FUNCTION GENERATION SYNTHESIS OF PLANAR MECHANISMS AS A MIXED PROBLEM OF CORRELATION OF CRANK ANGLES AND DEAD-CENTER DESIGN

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ABSTRACT

FUNCTION GENERATION SYNTHESIS OF PLANAR MECHANISMS AS A MIXED PROBLEM OF CORRELATION OF CRANK ANGLES AND DEAD-CENTER DESIGN

Kinematic synthesis of mechanisms is generally divided into three groups. One of them is function synthesis. In function synthesis, the design of correlation of crank angles and dead dead-center position stand out. These problems have been clearly defined and solved separately. But in some cases, problems may be encountered that require both correlation of crank angles and dead dead-center design. Such problems are called mixed function generation problems. In this thesis, an overview of these mixed function generation problems has been given and many problems have been solved analytically or semi-analytically. The solutions of all problems including three positions for the four-bar mechanism and the solution of a problem including four positions for a four-bar mechanism have been addressed. A problem including 3 positions and a problem including 4 positions for a slider-crank mechanism have been addressed. All solutions have been reduced to univariate equation and a fast solution has been found. Thus, link lengths can be found quickly by changing the problem inputs. Numerical solutions of all problems have been demonstrated using Excel.

ÖZET

KRANK AÇILARI KORELASYONU VE ÖLÜ-KONUM TASARIMININ KARMA PROBLEMİ İÇİN DÜZLEMSEL MEKANİZMALARIN İŞLEV SENTEZİ

Mekanizmaların kinematik sentezi genellikle üç gruba ayrılır. Bunlardan biri işlev sentezidir. İşlev sentezinde krank açılarının korelasyonu ve ölü konum tasarımı öne çıkmaktadır. Literatürde bu problemler ayrı ayrı açıkça tanımlanmış ve çözülmüştür. Ancak bazı durumlarda hem krank açılarının korelasyonu hem de ölü konum tasarımını gerektiren problemler ile karşılaşılabilir. Bu tür problemlere karma işlev sentezi problemleri denir. Bu tezde, bu karma işlev sentezi problemlerine genel bir bakış yapılmıştır ve analitik ya da yarı analitik olarak birçok problem çözülmüştür. Dört-kol mekanizması için üç konum içeren tüm problemlerin çözümleri ve dört-kol mekanizması için dört konum içeren bir problemin çözümü sunulmuştur. Krank-biyel mekanizması için bir tane 3 konum içeren problemin ve bir tane 4 konum içeren problemin çözümü sunulmuştur. Tüm çözümler tek değişkenli bir denkleme indirgenmiş ve hızlı bir çözüm bulunmuştur. Böylece problem girdileri değiştirilerek bağlantı uzunlukları hızlı bir şekilde bulunabilir. Tüm problemlerin sayısal çözümleri Excel kullanılarak gösterilmiştir.

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ABBREVIATIONS	EXPANSIONS
CCA	Correlation of crank angles
DCP	Dead-center position
EDCP	Extended dead-center position
FDCP	Folded dead-center position
OPT	Optimum transmission angle
DTA	Deviation of transmission angle from 90°
a	Input link length of a four-bar mechanism
b	Coupler link length of a four-bar mechanism
d	Output link length of a four-bar mechanism
c	Cosine
s	Sine
ϕ	Input angle of a four-bar mechanism
ψ	Output angle of a four-bar mechanism
$_E$	Extended dead-center position (Subscript)
$_F$	Folded dead-center position (Subscript)
$_D$	Dead-center position – folded or extended (Subscript)
I/O	Input-Output

CHAPTER 1

INTRODUCTION

Machines are comprised of several mechanisms. A mechanism is a constrained system of bodies designed to convert motions of, and forces on, one or several bodies into motions of, and forces on, the remaining bodies (IFToMM, 2022).

This thesis study issues function generation synthesis of planar mechanisms as a mixed problem of correlation of crank angles (CCA) and dead-center design. In the following subsections, kinematic synthesis of mechanisms is briefly presented. Then function generation synthesis is explained. After that motivation and aim of the thesis are stated. Finally, the outline of this thesis is presented.

