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THE DIFFERENCE OF HYPERHARMONIC NUMBERS VIA GEOMETRIC AND ANALYTIC METHODS

Çağatay Altuntaş, Haydar Göral, and Doğa Can Sertbaş

Abstract. Our motivation in this note is to find equal hyperharmonic numbers of different orders. In particular, we deal with the integerness property of the difference of hyperharmonic numbers. Inspired by finiteness results from arithmetic geometry, we see that, under some extra assumption, there are only finitely many pairs of orders for two hyperharmonic numbers of fixed indices to have a certain rational difference. Moreover, using analytic techniques, we get that almost all differences are not integers. On the contrary, we also obtain that there are infinitely many order values where the corresponding differences are integers.

1. Introduction

In this paper, we investigate the integerness property of the differences of hyperharmonic numbers. For this purpose, we apply geometric and analytic methods, and use a computer algebra toolbox to obtain several examples for hyperharmonic differences.

The n^{th} harmonic number is defined as the n^{th} partial sum of the harmonic series:

$$h_n = \sum_{k=1}^{n} \frac{1}{k}.$$

These numbers are equipped with various arithmetic and analytic properties so that there has been a constant focus on them. It is well known that for any $n > 1$, the n^{th} harmonic number is not an integer [24]. The difference $h_n - h_m$ is never an integer if $n > m \geq 1$ as well [15].

A generalization of harmonic numbers is the hyperharmonic numbers, introduced by Conway and Guy [7]. The n^{th} hyperharmonic number of *order* r is defined recursively as

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}$$

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for $r \geq 2$, where $h_n^{(1)} = h_n$ is the n^{th} harmonic number. They also presented a combinatorial identity that relates hyperharmonic numbers and harmonic numbers as follows:

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}).$$

This generalization has also plentiful properties, which attracts attention. For instance, the integerness problem for the hyperharmonic numbers has been studied by various authors. In 2007, Mező conjectured that there is no hyperharmonic integer for any integers $n, r \geq 2$ [17]. Moreover, he showed in the same paper that $h_n^{(r)}$ is non-integer for $n > 1$ and $r = 2, 3$.

This result was improved by Amrane and Belbachir in [2, 3], where they showed that $h_n^{(r)}$ is not an integer for any $n > 1$ and $r \leq 25$. They also gave a couple of (n, r) tuples where the corresponding hyperharmonic number is non-integer.

Then, these known results were extended by the second and the third authors [11]. For instance, it was shown that $h_n^{(r)}$ is non-integer for any $n > 1$ and $r \leq 20001$. Also, an asymptotic result was given as follows. Let

$$S(x) = |\{(n, r) \in [0, x] \times [0, x] : h_n^{(r)} \notin \mathbb{Z}\}|.$$

Then, one has

$$S(x) = x^2 + O\left(x^{\frac{2.475}{1.475}}\right)$$

so that the non-integer hyperharmonic numbers have full asymptotic density in the first quadrant. Later, the error term was improved in [1].

The generalized hyperharmonic numbers are another generalization in which different approaches can be applied to study their integerness. For instance, the interested reader may check [12] to see how topology can be used on the integerness of these numbers.

Despite all the results which support the conjecture of Mező, it was proven by the third author that there are infinitely many hyperharmonic integers [21].

From another point of view, one can consider the following problem which was also first proposed by Mező [17].

Problem 1.1. For which $n \neq m$ and $r \neq s$ does the equality

$$h_n^{(r)} = h_m^{(s)}$$

hold?

The motivation of this paper partially comes from this question and we give a partial answer. Moreover, we will show that the difference may be an integer, but it rarely happens.

Now, we state our first theorem.

Theorem A. *Let $n > m \geq 4$ and $\gcd(n-1, m-1) = 1$. Then for any rational number γ, there are only finitely many positive integer tuples (r, s) such that*

$$h_n^{(r)} - h_m^{(s)} = \gamma. \tag{1}$$

Moreover, equation (1) does not have any solutions when

$$(n, m) \in \{(3,2), (4,2), (4,3)\} \text{ and } \gamma \in \mathbb{Z}.$$

To prove the theorem, we will follow a geometric approach where we link our finiteness problem to a corresponding question in arithmetic geometry. In fact, one may relate to fundamental finiteness theorems in arithmetic geometry such as Mordell-Weil, Roth's, Siegel's and Falting's theorem [14].

Remark 1.2. Let $C(x) = |\{(n, m) \in [1, x]^2 : n, m \in \mathbb{Z}^{>0}, \gcd(n, m) = 1\}|$. Then, we have (see [4])

$$\lim_{x\to\infty} \frac{C(x)}{x^2} = \frac{6}{\pi^2}$$

so that a significant amount of tuples (n, m) in the rectangle $[1, x]^2$ are covered in the previous theorem.

Our second theorem states that the difference of hyperharmonic numbers can hardly be an integer, which is obtained by an analytic approach. In fact, we will give a careful count of the number of tuples (n, m, r, s) lying inside the four dimensional cube $[1, x]^4$ such that the corresponding hyperharmonic difference is non-integer. In particular, the non-integerness will be captured by a negative p-adic order for some p in a short interval.

Theorem B. *Let $T(x)$ be the number of tuples $(n, m, r, s) \in [1, x]^4$ so that the difference $h_n^{(r)} - h_m^{(s)}$ is not an integer. Then, for any $\epsilon > 0$ we have*

$$T(x) = x^4 + O_\epsilon\left(x^{\frac{59}{18}+\epsilon}\right),$$

where the implied constant depends only on ϵ. Moreover, if we assume the Riemann hypothesis, then we obtain

$$T(x) = x^4 + O\left(x^3 \log^3 x\right).$$

On the other hand, we are able to find infinitely many tuples $(n, m, r, s) \in \mathbb{Z}^4$ such that the corresponding difference $h_n^{(r)} - h_m^{(s)}$ is an integer. For instance, when $n = 6$, we have some values given in Table 1.

By Table 1, we see that $(r, s) = (20, 47501)$ is a solution for Problem 1.1 when $n = 6$ and $m = 2$. In particular, we will show in Section 4 that there are infinitely many solutions of this problem.

Now, throughout this paper, let $\mathbb{P}$ denote the set of prime numbers and for a given prime number p, let ν_p denote the p-adic order defined as follows. For a given integer n and a prime p, we define

$$\nu_p(n) = \begin{cases} a & \text{if } p^a \parallel n, \\ \infty & \text{if } n = 0, \end{cases}$$

TABLE 1. Several m, r, s values where the difference $h_n^{(r)} - h_m^{(s)}$ is an integer, when $n = 6$.

m	r	s	$h_n^{(r)} - h_m^{(s)}$
2	20	47501	0
3	15	161	296
4	5	4	151
5	6	1	338
6	723	3	1674946827908

and for a given rational number $\frac{a}{b}$, we set

$$\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b).$$

The p-adic order will be used frequently, particularly in the analytic point of view.

2. Geometric methods

In this section, we prove Theorem A using arithmetic geometry. The motivation of the theorem rises from the question: can a hyperharmonic difference be 0 or not? That is, we investigate whether

$$h_n^{(r)} = h_m^{(s)} \tag{2}$$

may hold or not. Now, before going any further, let us state the following lemma, which eases the computations with hyperharmonic numbers and will be used frequently throughout the paper.

Lemma 2.1. *For any positive integer n, define $f_n(x)$ as $\prod_{i=0}^{n-1}(x+i)$. Then, for any positive integer r, we have*

$$h_n^{(r)} = \frac{f_n'(r)}{n!}.$$

Proof. We have $\log f_n(x) = \sum_{i=0}^{n-1} \log(x+i)$ and by differentiating both sides we obtain

$$\frac{f_n'(x)}{f_n(x)} = \sum_{i=0}^{n-1} \frac{1}{x+i}.$$

It is known by [7] that the n^{th} hyperharmonic number of order r can be expressed as

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}).$$

As a result,

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1})$$

$$= \frac{r(r+1)\cdots(n+r-1)}{n!}\left(\frac{1}{r}+\frac{1}{r+1}+\cdots+\frac{1}{n+r-1}\right)$$
$$= \frac{f_n(r)}{n!}\frac{f_n^{'}(r)}{f_n(r)} = \frac{f_n^{'}(r)}{n!},$$

and we are done. □

Consequently, working with (2) can be done by looking for solutions of the equation

$$\frac{f_n^{'}(r)}{n!} = \frac{f_m^{'}(s)}{m!}. \tag{3}$$

To answer the question, we make use of [6, Theorem 1.1]. Now, (3) can be written as

$$p(x) = q(y) \tag{4}$$

for some polynomials $p(x), q(x) \in \mathbb{Q}[x]$ of degrees $n-1$ and $m-1$, respectively. Thus, one may consider the solutions of

$$F(x,y) := p(x) - q(y) = 0 \tag{5}$$

instead of (4). Let us say that the equation $F(x,y) = 0$ has infinitely many rational solutions with a bounded denominator if there is a positive integer δ such that (5) has infinitely many solutions $(x,y) \in \mathbb{Q}\times\mathbb{Q}$ where $\delta x, \delta y \in \mathbb{Z}$. Now, we state five *standard pairs* of polynomials $(p(x), q(x))$ over $\mathbb{Q}$ as in [6] as follows. Let a, b be non-zero rational numbers, m, n be positive integers and $g(x)$ be a non-zero polynomial.

1) *The first kind.* A pair

$$(x^k, ax^r g(x)^k)$$

or switched, $(ax^r g(x)^k, x^k)$ is a standard pair of the first kind, provided that $0 \le r < k$, $\gcd(r,k) = 1$ and $r + \deg g(x) > 0$.

2) *The second kind.* A pair

$$(x^2, (ax^2+b)g(x)^2)$$

or switched is a standard pair of the second kind.

Let $D_k(x,\alpha)$ be the k^{th} *Dickson polynomial of the first kind* defined as

$$D_k(x,\alpha) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k}{k-i}\binom{k-i}{i}(-\alpha)^i x^{k-2i}$$

with parameter $\alpha \in \mathbb{Q}$ (see [16]).

3) *The third kind.* A pair

$$\left(D_k\left(x, a^\ell\right), D_\ell\left(x, a^k\right)\right)$$

with $\gcd(k,\ell) = 1$ is a standard pair of the third kind.

4) *The fourth kind.* A pair

$$(a^{-k/2}D_k(x,a), -b^{-\ell/2}D_\ell(x,b))$$

with $\gcd(k,\ell)=2$ is a standard pair of the fourth kind.

5) *The fifth kind.* A pair

$$((ax^2-1)^3, 3x^4-4x^3)$$

or switched, is a standard pair of the fifth kind.

In fact, our concern will be whether the polynomials in (4) are standard pairs or not.

Remark 2.2. Given a standard pair $(p(x), q(x))$ over $\mathbb{Q}$ of any kind, (4) has infinitely many rational solutions with a bounded denominator (see [6, p. 2]).

The following theorem will be a key step towards Theorem A.

