad \exp\left(\frac{-i}{2\hbar}\left(\mu(t)\Big(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\Big)-\omega_0\frac{\cot\eta(t)}{|\epsilon(t)|^2}\right)\left|\mathbf{q}-\mathbf{x}^{(p)}(t)\right|^2\right)\\ &\quad \exp\left(\frac{i}{2\hbar}\left(2\mathbf{p}^{(p)}(t)\cdot\mathbf{q}+\omega_0\cot\eta(t)|\mathbf{q}'|^2\right)\right)\\ &\quad \exp\left(\frac{-i}{\hbar\sin\eta(t)|\epsilon(t)|}(\mathbf{q}-\mathbf{x}^{(p)}(t))\cdot\mathbf{q}'\right),\end{aligned}$$

where $\zeta_{dec}(t)$ is defined by (6.24) and

$$\epsilon(t)=\frac{x_1^{(h)}(t)}{x_0}+i(\omega_0x_0)x_2^{(h)}(t)=|\epsilon(t)|e^{i\eta(t)}, \tag{6.29}$$

with modulus and phase

$$|\epsilon(t)|=\sqrt{\frac{(x_1^{(h)}(t))^2}{x_0^2}+(\omega_0x_0)^2(x_2^{(h)}(t))^2},\qquad \eta(t)=\int_{t_0}^{t}\frac{\omega_0}{\mu(s)|\epsilon(s)|^2}ds. \tag{6.30}$$

Then,

$$\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0)=\mathcal{K}_{dec}(R_\theta(t)\mathbf{q},t;\mathbf{q}',t_0),$$

where $R_\theta(t)$ is the rotation matrix given by (6.4), and explicitly in terms of the solutions to the coupled systems (6.1) and (6.2), we get

$$\begin{aligned}\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t_0) &= \frac{-i\omega_0}{2\pi\hbar}\frac{1}{|\epsilon(t)|\sin\eta(t)}\exp\left(\frac{-i}{2\hbar}\int_{t_0}^{t}\zeta_{c}(s)ds\right)\\ &\quad \exp\left(\frac{-i}{2\hbar}\left(\mu(t)\Big(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\Big)-\omega_0\frac{\cot\eta(t)}{|\epsilon(t)|^2}\right)\left|\mathbf{q}-\mathbf{X}^{(p)}(t)\right|^2\right)\\ &\quad \exp\left(\frac{i}{2\hbar}\left(2\mathbf{P}^{(p)}(t)\cdot\mathbf{q}+\omega_0\cot\eta(t)|\mathbf{q}'|^2\right)\right)\\ &\quad \exp\left(\frac{-i}{\hbar\sin\eta(t)|\epsilon(t)|}\Big(R_\theta(t)(\mathbf{q}-\mathbf{X}^{(p)}(t))\Big)\cdot\mathbf{q}'\right),\end{aligned}$$

where $\zeta_c(t)$ is given by (6.16), and $\epsilon(t)$ given by (6.29) can be written also in terms of the homogeneous solution to the coupled classical system (6.1) as

$$\epsilon(t) = \frac{1}{x_0}\Big(\cos\theta(t)X_1^{(h)}(t) + \sin\theta(t)X_2^{(h)}(t)\Big) + i(\omega_0 x_0)\Big(-\sin\theta(t)X_1^{(h)}(t) + \cos\theta(t)X_2^{(h)}(t)\Big). \tag{6.31}$$

In general, the evolution of a state from an arbitrary time t' to t, is defined as

$$\Psi(\mathbf{q},t) = \hat{\mathcal{U}}_{gen}(t,t')\Psi(\mathbf{q},t') = \int_{\mathbb{R}^2} \mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t')\Psi(\mathbf{q}',t')d\mathbf{q}', \quad t_0 \le t' < t,$$

and it implies that

$$\begin{aligned}
\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t') &= \hat{U}_{gen}(t,t')\delta(\mathbf{q}-\mathbf{q}') \\
&= \hat{U}_{\theta}^{\dagger}(t,t')\mathcal{K}_{dec}(\mathbf{q},t;\mathbf{q}',t') \\
&= \hat{U}_{\theta}(t',t_0)\mathcal{K}_{dec}(R_{\theta}(t)\mathbf{q},t;\mathbf{q}',t').
\end{aligned}$$

After some calculations, we obtain the following result

$$\begin{aligned}
&\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t') = \frac{-i\omega_0}{2\pi\hbar|\epsilon(t)||\epsilon(t')|\sin\left(\eta(t)-\eta(t')\right)} \exp\left\{\frac{-i}{2\hbar}\int_{t_0}^{t}\zeta_c(s)ds\right\} \\
&\exp\left\{\frac{-i}{2\hbar}\left(\mu(t)\Big(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\Big) - \frac{\omega_0\cot\left(\eta(t)-\eta(t')\right)}{|\epsilon(t)|^2}\right)\left|R_{\theta}^{T}(t')\mathbf{q}-\mathbf{X}^{(p)}(t)\right|^2\right\} \\
&\exp\left\{\frac{i}{2\hbar}\left(\mu(t')\Big(B(t')-\frac{d}{dt}\ln|\epsilon(t')|\Big) - \frac{\omega_0\cot\left(\eta(t)-\eta(t')\right)}{|\epsilon(t')|^2}\right)\left|R_{\theta}^{T}(t')\mathbf{q}'-\mathbf{X}^{(p)}(t')\right|^2\right\} \\
&\exp\left\{\frac{i}{\hbar}\left[\mathbf{p}^{(P)}(t)\cdot(R_{\theta}^{T}(t')\mathbf{q}) - \mathbf{P}^{(p)}(t')\cdot(R_{\theta}^{T}(t')\mathbf{q}')\right]\right\} \\
&\exp\left\{\frac{-i}{\hbar}\frac{(R_{\theta}(t)(R_{\theta}^{T}(t')\mathbf{q}-\mathbf{X}^{(p)}(t)))\cdot(\mathbf{q}'-R_{\theta}^{T}(t')\mathbf{x}^{(p)}(t'))}{|\epsilon(t)||\epsilon(t')|\sin\left(\eta(t)-\eta(t')\right)}\right\}.
\end{aligned}$$

Usually, the propagator is interpreted as the probability amplitude of finding the particle at point q and time t, given that at the past it was at point q' and time t'. By construction, the propagator $\mathcal{K}_{gen}(\mathbf{q},t;\mathbf{q}',t')$ can be seen as a solution of the time-dependent Schrödinger equation in the variables q, t, with q', t' treated as parameters. It is the solution corresponding to Dirac-delta initial condition $\delta(\mathbf{q}-\mathbf{q}')$, which is highly singular, and due to this, the propagator as a "wave function" is not normalizable. In any case, the propagator like the evolution operator, contains all necessary knowledge for describing the dynamics of the quantum system.

6.3. Time-Evolution of Quantum States

In this section, for the generalized two-dimensional quantum parametric oscillator, we find time-evolution of eigenstates and coherent states explicitly.

6.3.1. Time-Evolution of Harmonic Oscillator Eigenstates

First, we solve IVP (6.13) by taking the initial function to be an eigenstate $\varphi_n(\mathbf{q})$ of the two-dimensional simple harmonic oscillator, whose Hamiltonian is $\hat{H}_0 = \sum_{j=1}^{2}(\hat{p}_j^2 + \omega_0^2\hat{q}_j^2)/2$. As known, these eigenstates correspond to eigenvalues $E_n = E_{n_1} + E_{n_2} = \hbar\omega_0(n_1 + n_2 + 1)$, and for $n = (n_1, n_2)$ we have

$$\varphi_n(\mathbf{q}) = \varphi_{n_1}(q_1)\varphi_{n_2}(q_2), \quad n_1, n_2 = 0, 1, 2, ...,$$

with

$$\varphi_{n_j}(q_j) = N_{n_j} e^{-\frac{\omega_0}{2\hbar}q_j^2} H_{n_j}\left(\sqrt{\frac{\omega_0}{\hbar}}q_j\right), \quad j = 1, 2,$$

where $H_{n_j}(\sqrt{\omega_0/\hbar}q_j)$ are Hermite polynomials and $N_{n_j} = (\omega_0/\pi\hbar)^{1/4}(2^{n_j}n_j!)^{-1/2}$ are the normalization constants. According to this, time-evolved eigenstates of the two dimensional oscillator (6.18) with Hamiltonian $\hat{H}_{dec}(t)$ are of the form

$$\Psi_n^0(\mathbf{q}, t) = \hat{U}_{dec}(t, t_0)\varphi_n(\mathbf{q}) = \prod_{j=1}^{2} \hat{U}_j(t, t_0)\varphi_{n_j}(q_j),$$

and using the equations (6.27) and (6.28), we obtain explicitly the wave functions

$$\begin{aligned}\Psi_n^0(\mathbf{q},t) &= \mathbf{N}_n\frac{1}{|\epsilon(t)|}\exp\Big(-\frac{iE_n}{\hbar\omega_0}\eta(t)\Big)\times\exp\Big(\frac{-i}{\hbar}\int_{t_0}^{t}\zeta_{dec}(s)ds\Big)\\ &\times\exp\Big\{\frac{i}{\hbar}\Big[\frac{-\mu(t)}{2}\Big(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\Big)\big|\mathbf{q}-\mathbf{x}^{(p)}(t)\big|^2+\mathbf{p}^{(p)}(t)\cdot\mathbf{q}\Big]\Big\}\\ &\times\exp\Big[-\frac{\omega_0}{2\hbar}\frac{\big|\mathbf{q}-\mathbf{x}^{(p)}(t)\big|^2}{|\epsilon(t)|^2}\Big]\mathbf{H}_n\Big(\sqrt{\frac{\omega_0}{\hbar}}\frac{\mathbf{q}-\mathbf{x}^{(p)}(t)}{|\epsilon(t)|},t\Big),\end{aligned}$$

and the corresponding probability densities

$$\rho_n^0(\mathbf{q},t)=\mathbf{N}_n^2\frac{1}{|\epsilon(t)|^2}\exp\Big[-\frac{\omega_0}{\hbar}\frac{\big|\mathbf{q}-\mathbf{x}^{(p)}(t)\big|^2}{|\epsilon(t)|^2}\Big]\times\mathbf{H}_n^2\Big(\sqrt{\frac{\omega_0}{\hbar}}\frac{\mathbf{q}-\mathbf{x}^{(p)}(t)}{|\epsilon(t)|},t\Big),$$

where $|\epsilon(t)|$ is as defined in (6.30), and we used the compact notations $\mathbf{N}_n = N_{n_1}N_{n_2}$ and

$$\mathbf{H}_n\Big(\sqrt{\frac{\omega_0}{\hbar}}\frac{\mathbf{q}}{|\epsilon(t)|},t\Big)\equiv\prod_{j=1}^{2}H_{n_j}\Big(\sqrt{\frac{\omega_0}{\hbar}}\Big(\frac{q_j}{|\epsilon(t)|}\Big)\Big).$$

Now, formally time-evolved solutions of the IVP (6.13) will be as expected

$$\Psi_n^\theta(\mathbf{q},t)=\hat{U}_\theta^\dagger(t,t_0)\Psi_n^0(\mathbf{q},t)=\Psi_n^0(R_\theta(t)\mathbf{q},t).$$

Then, in terms of solutions to the coupled systems of classical equations (6.1) and (6.2) we have

$$\begin{aligned}\Psi_n^\theta(\mathbf{q},t) &= \mathbf{N}_n\frac{1}{|\epsilon(t)|}\exp\Big(-\frac{iE_n}{\hbar\omega_0}\eta(t)\Big)\times\exp\Big\{\frac{-i}{\hbar}\int_{t_0}^{t}\zeta_c(s)ds\Big\}\\ &\times\exp\Big\{\frac{i}{\hbar}\Big[\frac{-\mu(t)}{2}\Big(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\Big)\big|\mathbf{q}-\mathbf{X}^{(p)}(t)\big|^2+\mathbf{P}^{(p)}(t)\cdot\mathbf{q}\Big]\Big\}\\ &\times\exp\Big\{-\frac{\omega_0}{2\hbar}\frac{\big|\mathbf{q}-\mathbf{X}^{(p)}(t)\big|^2}{|\epsilon(t)|^2}\Big\}\times\mathbf{H}_n\Big(\sqrt{\frac{\omega_0}{\hbar}}\frac{R_\theta(t)(\mathbf{q}-\mathbf{X}^{(p)}(t))}{|\epsilon(t)|},t\Big),\end{aligned} \quad (6.32)$$

and probability densities become

$$\rho_n^\theta(\mathbf{q},t) = \mathbf{N}_n^2 \frac{1}{|\epsilon(t)|^2} \exp\left\{-\frac{\omega_0}{\hbar}\frac{\left|\mathbf{q}-\mathbf{X}^{(p)}(t)\right|^2}{|\epsilon(t)|^2}\right\}\mathbf{H}_n^2\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{R_\theta(t)(\mathbf{q}-\mathbf{X}^{(p)}(t))}{|\epsilon(t)|}, t\right). \quad (6.33)$$