1.1. Kinematic Synthesis of Mechanisms

Kinematics of mechanisms can be studied under two main categories: kinematic analysis and kinematic synthesis. In kinematic analysis, the motions of links (position, velocity and acceleration) are determined for given inputs to the mechanism when the structure and all link lengths of the mechanism are known. In kinematic synthesis, the structure and all link lengths of the mechanism are determined when the motions of links typically for given inputs to the mechanism are known. Kinematic analysis and kinematic synthesis are reverse problems. There are some steps of kinematic synthesis such as type synthesis, number synthesis and dimensional synthesis.

Type synthesis is the choice of elements to constitute the mechanism, such as gears, linkages, belts, pulleys, etc. Number synthesis is used in the selection of number of links and joints. Type synthesis and number synthesis form constitute the structural synthesis.

After all these steps, it is necessary to determine the link dimensions of the mechanism according to the desired motion. Determining link dimensions is called dimensional synthesis. There are different tasks for dimensional synthesis: function generation, path generation, and motion generation (or rigid body guidance). In simple terms, function generator mechanisms produce a desired function. In function generation, the desired motion usually contains one parameter (e.g., angle of rocker link). Path generator mechanisms have a point that travels on a given path. In path generation of planar mechanisms, the desired motion contains two parameters (e.g., x and y coordinate of coupler point). Motion generator mechanisms follow a given rigid body motion. In motion generation of planar mechanisms, the desired motion typically contains three parameters (e.g., x and y coordinates of a coupler point and angle of the coupler link).

1.2. Function Generation Synthesis

In function synthesis, the goal is to design a mechanism that generates a desired function. In other words, the mechanism is designed such that the input and output of the mechanism approximately generate a desired input/output (I/O) relationship. The mechanisms synthesized in this way are called function generators. Planar four-bar mechanisms are widely used as function generators. Function generation synthesis of mechanisms is generally divided into two methods: CCA problems and dead-center design.

1.3. Motivation of the Thesis

In some applications, a mechanism may be required to provide both some I/O relationships and to provide one or both dead-center position (DCP). Nowadays, although such a problem can be quickly solved using a CAD program, an analytical or semi-analytical solution is very useful when the function generation problem is merely a small part of the design of a multi-loop mechanism. As you change the problem variables using an analytical or semi-analytical solution, the result can be displayed quickly. Thus, its effects on the multi-loop mechanism can be displayed. As an example, the design of a four-bar loop connected to the bucket of a loader mechanism of a construction machine is such a problem. The problems involving only CCA and only dead-center design have been clearly defined separately in literature. But the problems involving CCA and dead-center design together are not sufficiently worked out and their solutions are not presented.

1.4. Aim of the Thesis

The aim of this thesis is to examine the function generation synthesis of planar mechanisms as a mixed problem of CCA and dead-center design, to identify possible problems and to present a solution to some of them.

1.5. Outline of the Thesis

This thesis consists of 5 chapters: Introduction, Literature Survey, Mixed Function Generation Synthesis Problems for Planar Mechanisms, Numerical Examples and Conclusions. In Chapter 2, literature review about the birth of kinematic synthesis of mechanisms, CCA, dead-center design and mixed function generation synthesis problems are presented. In Chapter 3, mixed function generation synthesis problems for four-bar and slider-crank mechanisms for 3 and 4 positions are studied. In Chapter 4, numerical examples of problems in Chapter 3 are presented. The conclusions of the thesis are presented in Chapter 5. The results of the thesis are stated and possible problems for future studies are discussed.

CHAPTER 2

LITERATURE SURVEY

In this chapter, first a brief review on kinematic synthesis of mechanisms is presented. Then, literature survey on CCA, dead-center design and mixed function generation synthesis problems is presented.

2.1. Kinematic Synthesis of Mechanisms

In early ages, people achieved mechanical motion by their own muscle forces, but then the muscle power began to be insufficient. This situation has prompted people to invent and use tools and mechanisms by using their creativity.

The invention of the external combustion steam engine created a revolution in mechanization. The machine built by Thomas Savery, the first commercial example, is called the Savery Engine (Figure 2.1) (Savery, 1702). The area of use of this machine was to throw water out of a mine. It was not used for long periods because its efficiency was very low, but it led to future studies.