Theorem 2.3 ([6, Theorem 1.1])**.** *Let $p(x), q(x)$ be non-constant polynomials over $\mathbb{Q}$. Then, the following statements are equivalent.*

(i) *There are infinitely many rational solutions with a bounded denominator of equation* (4)*.*

(ii) *The polynomials p and q can be written as $p=\varphi\circ p_1\circ\lambda$ and $q=\varphi\circ q_1\circ\mu$ where $\lambda(x), \mu(x)$ are linear polynomials over $\mathbb{Q}$, $\varphi(x)\in\mathbb{Q}[x]$ and (p_1,q_1) is a standard pair over $\mathbb{Q}$ such that the equation $p_1(x)=q_1(y)$ has infinitely many rational solutions with a bounded denominator.*

Moreover, we need the following fact from [6] for our set up.

Fact 2.4 ([6, Remark 1.2.ii])**.** In Theorem 2.3(ii), if we have

$$\gcd(\deg p, \deg q)=1,$$

then $\deg\varphi=1$ and $(p_1(x), q_1(x))$ is a standard pair of the first or third kind over $\mathbb{Q}$.

Now, the next proposition will be a first step towards proving Theorem A.

Proposition 2.5. *For any positive integer n, let*

$$f_n(x) := \prod_{i=0}^{n-1}(x+i).$$

Suppose that $n>3$. Then, the polynomial $\frac{f_n'(x)}{n!}+\gamma$ cannot be written as

$$a(cx+d)^{n-1}+b$$

for any rational numbers a, b, c, d, γ with $a, c\neq 0$.

Proof. Let $n>3$ be a positive integer. We have

$$\begin{aligned} f_n(x) &= x(x+1)\cdots(x+n-1)\\ &= x^n + \big(1+2+\cdots+(n-1)\big)x^{n-1} + \left(\sum_{1\le i<j\le n-1} ij\right)x^{n-2}+\cdots\\ &\quad + \left(\sum_{i=1}^{n-1}\frac{(n-1)!}{i}\right)x^2 + (n-1)!x. \end{aligned}$$

One may verify that

$$\sum_{1\le i<j\le t} ij = \frac{(t-1)t(t+1)(3t+2)}{24}.$$

Then, we can write

$$f_n(x) = x^n + \frac{(n-1)n}{2}x^{n-1} + \frac{(n-2)(n-1)n(3n-1)}{24}x^{n-2} + \cdots$$
$$+ (n-1)!h_{n-1}x^2 + (n-1)!x.$$

Taking derivative with respect to x, we obtain

$$f_n'(x) = nx^{n-1} + \frac{(n-1)^2 n}{2}x^{n-2} + \frac{(n-2)^2(n-1)n(3n-1)}{24}x^{n-3} + \cdots$$
$$+ 2(n-1)!h_{n-1}x + (n-1)!.$$

Now, suppose that

$$\frac{f_n'(x)}{n!} + \gamma = a(cx+d)^{n-1} + b$$

holds for some rational numbers a, b, c, d and γ with $a, c \neq 0$. We have

$$\frac{f_n'(x)}{n!} = a(cx+d)^{n-1} + b - \gamma.$$

Recall that $n > 3$, so we can equate the coefficients of x^{n-1}, x^{n-2} and x^{n-3} on both sides as follows.

Coefficient of x^{n-1}. The equality

$$\frac{n}{n!} = ac^{n-1}$$

implies

$$ac^{n-1} = \frac{1}{(n-1)!}. \tag{6}$$

Coefficient of x^{n-2}. We have

$$\frac{(n-1)^2 n}{2n!} = (n-1)ac^{n-2}d$$

which gives

$$ac^{n-2}d = \frac{1}{2(n-2)!}. \tag{7}$$

Coefficient of x^{n-3}. The equation

$$\frac{(n-2)^2(n-1)n(3n-1)}{24n!} = a\binom{n-1}{2}c^{n-3}d^2$$

yields

$$ac^{n-3}d^2 = \frac{(n-2)^2(3n-1)}{12n!}. \tag{8}$$

Now, multiplying (7) with c gives

$$ac^{n-1}d = \frac{c}{2(n-2)!}. \tag{9}$$

By (6) we have $ac^{n-1} = \frac{1}{(n-1)!}$. Thus, using (9), we obtain

$$\frac{d}{(n-1)!} = \frac{c}{2(n-2)!}$$

so that we have

$$\frac{c}{d} = \frac{2}{n-1}. \tag{10}$$

Moreover, by (7) and (8), we have

$$\frac{ac^{n-2}d}{ac^{n-3}d^2} = \frac{1}{2(n-2)!}\frac{12n!}{(n-2)^2(3n-1)}.$$

Consequently, we can write

$$\frac{c}{d} = \frac{6(n-1)n}{(n-2)^2(3n-1)}. \tag{11}$$

Combining (10) and (11), one arrives at

$$\frac{2}{n-1} = \frac{6(n-1)n}{(n-2)^2(3n-1)}.$$

Therefore,

$$7n^2 - 13n + 4 = 0$$

must hold. However, $7n^2 - 13n + 4 > 0$ for any $n > 3$. This is a contradiction, and we conclude the result. □

The following proposition will be another key step towards our proof of Theorem A.

Proposition 2.6. *Let n be a positive integer and define*

$$f_n(x) := \prod_{i=0}^{n-1}(x+i).$$

Suppose that $n > 5$. Then, the polynomial $\frac{f_n'(x)}{n!} + \gamma$ cannot be written as

$$aD_{n-1}(cx+d,\alpha) + b,$$

where D_{n-1} is the $(n-1)^{th}$ Dickson polynomial of the first kind and a, b, c, d, α, γ with $a, c \neq 0$ are rational numbers.

Proof. Assume that $n > 5$ and write

$$\begin{aligned} f_n(x) &= x(x+1)\cdots(x+n-1) \\ &= x^n + \left(\sum_{i=1}^{n-1} i\right)x^{n-1} + \left(\sum_{1\le i<j\le n-1} ij\right)x^{n-2} + \left(\sum_{1\le i<j<k\le n-1} ijk\right)x^{n-3} \end{aligned}$$

$$+\left(\sum_{1\le i<j<k<\ell\le t} ijk\ell\right)x^{n-4}+\cdots+(n-1)!x.$$

One may check that

$$\sum_{1\le i<j\le t} ij=\frac{(t-1)t(t+1)(3t+2)}{24} \text{ and } \sum_{1\le i<j<k\le t} ijk=\frac{(t-2)(t-1)t^2(t+1)^2}{48}$$

for any positive integer $t\ge 3$. In addition,

$$\sum_{1\le i<j<k<\ell\le t} ijk\ell=\frac{(t-3)(t-2)(t-1)t(t+1)(15t^3+15t^2-10t-8)}{5760}$$

holds. Thus, we write

$$\begin{aligned}f_n(x)=x^n&+\frac{(n-1)n}{2}x^{n-1}+\frac{(n-2)(n-1)n(3n-1)}{24}x^{n-2}\\&+\frac{(n-3)(n-2)(n-1)^2n^2}{48}x^{n-3}\\&+\frac{(n-4)(n-3)(n-2)(n-1)n(15n^3-30n^2+5n+2)}{5760}x^{n-4}\\&+\cdots+(n-1)!x.\end{aligned}$$

Taking derivative, we have

$$\begin{aligned}f_n^{'}(x)=nx^{n-1}&+\frac{(n-1)^2n}{2}x^{n-2}+\frac{(n-2)^2(n-1)n(3n-1)}{24}x^{n-3}\\&+\frac{(n-3)^2(n-2)(n-1)^2n^2}{48}x^{n-4}\\&+\frac{(n-4)^2(n-3)(n-2)(n-1)n(15n^3-30n^2+5n+2)}{5760}x^{n-5}\\&+\cdots+(n-1)!.\end{aligned}$$

Now, suppose that

$$\frac{f_n'(x)}{n!}+\gamma=aD_{n-1}(cx+d,\alpha)+b$$

holds for some positive integer $n>5$ and for some rational numbers a, b, c, d, α, γ with $a,c,\alpha\ne 0$. Let us write

$$\frac{f_n'(x)}{n!}=aD_{n-1}(cx+d,\alpha)+b-\gamma. \tag{12}$$

We will equate the coefficients of the monomials x^{n-1}, x^{n-2}, x^{n-3}, x^{n-4} and x^{n-5} on both sides of (12). As we have

$$D_{n-1}(cx+d,\alpha)=\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{n-1}{n-1-i}\binom{n-1-i}{i}(-\alpha)^i(cx+d)^{n-1-2i},$$

determining the terms in the sum for $i = 0, 1, 2$ will be enough for our purposes. We have

$$
\begin{aligned}
&D_{n-1}(cx+d,\alpha)\\
={}& \frac{n-1}{n-1}\binom{n-1}{0}(-\alpha)^0(cx+d)^{n-1}\\
&+\frac{n-1}{n-2}\binom{n-2}{1}(-\alpha)(cx+d)^{n-3}\\
&+\frac{n-1}{n-3}\binom{n-3}{2}(-\alpha)^2(cx+d)^{n-5}+\cdots\\
&+\frac{n-1}{n-1-\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{n-1}{2}\rfloor}(-\alpha)^{\lfloor\frac{n-1}{2}\rfloor}(cx+d)^{n-1-2\lfloor\frac{n-1}{2}\rfloor},
\end{aligned}
$$

so that we can write

$$
\begin{aligned}
D_{n-1}(cx+d,\alpha) = (cx+d)^{n-1} &- (n-1)\alpha(cx+d)^{n-3}\\
&+\frac{(n-1)(n-4)}{2}\alpha^2(cx+d)^{n-5}+E_1(x).
\end{aligned}
$$

Furthermore, the polynomial $aD_{n-1}(cx+d,\alpha)+b-\gamma$ in (12) can be written as follows.

$$
\begin{aligned}
&aD_{n-1}(cx+d,\alpha)+b-\gamma\\
={}& ac^{n-1}x^{n-1}+(n-1)ac^{n-2}dx^{n-2}\\
&+(n-1)ac^{n-3}\left(\frac{n-2}{2}d^2-\alpha\right)x^{n-3}\\
&+(n-1)(n-3)ac^{n-4}d\left(\frac{n-2}{6}d^2-\alpha\right)x^{n-4}\\
&+\frac{(n-4)(n-1)}{2}ac^{n-5}\left[\frac{(n-3)(n-2)}{12}d^4-(n-3)d^2\alpha+\alpha^2\right]x^{n-5}\\
&+E_2(x).
\end{aligned}
$$

Now, we are set to equate the first five coefficients in (12).