Here, the expectation values of position and momentum at states $\Psi_n^\theta(\mathbf{q},t)$ are

$$\langle\hat{\mathbf{q}}\rangle_n^\theta(t) = \begin{pmatrix}\langle\hat{q}_1\rangle_n^\theta(t)\\ \langle\hat{q}_2\rangle_n^\theta(t)\end{pmatrix} = \mathbf{X}^{(p)}(t), \quad \langle\hat{\mathbf{p}}\rangle_n^\theta(t) = \begin{pmatrix}\langle\hat{p}_1\rangle_n^\theta(t)\\ \langle\hat{p}_2\rangle_n^\theta(t)\end{pmatrix} = \mathbf{P}^{(p)}(t), \quad (6.34)$$

showing that they don't depend on the wave number $n = (n_1, n_2)$ and are completely determined by the external forces. Then, the uncertainties in position and momentum are found as

$$(\Delta\hat{\mathbf{q}})_n^\theta(t) = \begin{pmatrix}(\Delta\hat{q}_1)_n^\theta(t)\\ (\Delta\hat{q}_2)_n^\theta(t)\end{pmatrix} = \sqrt{\frac{\hbar}{\omega_0}}|\epsilon(t)|\,\mathbf{\Lambda}(n_1, n_2, \theta(t)), \quad (6.35)$$

$$(\Delta\hat{\mathbf{p}})_n^\theta(t) = \begin{pmatrix}(\Delta\hat{p}_1)_n^\theta(t)\\ (\Delta\hat{p}_2)_n^\theta(t)\end{pmatrix} = \sqrt{\hbar\omega_0}\frac{\Sigma(t)}{|\epsilon(t)|}\,\mathbf{\Lambda}(n_1, n_2, \theta(t)), \quad (6.36)$$

where

$$\mathbf{\Lambda}(n_1, n_2, \theta(t)) = \begin{pmatrix}\left(\cos^2\theta(t)n_1 + \sin^2\theta(t)n_2 + 1/2\right)^{1/2}\\ \left(\sin^2\theta(t)n_1 + \cos^2\theta(t)n_2 + 1/2\right)^{1/2}\end{pmatrix},$$

$$\Sigma(t) = \sqrt{1 + \frac{|\epsilon(t)|^4}{\omega_0^2}\left[\frac{d\ln|\epsilon(t)|}{dt} - B(t)\right]^2},$$

and the uncertainty product becomes

$$(\Delta\hat{\mathbf{q}})(\Delta\hat{\mathbf{p}})_n^\theta(t) = \begin{pmatrix} (\Delta\hat{q}_1)(\Delta\hat{p}_1)_n^\theta(t) \\ (\Delta\hat{q}_2)(\Delta\hat{p}_2)_n^\theta(t) \end{pmatrix} = \hbar\,\Sigma(t) \begin{pmatrix} \cos^2\theta(t)n_1 + \sin^2\theta(t)n_2 + 1/2 \\ \sin^2\theta(t)n_1 + \cos^2\theta(t)n_2 + 1/2 \end{pmatrix}.$$

Clearly, uncertainties for some subcases can be easily recovered from above results. For example, in case $\theta(t) = 0$ one gets the uncertainties for the two-dimensional decoupled parametric oscillator. In case $\theta(t) \neq 0$, and $\mu(t) = 1$, $\omega^2(t) = \omega_0^2$, $B(t) = 0$, one gets the uncertainties for the simple harmonic oscillator in electromagnetic field as

$$(\Delta\hat{\mathbf{q}})_n^\theta(t) = \sqrt{\frac{\hbar}{\omega_0}}\mathbf{\Lambda}(n_1, n_2, \theta(t)), \quad (\Delta\hat{\mathbf{p}})_n^\theta(t) = \sqrt{\hbar\omega_0}\mathbf{\Lambda}(n_1, n_2, \theta(t)),$$

and we note that when $n_1 = n_2$, then $\mathbf{\Lambda}(n_1, n_2, \theta(t))$ becomes independent of $\theta(t)$.

Finally, it is not difficult to show that expectation value of angular momentum operator $\hat{L}$ at wave function $\Psi_n^\theta(\mathbf{q}, t)$ is

$$\langle\hat{L}\rangle_n(t) = \langle\Psi_n^\theta(\mathbf{q}, t)|\hat{L}|\Psi_n^\theta(\mathbf{q}, t)\rangle = X_1^{(p)}(t)P_2^{(p)}(t) - X_2^{(p)}(t)P_1^{(p)}(t),$$

and the matrix elements are

$$\langle\Psi_n^\theta(\mathbf{q}, t)|\hat{L}|\Psi_m^\theta(\mathbf{q}, t)\rangle = \Big(X_1^{(p)}(t)P_2^{(p)}(t) - X_2^{(p)}(t)P_1^{(p)}(t)\Big)\delta_{nm},$$

where δ_{nm} is the Kronecker delta. In particular, when there are no external fields ($D_j(t) = E_j(t) = 0,\ j = 1, 2$), for the angular momentum operator one has expectation $\langle\hat{L}\rangle_n(t) = 0$ and uncertainty

$$(\Delta\hat{L})_n(t) = \sqrt{\hbar^2\Big((n_1+1)^2(n_2+1)^2 + n_1^2 n_2^2\Big)\Big(X_1^{(h)}(t)P_2^{(h)}(t) - X_2^{(h)}(t)P_1^{(h)}(t)\Big)^2},$$

which is determined by the homogenous solutions of the classical equations and depends on the wave number $n = (n_1, n_2)$.

6.3.2. Time-Evolution of Glauber Coherent States

By taking the initial function to be a coherent state of the simple two-dimensional harmonic oscillator with Hamiltonian $\hat{H}_0$, we solve the IVP (6.13), that is

$$\phi_\alpha(\mathbf{q}) = \phi_{\alpha_1}(q_1)\phi_{\alpha_2}(q_2),$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\alpha_j = \alpha_j^{(1)} + i\alpha_j^{(2)}$, with $\alpha_j^{(1)}$, $\alpha_j^{(2)}$ being real constants, and

$$\phi_{\alpha_j}(q_j) = \Big(\frac{\omega_0}{\pi\hbar}\Big)^{\frac{1}{4}} \exp\big[-i\alpha_j^{(1)}\alpha_j^{(2)}\big] \exp\Big[i\alpha_j^{(2)}\sqrt{\frac{2\omega_0}{\hbar}}q_j\Big] \exp\Big[-\frac{\omega_0}{2\hbar}\Big(q_j - \sqrt{\frac{2\hbar}{\omega_0}}\alpha_j^{(1)}\Big)^2\Big]$$

for $j = 1, 2$. Then, time-evolved coherent states $\Phi_\alpha^0(\mathbf{q}, t) = \hat{U}_{dec}(t, t_0)\phi_\alpha(\mathbf{q})$ of the decoupled oscillator are found as

$$\begin{aligned}
\Phi_\alpha^0(\mathbf{q}, t) \;=\;& \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{\epsilon(t)} \exp\Big\{-\frac{1}{2}\Big(\frac{(\epsilon^*(t))^2}{|\epsilon(t)|^2}\alpha^2 + |\alpha|^2\Big)\Big\} \times \exp\Big\{-\frac{i}{\hbar}\int_{t_0}^{t} \zeta_{dec}(s)ds\Big\} \\
& \times \exp\Big\{\frac{1}{2\hbar}\Big[-\Big(i\mu(t)\Big(B(t) - \frac{d}{dt}\ln|\epsilon(t)|\Big) + \frac{\omega_0}{|\epsilon(t)|^2}\Big)\Big|\mathbf{q} - \mathbf{x}^{(p)}(t)\Big|^2 + 2i\mathbf{p}^{(p)}(t)\cdot\mathbf{q}\Big]\Big\} \\
& \times \exp\Big\{\sqrt{\frac{2\omega_0}{\hbar}}\frac{1}{\epsilon(t)}(\mathbf{q} - \mathbf{x}^{(p)}(t))\cdot\alpha\Big\},
\end{aligned}$$

where $\alpha^2 = \alpha.\alpha$, $|\alpha|^2 = \alpha.\alpha^*$, and we have

$$\rho_\alpha^0(\mathbf{q}, t) = |\Phi_\alpha^0(\mathbf{q}, t)|^2 = \Big(\frac{\omega_0}{\pi\hbar}\Big)\frac{1}{|\epsilon(t)|^2} \exp\Big\{-\frac{\omega_0}{\hbar}\frac{\big|\mathbf{q} - \langle\hat{\mathbf{q}}\rangle_\alpha^0(t)\big|^2}{|\epsilon(t)|^2}\Big\}.$$

Here, expectation values at $\Phi^0_\alpha(\mathbf{q},t)$ are obtained as

$$\langle \hat{\mathbf{q}} \rangle^0_\alpha(t) \equiv \begin{pmatrix} \langle \hat{q}_1 \rangle^0_{\alpha_1}(t) \\ \langle \hat{q}_2 \rangle^0_{\alpha_2}(t) \end{pmatrix} = \sqrt{\frac{2\hbar}{\omega_0}} \mathbf{C}^0_\alpha \mathbf{x}^{(h)}(t) + \mathbf{x}^{(p)}(t), \tag{6.37}$$

$$\langle \hat{\mathbf{p}} \rangle^0_\alpha(t) \equiv \begin{pmatrix} \langle \hat{p}_1 \rangle^0_{\alpha_1}(t) \\ \langle \hat{p}_2 \rangle^0_{\alpha_2}(t) \end{pmatrix} = \sqrt{\frac{2\hbar}{\omega_0}} \mathbf{C}^0_\alpha \mathbf{p}^{(h)}(t) + \mathbf{p}^{(p)}(t), \tag{6.38}$$

where the coefficient matrix $\mathbf{C}^0_\alpha$ is defined as

$$\mathbf{C}^0_\alpha = \begin{pmatrix} \frac{\alpha^{(1)}_1}{x_0} & \omega_0 x_0 \alpha^{(2)}_1 \\ \frac{\alpha^{(1)}_2}{x_0} & \omega_0 x_0 \alpha^{(2)}_2 \end{pmatrix}. \tag{6.39}$$

The uncertainties at coherent states $\Phi^0_\alpha(\mathbf{q},t)$ are

$$(\Delta\hat{\mathbf{q}})^0_\alpha(t) = \begin{pmatrix} (\Delta\hat{q}_1)^0_\alpha(t) \\ (\Delta\hat{q}_2)^0_\alpha(t) \end{pmatrix}, \quad (\Delta\hat{\mathbf{p}})^\theta_\alpha(t) = \begin{pmatrix} (\Delta\hat{p}_1)^0_\alpha(t) \\ (\Delta\hat{p}_2)^0_\alpha(t) \end{pmatrix},$$

where

$$(\Delta\hat{q}_j)^0_\alpha(t) = \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon(t)|, \quad (\Delta\hat{p}_j)^0_\alpha(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon(t)|}\Sigma(t), \quad j = 1, 2,$$

and the uncertainty product becomes

$$(\Delta\hat{\mathbf{q}})(\Delta\hat{\mathbf{p}})^0_\alpha(t) = \begin{pmatrix} (\Delta\hat{q}_1)(\Delta\hat{p}_1)^0_\alpha(t) \\ (\Delta\hat{q}_2)(\Delta\hat{p}_2)^0_\alpha(t) \end{pmatrix}, \quad (\Delta\hat{q}_j\Delta\hat{p}_j)^0_\alpha(t) = \frac{\hbar}{2}\Sigma(t), \quad j = 1, 2.$$