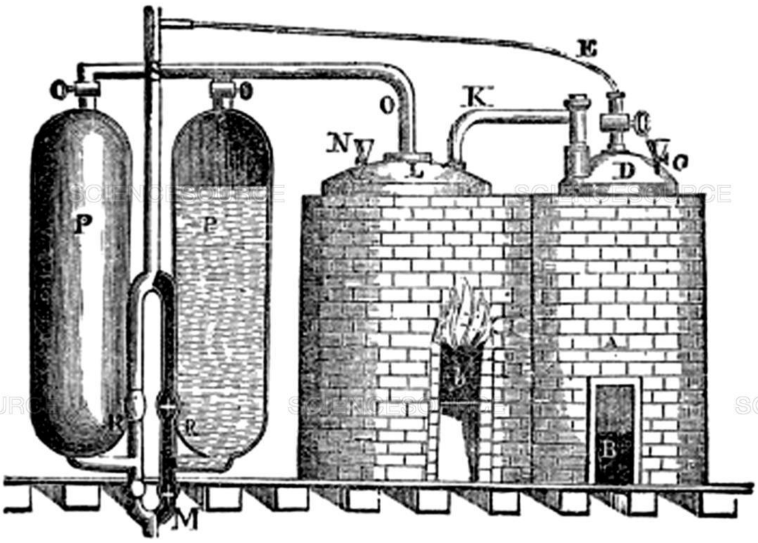


Figure 2.1. Savery engine (Source: Savery, 1702)

In the 1700s, Thomas Newcomen developed a new steam engine (Figure 2.2) (ASME, 1981). Although it had some mechanical advantages and a relatively safer steam engine, it did not achieve the desired efficiency and fuel consumption did not decrease.

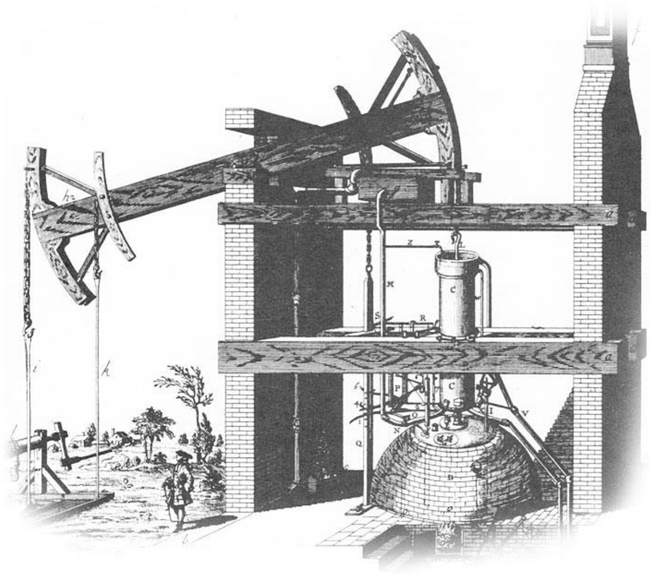


Figure 2.2. Newcomen's engine (Source: ASME, 1981)

While repairing a Newcomen's engine, as a result of his reviews, James Watt decided that he could improve the engine (ASME, 1986). Watt made a two-room design, one constantly hot and one constantly cold (Figure 2.3). Watt wanted to convert rotational shaft motion into translating motion and he invented Watt's straight-line motion mechanism.

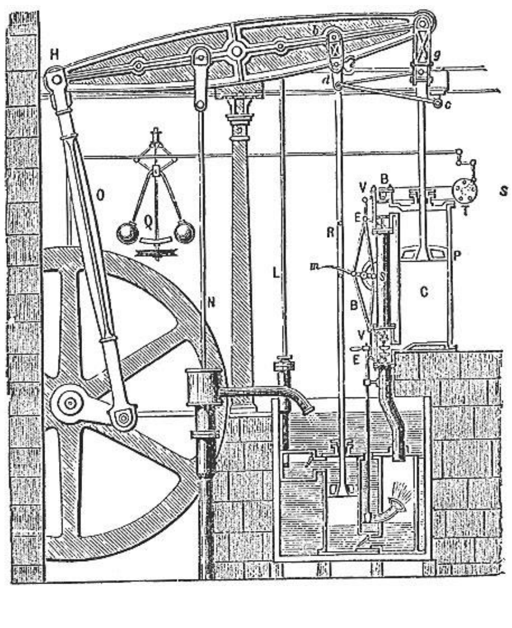


Figure 2.3. Watt's engine (Source: ASME, 1986)

After this straight-line mechanism by Watt, the subject began to attract the attention of some mathematicians. Scientists wanted to find mechanisms that follow straight-line. Perhaps this can be regarded as the birth of kinematic synthesis (Ceccarelli, 2008). Richard Roberts invented Robert’s mechanism, which follows an approximate straight line (Figure 2.4).