Coefficient of x^{n-1}. We have

$$ac^{n-1} = \frac{1}{(n-1)!}. \tag{13}$$

Coefficient of x^{n-2}. We write

$$\frac{(n-1)^2 n}{2n!} = (n-1)ac^{n-2}d$$

so that

$$ac^{n-2}d = \frac{1}{2(n-2)!}. \tag{14}$$

Coefficient of x^{n-3}. The equation

$$\frac{(n-2)^2(n-1)n(3n-1)}{24n!} = (n-1)ac^{n-3}\left(\frac{n-2}{2}d^2-\alpha\right)$$

implies

$$\frac{(n-2)(3n-1)}{24(n-1)(n-3)!} = ac^{n-3}\left(\frac{n-2}{2}d^2-\alpha\right). \tag{15}$$

Coefficient of x^{n-4}. The equality

$$\frac{(n-3)^2(n-2)(n-1)^2n^2}{48n!} = (n-1)(n-3)ac^{n-4}d\left(\frac{n-2}{6}d^2-\alpha\right)$$

gives

$$\frac{n}{48(n-4)!} = ac^{n-4}d\left(\frac{n-2}{6}d^2-\alpha\right). \tag{16}$$

Coefficient of x^{n-5}. In (12), on the left hand-side we have

$$k_1 = \frac{(n-4)^2(n-3)(n-2)(n-1)n(15n^3-30n^2+5n+2)}{5760n!}$$

and on the right hand-side we have

$$k_2 = \frac{(n-4)(n-1)}{2}ac^{n-5}\left[\frac{(n-3)(n-2)}{12}d^4-(n-3)d^2\alpha+\alpha^2\right].$$

We ignore any cancellations in this case and simply write

$$k_1 = k_2 \tag{17}$$

for the coefficients of x^{n-5} in short.

Now, using the equations above, we can write everything in terms of the number c. Multiplying (14) with c, we have

$$ac^{n-1}d = \frac{c}{2(n-2)!}.$$

Using (13) we get

$$d = \frac{n-1}{2}c.$$

Then, (15) can be written as

$$\frac{(n-2)(3n-1)}{24(n-1)(n-3)!} = ac^{n-3}\left(\frac{n-2}{2}\left(\frac{n-1}{2}c\right)^2-\alpha\right).$$

Consequently,

$$\begin{aligned} ac^{n-3}\alpha &= \frac{(n-2)(n-1)^2}{8}ac^{n-1} - \frac{(n-2)(3n-1)}{24(n-1)(n-3)!} \\ &\overset{(13)}{=} \frac{(n-2)(n-1)^2}{8}\frac{1}{(n-1)!} - \frac{(n-2)(3n-1)}{24(n-1)(n-3)!} \end{aligned}$$

$$= \frac{3(n-2)(n-1)^2 - (n-2)^2(3n-1)}{24(n-1)!}$$
$$= \frac{(n-2)(n+1)}{24(n-1)!}.$$

Next, by multiplying both sides with c^2, we obtain

$$ac^{n-1}\alpha \overset{(13)}{=} \frac{\alpha}{(n-1)!} = \frac{c^2(n-2)(n+1)}{24(n-1)!}.$$

Thus, we see that

$$\alpha = \frac{c^2(n-2)(n+1)}{24}. \tag{18}$$

Before we proceed to the last step of the proof, notice that one can also use (16) to obtain (18).

Finally, we write d and α in terms of c in (17) as follows:

$$\begin{aligned}
k_2 &= \frac{(n-4)(n-1)}{2} ac^{n-5}\left[\frac{(n-3)(n-2)}{12} d^4 - (n-3)d^2\alpha + \alpha^2\right] \\
&= \frac{(n-4)(n-1)}{2} ac^{n-5}\left[\begin{array}{l} \frac{(n-3)(n-2)}{12}\left(\frac{n-1}{2}c\right)^4 \\ -(n-3)\left(\frac{n-1}{2}c\right)^2\left(\frac{c^2(n-2)(n+1)}{24}\right) \\ +\left(\frac{c^2(n-2)(n+1)}{24}\right)^2 \end{array}\right] \\
&= \frac{(n-4)(n-2)(n-1)}{2} ac^{n-1}\left[\begin{array}{l} \frac{(n-3)(n-1)^4}{192} - \frac{(n-3)(n-1)^2(n+1)}{96} \\ + \frac{(n-2)(n+1)^2}{576} \end{array}\right] \\
&\overset{(13)}{=} \frac{(n-4)(n-2)(n-1)}{2} \frac{1}{(n-1)!}\left[\begin{array}{l} \frac{(n-3)(n-1)^4}{192} - \frac{(n-3)(n-1)^2(n+1)}{96} \\ + \frac{(n-2)(n+1)^2}{576} \end{array}\right] \\
&= \frac{n-4}{2(n-3)!}\left(\frac{3n^5 - 27n^4 + 79n^3 - 78n^2 + 12n + 7}{576}\right) \\
&= k_1 = \frac{(n-4)^2(n-3)(n-2)(n-1)n(15n^3 - 30n^2 + 5n + 2)}{5760n!}.
\end{aligned}$$

In fact, we obtain that

$$\begin{aligned}
&5(3n^5 - 27n^4 + 79n^3 - 78n^2 + 12n + 7) \\
&= (n-4)(n-3)(15n^3 - 30n^2 + 5n + 2)
\end{aligned}$$

which in turn yields

$$3n^2 + 14n + 11 = 0.$$

However, the polynomial

$$3n^2 + 14n + 11$$

is always positive for any $n > 5$, a contradiction. This completes the proof. $\square$

Now, to give our results via a geometric approach, we continue by recalling some basic definitions from arithmetic geometry. The interested reader may consult [14, 23].

Let k be a field. We define the usual *affine plane* as

$$\mathbb{A}^2(k) = \{(x, y) : x, y \in k\}.$$

For any positive integer n, the affine space $\mathbb{A}^n(k)$ is defined similarly. Now, suppose that $a, b, c, x, y, z \in k$ such that the vectors (a, b, c) and (x, y, z) are not the zero vector $(0, 0, 0)$. Then, define a relation $\sim$ as follows: $(x, y, z) \sim (a, b, c)$ if and only if there exists $\lambda \in k^*$ such that $x = \lambda a$, $y = \lambda b$, $z = \lambda c$. This relation is an equivalence relation so that we have the equivalence classes

$$[x, y, z] = \{(a, b, c) : a, b, c \in k, (a, b, c) \neq (0, 0, 0) \text{ and } (x, y, z) \sim (a, b, c)\}.$$

Then, the projective plane over k is defined by

$$\mathbb{P}^2(k) = \{[x, y, z] : x, y, z \in k, (x, y, z) \neq (0, 0, 0)\}.$$

Note that if $z \neq 0$, then we have $(x, y, z) \sim (\frac{x}{z}, \frac{y}{z}, 1)$. Therefore, we can write

$$\mathbb{P}^2(k) = \{[x, y, 1] : x, y \in k\} \cup \{[a, b, 0] : a, b \in k\}.$$

The points in the set $\{[a, b, 0] : a, b \in k\}$ above are called the points at infinity.

A curve in the affine plane $\mathbb{A}^2(k)$ is defined by the set of $\bar{k}$-solutions of a polynomial in $k[x, y]$. To define a curve in the projective plane, we need a homogeneous polynomial. We say that a polynomial $F(x, y, z)$ is *homogeneous* of degree d if for any monomial $x^p y^r z^s$ in F, we have $p + r + s = d$. Now, a projective curve C in the projective plane $\mathbb{P}^2(k)$ is defined as the set of $\bar{k}$-solutions of a non-constant homogeneous polynomial $F(x, y, z)$ in $k[x, y, z]$. We will simply write $\mathbb{A}^2$ and $\mathbb{P}^2$ for affine and projective planes when k is understood from the context. Now, let us consider a curve $C : f(x, y) = 0$ in the affine space. We extend C to a curve $\widehat{C}$ in the projective plane as follows. Let d be the highest degree of the monomials in $f(x, y)$. Then, we define

$$\widehat{C} : F(x, y, z) = z^d f\left(\frac{x}{z}, \frac{y}{z}\right) = 0.$$

The curve $\widehat{C}$ is called the *projectivization* of C. Note that if (x, y) is a point on C, then $[x, y, 1]$ is a point on the curve $\widehat{C}$. We say that an affine curve

$$C : f(x, y) = 0$$

is *singular* at a point $P \in C$ if

$$\frac{\partial f}{\partial x}(P) = f_x(P) = 0 \text{ and } \frac{\partial f}{\partial y}(P) = f_y(P) = 0.$$

Similarly, we say that the projective curve $C' : F(x, y, z) = 0$ is singular at a point $Q \in C'$ if the partial derivatives F_x, F_y and F_z vanish at Q. Otherwise, we say that C' is non-singular, or *smooth*, at the point Q. If the curve C' is smooth at every point, then C' is called a smooth curve. Note that the same definitions apply to affine curves.

From now on, let us take $k = \mathbb{C}$. Suppose that C is an affine curve and P is a point on the curve. If the coordinates of P are integers, then we say that P is an integral point on the curve and if the coordinates are rational numbers, then we say that P is a rational point on the curve. The set of integral and rational points on C are denoted by $C(\mathbb{Z})$ and $C(\mathbb{Q})$, respectively. Moreover, we say that a projective curve C given by the polynomial equation $F(x, y, z) = 0$ is a *rational curve* if $F(x, y, z) \in \mathbb{Q}[x, y, z]$.

In addition, for a given curve C we have a numerical invariant g called *genus*, a non-negative integer, in which its derivation relies on the number of singularities of C (see [10, Chapter 8]).

On the other hand, whenever we have a smooth projective curve C defined over $\mathbb{Q}$ of degree d, we have

$$g = \frac{(d-1)(d-2)}{2}.$$

This is called the *genus-degree formula*. In 1929, *Siegel* (see [22]) proved that if C is a smooth rational curve with genus $g > 0$, then $C(\mathbb{Z})$ is finite. In 1983, the result was improved for genus $g > 1$ by *Faltings* (see [9]). He proved that if C is a smooth rational curve with genus $g > 1$, then $C(\mathbb{Q})$ is finite. This was also known as the Mordell Conjecture.

Now, suppose that we have

$$h_n^{(r)} - h_m^{(s)} = a$$

for some integers n, m, r, s and a rational number a. By Lemma 2.1, we can write

$$h_n^{(r)} - h_m^{(s)} = \frac{f_n'(r)}{n!} - \frac{f_m'(s)}{m!} = a. \tag{19}$$

Without loss of generality, we may assume $n \geq m \geq 2$. Then, we can rewrite (19) as

$$f_n'(r) - d \cdot f_m'(s) = n!a$$

with

$$d = n(n-1) \cdots (m+1). \tag{20}$$

Now, let us define a curve in the affine plane $\mathbb{A}^2$ by

$$C_{n,m,a} : f(r, s) = f_n'(r) - d \cdot f_m'(s) - n!\ a = 0.$$

Recall that $f_n'(r)$ is of degree $n - 1$ and $f_m'(s)$ is of degree $m - 1$. Therefore, we can define the projectivization $\widehat{C}_{n,m,a}$ of $C_{n,m,a}$ in the projective plane $\mathbb{P}^2$

as follows.

$$\widehat{C}_{n,m,a} : F(r,s,t) = t^{n-1} f\left(\frac{r}{t}, \frac{s}{t}\right) = 0.$$

Consequently, we obtain that

$$\begin{aligned} F(r,s,t) = &\ \left(a_{n-1}r^{n-1} + a_{n-2}r^{n-2}t + \cdots + a_1 r t^{n-2} + a_0 t^{n-1}\right) \\ &- \left(b_{n-2}s^{n-2}t + b_{n-3}s^{n-3}t^2 + \cdots + b_1 s t^{n-2} + b_0 t^{n-1}\right) - (n!a)t^{n-1} \\ = &\ 0 \end{aligned}$$

for some positive integers a_i, b_j with $i = 0, \ldots, n-1$, and $j = 0, \ldots, n-2$ (see Propositions 2.5, 2.6).