Now, time-evolved coherent states of the generalized two-dimensional oscillator are

$$\Phi_\alpha^\theta(\mathbf{q},t) = \hat{\mathcal{U}}_{gen}(t,t_0)\phi_\alpha(\mathbf{q}) = \hat{U}_\theta^\dagger(t,t_0)\Phi_\alpha^0(\mathbf{q},t) = \Phi_\alpha^0(R_\theta(t)\mathbf{q},t),$$

and in terms of solutions to the classical systems (6.1) and (6.2), we get

$$\begin{aligned}
\Phi_\alpha^\theta(\mathbf{q},t) \quad = \quad & \sqrt{\frac{\omega_0}{\pi\hbar}}\frac{1}{\epsilon(t)}\exp\Big\{-\frac{1}{2}\Big(\frac{(\epsilon^*(t))^2}{|\epsilon(t)|^2}\alpha^2+|\alpha|^2\Big)\Big\}\times\exp\Big\{-\frac{i}{\hbar}\int_{t_0}^{t}\zeta_c(s)ds\Big\}\\
&\times\exp\Big\{\frac{1}{\hbar}\Big[-\Big(i\mu(t)\Big(B(t)-\frac{d}{dt}\ln|\epsilon(t)|\Big)+\frac{\omega_0}{|\epsilon(t)|^2}\Big)\Big|\mathbf{q}-\mathbf{X}^{(p)}(t)\Big|^2+2i\mathbf{P}^{(p)}(t)\cdot\mathbf{q}\Big]\Big\}\\
&\times\exp\Big\{\sqrt{\frac{2\omega_0}{\hbar}}\frac{1}{\epsilon(t)}R_\theta(t)(\mathbf{q}-\mathbf{X}^{(p)}(t))\cdot\alpha\Big\}. \qquad (6.40)
\end{aligned}$$

Then, the probability densities become

$$\rho_\alpha^\theta(\mathbf{q},t) = \Big(\frac{\omega_0}{\pi\hbar}\Big)\frac{1}{|\epsilon(t)|^2}\exp\Big\{-\frac{\omega_0}{\hbar}\frac{\big|\mathbf{q}-\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t)\big|^2}{|\epsilon(t)|^2}\Big\}, \qquad (6.41)$$

with squeezing coefficient $|\epsilon(t)|$ given by (6.31). We note that, since (6.31) is equal to (6.29), then $\epsilon(t)$ does not depend on $\theta(t)$, thus uncertainties at $\Phi_\alpha^0(\mathbf{q},t)$ and at $\Phi_\alpha^\theta(\mathbf{q},t)$ are same. On the other hand, expectation values at $\Phi_\alpha^\theta(\mathbf{q},t)$ depend on $\theta(t)$, and are determined as

$$\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t) = R_\theta^T(t)\langle\hat{\mathbf{q}}\rangle_\alpha^0(t), \quad \langle\hat{\mathbf{p}}\rangle_\alpha^\theta(t) = R_\theta^T(t)\langle\hat{\mathbf{p}}\rangle_\alpha^0(t),$$

where $\langle\hat{\mathbf{q}}\rangle_\alpha^0(t)$, $\langle\hat{\mathbf{p}}\rangle_\alpha^0(t)$ are given by (6.37) and (6.38), respectively. In terms of the classical solutions to systems (6.1) and (6.2), the expectation values are obtained as

$$\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t) = \sqrt{\frac{2\hbar}{\omega_0}}\mathbf{C}_\alpha^\theta(t)\mathbf{X}^{(h)}(t)+\mathbf{X}^{(p)}(t)$$

and

$$\langle \hat{\mathbf{p}} \rangle_\alpha^\theta(t) = \sqrt{\frac{2\hbar}{\omega_0}} \mathbf{C}_\alpha^\theta(t) \mathbf{P}^{(h)}(t) + \mathbf{P}^{(p)}(t),$$

where $\mathbf{C}_\alpha^\theta(t)$ is the similarity matrix

$$\mathbf{C}_\alpha^\theta(t) = R_\theta^T(t) \mathbf{C}_\alpha^0 R_\theta(t), \tag{6.42}$$

with $\mathbf{C}_\alpha^0 = \mathbf{C}_\alpha^\theta(t_0)$ being the matrix given by (6.39).

Thus, time-evolved coherent states of the generalized quantum oscillator in the given external fields, are two-dimensional squeezed Gaussian wave packets that follow the trajectory of the classical particles. In general, they do not preserve the minimum uncertainty and their squeezing properties are controlled by the squeezing coefficient $|\epsilon(t)|$, which depends on the choice of the parameters $\mu(t), \omega^2(t)$ and $B(t)$. On the other hand, the displacement properties of coherent states depend also on parameters $D_j(t), E_j(t),\ j = 1, 2$, and the rotation angle $\theta(t)$.

Lastly, we write the expectation values of angular momentum at coherent states (6.40), when there are no external fields, as

$$\begin{aligned} \langle \hat{L} \rangle_\alpha(t) &= 2\hbar(\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)}) \Big(X_1^{(h)}(t) P_2^{(h)}(t) - X_2^{(h)}(t) P_1^{(h)}(t) \Big) \\ &= \frac{2\hbar}{\omega_0} (\det C_\alpha^0) \Big(X_1^{(h)}(t) P_2^{(h)}(t) - X_2^{(h)}(t) P_1^{(h)}(t) \Big), \end{aligned}$$

where $\mathbf{C}_\alpha^0$ is given by (6.39). In that case uncertainties become

$$(\Delta \hat{L})_\alpha(t) = \sqrt{\hbar^2 (|\alpha_1|^2 + |\alpha_2|^2) \Big(X_1^{(h)}(t) P_2^{(h)}(t) - X_2^{(h)}(t) P_1^{(h)}(t) \Big)^2}.$$

Similarly, in the presence of external fields one can compute expectations and uncertainties of angular momentum by straightforward calculations.

6.4. Quantum Dynamical Invariants

In this section, time-dependent linear and quadratic invariants for the quantum system are constructed using the evolution operator formalism. As known, if time-development of a given quantum system is described by the unitary evolution operator $\hat{U}(t, t_0)$, then any operator of the form $\hat{A}(t) = \hat{U}(t, t_0)\hat{A}(t_0)\hat{U}^\dagger(t, t_0)$ is an integral of motion or a dynamical invariant. Using these dynamical invariants we establish relation between the present results and those obtained by the MMT- and the LR- approaches.

6.4.1. Linear Invariants

For the generalized two-dimensional oscillator with Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ given by (6.14), using the evolution operator (6.25), one can find dynamical invariants

$$\hat{\mathbf{A}}_\theta(t) = \begin{pmatrix} \hat{A}_{\theta,1}(t) \\ \hat{A}_{\theta,2}(t) \end{pmatrix}, \quad \hat{\mathbf{A}}^\dagger_\theta(t) = \begin{pmatrix} \hat{A}^\dagger_{\theta,1}(t) \\ \hat{A}^\dagger_{\theta,2}(t) \end{pmatrix} \tag{6.43}$$

defined as

$$\hat{\mathbf{A}}_\theta(t) = \hat{\mathcal{U}}_{gen}(t, t_0)\hat{\mathbf{a}}\,\hat{\mathcal{U}}^\dagger_{gen}(t, t_0),$$
$$\hat{\mathbf{A}}^\dagger_\theta(t) = \hat{\mathcal{U}}_{gen}(t, t_0)\hat{\mathbf{a}}^\dagger\,\hat{\mathcal{U}}^\dagger_{gen}(t, t_0),$$

where

$$\hat{\mathbf{a}} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \sqrt{\frac{\omega_0}{2\hbar}}\hat{\mathbf{q}} + \sqrt{\frac{\hbar}{2\omega_0}}\nabla, \tag{6.44}$$

$$\hat{\mathbf{a}}^\dagger = \begin{pmatrix} \hat{a}^\dagger_1 \\ \hat{a}^\dagger_2 \end{pmatrix} = \sqrt{\frac{\omega_0}{2\hbar}}\hat{\mathbf{q}} - \sqrt{\frac{\hbar}{2\omega_0}}\nabla \tag{6.45}$$

are respectively the non-Hermitian lowering and raising Dirac operators for the standard two-dimensional harmonic oscillator $\hat{H}_0 = -(\hbar^2/2)\nabla^2 + (\omega_0^2/2)\hat{\mathbf{q}}^2$. Explicit calculations give us dynamical invariants, that are linear in position and momentum,

$$\hat{\mathbf{A}}_\theta(t) = \frac{-i}{\sqrt{2\omega_0\hbar}}\left[\mu(t)\Big(\dot{\epsilon}(t) - B(t)\epsilon(t)\Big)\hat{\mathbf{Q}} - \epsilon(t)\hat{\mathbf{P}}\right], \tag{6.46}$$

and

$$\hat{\mathbf{A}}^\dagger_\theta(t) = \frac{i}{\sqrt{2\omega_0\hbar}}\left[\mu(t)\Big(\dot{\epsilon}^*(t) - B(t)\epsilon^*(t)\Big)\hat{\mathbf{Q}} - \epsilon^*(t)\hat{\mathbf{P}}\right], \tag{6.47}$$

where

$$\hat{\mathbf{Q}} = R_\theta(t)\left(\hat{\mathbf{q}} - \mathbf{X}^{(p)}(t)\right), \quad \hat{\mathbf{P}} = R_\theta(t)\left(\hat{\mathbf{p}} - \mathbf{P}^{(p)}(t)\right),$$

and $\epsilon(t)$ is defined by (6.31). Here, $\epsilon(t)$ is a complex solution of equation (6.9), that is

$$\ddot{\epsilon}(t) + \frac{\dot{\mu}}{\mu}\dot{\epsilon}(t) + \left(\omega^2(t) - \left(\dot{B}(t) + B^2(t) + \frac{\dot{\mu}}{\mu}B(t)\right)\right)\epsilon(t) = 0 \tag{6.48}$$

and it satisfies the IC's

$$\epsilon(t_0) = 1, \quad \dot{\epsilon}(t_0) = B(t_0) + \frac{i\omega_0}{\mu(t_0)}. \tag{6.49}$$

Therefore, using the Wronskian $W(t) = W(\epsilon(t), \epsilon^*(t)) = \epsilon(t)\dot{\epsilon}^*(t) - \epsilon^*(t)\dot{\epsilon}(t) = -2i\omega_0/\mu(t)$, one can show that these linear invariants (6.46) satisfy commutation relations

$$[\hat{A}_{\theta,i}(t), \hat{A}^\dagger_{\theta,j}(t)] = \delta_{ij}, \quad i, j = 1, 2,$$

and can be seen also as generalized lowering and rising operators.

Moreover, coherent states $\Phi^\theta_\alpha(q_1, q_2, t)$, $\alpha = (\alpha_1, \alpha_2)$ found in (6.40) by construction are eigenstates of $\hat{A}_{\theta,j}(t)$ corresponding to complex eigenvalues $\alpha_j,\ j = 1, 2$. Indeed, if $\phi_{\alpha_j}(q_j)$ are eigenstates of $\hat{a}_j$ so that $\hat{a}_j\phi_{\alpha_j}(q_j) = \alpha_j\phi_{\alpha_j}(q_j)$, then

$$\hat{\mathcal{U}}_{gen}(t,t_0)\hat{a}_j\hat{\mathcal{U}}^{\dagger}_{gen}(t,t_0)\hat{\mathcal{U}}_{gen}(t,t_0)\phi_{\alpha_1}(q_1)\phi_{\alpha_2}(q_2) = \alpha_j\hat{\mathcal{U}}_{gen}(t,t_0)\phi_{\alpha_1}(q_1)\phi_{\alpha_2}(q_2), \quad j = 1, 2,$$

from which it follows

$$\hat{A}_{\theta,j}(t)\Phi^{\theta}_{\alpha}(q_1,q_2,t) = \alpha_j\Phi^{\theta}_{\alpha}(q_1,q_2,t), \quad j = 1, 2.$$

Now we consider (Malkin, Man'ko & Trifonov, 1970), where Malkin, Man'ko and Trifonov study the problem of the N-dimensional nonstationary harmonic oscillator and the problem of a charged particle in an axially symmetric and a uniform time-dependent electromagnetic field. MMT-approach for solving problems described by a Schrödinger operator $\hat{S}(t) = i\hbar\partial_t - \hat{H}(t)$ is based on finding all independent linear in position and momentum invariants. In that context, an invariant is defined as an operator $\hat{A}(t)$ that commutes with $\hat{S}(t)$, that is $[\hat{A}(t), \hat{S}(t)] = 0$.