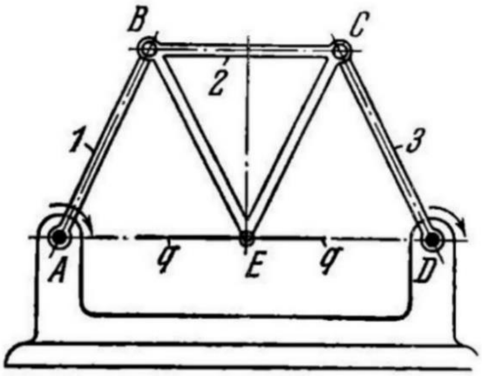


Figure 2.4. Robert’s straight-line mechanism (Source: Artobolevsky, 1975)

Franz Reuleaux, Alexander B. W. Kennedy, and Ludwig E. H. Burmester worked on the analysis and synthesis of mechanism using geometry. Reuleaux made models of many mechanisms (Figure 2.5). Kennedy translated Reuleaux’s book into English.

Figure 2.5. Robert’s straight-line mechanism from Reaulaux's Collection in Cornell University (Source: ASME, 2004)

Franz Grashof created a mathematical model that found under what conditions four-bar mechanisms would be crank-rocker, double-crank or double-rocker. Pafnuty L. Chebyshev developed analytical methods for the analysis and synthesis of mechanisms. Chebyshev Polynomials are used to provide optimal spacing of precision points for function and path generation. Samuel Roberts and Chebyshev developed the theorem stating that there are three four-bar mechanisms that produce the same coupler curve. Later on, based on Chebyshev's work Artobolevskii (1944) and Levitskii (1946, 1950) have developed many methods for function and path generation synthesis.

Ferdinand Freudenstein created the algebraic model known today as Freudenstein equation for the synthesis of functions of the planar four-bar mechanism in his doctoral dissertation. Freudenstein and his student George N. Sandor were pioneers of computer-aided mechanism synthesis. They performed computer-aided calculations of link lengths to perform function, path and motion generation. In light of his work and the academic family he left behind, Freudenstein is known as the "Father of Modern Kinematics" (Ceccarelli, 2011).

Richard Hartenberg and his student Jacques Denavit found the Denavit-Hartenberg (DH) Representation and its parameters (Denavit & Hartenberg, 1964). In the following years, Arthur G. Erdman and his team and J. Michael McCarthy and his team made computer-aided programs to synthesize and analyze mechanisms.

2.2. Correlation of Crank Angles

As mentioned in Chapter 1.2, function generation synthesis of mechanisms can be divided into two: CCA problems and dead-center design.

If the problem is modeled in such a way that the input and output links of the four-bar mechanism must provide certain angle values relative to the fixed link, Söylemez calls the problem as CCA (Söylemez, 2018). Early methods of function generation pertaining to CCA have been graphical (Burmester, 1888) (Koestsier, 1989) (Hain, 1967) (Svoboda, 1948) (Beyer, 1953). The developed geometric synthesis procedures were time-consuming and required some skill. With the development of computers, analytical solutions became available very quickly. At mid-20th century, Levitskii Freudenstein and many others opened a new era in kinematic synthesis by solving the problem of CCA analytically (Levitskii, 1946) (Levitskii, 1950) (Freudenstein, 1954).

2.3. Dead-Center Design

In four-bar mechanisms of crank-rocker and double-rocker proportions, the output rocker link oscillates between two angle limits. The positions of the four-bar mechanism when the rocker is at a limit are called the DCP. The four-bar mechanism has two DCPs: the folded DCP (FDCP) and extended DCP (EDCP). The crank and coupler become collinear in both DCPs in extended form or folded on top of each other (Söylemez, 2018).