At this point, we can associate our work on hyper harmonic differences with arithmetic geometry. If, for some rational number a and a integer tuple (m, n), the corresponding algebraic curve $C_{n,m,a}$ is smooth and its genus is greater than 0, then $C(\mathbb{Z})$ is finite by Siegel's Theorem [22]. In fact, there are only finitely many positive integer tuples (r, s) which satisfy

$$h_n^{(r)} = h_m^{(s)}.$$

Next, let us show that the curve $\widehat{C}_{n,m,a}$ is singular whenever $n - m > 1$.

Proposition 2.7. *Let $n > m$ be two positive integers and a be a rational number. Then,*

$$n - m = 1 \text{ if and only if the projective curve } \widehat{C}_{n,m,a} \text{ is smooth at infinity.}$$

Proof. First, assume that $n - m = 1$. In order to define the projective curve $\widehat{C}_{n,m,a}$, let us write

$$h_n^{(r)} - h_m^{(s)} = \frac{f_n'(r)}{n!} - \frac{f_m'(s)}{m!} = a.$$

Moreover, we define the affine curve

$$C_{n,m,a} : f(r,s) = f_n'(r) - n f_{n-1}'(s) - n!\ a = 0$$

as $m = n - 1$. Then, we can define the projective curve $\widehat{C}_{n,m,a}$ as above:

$$\widehat{C}_{n,m,a} : F(r,s,t) = t^{n-1} f\left(\frac{r}{t}, \frac{s}{t}\right) = 0.$$

Now, we check which point at infinity lie on the curve $\widehat{C}_{n,m,a}$. Let $F(P) = 0$ for some $P = [r_0, s_0, 0]$. Then, we have

$$a_{n-1} r_0^{n-1} = 0,$$

but as $a_{n-1} = n \neq 0$, we obtain that $r_0 = 0$. Consequently, we get $P = [0, 1, 0]$ as the only point at infinity lying on the curve. Next, observe that

$$\begin{aligned} F_r &= (n-1)a_{n-1}r^{n-2} + (n-2)a_{n-2}r^{n-3}t + \cdots + a_1 t^{n-2} \\ F_s &= -(n-2)b_{n-2}s^{n-3}t - \cdots - b_1 t^{n-2} \text{ and} \\ F_t &= (n-1)(a_0 - b_0 - n!a)t^{n-2} + \cdots + (a_{n-2}r^{n-2} - b_{n-2}s^{n-2}). \end{aligned}$$

Furthermore, as $n > m > 0$ and $F_t(0,1,0) = -b_{n-2} = -n(n-1) \neq 0$ we conclude that the curve $\widehat{C}_{n,m,a}$ is smooth at the point $[0,1,0]$.

Conversely, suppose that the projective curve is smooth at infinity. Let $n - m = \ell > 1$ and write

$$f_n'(r) = a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 \text{ and } f_m'(s) = b_{m-1}s^{m-1} + \cdots + b_1 s + b_0$$

for some $a_i, b_j \in \mathbb{Z}$ for $i = 0,1,\ldots,n-1$ and $j = 0,1,\ldots,m-1$. Also, let us write $d_j = d \cdot b_j$, $j = 0,1,\ldots,m-1$ for simplicity, where d is defined as in (20). We have

$$\begin{aligned} F(r,s,t) = &\left(a_{n-1}r^{n-1} + a_{n-2}r^{n-2}t + \cdots + a_1 r t^{n-2} + a_0 t^{n-1}\right) \\ &- \left(d_{m-1}s^{m-1}t^{\ell} + d_{m-2}s^{m-2}t^{\ell+1} + \cdots + d_1 s t^{n-2} + d_0 t^{n-1}\right) \\ &- (n!a)t^{n-1} \\ = &\ 0. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} F_r &= (n-1)a_{n-1}r^{n-2} + (n-2)a_{n-2}r^{n-3}t + \cdots + a_1 t^{n-2} \\ F_s &= -(m-1)d_{m-1}s^{m-2}t^{\ell} - (m-2)d_{m-2}s^{m-3}t^{\ell+1} \\ &\quad - \cdots - d_1 t^{n-2} \text{ and} \\ F_t &= (n-1)(a_0 - d_0 - n!a)t^{n-2} + (n-2)(a_1 r - d_1 s)t^{n-3} \\ &\quad + \cdots + a_{n-2}r^{n-2}. \end{aligned} \tag{21}$$

Notice that the point $P = [0,1,0]$ is on the curve $\widehat{C}_{n,m,a}$ as

$$F(0,1,0) = 0.$$

Moreover, by (21), we have

$$F_r(P) = F_s(P) = F_t(P) = 0$$

since $\ell > 1$. Thus, we obtain that the curve $\widehat{C}_{n,m,a}$ is singular at one of the points at infinity. Therefore, we must have $n - m = 1$ and this completes the proof. □

Now, we are set to prove the first part of Theorem A.

Proof of Theorem A. Let $n > m \geq 4$ be two positive integers with $\gcd(n-1, m-1) = 1$ and γ be any rational number such that the following equation is satisfied.

$$h_n^{(r)} - h_m^{(s)} = \gamma. \tag{22}$$

Case 1. $n \geq 6$.

In this case, we make use of Theorem 2.3. Observe that (22) can be rewritten as

$$\frac{f_n'(r)}{n!} = \frac{f_m'(s)}{m!} + \gamma$$

so that we have a polynomial equation

$$p(r) = q(s) \tag{23}$$

with $p(x), q(x) \in \mathbb{Q}[x]$. Moreover, notice that the polynomials $f_n^{'}(r)$ and $f_m^{'}(s)$ are of degrees $n-1$ and $m-1$, respectively. Hence, since $n > m \geq 4$, the polynomials p and q are non-constant.

Now, suppose that equation (23) has infinitely many integer solutions. Thus, it has infinitely many rational solutions with a bounded denominator. Then by Theorem 2.3, the polynomials p and q can be decomposed as $p = \varphi \circ p_1 \circ \lambda$ and $q = \varphi \circ q_1 \circ \mu$ where

- $\lambda(x), \mu(x) \in \mathbb{Q}[x]$ are linear,
- $\varphi(x) \in \mathbb{Q}[x]$ and
- (p_1, q_1) is a standard pair over the rationals

such that the equation $p_1(x) = q_1(y)$ has infinitely many rational solutions with a bounded denominator. In this case, since $\gcd(n-1, m-1) = \gcd(\deg p, \deg q) = 1$ we know by Fact 2.4 that $\deg \varphi = 1$ and $(p_1(x), q_1(x))$ must be a standard pair of the first or third kind over $\mathbb{Q}$. Moreover, since the polynomials φ, λ, μ are all linear, we have

$$\deg p_1 = \deg p = n-1 \text{ and } \deg q_1 = \deg q = m-1.$$

If the pair $(p_1(x), q_1(x))$ is of the first kind, recall that they must be of the form

$$(x^k, ax^r g(x)^k)$$

or switched $(ax^r g(x)^k, x^k)$ for some non-zero rational a and for some non-zero polynomial $g(x)$ over the rationals, provided that $0 \leq r < k$, $\gcd(r, k) = 1$ and $r + \deg g(x) > 0$ are satisfied. Moreover, let

$$\varphi(x) = ax + b, \lambda(x) = c_1 x + d_1 \text{ and } \mu(x) = c_2 x + d_2$$

for some rational numbers a, b, c_1, c_2, d_1, d_2 with $a, c_1, c_2 \neq 0$. Furthermore, either $p_1(x) = x^{n-1}$ or $q_1(x) = x^{m-1}$ must hold. If $p_1(x) = x^{n-1}$, then we write

$$p(x) = (\varphi \circ p_1 \circ \lambda)(x) = a(c_1 x + d_1)^{n-1} + b \tag{24}$$

and if $q_1(x) = x^{m-1}$, then we have

$$q(x) = (\varphi \circ q_1 \circ \mu)(x) = a(c_2 x + d_2)^{m-1} + b. \tag{25}$$

However, as $n \geq 6$ and $m \geq 4$, a decomposition as in (24) or (25) is not possible by Proposition 2.5.

If (p_1, q_1) is a standard pair of the third kind, let us write $p_1(x) = D_{n-1}(x, \alpha)$ where α is a non-zero parameter in $\mathbb{Q}$. (In fact, we must have $\alpha = a^{m-1}$ for some $0 \neq a \in \mathbb{Q}$ but we proved the general case.) To add, let $\varphi(x) = ax + b$ and $\lambda(x) = cx + d$ for some rational numbers a, b, c, d with $a, c \neq 0$ such that we have

$$p(x) = (\varphi \circ p_1 \circ \lambda)(x) = aD_{n-1}(cx + d, \alpha) + b. \tag{26}$$

In this case, a decomposition that we give in (26) is impossible by Proposition 2.6. Hence, (23) has finitely many rational solutions with a bounded denominator by Theorem 2.3. In particular, there are finitely many positive integer tuples (r, s) such that (22) holds.

Case 2. $n = 5$ and $m = 4$.

Suppose that we have

$$h_5^{(r)} - h_4^{(s)} = \frac{f_5'(r)}{5!} - \frac{f_4'(s)}{4!} = \gamma. \tag{27}$$

Then, we can rewrite this as an affine curve:

$$C_{5,4,\gamma}\ : f(r,s) = 5r^4 + 40r^3 + 105r^2 + 100r - 20s^3 - 90s^2 - 110s - 6 - 120\gamma = 0.$$

Furthermore, we have

$$\frac{\partial f}{\partial r} = 20r^3 + 120r^2 + 210r + 100 \text{ and } \frac{\partial f}{\partial s} = -60s^2 - 180s - 110.$$

Now, the equations $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial s} = 0$ give us the following set of points:

$$P_1\left(-2, \frac{\sqrt{15}-9}{6}\right), \qquad P_2\left(-2-\sqrt{3/2}, \frac{\sqrt{15}-9}{6}\right),$$

$$P_3\left(-2+\sqrt{3/2}, \frac{\sqrt{15}-9}{6}\right), \qquad P_4\left(-2, -\frac{\sqrt{15}+9}{6}\right),$$

$$P_5\left(-2-\sqrt{3/2}, -\frac{\sqrt{15}+9}{6}\right), \qquad P_6\left(-2+\sqrt{3/2}, -\frac{\sqrt{15}+9}{6}\right).$$

Moreover, we have

$$f(P_1) = 0 \text{ if } \gamma = \frac{36+25\sqrt{15}}{1080}, \ f(P_2) = 0 \text{ if } \gamma = \frac{100\sqrt{15}-261}{4320},$$

$$f(P_3) = 0 \text{ if } \gamma = \frac{100\sqrt{15}-261}{4320}, \ f(P_4) = 0 \text{ if } \gamma = \frac{36-25\sqrt{15}}{1080},$$

$$f(P_5) = 0 \text{ if } \gamma = -\frac{261+100\sqrt{15}}{4320}, \ f(P_6) = 0 \text{ if } \gamma = -\frac{261+100\sqrt{15}}{4320}.$$

Hence, the affine curve $C_{5,4,\gamma}$ is smooth as γ is chosen to be rational. Now, let

$$\widehat{C}_{5,4,\gamma} : F(r,s,t) = t^4 f\left(\frac{r}{t}, \frac{s}{t}\right) = 0$$

be the projectivization of the curve $C_{5,4,\gamma}$. By Proposition 2.7, as we have $5 - 4 = 1$, the projective curve $\widehat{C}_{5,4,\gamma}$ is smooth at infinity so that we have a non-singular curve.