We note that the Hamiltonian in (Malkin, Man'ko & Trifonov, 1970) don't contain damping and external forces so that it is a particular case of Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ given by (6.14). Then, if in (Malkin, Man'ko & Trifonov, 1970) one takes $\epsilon(t)$ to satisfy (6.48) for $\mu(t) = 1$ and $B(t) = 0$ with the specific IC's (6.49), it will coincide with $\epsilon(t)$ defined in the present work. Also, one can write

$$\begin{pmatrix}\hat{A}(t)\\ \hat{B}(t)\end{pmatrix} = \frac{1}{2\sqrt{e}}\begin{pmatrix}-i & 1\\ 1 & -i\end{pmatrix}\begin{pmatrix}\hat{A}_{\theta,1}(t)\\ \hat{A}_{\theta,2}(t)\end{pmatrix}, \quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}-i & 1\\ 1 & -i\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}, \tag{6.50}$$

which shows that the invariants $\hat{A}(t), \hat{B}(t)$ found in (Malkin, Man'ko & Trifonov, 1970) can be written as linear combinations of our invariants $\hat{A}_{\theta,1}(t), \hat{A}_{\theta,2}(t),\ j = 1.2$. Lastly, if one takes α and β as defined in (6.50), then coherent states $|\alpha, \beta; t\rangle$ found in (Malkin, Man'ko & Trifonov, 1970) will coincide with coherent states (6.40) found in this work.

6.4.2. Quadratic Invariants

For the quantum system described by Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ given by (6.14), using the evolution operator and $\hat{H}_0 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1$, we can define a quadratic Hermitian invariant

$$\hat{I}_\theta(t) = \hat{\mathcal{U}}_{gen}(t, t_0)\hat{H}_0 \hat{\mathcal{U}}^\dagger_{gen}(t, t_0). \tag{6.51}$$

This invariant can be expressed in terms of the linear invariants (6.46) and (6.47) as follows

$$\hat{I}_\theta(t) = \hat{A}^\dagger_{\theta,1}(t)\hat{A}_{\theta,1}(t) + \hat{A}^\dagger_{\theta,2}(t)\hat{A}_{\theta,2}(t) + 1.$$

We note that, the invariants (6.46) and (6.47) can be written also in the form

$$\begin{pmatrix} \hat{A}_{\theta,1}(t) \\ \hat{A}_{\theta,2}(t) \end{pmatrix} = \frac{e^{i\eta(t)}}{\sqrt{2\omega_0 \hbar}} \left[\left(\frac{\omega_0}{|\epsilon(t)|} + i|\epsilon(t)|\mu(t)\Big(B(t) - \frac{d}{dt} \ln |\epsilon(t)|\Big) \right) \begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{pmatrix} + i|\epsilon(t)| \begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \end{pmatrix} \right]$$

and

$$\begin{pmatrix} \hat{A}^\dagger_{\theta,1}(t) \\ \hat{A}^\dagger_{\theta,2}(t) \end{pmatrix} = \frac{e^{-i\eta(t)}}{\sqrt{2\omega_0 \hbar}} \left[\left(\frac{\omega_0}{|\epsilon(t)|} - i|\epsilon(t)|\mu(t)\Big(B(t) - \frac{d}{dt} \ln |\epsilon(t)|\Big) \right) \begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{pmatrix} - i|\epsilon(t)| \begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \end{pmatrix} \right],$$

where $\sigma(t) = |\epsilon(t)|$ satisfies the Ermakov-Pinney nonlinear differential equation

$$\ddot{\sigma}(t) + \frac{\dot{\mu}}{\mu}\dot{\sigma}(t) + \left(\omega^2(t) - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu} B\right)\right)\sigma(t) = \frac{1}{\mu^2 \sigma^3(t)}, \tag{6.52}$$

with initial conditions

$$\sigma(t_0) = 1, \quad \dot{\sigma}(t_0) = B(t_0). \tag{6.53}$$

Then, the quadratic invariant becomes

$$\hat{I}_\theta(t) = \frac{1}{2\omega_0\hbar}\sum_{j=1}^{2}\left\{\frac{\omega_0^2}{|\epsilon(t)|^2}\hat{Q}_j^2 + \left[|\epsilon(t)|\mu(t)\Big(B(t) - \frac{d}{dt}\ln|\epsilon(t)|\Big)\hat{Q}_j + |\epsilon(t)|\hat{P}_j\right]^2\right\}, \qquad (6.54)$$

and it is special in the sense that $|\epsilon(t)|$ is a particular solution of the Ermakov-Pinney equation (6.52) satisfying the initial conditions (6.53). Now, since the following commutation relations hold

$$[\hat{A}_{\theta,i}(t), \hat{A}^\dagger_{\theta,j}(t)] = \delta_{ij}, \quad [\hat{I}_\theta(t), \hat{A}_{\theta,j}(t)] = -\hat{A}_{\theta,j}(t), \quad [\hat{I}_\theta(t), \hat{A}^\dagger_{\theta,j}(t)] = \hat{A}^\dagger_{\theta,j}(t) \quad j = 1, 2,$$

then the eigenvalues and eigenstates of the invariant $\hat{I}_\theta(t)$ can be found by the same algebraic procedure as for the simple harmonic oscillator. Here, $\hat{H}_0\varphi_n(\mathbf{q}) = E_n\varphi_n(\mathbf{q})$, so that by construction of (6.51) we have $\hat{I}_\theta(t)\Psi_n^\theta(\mathbf{q}, t) = E_n\Psi_n^\theta(\mathbf{q}, t)$, showing that time-evolved wave function solutions of the Scrödinger equation found as $\Psi_n^\theta(\mathbf{q}, t) = \hat{\mathcal{U}}_{gen}(t, t_0)\varphi_n(\mathbf{q})$ in (6.32) are eigenstates of the invariant $\hat{I}_\theta(t)$ corresponding to eigenvalues $E_n = \hbar\omega_0(n_1 + n_2 + 1)$.

In the work of Lewis and Riesenfeld (Lewis & Riesenfeld, 1969), for a quantum system described by an explicitly time-dependent Hamiltonian $\hat{H}(t)$, a dynamical invariant is defined to be an operator $\hat{I}(t)$ satisfying $i\hbar\partial_t\hat{I}(t) - [\hat{H}(t), \hat{I}(t)] = 0$. As known LR-approach for solving nonstationary quantum oscillators is based on finding Hermitian quadratic invariant of the form (6.54). Then, eigenstates of the quadratic invariant constructed by the LR-technique and solutions of the Scrödinger equation usually differ by a time-dependent phase factor. We note that, in (Lewis & Riesenfeld, 1969) the Hamiltonian describing a charged particle in a time-dependent electromagnetic field is a particular case of Hamiltonian $\hat{\mathcal{H}}_{gen}(t)$ given by (6.14). For more recent and related results based on linear and quadratic invariants one can see also (Abdalla & Choi, 2007).

6.5. Cauchy-Euler Type Quantum Oscillator in Time-Variable Magnetic and Electric Fields

Now, we introduce and discuss an exactly solvable quantum model described by the Hamiltonian

$$\begin{aligned}\hat{\mathcal{H}}_{gen}(t) &= \sum_{j=1}^{2}\left[\frac{1}{2t^{\gamma}}\hat{p}_j^2 + \frac{B(t)}{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j) + \frac{\omega_0^2 t^{\gamma-2}}{2}\hat{q}_j^2\right] \\ &+ E_0\, t^{\gamma}\sin(\Omega_E \ln t)\hat{q}_1 + E_0\, t^{\gamma}\cos(\Omega_E \ln t)\hat{q}_2 + \frac{\lambda_0}{t}(\hat{q}_1\hat{p}_2 - \hat{q}_2\hat{p}_1). \quad (6.55)\end{aligned}$$

In this model, for $t \geq t_0$, $t_0 = 1$, we have time-dependent increasing mass $\mu(t) = t^{\gamma}$ for damping parameter $\gamma \geq 1$, and decreasing frequency $\omega^2(t) = \omega_0^2/t^2$, $\omega_0 > 0$. Then, to preserve the Cauchy-Euler structure of the oscillator we take $B(t) = -\Omega_B \tan(\Omega_B \ln t)/t$, where $\Omega_B = \sqrt{\omega_B^2 - (\gamma-1)^2/4}$ and $\omega_B^2 > (\gamma-1)^2/4$. In addition, we consider external electric fields $E_1(t) = E_0\, t^{\gamma}\sin(\Omega_E \ln t)$, $E_2(t) = E_0\, t^{\gamma}\cos(\Omega_E \ln t)$ with E_0, Ω_E- real constants, that are oscillating in time with increasing amplitude and decreasing frequency. The last term in (6.55) is the angular momentum with Larmor type frequency $\lambda(t) = \lambda_0/t$, λ_0–real constant, that depends on time and tends to zero when time increases.

In what follows, first we write the solutions to the corresponding coupled system of classical equations of motion. Then, we describe in detail time-evolved eigenfunctions and coherent states.

A. The classical problem

For the quantum evolution problem with Hamiltonian (6.55), the corresponding coupled system of classical equations of motion is of the form

$$\begin{pmatrix}\ddot{X}_1\\ \ddot{X}_2\end{pmatrix} + \begin{pmatrix}\gamma/t & 2\lambda_0/t\\ -2\lambda_0/t & \gamma/t\end{pmatrix}\begin{pmatrix}\dot{X}_1\\ \dot{X}_2\end{pmatrix} + \begin{pmatrix}\frac{\omega_0^2+\omega_B^2-\lambda_0^2}{t^2} & \frac{\lambda_0(\gamma-1)}{t^2}\\ -\frac{\lambda_0(\gamma-1)}{t^2} & \frac{\omega_0^2+\omega_B^2-\lambda_0^2}{t^2}\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix} = \begin{pmatrix}-E_0\sin(\Omega_E \ln t)\\ -E_0\cos(\Omega_E \ln t)\end{pmatrix}. \quad (6.56)$$

For $E_0 = 0$, system (6.56) with initial conditions (6.11) has homogeneous solution $\mathbf{X}^{(h)}(t) \equiv$

$R_\theta^T(t)\mathbf{x}^{(h)}(t)$, explicitly found as

$$\mathbf{X}^{(h)}(t) = R_\theta^T(t)\begin{pmatrix} \frac{\sqrt{\omega_0^2+\omega_B^2}}{\Omega_g} t^{-(\gamma-1)/2}\cos\left(\Omega_g \ln t - \delta_g\right) \\ \frac{1}{\Omega_g} t^{-(\gamma-1)/2}\sin\left(\Omega_g \ln t\right) \end{pmatrix}, \ t \geq 1, \tag{6.57}$$

where $\Omega_g = \sqrt{\omega_0^2 + \omega_B^2 - (\gamma-1)^2/4}$ is the oscillator frequency and $\delta_g = \arctan((\gamma - 1)/2\Omega_g)$.

For $E_0 \neq 0$, particular solution is $\mathbf{X}^{(p)}(t) \equiv R_\theta^T(t)\mathbf{x}^{(p)}(t)$ and explicitly we have

$$\mathbf{X}^{(p)}(t) = R_\theta^T(t)\begin{pmatrix} A_1^{(h)} t^{-(\gamma-1)/2}\cos(\Omega_g \ln t - \delta_1^{(h)}) - \frac{E_0}{\sqrt{a^2+b^2}}\cos\left((\Omega_E + \lambda_0)\ln t - \delta_p\right) \\ A_2^{(h)} t^{-(\gamma-1)/2}\sin(\Omega_g \ln t - \delta_2^{(h)}) + \frac{E_0}{\sqrt{a^2+b^2}}\sin\left((\Omega_E + \lambda_0)\ln t - \delta_p\right) \end{pmatrix}, \quad t \geq 1, \tag{6.58}$$

where $a = (\omega_0^2 + \omega_B^2) - (\Omega_E + \lambda_0)^2$, $b = (1-\gamma)(\Omega_E + \lambda_0)$, $\delta_p = \text{arccot}(b/a)$ and $A_j^{(h)}, \delta_j^{(h)}, \ j = 1, 2$ are constants of the transient part such that $\mathbf{X}^{(p)}(t)$ satisfies the initial conditions (6.12). Here, rotation angle is $\theta(t) = \lambda_0 \ln t$ and the rotation matrix becomes

$$R_\theta(t) = \begin{pmatrix} \cos(\lambda_0 \ln t) & \sin(\lambda_0 \ln t) \\ -\sin(\lambda_0 \ln t) & \cos(\lambda_0 \ln t) \end{pmatrix}, \quad t \geq 1,$$

where the sign of λ_0 determines the direction of rotation.