The design of a mechanism according to the swing angle of the output rocker link and usually the amount of crank rotation corresponding to this swing angle is called dead-center design (Söylemez, 2018). The dead-center design problem is generally accompanied by transmission angle optimization. Dead-center design with transmission-angle optimization of planar four-bar mechanisms was first addressed by Alt, then improved by Meyer zur Capellen and Volmer (Alt, 1925) (Meyer zur Capellen, 1956) (Volmer, 1957). Freudenstein and Primrose obtained a closed form analytical solution which was determined as the root of a degree 3 univariate polynomial equation (Freudenstein & Primrose, 1972). Meyer zur Capellen and Volmer studied a similar problem for slider-crank mechanisms graphically and the link lengths are expressed using the initial crank angle as the parameter. Söylemez obtained a closed form analytical solution which yields a unique solution (Söylemez, 2002).

2.4. Mixed Function Generation Synthesis

Many classical textbooks do not cover the problem of dead-center design (Sandor & Erdman, 1984) (Hartenberg & Denavit, 1964) (McCarthy, 2010). Hall and Norton separately consider the dead-center design and the problems of the CCA (Hall & Goodman, 1961) (Norton, 2004). Mallik et al. present dead-center design as the fourth type of synthesis problem in addition to function, path and motion generation problems (Mallik et al., 1994). Pennestrì and Valentini summarized the methods of analytical function synthesis for four-bar and slider-crank mechanisms for two and three positions, including dead-center design problems (Pennestrì & Valentini, 2009). Mutlu achieved the design of the crank-rocker mechanisms via analytical methods, based on a closed-form solution for specific design cases (Mutlu, 2021).

In certain applications, a function generator needs to be designed to satisfy both CCA and DCP design problems. There are a few recent publications that deal with both CCA and DCP design together. Kiper and Erez presented an analytical solution to the problem of synthesis of four-bar mechanisms for 2 CCA positions and FDCP as the third position (Kiper & Erez, 2019). Kiper et al. have worked on slider-crank mechanism design for 3 CCA positions and a DCP (folded or extended), where the problem is reduced to a degree 8 univariate polynomial equation (Kiper et al., 2020). Kadak and Kiper have presented the necessary formulation for the synthesis of a four-bar mechanism for 3 CCA positions and a DCP, where the problem is reduced to the solution of a degree 6 univariate polynomial equation (Kadak & Kiper, 2021).

CHAPTER 3

MIXED FUNCTION GENERATION SYNTHESIS PROBLEMS FOR PLANAR MECHANISMS

In this Chapter, firstly, all possible mixed function generation synthesis problems for four-bar mechanism are defined and derivations of common formulas to be used in the subsections are presented. Then, solutions of almost all problems for four-bar mechanism including 3 positions (Section 3.1), one of the problems for four-bar mechanism including 4 positions (Section 3.2), are presented, and finally, a solution of a problem for slider-crank mechanism including 3 positions (Section 3.3), and 4 positions (Section 3.4), are presented in separated subsections.

Both the input and output angles of the four-bar mechanism are specified for CCA. For DCPs, it is enough to specify only one of the input or output angles. For a DCP, either the input or the output link may be aligned with the coupler link. As a result, the link lengths of the mechanism are desired.

For a planar four-bar mechanism, since the link length dimensions can be scaled as desired without affecting the I/O relationship between the angles, without loss of generality one of the link length values can be assumed, while the remaining three will be unknown.

In problems including 3 positions, generally there are three equations and three unknowns, and the problems can be solved quite simply. But in problems including 4 and 5 positions, the number of unknown link lengths is 3, while the number of equations is 4 or 5. In these problems, the angle(s) of the reference line from which the input and/or the output angle is measured is(are) used as the extra unknowns. Such an extra unknown angle shall be called the offset angle. In problems including 4 positions, it is sufficient to use an offset angle, while in problems including 5 positions, two offset angles should be used. Table 3.1 shows all the problems including 3 positions.

Table 3.1. Possible problems including 3 positions

#	Position 1	Position 2	Position 3
1	CCA	CCA	CCA
2	CCA	CCA	OPT
3	CCA	CCA	DCP
4	CCA	DCP	OPT
5	CCA	EDCP	FDCP
6	CCA	EDCP	EDCP
7	CCA	FDCP	FDCP
8	EDCP	FDCP	OPT