Thus, the curve satisfies the genus degree formula. Namely, it has genus

$$g = \frac{(4-1)(4-2)}{2} = 3 > 0.$$

Hence, by Siegel's Theorem [22], we conclude that there can be only finitely many positive integer tuples (r, s) which satisfy equation (27). This completes the proof of the first part.

Now, we can prove the rest of Theorem A. When $(n, m) = (3, 2)$ we have

$$h_3^{(r)} - h_2^{(s)} = \frac{1}{2}r^2 + (r - s) + \frac{1}{3} - \frac{1}{2} = \frac{1}{2}r^2 + (r - s) - \frac{1}{6}$$

so that the difference is an integer only if $\frac{3r^2-1}{6}$ is an integer. As $3 \nmid 3r^2 - 1$ for any positive integer r, the result follows.

When $(n, m) = (4, 2)$, we get

$$h_4^{(r)} - h_2^{(s)} = \frac{4r^3 + 18r^2 + 22r + 6}{24} - \frac{2s + 1}{2}$$

such that $\nu_p\left(h_4^{(r)}\right) = -2$ because $\nu_2\left(4r^3 + 18r^2 + 22r + 6\right) = 1$ for any r. To add, as $\nu_2(\frac{2s+1}{2}) = -1$ for any positive integer s, the difference $h_4^{(r)} - h_2^{(s)}$ is never an integer.

When $(n, m) = (4, 3)$, we obtain that

$$h_4^{(r)} - h_3^{(s)} = \frac{4r^3 + 18r^2 + 22r + 6}{24} - \frac{3s^2 + 6s + 2}{6}.$$

Also, note that $\nu_2(\frac{3s^2+6s+2}{6}) \geq -1$. Again, as we get $\nu_2(\frac{4r^3+18r^2+22r+6}{24}) = -2$, we obtain the result. □

Remark 2.8. Note that any equation

$$p(x) = q(y)$$

having infinitely many rational solutions with bounded denominator does not imply that it has infinitely many integer solutions. For instance, the equation $h_n^{(r)} = s$ always has a solution with a bounded denominator which is positive. That is, if we fix a positive integer n, then we can find some positive rational number s for any given positive integer r. As we have

$$h_n^{(r)} = \frac{f_n'(r)}{n!},$$

we can say that there are infinitely many rational solutions (r, s) for the equation such that $(n!) \cdot r, (n!) \cdot s \in \mathbb{Z}$. In particular, we can choose some positive integer $n > 1$. However, it is known by [11] that $h_n^{(r)} \notin \mathbb{Z}$ for any r and $1 < n \leq 32$. Thus, even though the equation $h_n^{(r)} = s$ has infinitely many positive rational solutions with bounded denominator for any $1 < n \leq 32$, there is no positive integer solution to the equation.

Remark 2.9. When $n = 4$ and $m = 3$, we have the following:

$$h_4^{(r)} = \frac{4r^3 + 18r^2 + 22r + 6}{24}$$

$$h_3^{(s)} = \frac{3s^2+6s+2}{6} = \frac{12s^2+24s+8}{24}.$$

Then, the difference

$$h_4^{(r)} - h_3^{(s)} = \frac{4r^3+6(3r^2-2s^2)+2(11r-12s)-2}{24}$$

and $h_4^{(r)} - h_3^{(s)} = 0$ implies that $4r^3+6(3r^2-2s^2)+2(11r-12s)-2=0$. By SageMath [18], the latter equation gives an elliptic curve of genus 1. Now, we have $2r^3+9r^2+11r-1=6s^2+12s$. Setting $r=3r_0, s=3s_0$ we get

$$2(3r_0)^3+9(3r_0)^2+11(3r_0)-1 = 6(3s_0)^2+12(3s_0),$$
$$54r_0^3+81r_0^2+33r_0-1 = 54s_0^2+36s_0,$$
$$r_0^3+\frac{3}{2}r_0^2+\frac{11}{18}r_0-\frac{1}{54} = s_0^2+\frac{2}{3}s_0$$

so that $C(\mathbb{Q}) \simeq \{0\} \bigoplus \mathbb{Z} \simeq \mathbb{Z}$ is generated by the point $\langle(-\frac{1}{6},-\frac{1}{6})\rangle$ using SageMath [18]. Recall that this curve has genus 1. However, $h_4^{(r)} - h_3^{(s)}$ is never 0 as $h_4^{(r)} - h_3^{(s)} \notin \mathbb{Z}$ by the last part of Theorem A.

Remark 2.10. Using SageMath [18], we found that the set

$$\{(n,m) \in \mathbb{Z}^2 : m \le n \le 20, \text{ the curve } C_{n,m,a} \text{ has genus } 1\}$$

consists of only $(4,3),(5,3),(7,7)$.

3. Analytic and algebraic approach

In this section, we first use analytic methods to prove our second theorem. Then, we close the section by mentioning some algebraic facts.

3.1. Analytic methods

To begin with, let us recall Lemma 2.1.

Lemma 2.1. *For any positive integer n, define $f_n(x)$ as $\prod_{i=0}^{n-1}(x+i)$. Then, for any positive integer r, we have*

$$h_n^{(r)} = \frac{f_n'(r)}{n!}.$$

Now, we continue with the following observation, which will also be used in Section 4, *Elementary and Algebraic Methods.*

Proposition 3.1. *If m is a positive integer and $p > m$ is a prime number, then*

$$h_p^{(r)} - h_m^{(s)} \notin \mathbb{Z}$$

for any positive integers r and s.

Proof. Notice that

$$\begin{aligned} f_p(x) &= x(x+1)\cdots(x+p-1) \\ &\equiv x^p - x \text{ in } \mathbb{F}_p[x]. \end{aligned}$$

Therefore, $f_p^{'}(x) \equiv -1$ in $\mathbb{F}_p[x]$ so that if we write $f_p^{'}(r) = \sum_{k=0}^{n-1} a_k r^k$, then all the coefficients a_k will be divisible by p except a_0. Moreover, we know by Lemma 2.1 that $h_p^{(r)}$ can be written as $\frac{f_p^{'}(r)}{p!}$. Thus, $\nu_p(h_p^{(r)}) < 0$ for any $r \in \mathbb{Z}_{>0}$. On the other hand, since $p > m$ we have $\nu_p\left(h_m^{(s)}\right) = \nu_p\left(\frac{f_m^{'}(s)}{m!}\right) \geq 0$. As a result, $h_p^{(r)} - h_m^{(s)} \notin \mathbb{Z}$. □

Now, we will use some of the arguments and notations given in [11] to prove the non-integerness of the hyperharmonic difference. Let $I(n,r)$ be the set $\{r, r+1, \ldots, n+r-1\}$ for any positive integers n, r. For any prime p, let $I_p(n,r)$ be the set of all multiples of p in $I(n,r)$. Also, note that if p is a prime less than or equal to n and $|I_p(n,r)| = 1$, then $h_n^{(r)} \notin \mathbb{N}$ as $\nu_p\left(h_n^{(r)}\right) < 0$ (see [11, Proposition 6]). The latter argument will also be covered in the proof of Proposition 3.3.

Fact 3.2. For any $\alpha \in \mathbb{R}^{>0}$, there exists a constant $x_\alpha \in \mathbb{R}$ depending on α such that for all $x \geq x_\alpha$, there lies a prime in the interval $((1-\alpha)x, x]$.

We will use Fact 3.2 to obtain a prime p which satisfies $|I_p(n,r)| = 1$ for some $n, r \in \mathbb{Z}^{>0}$. Notice that the above fact can be obtained using the prime number theorem.

Proposition 3.3. *Suppose that two positive integers m, r are given. Then, there exists a positive integer n_c, depending on m, r such that for all $n \geq n_c$, the difference $h_n^{(r)} - h_m^{(s)}$ is never an integer for any positive integer s.*

Proof. Let n_c be a sufficiently large positive integer so that $\left(\frac{2n}{3}, n\right] \cap \mathbb{P}$ is non-empty for any $n \geq n_c$ by Fact 3.2. If necessary, choose n_c such that $n_c \geq \max\{\frac{3m}{2}, 3r-3\}$ also holds. Let $n \geq n_c$. Then, $n \geq 3r-3$ implies that $\frac{2n}{3} \geq \frac{n+r-1}{2}$ and

$$\left(\frac{2n}{3}, n\right] \subseteq \left(\frac{n+r-1}{2}, n\right].$$

Therefore, there exists a prime p in the interval $\left(\frac{n+r-1}{2}, n\right]$.

Note that $n+r-1 < 2p$ and since $p < n$ we have $r-1 < p$. Thus, $I_p(n,r) = \{p\}$ and $|I_p(n,r)| = 1$. Now, observe that

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}) = \frac{\sum_{i=r}^{n+r-1} A_i}{n!}$$

where $A_i = \frac{\mathrm{Per}(n+r-1,r-1)}{i}$ for $i \in \{r, \dots, n+r-1\}$. Then, consider the difference

$$h_n^{(r)} - h_m^{(s)} = \frac{\sum_{i=r}^{n+r-1} A_i}{n!} - \frac{\sum_{j=s}^{m+s-1} B_j}{m!}$$

with $B_j = \frac{\mathrm{Per}(m+s-1,s-1)}{j}$ for $j \in \{s, \dots, m+s-1\}$. Recall that $|I_p(n,r)| = 1$, so except for A_p in $h_n^{(r)}$, A_j is divisible by p for any $j = r, \dots, n+r-1$. Consequently, we get $\nu_p\left(h_n^{(r)}\right) < 0$. Also, we have $n \geq \frac{3m}{2}$ hence $\frac{2n}{3} \geq m$ and $n \geq p > m$ hold. As a result, $\nu_p\left(h_m^{(s)}\right) \geq 0$. Thus, the difference has a negative p-adic order and the proof is done. □

Remark 3.4. Let n, m be positive integers and p be a prime number with $n > p > m$. If $|I_p(n,r)| = 1$ for some positive integer r, then $h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}$ for any positive integer s.

Proof. Since $|I_p(n,r)| = 1$, we have $\nu_p\left(h_n^{(r)}\right) < 0$ by Proposition 3.3 above. Also, as $p > m$, we have $\nu_p\left(h_m^{(s)}\right) \geq 0$ so that $\nu_p\left(h_n^{(r)} - h_m^{(s)}\right) < 0$. □

As a consequence of Remark 3.4, we can state the following proposition.