B. Time-evolution of the wave functions $\Psi_n^\theta(\mathbf{q}, t)$

For the wave functions $\Psi_n^\theta(\mathbf{q}, t)$ the probability densities are given by Eq.(6.33), that is

$$\rho_n^\theta(\mathbf{q}, t) = \mathbf{N}_n^2 \frac{1}{|\epsilon(t)|^2}\exp\left\{-\frac{\omega_0}{\hbar}\frac{\left|\mathbf{q} - \mathbf{X}^{(p)}(t)\right|^2}{|\epsilon(t)|^2}\right\}\mathbf{H}_n^2\left(\sqrt{\frac{\omega_0}{\hbar}}\frac{R_\theta(t)(\mathbf{q} - \mathbf{X}^{(p)}(t))}{|\epsilon(t)|}, t\right), \ \ n = (n_1, n_2),$$

where $\mathbf{X}^{(p)}(t)$ is found in (6.58), and squeezing coefficient is

$$|\epsilon(t)| = \frac{t^{-(\gamma-1)/2}}{\Omega_g}\sqrt{(\omega_0^2 + \omega_B^2)\cos^2\left(\Omega_g \ln t - \delta_g\right) + \omega_0^2 \sin^2\left(\Omega_g \ln t\right)}, \tag{6.59}$$

which is smooth and oscillatory for $t \geq 1$. Then, for a given $\omega_0 > 0$ and $\gamma \geq 1$, the frequency $\Omega_g = \sqrt{\omega_0^2 + \omega_B^2 - (\gamma-1)^2/4}$ of oscillations in $|\epsilon(t)|$ can be increased by increasing the value of ω_B in parameter $B(t)$. When $\gamma = 1$, amplitude is fixed and one has $|\epsilon(t)| \to 1$ as $\omega_B \to 0$. However, when $\gamma > 1$, the amplitude of oscillations in $|\epsilon(t)|$ decreases and approaches zero as time increases.

In Fig.6.1, we plot the probability density $\rho_n^\theta(\mathbf{q}, t)$ with $n = (1, 2)$ at three different times. For this, we take $\gamma = 2$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$, Larmor type frequency $\lambda(t) = 7/t$, and $E_0 = 0$, so that there are no external electric fields. These plots show how the width and amplitude of the wave packets change with time and how they are rotated with angle $\theta(t) = 7\ln t$ under the influence of the magnetic field. Uncertainties of position and momentum at time-evolved wave functions $\Psi_n^\theta(\mathbf{q}, t)$

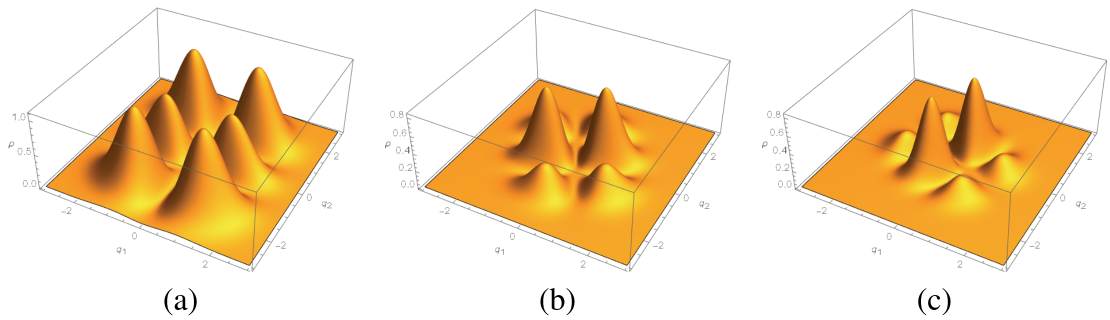


Figure 6.1. Probability density $\rho_n^\theta(\mathbf{q}, t)$ for $n = (1, 2)$, $\gamma = 2$, $\hbar = 1$, $\omega_0 = 1$, $\theta(t) = 7\ln t$, $E_0 = 0$, at times (a) $t_0 = 1$, (b) $t = 1.07$, (c) $t = 1.85$.

are found by (6.35) and (6.36), respectively,

$$(\Delta\hat{\mathbf{q}})_n^\theta(t) = \sqrt{\frac{\hbar}{\omega_0}}|\epsilon(t)|\,\mathbf{\Lambda}(n_1, n_2, \theta(t)), \qquad (\Delta\hat{\mathbf{p}})_n^\theta(t) = \sqrt{\hbar\omega_0}\frac{\Sigma(t)}{|\epsilon(t)|}\,\mathbf{\Lambda}(n_1, n_2, \theta(t)),$$

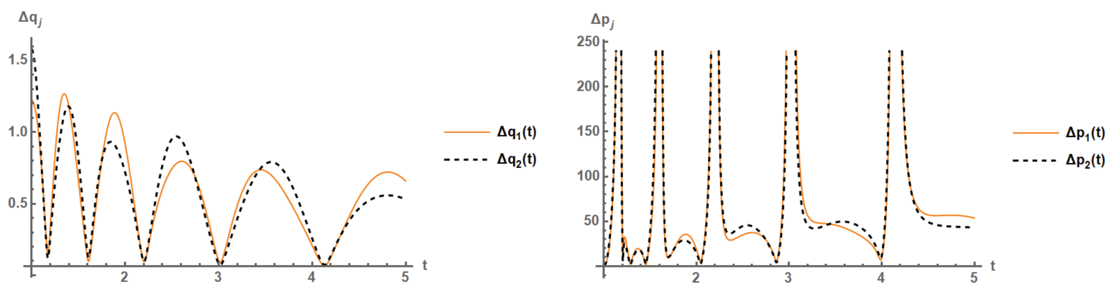
where for this model we obtain the vector valued function

$$\mathbf{\Lambda}(n_1, n_2, \theta(t)) = \begin{pmatrix} \left(\cos^2(\lambda_0\ln t)n_1 + \sin^2(\lambda_0\ln t)n_2 + 1/2\right)^{\frac{1}{2}} \\ \left(\sin^2(\lambda_0\ln t)n_1 + \cos^2(\lambda_0\ln t)n_2 + 1/2\right)^{\frac{1}{2}} \end{pmatrix},$$

and coefficient

$$\begin{aligned}\Sigma(t) &= \Bigg\{1+\frac{1}{4\omega_0^2 t^2}\Bigg[\Big(2\Omega_B\tan(\Omega_B\ln t)-\gamma+1\Big)\\ &\times\Big((\omega_0^2+\omega_B^2)\cos^2(\Omega_g\ln t-\delta_g)+\omega_0^2\sin^2(\Omega_g\ln t)\Big)\\ &+\Omega_g\Big(-(\omega_0^2+\omega_B^2)\sin(2(\Omega_g\ln t-\delta_g))+\omega_0^2\sin(2\Omega_g\ln t)\Big)\Bigg]^2\Bigg\}^{1/2}. \qquad (6.60)\end{aligned}$$

Clearly, $\boldsymbol{\Lambda}(n_1, n_2, \theta(t))$ carries the dependence of the uncertainties on the wave numbers n_1, n_2 and the rotation angle $\theta(t)$, while $|\epsilon(t)|$ and $\Sigma(t)$ depend only on parameters $\mu(t)$, $\omega^2(t)$ and $B(t)$. We note that for $\gamma = 1$ and $\omega_B \to 0$, one has $\Sigma(t) \to 1$. Otherwise the coefficient $\Sigma(t)$ has singularities due to the singularities in $B(t)$, and this affects the uncertainties in momentum. Using the same parameters as in Fig. 6.1, then in Fig. 6.2 we plot uncertainties in position and momentum at wave function $\Psi_n^\theta(\mathbf{q}, t)$ for $n = (1, 2)$. As can be seen in Fig. 6.2-(a), uncertainty in position is smooth, oscillatory and approaches to zero as time increases. However, singularities appear in uncertainties of momentum since they depend on the coefficient $\Sigma(t)$ found by (6.60).



(a) (b)

Figure 6.2. Uncertainties in position and momentum for $n = (1, 2)$ and $\gamma = 2$, $\hbar = 1$, $\omega_0 = 1$: (a) $(\Delta\hat{q}_j)^\theta_{n_j}(t)$, $j = 1, 2$ for $t \in [1, 5]$, (b) $(\Delta\hat{p}_j)^\theta_{n_j}(t)$, $j = 1, 2$ for $t \in [1, 5]$.

Now, we discuss possible trajectories of the wave packets in the two-dimensional coordinate space, that are determined by the expectation values of position at state $\Psi_n^\theta(\mathbf{q}, t)$. According to the general results found in (6.34), if there are no external fields wave packets are localized at $(q_1, q_2) = (0, 0)$ in $\mathbb{R}^2$, as in Fig.6.1. However, in the presence of external fields wave packets will move along the trajectory $\langle \hat{\mathbf{q}} \rangle_n^\theta(t) = \mathbf{X}^{(p)}(t)$ in $\mathbb{R}^2$, which for this model is given by (6.58). Then, depending on parameter $\gamma \geq 1$ in (6.58), we consider two cases:

i) For $\gamma = 1$, we have the trajectory

$$\langle \hat{\mathbf{q}} \rangle_n^\theta(t) = R_\theta^T(t) \begin{pmatrix} \frac{E_0}{|a|\Omega_g}\Big((\Omega_E + \lambda_0)\sin(\Omega_g \ln t) - \Omega_g \sin\big((\Omega_E + \lambda_0)\ln t\big)\Big) \\ \frac{E_0}{|a|}\Big(\cos(\Omega_g \ln t) - \cos\big((\Omega_E + \lambda_0)\ln t\big)\Big) \end{pmatrix}, \quad t \geq 1,$$

where $\Omega_g = \sqrt{\omega_0^2 + \omega_B^2}$ is the oscillator frequency and Ω_E, λ_0 are frequencies due to the external fields. When $(\Omega_E + \lambda_0) = \Omega_g$, then one has balance between frequencies and particle is localized at the origin for any time. When $(\Omega_E + \lambda_0)/\Omega_g$ is a rational number, the trajectory $\langle \hat{\mathbf{q}} \rangle_n^\theta(t)$ is a closed plane curve. In this case, a particle moving along the trajectory returns to its starting point after some time, whatever the starting point is, and then retraces the same curve. On the other hand, when $(\Omega_E + \lambda_0)/\Omega_g$ is not rational, the curve will never close and the particle will pass through every point of a bounded region containing the origin in $\mathbb{R}^2$, eventually filling it. Clearly, we have non-uniform motion with smoothly decreasing speed.

ii) For $\gamma > 1$, since transient part of $\mathbf{X}^{(p)}(t)$ quickly tends to zero, after some time we have

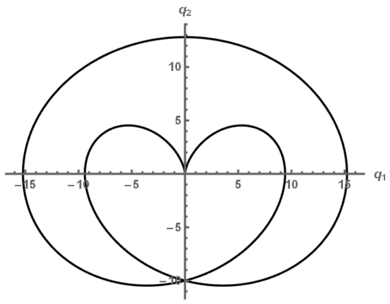
$$\langle \hat{\mathbf{q}} \rangle_n^\theta(t) \approx R_\theta^T(t) \begin{pmatrix} -\frac{E_0}{\sqrt{a^2+b^2}} \cos\big((\Omega_E + \lambda_0)\ln t - \delta_p\big) \\ \frac{E_0}{\sqrt{a^2+b^2}} \sin\big((\Omega_E + \lambda_0)\ln t - \delta_p\big) \end{pmatrix} = \begin{pmatrix} -\frac{E_0}{\sqrt{a^2+b^2}} \cos(\Omega_E \ln t - \delta_p) \\ \frac{E_0}{\sqrt{a^2+b^2}} \sin(\Omega_E \ln t - \delta_p) \end{pmatrix}.$$

Then, the particle exhibits again a non-uniform motion with decreasing speed and with λ_0 contributing to the phase and radius of the orbit. In that case the trajectory is not closed since usually it does not repeat, but in the long time limit it converges to a circular orbit given by (6.61). □

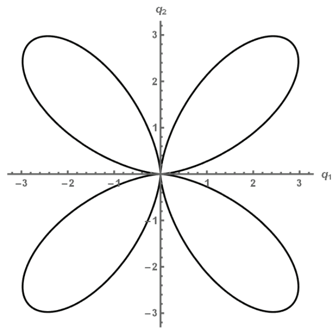
As an example, for $\gamma = 1$ in Fig.6.3 we plot the trajectory

$$\langle \hat{\mathbf{q}} \rangle_n^\theta(t) = \frac{E_0}{|a|} R_\theta^T(t) \begin{pmatrix} \frac{15+\lambda_0}{10} \sin(10 \ln t) - \sin((15 + \lambda_0)\ln t) \\ \cos(10 \ln t) - \cos((15 + \lambda_0)\ln t) \end{pmatrix}, \quad \theta(t) = \lambda_0 \ln t,$$

starting at the origin and with parameters $\Omega_g = 10$, $\Omega_E = 15$, $E_0 = 800$, $a = 100 - (15 + \lambda_0)^2$. In Fig.6.3-(a) we see the plot for $\theta(t) = 0$, which is a closed curve since $\Omega_E/\Omega_g = 3/2$ is rational. And in Fig.6.3-(b) we show this curve under rotation with rotation angle $\theta(t) = 15 \ln t$.