Proposition 3.5. *Let $n, m, r \in \mathbb{Z}^{>0}$ be given and there exist integers $a, b \geq 1$ and $p, q \in \mathbb{P}^{>m}$ such that one of the conditions*

$$(a-1)n \leq r < an, \quad \frac{n+r}{a+1} < p < n \quad \text{or}$$

$$\frac{bn}{2} < r \leq bn, \quad \frac{n+r}{b+2} < q < \frac{r}{b} \tag{28}$$

holds. Then for any positive integer s, we have

$$h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}.$$

Proof. In either case, we will show that

$$|I_p(n,r)| = |I_q(n,r)| = 1$$

and since $p, q > m$ we obtain the result via Remark 3.4. For the first case, let us show that $I_p(n,r) = \{ap\}$. We have $p < n$ so that $(a-1)p < (a-1)n \leq r$. Also, as $\frac{n+r}{a+1} < p$, we get $n + r < (a+1)p$ and $r < ap + p - n < ap$. Moreover, $ap < n + r$ holds because otherwise we get $ap > p + r$ or $(a-1)p > r$ which is a contradiction. Thus, $(a-1)p < r < ap < n + r < (a+1)p$ holds, so $|I_p(n,r)| = 1$ and the first part is done.

For the second case, we will show that $I_q(n,r) = \{(b+1)q\}$ and the result will follow. Observe that we have $q < n$ and since $bq < r$, we get

$$bq + q = (b+1)q < q + r < n + r.$$

Also, $n + r < (b + 2)q$ implies that $(b + 1)q = (b + 2)q - q > n + r - q > r$. Therefore, the inequality $bq < r < (b + 1)q < n + r < (b + 2)q$ gives that $|I_q(n, r)| = 1$ and we obtain the result. □

Next, our observations give rise to the following proposition, which enables us to locate the intervals containing r, in which the difference $h_n^{(r)} - h_m^{(s)}$ is not an integer.

Proposition 3.6. *Let n and m be positive integers and p be a prime number where $m < p < n$ and $\frac{n}{2} < p$. Then, for any $r \in \big((t-1)p, (t+1)p - n\big]$ for some positive integer t, we have $h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}$ for any positive integer s.*

Proof. Let t be a positive integer and $r \in \big((t-1)p, (t+1)p - n\big]$ so that we have $(t-1)p < r$. Also, $tp - p < r$ gives that $tp < p + r < n + r$. Lastly, $r \leq (t+1)p - n$ implies that $n + r - 1 < (t+1)p$ and $r < tp$. Thus, we obtained that $|I_p(n, r)| = 1$ and as $p > m$, Remark 3.4 gives the result. □

Remark 3.7. Suppose that n is a positive integer and p is a prime number where $\frac{n}{2} < p < n$. Then, we have

$$\nu_p\left(h_n^{(r)}\right) \geq 0$$

if and only if $r \in \big((t+1)p - n, tp\big]$ for some positive integer t.

Proof. Let n be a positive integer, p be a prime where $\frac{n}{2} < p < n$ holds. Define the intervals

$$I_t := \big((t+1)p - n, tp\big] \text{ and } J_t := \big((t-1)p, (t+1)p - n\big]$$

for $t \in \mathbb{Z}^{>0}$. Observe that

$$I_t \cup J_t = \big((t-1)p, tp\big]$$

such that we obtain a partition of $\mathbb{Z}^{>0}$. Now, let r be a positive integer. Then, there exists a positive integer t for which we have $r \in I_t \cup J_t$. If $r \in J_t$, then we know by the proof of Proposition 3.6 that $\nu_p\left(h_n^{(r)}\right) < 0$. Moreover, if $r \in I_t = \big((t+1)p - n, tp\big]$ we have $(t+1)p - n = tp + p - n < r$ and hence $tp + p - 1 < n + r - 1$. Thus, we get

$$r \leq tp \text{ and } tp + p = (t+1)p \leq n + r - 1.$$

In addition, as $|I(n, r)| = n < 2p$ we obtain that $I_p(n, r) = \{tp, (t+1)p\}$. Now, let us write

$$h_n^{(r)} = \frac{\sum_{i=r}^{n+r-1} A_i}{n!}$$

with $A_i = \frac{\mathrm{Per}(n+r-1, r-1)}{i}$ for $i \in \{r, \ldots, n + r - 1\}$. Finally, observe that each A_i is divisible by p. Moreover, we have $\nu_p(n!) = 1$ by the assumption. Thus, we have

$$\nu_p\left(h_n^{(r)}\right) \geq 0$$

which completes the proof. □

Notice that the prime p above can be taken as the greatest prime that is less than n. Thus, we obtain the following remark.

Remark 3.8. Given any integer n, let $p^{\langle n\rangle}$ denote the greatest prime that is less than n. Then,

$$\nu_{p^{\langle n\rangle}}\left(h_n^{(r)}\right) \geq 0$$

if and only if $r \in \big((t+1)p^{\langle n\rangle} - n, tp^{\langle n\rangle}\big]$ for some positive integer t.

Now, using Proposition 3.5 and Remark 3.8 we can take the first step towards Theorem B as follows. Similar ideas can also be found in [20, Chapter 3].

Theorem 3.9. *Let $\Phi(x) = o(x)$ be a monotonically increasing positive function such that the interval $(x - \Phi(x), x]$ contains a prime number for any sufficiently large real number x. Suppose also that $x - 2\Phi(x)$ and $\frac{x^2}{\Phi(x)}$ are monotonically increasing for any sufficiently large real number x. Then, for any constant $C \in (0, \frac{1}{3})$, there exists a positive integer n_0 depending on C such that if $n \geq n_0$, $r \leq C\frac{n^2}{\Phi(n)}$ and $m \leq n - 3\Phi(n)$ hold, then we have $h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}$ for any positive integer s.*

Proof. Let $C \in (0, \frac{1}{3})$ and suppose that there exists a real number $x_0 > 0$ such that for any $x \geq x_0$, $(x - \Phi(x), x] \cap \mathbb{P} \neq \emptyset$ holds where $\Phi(x)$ is a monotonically increasing positive function with $\Phi(x) = o(x)$. Moreover, let k_0 be a sufficiently large integer such that for any $x \geq k_0$, the functions $x - 2\Phi(x)$ and $\frac{x^2}{\Phi(x)}$ are monotonically increasing.

Now, let $n_0 \geq \max\{x_0, k_0\}$ be a sufficiently large integer depending on C satisfying

$$n_0 - 2\Phi(n_0) > x_0 \text{ and } C\frac{{n_0}^2}{\Phi(n_0)} \geq 1$$

in which $n - 2\Phi(n)$ and $C\frac{n^2}{\Phi(n)}$ are increasing for any $n \geq n_0$. Next, assume that $n \geq n_0$. Let p be the greatest prime that is less than or equal to n, so we have $p \in (n - \Phi(n), n]$ and $p > \frac{n}{2}$. By Proposition 3.6, if $r \in \big((t-1)p, (t+1)p - n\big]$ for some positive integer t, then as $p > n - \Phi(n) > n - 3\Phi(n) \geq m$ and $p > \frac{n}{2}$ hold, the difference $h_n^{(r)} - h_m^{(s)}$ is not an integer for any $s \in \mathbb{Z}^{>0}$. Therefore, it is enough to check the intervals $\big((t+1)p - n, tp\big]$ for $t \in \mathbb{Z}^{>0}$.

Let $r \leq C\frac{n^2}{\Phi(n)}$ where n is a sufficiently large integer. As r is bounded, we can bound such integers t. Thus, let us set

$$t_0 = t_0(n) = \left\lfloor C\frac{n^2}{\Phi(n)p}\right\rfloor + 1. \tag{29}$$

We will show that for any positive integer $t \in \{1, 2, \ldots, t_0\}$ if $r \in \big((t+1)p - n, tp\big]$, then we have $h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}$, using the inequalities in (28) of Proposition 3.5. Hence, we can cover all the values of $r \leq C\frac{n^2}{\Phi(n)}$. Note that $r \leq tp < tn$ holds. Now, as $\Phi(n) = o(n)$ and $t \geq 1$, the inequality

$$\frac{\Phi(n)}{n} \leq \frac{1}{4} \leq \frac{t}{2t+2}$$

holds for any sufficiently large n. Then, $p > n - \Phi(n) \geq n \cdot \frac{t+2}{2t+2}$ must hold as $t \geq 1$. Now, $(t+1)p - n > \frac{tn}{2}$ gives that $r > \frac{tn}{2}$ for any $r \in \big((t+1)p - n, tp\big]$ and $t \in \{1, 2, \ldots, t_0\}$. Thus, we get $\frac{tn}{2} < r \leq tn$ and the first inequality in (28) of Proposition 3.5 holds. Next, we will find a prime in the interval $\left(\frac{n+r}{t+2}, \frac{r}{t}\right)$ for any $t \in \{1, 2, \ldots, t_0\}$ and the second inequality in (28) of Proposition 3.5 will be covered. As $n - \Phi(n) < p \leq n$ is satisfied,

$$\left(\frac{t+1}{t+2}n, n - 2\Phi(n)\right] \subseteq \left(\frac{n+r}{t+2}, \frac{r}{t}\right) \tag{30}$$

since $t \geq 1$. Also, $(n - 2\Phi(n) - \Phi(n - 2\Phi(n)), n - 2\Phi(n)] \cap \mathbb{P} \neq \emptyset$ must hold. Moreover, $\Phi(n) = o(n)$ is a monotonically increasing function so that $\Phi(n) \geq \Phi(n - 2\Phi(n))$ and consequently,

$$n - 3\Phi(n) \leq n - 2\Phi(n) - \Phi(n - 2\Phi(n))$$

holds. Then,

$$(n - 3\Phi(n), n - 2\Phi(n)] \supseteq (n - 2\Phi(n) - \Phi(n - 2\Phi(n)), n - 2\Phi(n)]$$

and, since $(n - 2\Phi(n) - \Phi(n - 2\Phi(n)), n - 2\Phi(n)] \cap \mathbb{P} \neq \emptyset$ holds, we get

$$(n - 3\Phi(n), n - 2\Phi(n)] \cap \mathbb{P} \neq \emptyset. \tag{31}$$

Note that $p \in (n - \Phi(n), n]$, thus $p \notin (n - 3\Phi(n), n - 2\Phi(n)]$. Moreover, let us set

$$A = \frac{3C+1}{6} > C.$$

We have $\Phi(n) = o(n)$, thus

$$\frac{\Phi(n)}{n} < 1 - 3A = \frac{1-3C}{2}$$

holds for sufficiently large n depending on C. Then, $3An < n - \Phi(n) < p$ so that

$$A\frac{n^2}{\Phi(n)p} < \frac{n}{3\Phi(n)}. \tag{32}$$

Since $n - \Phi(n) < p \leq n$ and $\Phi(n) = o(n)$, the function $\frac{n^2}{\Phi(n)p}$ is also increasing. Thus, for any sufficiently large n,

$$C\frac{n^2}{\Phi(n)p} + 3 < A\frac{n^2}{\Phi(n)p} \tag{33}$$

holds since $C < A < \frac{1}{3}$ and n is a sufficiently large number depending on C. The inequalities (29), (32), (33) yield that

$$t_0 \leq C\frac{n^2}{\Phi(n)p} + 1 < \frac{n}{3\Phi(n)} - 2.$$

Furthermore, recall that we have $t \in \{1, 2, \ldots, t_0\}$ and the above inequality implies that $\left(1 - \frac{1}{t+2}\right) n < n - 3\Phi(n)$. Then, by (30) and (31) we have

$$(n - 3\Phi(n), n - 2\Phi(n)] \subseteq \left(\frac{t+1}{t+2} \cdot n, n - 2\Phi(n)\right] \subseteq \left(\frac{n+r}{t+2}, \frac{r}{t}\right).$$

Hence,

$$\left(\frac{n+r}{t+2}, \frac{r}{t}\right) \cap \mathbb{P} \neq \emptyset$$

and the second condition of (28) of Proposition 3.5 is covered by some prime q lying in the interval $(n - 3\Phi(n), n - 2\Phi(n)]$. Finally, since $m < n - 3\Phi(n) < q$, we get that $h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}$ and the proof is complete. □

Remark 3.10. The function $\Phi(x)$ in Theorem 3.9 above can be taken as $x^{0.525}$ (see [5]).

As a result, we obtain the following corollary.

Corollary 3.11. *For any constant $C \in (0, \frac{1}{3})$, there exists a positive integer n_0 depending on C such that if $n \geq n_0$, $r \leq Cn^{1.475}$ and $m \leq n - 3n^{0.525}$, then $h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}$ for any positive integer s.*

The following fact on the difference of consecutive primes is the last step towards Theorem B.

Fact 3.12. Let p_k denote the k^{th} prime number. Then, for any real number $\epsilon > 0$, we have

$$\sum_{p_k \leq x} (p_{k+1} - p_k)^2 \ll_\epsilon x^{\frac{23}{18}+\epsilon}$$

by [13]. Moreover, if we assume the Riemann hypothesis, then by [19] we have

$$\sum_{p_k \leq x} (p_{k+1} - p_k)^2 \ll x \log^3 x.$$

Now, we are ready to prove Theorem B.

Theorem B. *Let $T(x)$ be the number of tuples $(n, m, r, s) \in [1, x]^4$ so that the difference $h_n^{(r)} - h_m^{(s)}$ is not an integer. Then, for any $\epsilon > 0$ we have*

$$T(x) = x^4 + O_\epsilon\left(x^{\frac{59}{18}+\epsilon}\right),$$

where the implied constant depends only on ϵ. Moreover, if we assume the Riemann hypothesis, then we obtain

$$T(x) = x^4 + O\left(x^3 \log^3 x\right).$$

Proof. Let us define

$$D(x) := |\{(n,m,r,s) \in [1,x]^4 : m \leq n, n_0 \leq n, h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}\}|$$

and

$$E_n(x) := |\{(m,r,s) \in [1,x]^3 : m \leq n, h_n^{(r)} - h_m^{(s)} \in \mathbb{Z}\}|$$

for each $n \leq x$. Observe that we only count half of the tuples (n,m,r,s) inside $[1,x]^4$ as $m \leq n$. So, we can write

$$D(x) + \sum_{n_0 \leq n \leq x} E_n(x) = \frac{1}{2}x^4 + O(x^3). \tag{34}$$

Also, note that we have

$$T(x) = 2D(x) + O(x^3)$$

as the cases $n \geq m$ and $m \geq n$ are symmetric.

Now, we can write $E_n(x)$ as follows.

$$\begin{aligned} E_n(x) &= O\Bigg(\sum_{s \leq x} \sum_{r \leq x} \sum_{\substack{m \leq n \\ h_n^{(r)} - h_m^{(s)} \in \mathbb{Z}}} 1\Bigg) \\ &= O\Bigg(\sum_{s \leq x} \sum_{r \leq x} \sum_{\substack{m < p^{\langle n \rangle} \\ h_n^{(r)} - h_m^{(s)} \in \mathbb{Z}}} 1 + \sum_{s \leq x} \sum_{r \leq x} \sum_{\substack{p^{\langle n \rangle} \leq m \leq n \\ h_n^{(r)} - h_m^{(s)} \in \mathbb{Z}}} 1\Bigg). \end{aligned} \tag{35}$$

By Remark 3.8, we know that

$$\nu_{p^{\langle n \rangle}}(h_n^{(r)}) \geq 0$$

if and only if $r \in ((t+1)p^{\langle n \rangle} - n, tp^{\langle n \rangle}]$ holds for some positive integer t. Moreover, there are at most $\left\lfloor \frac{x}{p^{\langle n \rangle}} \right\rfloor$ many such values of t as $r \leq x$. In addition, let us set

$$\Delta(n) = n - p^{\langle n \rangle}$$

whenever n is not prime. Notice that if n is prime, then by Proposition 3.1, the difference is never an integer. Now, observe that the number of integers in the interval $((t+1)p^{\langle n \rangle} - n, tp^{\langle n \rangle}]$ is bounded by $\Delta(n)$.

Moreover, for a fixed positive integer n, if there is a tuple $(m,r,s) \in E_n(x)$ with $m < p^{\langle n \rangle}$, then $\nu_{p^{\langle n \rangle}}\left(h_n^{(r)}\right) \geq 0$ holds. That is because we have $h_n^{(r)} - h_m^{(s)} \in \mathbb{Z}$, which implies that $\nu_{p^{\langle n \rangle}}\left(h_n^{(r)} - h_m^{(s)}\right) \geq 0$ and as $m < p^{\langle n \rangle}$, we have $\nu_{p^{\langle n \rangle}}\left(h_m^{(s)}\right) \geq 0$.

Now, consider the first summand in the last error term at (35). We have

$$\sum_{s \leq x} \sum_{r \leq x} \sum_{\substack{m < p^{\langle n \rangle} \\ h_n^{(r)} - h_m^{(s)} \in \mathbb{Z}}} 1 \leq \sum_{r \leq x} \sum_{s \leq x} \sum_{\substack{m < p^{\langle n \rangle} \\ \nu_{p^{\langle n \rangle}}\left(h_n^{(r)}\right) \geq 0}} 1$$

$$< \sum_{\substack{r\le x \\ \nu_{p^{\langle n\rangle}}\left(h_n^{(r)}\right)\ge 0}} \sum_{s\le x} p^{\langle n\rangle} \le \sum_{s\le x} \frac{x}{p^{\langle n\rangle}} \Delta(n) p^{\langle n\rangle} \le x^2 \Delta(n).$$

For the second summand in the last error term in (35), we have

$$\sum_{s\le x} \sum_{r\le x} \sum_{\substack{p^{\langle n\rangle}\le m\le n \\ h_n^{(r)}-h_m^{(s)}\in\mathbb{Z}}} \le \sum_{s\le x}\sum_{r\le x} \big(\Delta(n)+1\big) \le x^2\Delta(n)+x^2.$$

Therefore, both summands yield $O(x^2\Delta(n))$ as n is not prime.

Consequently, we have

$$\sum_{n\le x} E_n(x) = O\left(x^2 \sum_{n\le x} \Delta(n)\right). \tag{36}$$

Furthermore, observe that if

$$n \in (p_k, p_{k+1}]$$

with p_k being the k^{th} prime number for some positive integer k, then

$$\Delta(n) \le p_{k+1} - p_k$$

holds. Thus,

$$\sum_{n\in(p_k,p_{k+1}]} \Delta(n) \le (p_{k+1}-p_k)^2$$

so that we get

$$\sum_{n\le x} \Delta(n) \le \sum_{p_k\le x} \sum_{n\in(p_k,p_{k+1}]} \Delta(n) \le \sum_{p_k\le x} (p_{k+1}-p_k)^2. \tag{37}$$

By Fact 3.12, we have

$$\sum_{p_k\le x} (p_{k+1}-p_k)^2 \ll_\epsilon x^{\frac{23}{18}+\epsilon}$$

for any real number $\epsilon > 0$. Hence, (36) can be written as

$$\sum_{n\le x} E_n(x) = O\left(x^2 \sum_{n\le x} \Delta(n)\right) = O_\epsilon\left(x^{\frac{59}{18}+\epsilon}\right).$$

Consequently, feeding this result into (34), we obtain that

$$D(x) = \frac{1}{2}x^4 + O_\epsilon\left(x^{\frac{59}{18}+\epsilon}\right).$$

This implies that

$$T(x) = x^4 + O_\epsilon\left(x^{\frac{59}{18}+\epsilon}\right)$$

and the first part of the proof is done. Moreover, if we assume the Riemann hypothesis, then by Fact 3.12 we have

$$\sum_{p_k \le x} (p_{k+1} - p_k)^2 \ll x \log^3 x.$$

This together with (36) and (37) gives

$$\sum_{n \le x} E_n(x) = O\left(x^2 \sum_{n \le x} \Delta(n) \right) = O\left(x^3 \log^3 x\right).$$

Hence, we argue as in the first part and obtain that

$$T(x) = x^4 + O\left(x^3 \log^3 x\right).$$

The proof is now complete. □

Remark 3.13. If we assume the Cramér's conjecture, then the function $\Phi(x)$ in Theorem 3.9 can be taken as $C \log^2 x$ for some positive number C (see [8]). Then, the error term in Theorem B can be reduced to $O\left(x^3 \log^2 x\right)$.

3.2. Some algebraic remarks

Here, we analyze the integerness properties of the differences of hyperharmonic numbers with different orders in an algebraic way.

Recall that we have

$$h_n^{(r)} = \frac{f_n'(r)}{n!},$$

where $f_n(x) = \prod_{i=0}^{n-1}(x+i)$. Then, [11, Theorem 23] can be restated follows:

Theorem. *Suppose that $n = kp^\alpha$ is an odd integer where k, α are positive integers, p is a prime and r is a given positive integer. Put $a = \frac{k-1}{2}, c = \lceil \frac{r}{p^\alpha} \rceil$. If $\nu_p\left(f_k'(c)\right) \le \nu_p(k!)$, then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Moreover, if c is not a root of $f_k'(x)$ modulo p, then $h_n^{(r)} \notin \mathbb{Z}$.*

In particular, for the polynomial

$$F_k(x) := \sum_{i=-a}^{a} \left[\prod_{j=-a}^{a} (x-j) \right] \frac{1}{x-i}$$

given in [11, Theorem 23], its shift $F_k(x+a)$ is $f_k'(x)$. Observe that

$$F_k(x+a) = \sum_{i=0}^{k-1} \left[\prod_{j=0}^{k-1} (x+j) \right] \frac{1}{x+i}.$$

Now, as

$$f_k(x) = \prod_{i=0}^{k-1}(x+i)$$

we deduce that

$$\frac{f_k'(x)}{f_k(x)} = \sum_{j=0}^{k-1} \frac{1}{x+j}.$$

Therefore, we obtain that

$$F_k(x+a) = \sum_{i=0}^{k-1} \left[\prod_{j=0}^{k-1} (x+j) \right] \frac{1}{x+i} = f_k(x) \cdot \frac{f_k'(x)}{f_k(x)} = f_k'(x).$$

Under this set up, we can present the following remark.