(a)



(b)

Figure 6.3. Trajectory of the wave packets $|\Psi_n^\theta(\mathbf{q}, t)|^2$ for any n and $\gamma = 1$, $\omega_0 = 1, \hbar = 1$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = 3\sqrt{11}$, $E_1(t) = 800t\sin(15\ln t)$, $E_2(t) = 800t\cos(15\ln t)$, $t \in [1, 4]$: (a) when $\lambda(t) = 0$, (b) when $\lambda(t) = 15/t$.

Another example for $\gamma = 1$ is given in Fig.6.4, where we plot the trajectory

$$\langle \hat{\mathbf{q}} \rangle_n^\theta(t) = \frac{E_0}{|a|} R_\theta^T(t) \begin{pmatrix} \frac{20\pi+\lambda_0}{10}\sin(10\ln t) - \sin((20\pi+\lambda_0)\ln t) \\ \cos(10\ln t) - \cos((20\pi+\lambda_0)\ln t) \end{pmatrix}, \quad \theta(t) = \lambda_0 \ln t,$$

with parameters $\Omega_g = 10$, $\Omega_E = 20\pi$, $E_0 = 5.10^3$ and $a = 100 - (20\pi + \lambda_0)^2$. In Fig.6.4-(a) we have $\theta(t) = 0$, and note that $\Omega_E/\Omega_g = 2\pi$ is irrational so that the curve is not closed. Particle will start from the origin, and then it will pass through every point of a bounded region in $\mathbb{R}^2$ as $t \to \infty$. In Fig.6.4-(b) we take $\theta(t) = 20\ln t$ and see particle motion along another open trajectory confined to a bounded region.

C. Time-evolution of coherent states $\Phi_\alpha^\theta(\mathbf{q}, t)$

At coherent states $\Phi_\alpha^\theta(\mathbf{q}, t)$ probability densities are two-dimensional Gaussian wave pack-

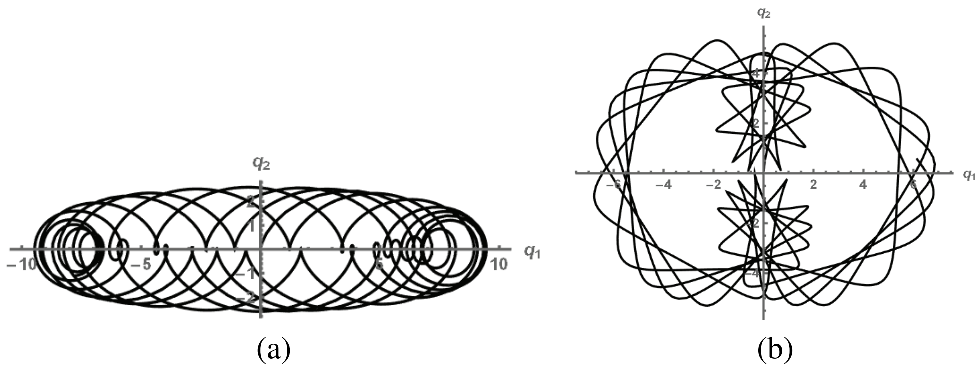


Figure 6.4. Trajectory of the wave packets $|\Psi_n^\theta(\mathbf{q},t)|^2$ for any n, with $\gamma = 1$, $\omega_0 = 1, \hbar = 1$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = 3\sqrt{11}$, $E_1(t) = 5.10^3 t\sin(20\pi\ln t)$, $E_2(t) = 5.10^3 t\cos(20\pi\ln t)$, $t \in [1,20]$: (a) when $\lambda(t) = 0$, (b) when $\lambda(t) = 20/t$.

ets given by (6.41), i.e.

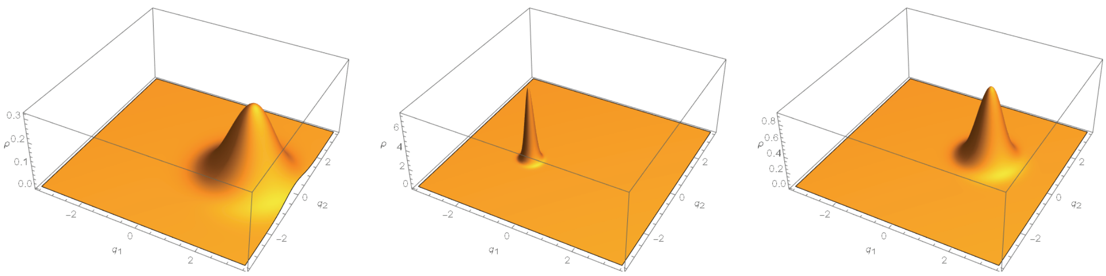
$$\rho_\alpha^\theta(\mathbf{q},t) = \left(\frac{\omega_0}{\pi\hbar}\right)\frac{1}{|\epsilon(t)|^2}\exp\left\{-\frac{\omega_0}{\hbar}\frac{\left|\mathbf{q}-\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t)\right|^2}{|\epsilon(t)|^2}\right\}, \quad \alpha = (\alpha_1,\alpha_2) \in \mathbf{C}^2,$$

where for this model squeezing coefficient is explicitly given by (6.59). As an example, in Fig. 6.5 we plot probability density for $\alpha = (20\sqrt{2}/\sqrt{401}, 10i)$, $\gamma = 2$, $\hbar = 1, \omega_0 = 1, \lambda(t) = 10/t$, and squeezing parameter $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$, at different times $t = 1,\ 1.2,\ 2$. These plots show the changes in width and amplitude of the wave packet that follows a trajectory

$$\langle\hat{\mathbf{q}}\rangle_\alpha^\theta(t) = R_\theta^T(t)\begin{pmatrix} 2t^{-1/2}\cos(10\ln t - \arctan(1/20)) \\ \sqrt{2}t^{-1/2}\sin(10\ln t) \end{pmatrix} \tag{6.61}$$

with rotation angle $\theta(t) = 10\ln t$, and in case $E_0 = 0$. Explicitly, the corresponding uncertainties are found according to (6.62), that is

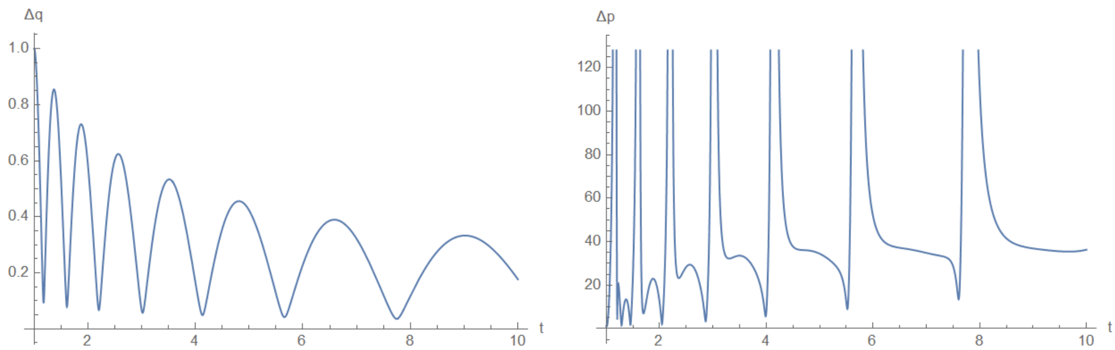
$$(\Delta\hat{q}_j)_\alpha^0(t) = \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon(t)|, \quad (\Delta\hat{p}_j)_\alpha^0(t) = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon(t)|}\Sigma(t), \quad j = 1,2,$$



(a) (b) (c)

Figure 6.5. Probability density $\rho_\alpha^\theta(\mathbf{q}, t)$ for $\alpha = (20\sqrt{2}/\sqrt{401}, 10i)$, $\lambda(t) = 10/t$, $\gamma = 2$, $\hbar = 1, \omega_0 = 1,\ E_0 = 0$ at times: (a) $t = t_0 = 1$, (b) $t = 1.2$, (c) $t = 2$.

where the coefficients $|\epsilon(t)|$ and $\Sigma(t)$ are given by (6.59) and (6.60), respectively. Clearly, uncertainties do not depend on α and $\theta(t)$, and they are equal in both directions. Fig.6.6 shows $(\Delta\hat{q}_j)_{\alpha_j}(t)$ and $(\Delta\hat{p}_j)_{\alpha_j}(t)$ for each $j = 1, 2$, where $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$ as in Fig.6.5. We note that uncertainties in position are smooth, oscillatory and approach zero as $t \to \infty$, but uncertainties in momentum have singularities due to the singularities in $B(t)$, as we see in Fig.6.6-(b).



(a) (b)

Figure 6.6. Uncertainties for $\gamma = 2$, $\hbar = 1$, $\omega_0 = 1$: (a) $(\Delta\hat{q}_j)_{\alpha_j}(t)$, $j = 1, 2$, (b) $(\Delta\hat{p}_j)_{\alpha_j}(t)$, $j = 1, 2$, $t \in [1, 10]$.

Now, we recall that the center of the wave packet $\rho_\alpha^\theta(\mathbf{q}, t)$ in the two-dimensional coordinate space follows the classical trajectory

$$\langle\hat{\mathbf{q}}\rangle^{\theta}_{\alpha}(t) = R^T_{\theta}(t)\langle\hat{\mathbf{q}}\rangle^{0}_{\alpha}(t) = \sqrt{\frac{2\hbar}{\omega_0}}\mathbf{C}^{\theta}_{\alpha}(t)\mathbf{X}^{(h)}(t) + \mathbf{X}^{(p)}(t), \tag{6.62}$$

and for this model $\mathbf{C}^{\theta}_{\alpha}(t)$ is defined by (6.42) with $\theta(t) = \lambda_0 \ln t$, $\mathbf{X}^{(h)}(t)$ is given by (6.57) and $\mathbf{X}^{(p)}(t)$ is given by (6.58). In particular, when $\theta(t) = 0$ and there are no external electric fields ($E_0 = 0$), then the trajectory will be $\langle\hat{\mathbf{q}}\rangle^{0}_{\alpha}(t) = \sqrt{2\hbar/\omega_0}\,\mathbf{C}^{0}_{\alpha}\,\mathbf{x}^{(h)}(t)$, which can be written explicitly as

$$\langle\hat{\mathbf{q}}\rangle^{0}_{\alpha}(t) = \sqrt{\frac{2\hbar}{\omega_0}}\begin{pmatrix}\alpha^{(1)}_1 & \omega_0\alpha^{(2)}_1\\ \alpha^{(1)}_2 & \omega_0\alpha^{(2)}_2\end{pmatrix}\begin{pmatrix}\frac{\sqrt{\omega_0^2+\omega_B^2}}{\Omega_g}t^{-(\gamma-1)/2}\cos\left(\Omega_g \ln t - \delta_g\right)\\ \frac{1}{\Omega_g}t^{-(\gamma-1)/2}\sin\left(\Omega_g \ln t\right)\end{pmatrix}. \tag{6.63}$$

In Eq.(6.63) depending on the values of $\gamma \geq 1$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2$, we note the following possibilities:

a) For $\gamma = 1$, the trajectory could be a line segment, a circle or an ellipse in $\mathbb{R}^2$, centered at the origin. In case $det(C^0_\alpha) \equiv \omega_0(\alpha^{(1)}_1\alpha^{(2)}_2 - \alpha^{(1)}_2\alpha^{(2)}_1) = 0$, wave packet oscillates along a line segment. If $\alpha^{(2)}_1 = \alpha^{(1)}_2 = 0$ and $|\alpha^{(2)}_2| = \sqrt{1 + (\omega_B^2/\omega_0^2)}\,|\alpha^{(1)}_1|$ or similarly if $\alpha^{(1)}_1 = \alpha^{(2)}_2 = 0$ and $|\alpha^{(2)}_1| = \sqrt{1 + (\omega_B^2/\omega_0^2)}\,|\alpha^{(1)}_2|$, then we have a circular motion. Otherwise, the motion is along an ellipse, and in any case it is a non-uniform motion.

b) For $\gamma > 1$, wave packet moves towards the origin due to damping affects. In case $det(C^0_\alpha) = 0$, it oscillates forth and back along a line segment passing through the origin, with decreasing amplitude and approaching the origin. If $det(C^0_\alpha) \neq 0$, wave packet moves inward usually along a spiral like trajectory as time increases.

It follows that when $\theta(t) \neq 0$, the rotated trajectories (except the circular ones) will be more complicated, as one can see in the following plots. For example, we consider the trajectory (6.62) for $\gamma = 1$,

$$\langle\hat{\mathbf{q}}\rangle^{\theta}_{\alpha}(t) = R^T_{\theta}(t)\begin{pmatrix}\cos(10\ln t)\\ \frac{1}{2}\sin(10\ln t)\end{pmatrix}, \qquad \theta(t) = \lambda_0 \ln(t),$$

which is an ellipse for $\theta = 0$ and $E_0 = 0$, as we see in Fig.6.7-(a). In Fig.6.7-(b) we see the rotated ellipse for $\theta(t) = 25 \ln t$, $\lambda_0 = 25$. Then, in Fig.6.7-(c), we plot the trajectory

$$\langle \hat{\mathbf{q}} \rangle_\alpha^\theta(t) = R_\theta^T(t) \left[\begin{pmatrix} \cos(10 \ln t) \\ \frac{1}{2} \sin(10 \ln t) \end{pmatrix} + \begin{pmatrix} 3 \sin(10 \ln t) - \sin(30 \ln t) \\ \cos(10 \ln t) - \cos(30 \ln t) \end{pmatrix} \right],$$

with rotation angle $\theta(t) = 25 \ln t$ and under the influence of electric fields. In that case, the trajectory depends also on the particular solution of the classical system, and since the ratio $(\Omega_E + \lambda_0)/\Omega_g = 3$ is a rational number the trajectory is closed.

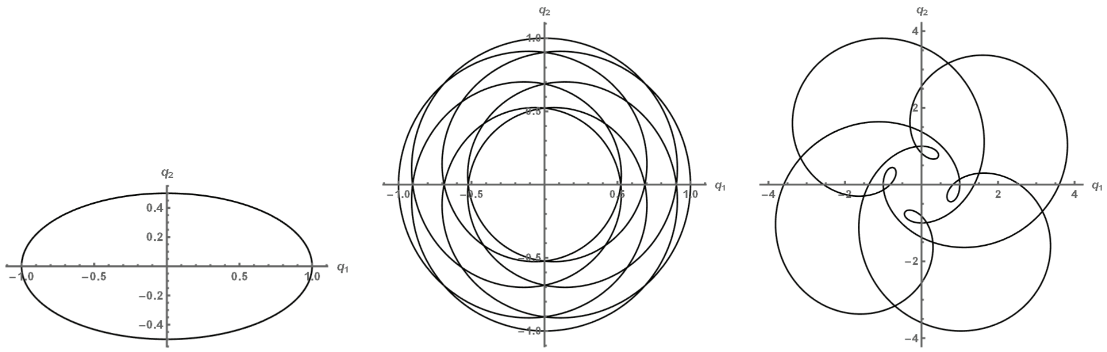


Figure 6.7. Trajectories of $\rho_\alpha^\theta(\mathbf{q}, t)$ with $\gamma = 1$, $\alpha = (\sqrt{2}/2, 5\sqrt{2}i/2)$, $B(t) = -3\sqrt{11} \tan(3\sqrt{11} \ln t)/t$, $\omega_B = 3\sqrt{11}$, $\hbar = \omega_0 = 1$: (a) $\lambda(t) = 0$, $E_0 = 0$, $t \in [1, 2]$, (b) $\lambda(t) = 25/t$, $E_0 = 0$, $t \in [1, 4]$, (c) $\lambda(t) = 25/t$, $E_1(t) = 800t \sin(5 \ln t)$, $E_2(t) = 800t \cos(5 \ln t)$, $t \in [1, 4]$.

As another example, for $\gamma = 2$ in Fig.6.8 we plot

$$\langle \hat{\mathbf{q}} \rangle_\alpha^\theta(t) = 2\sqrt{2} R_\theta^T(t) \begin{pmatrix} t^{-1/2} \sin(10 \ln t) \\ t^{-1/2} \sin(10 \ln t) \end{pmatrix}, \qquad \theta(t) = \lambda_0 \ln(t), \tag{6.64}$$

where in Fig.6.8-(a) we have $\theta = 0$ and $det(C_\alpha^0) = 0$, so that the wave packet oscillates along a straight line and approaches the origin as time increases. Fig.6.8-(b) shows the trajectory given by Eq.(6.64) with rotation angle $\theta(t) = 20 \ln t$. Then, in Fig.6.8-(c) we plot

$$\langle \hat{\mathbf{q}} \rangle_\alpha^\theta(t) = 2\sqrt{2} R_\theta^T(t) \begin{pmatrix} t^{-1/2}\sin(10\ln t) \\ t^{-1/2}\sin(10\ln t) \end{pmatrix} + \frac{E_0}{\sqrt{a^2+b^2}} \begin{pmatrix} -\cos(5\ln t - \mathrm{arccot}(b/a)) \\ \sin(5\ln t - \mathrm{arccot}(b/a)) \end{pmatrix},$$

where $a = 401/4 - (5+\lambda_0)^2$, $b = -(5+\lambda_0)$, for $\theta(t) = 20\ln t$, and in the presence of electric fields.

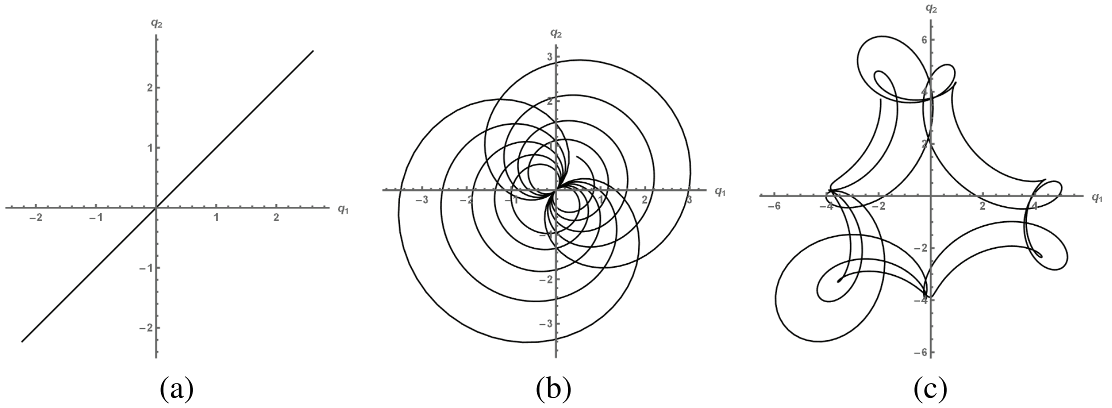
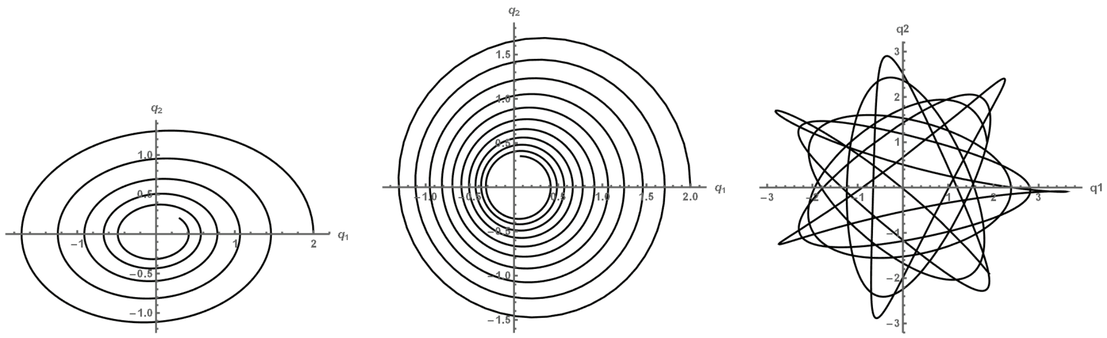


Figure 6.8. Trajectories of $\rho_\alpha^\theta(\mathbf{q}, t)$ with $\gamma = 2$, $\alpha = (20i, 20i)$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$, $\hbar = \omega_0 = 1$, $t \in [1, 20]$: (a) $\lambda(t) = 0$, $E_0 = 0$. (b) $\lambda(t) = 20/t$, $E_0 = 0$. (c) $\lambda(t) = 20/t$, $E_1(t) = 2.10^3 t\sin(5\ln t)$, $E_2(t) = 2.10^3 t\cos(5\ln t)$.

Finally, for $\gamma = 2$ in Fig.6.9 we show the trajectory given by Eq.(6.61). In Fig.6.9-(a) we take $\theta(t) = 0$, $E_0 = 0$ and since $det(C_\alpha^0) \neq 0$, the wave packet initially located at $(q_1, q_2) = (2\cos(\arctan(1/20)), 0)$ follows spiral like path approaching the origin as time increases. In Fig.6.9-(b) we have $\theta(t) = 10\ln t$, $E_0 = 0$ and wave packet again moves inward along a spiral. Then, in Fig.6.9-(c) we display the trajectory

$$\begin{aligned} \langle \hat{\mathbf{q}} \rangle_\alpha^\theta(t) &= R_\theta^T(t) \begin{pmatrix} 2t^{-1/2}\cos(10\ln t - \arctan(1/20)) \\ \sqrt{2}t^{-1/2}\sin(10\ln t) \end{pmatrix} \\ &\quad + \frac{E_0}{\sqrt{a^2+b^2}} \begin{pmatrix} -\cos(15\ln t - \mathrm{arccot}(b/a)) \\ \sin(15\ln t - \mathrm{arccot}(b/a)) \end{pmatrix}, \end{aligned}$$

of $\rho_\alpha^\theta(\mathbf{q}, t)$ for $\theta(t) = 10 \ln t$, $a = 401/4 - (15 + \lambda_0)^2$, $b = -(15 + \lambda_0)$ and electric fields $E_1(t) = 10^3 t \sin(15 \ln t)$, $E_2(t) = 10^3 t \cos(15 \ln t)$.



(a) (b) (c)

Figure 6.9. Trajectories of $\rho_\alpha^\theta(\mathbf{q}, t)$ with $\gamma = 2$, $\alpha = (20\sqrt{2}/\sqrt{401}, 10i)$, $B(t) = -3\sqrt{11}\tan(3\sqrt{11}\ln t)/t$, $\omega_B = \sqrt{397}/2$, $\hbar = \omega_0 = 1$, $t \in [1, 25]$: (a) $\lambda(t) = 0$, $E_0 = 0$. (b) $\lambda(t) = 10/t$, $E_0 = 0$. (c) $\lambda(t) = 10/t$, $E_1(t) = 10^3 t \sin(15 \ln t)$, $E_2(t) = 10^3 t \cos(15 \ln t)$.

Briefly saying, we have discussed squeezing properties of the wave packets due to influence of parameters $B(t)$ and $\gamma \geq 1$. Then, the trajectories of the wave packets in coordinate space were investigated according to the value of the damping parameter $\gamma \geq 1$. For coherent states, we've seen that their center follows the path of the classical particle in the two-dimensional configuration space and that the shape of the trajectory is closely related with the choice of $\alpha = (\alpha_1, \alpha_2)$. Lastly, according to their presence, the effects of magnetic and electric fields were illustrated by considering three different cases: (a) $\lambda_0 = 0$, $E_0 = 0$, (b) $\lambda_0 \neq 0$, $E_0 = 0$ and (c) $\lambda_0 \neq 0$, $E_0 \neq 0$.

CHAPTER 7

CONCLUSION

We considered the most general one-dimensional quantum parametric oscillator, whose Hamiltonian $\hat{H}_g(t)$ can be written as a linear combination of generators of the finite dimensional SU(1,1) and Heisenberg-Weyl Lie algebras. Then, we were able to write the displacement operator $\hat{D}(\alpha)$, squeeze operator $\hat{S}(z)$, and the evolution operator $\hat{U}_g(t, t_0)$, all being unitary as finite products of exponential operators, which are generators of the corresponding Lie groups. Based on these representations, we found the exact time-evolution of the nonclassical states, such as squeezed coherent states, even-odd coherent states and even-odd displaced squeezed states. We obtained their probability densities, expectations and uncertainties, and this allowed us to determine explicitly how the displacement of the wave packets depend on the complex parameter α and on all parameters of the Hamiltonian, and how squeezing properties depend on the complex parameter $z = re^{i\theta}$, and the time-dependent parameters $\mu(t), \omega(t)$ and $B(t)$. As an application of these results, we introduced an exactly solvable model as a generalization of the Caldirola-Kanai type quantum oscillator, which has an exponentially increasing mass $\mu(t) = e^{\gamma t}$, $\gamma > 0$, by adding to it a special mixed term $B(t)(\hat{q}\hat{p} + \hat{p}\hat{q})/2$ and a linear term $E(t)\hat{q}$. We chose the mixed term parameter $B(t)$ for which the structure of the corresponding classical equation in position space is preserved. For given frequency $\omega_0 > 0$, squeezing properties of the wave packets depend both on $r > 0$ and $B(t)$. However, the parameters $r > 0$ and $\theta \in [0, 2\pi)$ of the squeeze operator $\hat{S}(r, \theta)$ can be used to control only the amplitude and phase of the oscillating widths of the wave packets, while squeezing parameter $B(t)$ can be used to control not only their amplitude and phase, but also their frequency.