Remark 3.14. Let $n = kp^\alpha$ be an odd integer where k, α are positive integers and p be a prime number. Also, let r be given. Set $a = \frac{k-1}{2}$ and $c = \lceil \frac{r}{p^\alpha} \rceil$. Then, for any $m < p$, if $\nu_p \left(f_k'(c) \right) \leq \nu_p(k!)$ holds, then we have

$$h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}$$

for any positive integer s. Furthermore, for any positive integers $m < p$ and s, if c is not a root of $f_k'(x)$ modulo p, then

$$h_n^{(r)} - h_m^{(s)} \notin \mathbb{Z}.$$

Proof. For the first part, suppose that $\nu_p(f_k'(c)) \leq \nu_p(k!)$ and m is an integer less than p. Then by [11, Theorem 23], we have $\nu_p \left(h_n^{(r)} \right) < 0$. However, for any positive integer s, we have

$$h_m^{(s)} = \frac{f_m'(s)}{m!}$$

with $\nu_p \left(h_m^{(s)} \right) \geq 0$ as $m < p$. Thus, the first part of the proof is done. Now, for the second part, suppose that c is not a root of $f_k'(x)$ modulo p, namely, we have

$$f_k'(c) \not\equiv 0 \pmod{p}.$$

Thus, $\nu_p(f_k'(c)) = 0 \leq \nu_p(k!)$ as k is an integer. Then, as $m < p$ and s is any integer, we conclude the result by the first part of the theorem. ☐

4. Integer hyperharmonic differences and the problem of Mező

In this section, we show that the difference $h_n^{(r)} - h_n^{(s)}$ can be integers infinitely often for some positive integers $r \neq s$ and n as follows.

Proposition 4.1. *For any integer $n > 1$, the difference $h_n^{(r)} - h_n^{(s)}$ is an integer whenever $r \equiv s \pmod{n!}$ for some positive integers r and s. In addition, for any prime number $p \geq 5$ if $r \equiv s \left(\bmod \frac{(p-1)!}{2} \right)$, then $h_p^{(r)} - h_p^{(s)} \in \mathbb{Z}$.*

Proof. Recall that we have

$$h_n^{(r)} = \frac{f_n'(r)}{n!},$$

where $f_n(x)$ is defined as $\prod_{i=0}^{n-1}(x+i)$. Then, since $f_n(x)$ is a polynomial of degree n, we can write

$$f_n'(r) = \sum_{i=0}^{n-1} a_i r^i$$

for some positive integers $a_0, a_1, \ldots, a_{n-1}$. In particular, we have

$$a_0 = (n-1)!, \quad a_{n-2} = \frac{n(n-1)^2}{2} \text{ and } a_{n-1} = n.$$

Thus,

$$\begin{aligned} h_n^{(r)} - h_n^{(s)} &= \frac{f_n'(r) - f_n'(s)}{n!} \\ &= \sum_{k=0}^{n-1} \frac{a_k(r^k - s^k)}{n!} \\ &= \sum_{k=1}^{n-1} \frac{a_k(r-s)(r^{k-1} + \cdots + s^{k-1})}{n!} \end{aligned}$$

which is an integer whenever $r \equiv s \pmod{n!}$ holds. Now, we prove the last part of the theorem. By Proposition 3.1 if we write $f_p'(r) = \sum_{k=0}^{p-1} a_k r^k$, then we know that all the coefficients a_k will be divisible by p except $a_0 = (p-1)!$. Thus,

$$\begin{aligned} h_p^{(r)} - h_p^{(s)} &= \sum_{k=0}^{p-1} \frac{a_k(r^k - s^k)}{p!} \\ &= \sum_{k=1}^{p-1} \frac{pb_k(r-s)(r^{k-1} + \cdots + s^{k-1})}{p!} \\ &= \sum_{k=1}^{p-1} \frac{b_k(r-s)(r^{k-1} + \cdots + s^{k-1})}{(p-1)!} \qquad (38) \end{aligned}$$

for some positive integers b_k. If $r \equiv s \left(\bmod \frac{(p-1)!}{2}\right)$, then we have $r - s = t \cdot \frac{(p-1)!}{2}$ for some $t \in \mathbb{Z}$. This indicates that

$$h_p^{(r)} - h_p^{(s)} = \sum_{k=1}^{p-1} \frac{b_k t(r^{k-1} + \cdots + s^{k-1})}{2}.$$

Note that $h_p^{(r)} - h_p^{(s)} \in \mathbb{Z}$ when t is even. So, assume that t is odd. By the congruence $r \equiv s \left(\bmod \frac{(p-1)!}{2}\right)$ we know that r and s have the same parity, as $p \geq 5$. Therefore, $h_p^{(r)} - h_p^{(s)} \in \mathbb{Z}$, if r is even. So, assume also that r is odd. In that case, the sum $(r^{k-1} + \cdots + s^{k-1})$ is even when k is even, as there are k-many terms in the sum. Thus, it is enough to show that the sum $\sum_{i=1}^{(p-1)/2} b_{2i-1}$ is even. Instead, we will prove each odd indexed b_k is even.

Observe that $f_p(x) = \prod_{i=0}^{p-1}(x+i) \equiv x^{\ell+1}(x+1)^{\ell} \pmod 2$, where $\ell = \frac{p-1}{2} \geq 2$. Hence,

$$\begin{aligned}
f_p'(x) &\equiv (\ell+1)x^{\ell}(x+1)^{\ell} + \ell x^{\ell+1}(x+1)^{\ell-1} && \pmod 2\\
&\equiv x^{\ell}(x+1)^{\ell-1}((\ell+1)(x+1)+\ell x) && \pmod 2\\
&\equiv x^{\ell}(x+1)^{\ell-1}(\ell+x+1) && \pmod 2\\
&\equiv \begin{cases} x^{\ell+1}(x+1)^{\ell-1}, & \text{if } \ell \text{ is odd}\\ x^{\ell}(x+1)^{\ell}, & \text{if } \ell \text{ is even}\end{cases} && \pmod 2\\
&\equiv \begin{cases} \displaystyle\sum_{i=0}^{\ell-1}\binom{\ell-1}{i}x^{\ell+1+i}, & \text{if } \ell \text{ is odd}\\ \displaystyle\sum_{i=0}^{\ell}\binom{\ell}{i}x^{\ell+i}, & \text{if } \ell \text{ is even}\end{cases} && \pmod 2\,. \qquad (39)
\end{aligned}$$

In either case, the polynomial $f_p'(x)$ is equivalent to $\sum_{i=0}^{2c}\binom{2c}{i}x^{2d+i}$ modulo 2 for some positive integers c and d, as $\ell \geq 2$. Also notice that

$$f_p'(x) = \sum_{k=0}^{p-1} a_k x^k = (p-1)! + \sum_{k=1}^{p-1} p b_k x^k \equiv \sum_{k=1}^{p-1} b_k x^k \pmod 2\,,$$

since $p \geq 5$ is a prime number. Therefore, by congruence (39), we see that b_k is even for all odd $k \leq \ell$. Moreover, for each odd $k \geq \ell+1$, we have

$$b_k \equiv \binom{2c}{i} \pmod 2\,, \qquad (40)$$

for some positive odd integer i. Since i is odd, $\binom{2c}{i} = \frac{2c}{i} \cdot \binom{2c-1}{i-1} \in \mathbb{Z}$ and $\binom{2c-1}{i-1} \in \mathbb{Z}$, we deduce that b_k is even by congruence (40). In conclusion, whenever $r \equiv s \left(\bmod \frac{(p-1)!}{2}\right)$, we have $h_p^{(r)} - h_p^{(s)} \in \mathbb{Z}$. □

Remark 4.2. For primes $p = 2, 3$, a variation of Proposition 4.1 can be obtained as follows: note that for any positive integer r we have $h_2^{(r)} = r + \frac{1}{2}$. Therefore $h_2^{(r)} - h_2^{(s)}$ is integer for any $r, s \in \mathbb{Z}_{>0}$. Also, by equation (38) one can easily say that the difference $h_3^{(r)} - h_3^{(s)}$ is an integer if and only if r and s have the same parity.

Finally, we present our answer to Problem 1.1: *For which $n \neq m$ and $r \neq s$ does the equality*

$$h_n^{(r)} = h_m^{(s)}$$

hold?

Remark 4.3. For $m = 2$, we have

$$h_m^{(s)} = h_2^{(s)} = s + \frac{1}{2}$$

for $s \in \mathbb{Z}^{>0}$. Observe that if for some $n, r \in \mathbb{Z}^{>0}$, the hyperharmonic number $h_n^{(r)}$ is a half-integer, namely

$$h_n^{(r)} \in \mathbb{Z} + \frac{1}{2},$$

then we can find an appropriate s so that

$$h_n^{(r)} = h_2^{(s)}$$

holds. Recall by the last part of the proof of Theorem A that the equality cannot hold for $n = 3, 4$. Also, by Proposition 3.1, $h_5^{(r)} - h_2^{(s)}$ cannot be an integer. However, for $n = 6$, using the computer algebra system SageMath [18] we obtained some values of r where $h_6^{(r)}$ is a half-integer. In this case, finding one such example is enough to find infinitely many values of r where $h_6^{(r)}$ is also a half-integer by Proposition 4.1. That is,

$$h_6^{(r+k\cdot(6!))} \text{ is a half-integer since } h_6^{(r)} - h_6^{(r+k\cdot(6!))} \in \mathbb{Z}$$

for $k \in \mathbb{Z}^{\geq 0}$. Thus, the equality

$$h_6^{(r)} = h_2^{(s)} \tag{41}$$

in fact holds and there are infinitely many examples where some of them are illustrated in Table 2.

TABLE 2. Several r and s values which are the solutions of (41).

r	s	$h_6^{(r)} = h_2^{(s)}$
20	47501	95003/2
55	5228670	10457341/2
75	23275838	46551677/2
100	94231673	188463347/2

Moreover, for $n = 6$ and $m = 3$ we can find infinitely many (r, s) tuples such that

$$h_6^{(r)} - h_3^{(s)} \in \mathbb{Z}$$

holds. In particular, for $r = 15$ we have $h_6^{(15)} = 80507/6$ and for $s = 1$ we have $h_3^{(1)} = 11/6$. Therefore, we get

$$h_6^{(15)} - h_3^{(1)} = 13416.$$

In fact, the set $\{(15, 2k + 1) : k \in \mathbb{Z}^{\geq 0}\}$ of tuples (r, s) yield

$$h_6^{(r)} - h_3^{(s)} \in \mathbb{Z},$$

by Proposition 4.1. Similarly, when $m = 4$, we can find infinitely many tuples

$$(r, s) \in \{(5, 4 + k \cdot (4!)) : k \in \mathbb{Z}^{\geq 0}\}$$

so that the corresponding difference is also an integer. Furthermore, for $m = 5$, the set $\{(6, 1 + k \cdot (4!)) : k \in \mathbb{Z}^{\geq 0}\}$ of tuples (r, s) yield $h_6^{(r)} - h_5^{(s)} \in \mathbb{Z}$.

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Çağatay Altuntaş
Department of Mathematics Engineering
Faculty of Science and Literature
Istanbul Technical University
Ayazağa Kampüsü, 34469, Sariyer/Istanbul, Turkey
Email address: caltuntas@itu.edu.tr

Haydar Göral
Department of Mathematics
Izmir Institute of Technology
35430 Urla, Izmir, Turkey.
Email address: haydargoral@iyte.edu.tr

Doğa Can Sertbaş
Department of Mathematics
Faculty of Sciences and Literature
Çukurova University
01330 Adana, Turkey
Email address: dsertbas@cu.edu.tr