Then, we introduced an IBVP for a generalized quantum oscillator with time-variable coefficients, which was defined on a domain with a time-dependent boundary $s(t) < q < \infty$, $0 < t < T$. We showed that this problem can be solved analytically if the moving boundary is written as a linear combination of two linearly independent homogenous and a particular solution of the corresponding classical equation of motion in position space. We found solutions of the IBVP for a generalized quantum oscillator with a homogenous Dirichlet boundary condition $\Psi(s(t), t) = 0$. Furthermore, by comparing the results with the fixed boundary problems, we concluded that the moving boundary causes a shift in position coordinate and generates extra time-dependent exponentials in the solution contributing to the phase factor. We also examined an exactly solvable Caldirola-Kanai model to be able to analyze the effects of the moving boundary in detail. In addition to this, we introduced an IBVP for a generalized quantum oscillator with a Robin boundary condition $\partial\Psi(s(t), t)/\partial q - (i\hbar)\beta(t)\Psi(s(t), t) = 0$, and showed that the exact solution can be found when $s(t)$ and $\beta(t)$ are general solutions of the corresponding classical equations in position and momentum spaces, respectively.

We also considered the evolution problem for an N-dimensional generalized quantum parametric oscillator. By obtaining the exact form of the evolution operator, we found time development of the eigenstates and coherent states. Since properties of these states depend on solutions of the corresponding classical equations, we introduced exactly solvable models for which the oscillator structure in position space is preserved. Precisely, we found all nonzero squeezing parameters $B_j(t)$ for which frequency modification in q_j-direction remains constant, so that position uncertainties are always smooth, periodic and oscillatory. Moreover, we realized that when there are no external forces, the center of the wave packets in position space follows the classical Lissajous trajectories, and in general, when there are external forces, motion could be bounded or unbounded in space depending on the driving frequencies in each direction.

Finally, we solved an IVP for time-dependent Schrödinger equation for a generalized two-dimensional quantum parametric oscillator in the presence of time-varying magnetic and electric fields using the evolution operator method. We found the evolution operator and the propagator exactly in terms of solutions to the corresponding system of coupled classical equations of motion. Then, we applied the evolution operator to initial states such as the eigenstates and coherent states of the simple two-dimensional harmonic oscillator, and described explicitly the propagation of the time-dependent wave functions. In addition, by the evolution operator formalism, we constructed linear and quadratic invariants for the generalized two-dimensional quantum oscillator. As an exactly solvable model we introduced a two-dimensional Cauchy-Euler type quantum parametric oscillator with smoothly decreasing Larmor type frequency in oscillating external electric field. After solving the problem at classical level, we evaluated the probability densities, uncertainties and expectations at time-evolved eigenstates and coherent states and studied their behavior in details. That gave us more insight into how one can control the dynamics of the system by varying the parameters of damping and squeezing terms and by choosing proper external forces.

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APPENDIX A

THE UNCERTAINTIES AT TIME-EVOLVED EVEN-ODD COHERENT STATES

In this part, we find the uncertainties of position and momentum at time-evolved even-odd coherent states explicitly.

First, we start with expectation values at time-evolved even coherent states. From the definition of expectation value at a state, for any $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$\begin{aligned}\langle \hat{q} \rangle_{\alpha}^{e}(t) &= \langle \Phi_{\alpha}^{e}(q,t)|\hat{q}|\Phi_{\alpha}^{e}(q,t)\rangle = \langle \hat{U}_g(t,t_0)\Phi_{\alpha}^{e}(q)|\hat{q}|\hat{U}_g(t,t_0)\Phi_{\alpha}^{e}(q)\rangle \\ &= \langle \Phi_{\alpha}^{e}(q)|\hat{U}_g^{\dagger}(t,t_0)\hat{q}\hat{U}_g(t,t_0)|\Phi_{\alpha}^{e}(q)\rangle\end{aligned}$$

and

$$\begin{aligned}\langle \hat{p} \rangle_{\alpha}^{e}(t) &= \langle \Phi_{\alpha}^{e}(q,t)|\hat{p}|\Phi_{\alpha}^{e}(q,t)\rangle = \langle \hat{U}_g(t,t_0)\Phi_{\alpha}^{e}(q)|\hat{p}|\hat{U}_g(t,t_0)\Phi_{\alpha}^{e}(q)\rangle \\ &= \langle \Phi_{\alpha}^{e}(q)|\hat{U}_g^{\dagger}(t,t_0)\hat{p}\hat{U}_g(t,t_0)|\Phi_{\alpha}^{e}(q)\rangle.\end{aligned}$$

Using the following formulas, which are found in (Atılgan Büyükaşık & Çayiç, 2016)

$$\begin{aligned}\hat{U}_g^{\dagger}(t,t_0)\hat{q}\hat{U}_g(t,t_0) &= x_1(t)\hat{q} + x_2(t)\hat{p} + x_p(t), \\ \hat{U}_g^{\dagger}(t,t_0)\hat{p}\hat{U}_g(t,t_0) &= p_1(t)\hat{p} + p_2(t)\hat{p} + p_p(t),\end{aligned}$$

and $\langle \Phi_{\alpha}^{e}(q)|\hat{q}|\Phi_{\alpha}^{e}(q)\rangle = 0$, $\langle \Phi_{\alpha}^{e}(q)|\hat{p}|\Phi_{\alpha}^{e}(q)\rangle = 0$, we obtain

$$\langle \hat{q} \rangle_{\alpha}^{e}(t) = x_p(t), \quad \langle \hat{p} \rangle_{\alpha}^{e}(t) = p_p(t).$$

Then, we find expectation value of square of position as follows

$$\begin{aligned}
\langle\hat{q}^2\rangle^e_\alpha(t) &= \langle\Phi^e_\alpha(q,t)|\hat{q}^2|\Phi^e_\alpha(q,t)\rangle = \langle\hat{U}_g(t,t_0)\Phi^e_\alpha(q)|\hat{q}^2|\hat{U}_g(t,t_0)\Phi^e_\alpha(q)\rangle \\
&= \langle\Phi^e_\alpha(q)|\hat{U}^\dagger_g(t,t_0)\hat{q}^2\hat{U}_g(t,t_0)|\Phi^e_\alpha(q)\rangle \\
&= \langle\Phi^e_\alpha(q)|\left(\hat{U}^\dagger_g(t,t_0)\hat{q}\hat{U}_g(t,t_0)\right)^2|\Phi^e_\alpha(q)\rangle
\end{aligned}$$

Since

$$\begin{aligned}
\left(\hat{U}^\dagger_g(t,t_0)\hat{q}\hat{U}_g(t,t_0)\right)^2 &= x_1^2(t)\hat{q}^2 + x_1(t)x_2(t)\left(\hat{q}\hat{p}+\hat{p}\hat{q}\right) + x_2^2(t)\hat{p}^2 \\
&\quad +2\left(x_1(t)\hat{q}+x_2(t)\hat{p}\right)x_p(t) + x_p^2(t),
\end{aligned}$$

using the following expectation values at time-evolved even coherent states

$$\begin{aligned}
&\langle\Phi^e_\alpha(q)|\hat{q}^2|\Phi^e_\alpha(q)\rangle = \frac{\hbar}{2\omega_0}\left(1+2|\alpha|^2\tanh|\alpha|^2 + 2\left((\alpha^*)^2+\alpha^2\right)\right), \\
&\langle\Phi^e_\alpha(q)|\hat{p}^2|\Phi^e_\alpha(q)\rangle = \frac{\omega_0\hbar}{2}\left(1+2|\alpha|^2\tanh|\alpha|^2 - 2\left((\alpha^*)^2+\alpha^2\right)\right), \\
&\langle\Phi^e_\alpha(q)|\hat{q}\hat{p}+\hat{p}\hat{q}|\Phi^e_\alpha(q)\rangle = i\hbar\left((\alpha^*)^2-\alpha^2\right),
\end{aligned}$$

we get

$$\begin{aligned}
\langle\hat{q}^2\rangle^e_\alpha(t) &= \frac{\hbar}{2\omega_0}\left(1+2|\alpha|^2\tanh|\alpha|^2 + 2\left((\alpha^*)^2+\alpha^2\right)\right)x_1^2(t) \\
&\quad +i\hbar\left((\alpha^*)^2-\alpha^2\right)x_1(t)x_2(t) \\
&\quad +\frac{\omega_0\hbar}{2}\left(1+2|\alpha|^2\tanh|\alpha|^2 - 2\left((\alpha^*)^2+\alpha^2\right)\right)x_2^2(t) + x_p^2(t) \\
&= \frac{\hbar}{2\omega_0}\alpha^2\left(x_1^2(t) - 2i\omega_0 x_1(t)x_2(t) - \omega_0^2 x_2^2(t)\right) \\
&\quad +\frac{\hbar}{2\omega_0}(\alpha^*)^2\left(x_1^2(t) + 2i\omega_0 x_1(t)x_2(t) - \omega_0^2 x_2^2(t)\right) \\
&\quad +\frac{\hbar}{2\omega_0}\left(1+2|\alpha|^2\tanh|\alpha|^2\right)\left(x_1^2(t)+\omega_0^2 x_2^2(t)\right) + x_p^2(t).
\end{aligned}$$

We know that $\epsilon(t) = x_1(t) + i\omega_0 x_2(t)$, so

$$\langle \hat{q}^2 \rangle^e_\alpha(t) = \frac{\hbar}{2\omega_0}\left[(\alpha\epsilon^*(t))^2 + (\alpha^*\epsilon(t))^2 + |\epsilon(t)|^2\left(1 + 2|\alpha|^2\tanh|\alpha|^2\right)\right] + x_p^2(t).$$

By following the same steps, we can find expectation value of square of momentum as

$$\begin{aligned}\langle \hat{p}^2 \rangle^e_\alpha(t) &= \frac{\hbar}{2\omega_0}\left[(\alpha\mu(t)(\dot{\epsilon}^*(t) - B(t)\epsilon(t)))^2 + (\alpha^*\mu(t)(\dot{\epsilon}(t) - B(t)\epsilon(t)))^2\right. \\ &\quad \left. + \left(1 + 2|\alpha|^2\tanh|\alpha|^2\right)\frac{1}{|\epsilon(t)|^2}\left(\omega_0^2|\epsilon(t)|^4\left(\frac{d\ln|\epsilon(t)|}{dt} - B(t)\right)^2\right)\right] + p_p^2(t).\end{aligned}$$

Therefore, uncertainties of position and momentum at time-evolved even coherent states become

$$\begin{aligned}(\Delta\hat{q})^e_\alpha(t) &= \sqrt{\langle \hat{q}^2 \rangle^e_\alpha(t) - (\langle \hat{q} \rangle^e_\alpha(t))^2} = \sqrt{\frac{\hbar}{2\omega_0}}|\epsilon(t)|\sqrt{\Pi^e_q(t)}, \\ (\Delta\hat{p})^e_\alpha(t) &= \sqrt{\langle \hat{p}^2 \rangle^e_\alpha(t) - (\langle \hat{p} \rangle^e_\alpha(t))^2} = \sqrt{\frac{\omega_0\hbar}{2}}\frac{1}{|\epsilon(t)|}\sqrt{\Pi^e_p(t)},\end{aligned}$$

where

$$\begin{aligned}\Pi^e_q(t) &= 1 + 2|\alpha|^2\tanh|\alpha|^2 + \frac{2}{|\epsilon(t)|^2}\Re(\alpha\epsilon^*(t))^2, \\ \Pi^e_p(t) &= \left(1 + \frac{\mu^2(t)|\epsilon(t)|^4}{\omega_0^2}\left(\frac{d\ln|\epsilon(t)|}{dt} - B(t)\right)^2\right)\left(1 + 2|\alpha|^2\tanh|\alpha|^2\right) \\ &+ \frac{2|\epsilon(t)|^2}{\omega_0^2}\Re\left[\alpha^2(\mu(t)(\dot{\epsilon}^*(t) - B(t)\epsilon^*(t)))^2\right].\end{aligned}$$

By the same way, uncertainties at time-evolved odd coherent states can be found.

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