For the 1st problem, where 3 CCA positions are given, the well-known solution of Freudenstein gives a fast solution (Freudenstein, 1954). If 2 CCA positions are given, either any DCP is given as the third position or optimum transmission angle (OPT) is required (2^{nd} and 3^{rd} rows in Table 3.1). In the 3rd problem, any DCP is given regardless of FDCP and EDCP, because there is a unified solution as it will be apparent in the following sections. If only one CCA position is given, there are two cases. In the first case, any DCP is given as a second desired position and OPT is required (4^{th} row in Table 3.1). In this problem, any DCP is given regardless of FDCP and EDCP. In the second case, two DCPs are given as the second and third position (5^{th}, 6^{th} and 7^{th} rows in Table 3.1). In the 5th problem, different types of DCPs are given, whereas in the 6th and 7th problems, same types of DCPs are given. The solutions to the 6th and 7th problems turn out to be the same, regardless of whether the DCPs are two EDCPs or two FDCPs. Therefore, the solution of the 6^{th} and 7^{th} problems is given in a single section (Section 3.6). If no CCA is given, 2 DCPs are given and OPT is required (8th row in Table 3.1). This last problem is the well-known Alt's transmission angle optimization problem for DCP design (Alt, 1925). It is not possible to talk about OPT in problems where both of the same types of DCPs are given. Because in the case where two FDCPs are given, the transmission angle will be 0° in one of the two DCPs, or in the case where two EDCPs are provided, the transmission angle will be 180° in one of the two DCPs.

In problems including a DCP, only one of the input or output angles is specified. The solution of the problem is different depending on whether the input angle or output angle is given for the DCP. Therefore, the problem has been defined and solved for both cases.

If a problem contains only one DCP (3^{rd} and 4^{th} problems), there are 2 solutions:

- input angle is given for DCP
- output angle is given for DCP

If a problem contains two DCP (5^{th}, 6^{th}, 7^{th} and 8^{th} problems), there are 3 solutions:

- input angles are given for both DCPs
- output angles are given for both DCPs
- input angle is given for a DCP and output angle is given for other DCP

But in problems where the same type of DCPs are given (6^{th} and 7^{th}), the behavior of the two rocker links are the same. Therefore, it is important whether the two angles given are given for the same link or a different one.

Two different methods were used to optimize the transmission angle. Therefore, these problems have been defined and solved in two different ways (2^{nd}, 3^{rd} and 8^{th} problems). However, in the 8^{th} problem, the two solutions are the same. Totally 18 problems have been identified and solved for problems including 3 positions.

In path and motion generation, there is an order of given points or positions according to the input joint variable. Since the input joint variable values are well defined in function generation problems, there is no order problem.

There are 2 configurations (or assembly modes) of four-bar and slider-crank mechanisms for a given input variable value. In both configurations, the I/O equation is the same, so the synthesis method does not distinguish between different configurations. After the synthesis is performed, it is necessary to make sure that the desired function is generated for the same configuration of the mechanism in all design positions. To avoid such a problem, after the synthesis procedure, the mechanism should be analyzed, and better simulated, for the desired range of motion.

Problems for 4 positions are obtained by adding one more CCA position to problems for 3 positions (first column in Table 3.2). The above mentioned explanations for problems including 3 positions also apply to problems for 4 positions as well. In addition, an offset angle should be used here. Adding the offset angle to the input or output angle changes the solution to the problem.

Table 3.2. Possible problems including 4 positions

#	Position 1	Position 2	Position 3	Position 4
1	CCA	CCA	CCA	CCA
2	CCA	CCA	CCA	OPT
3	CCA	CCA	CCA	DCD
4	CCA	CCA	DCP	OPT
5	CCA	FDCP	EDCP	CCA
6	CCA	FDCP	EDCP	OPT
7	CCA	FDCP	FDCP	CCA
8	CCA	EDCP	EDCP	CCA

In Table 3.3, problems for 5 positions are listed by adding one more CCA position to problems for 4 positions. In addition, two offset angles should be used here.

Table 3.3. Possible problems including 5 positions

#	Position 1	Position 2	Position 3	Position 4	Position 5
1	CCA	CCA	CCA	CCA	CCA
2	CCA	CCA	CCA	CCA	OPT
3	CCA	CCA	CCA	CCA	DCD
4	CCA	CCA	CCA	DCP	OPT
5	CCA	CCA	FDCP	EDCP	CCA
6	CCA	CCA	FDCP	EDCP	OPT
7	CCA	CCA	FDCP	FDCP	CCA
8	CCA	CCA	EDCP	EDCP	CCA

3.1. Common Formulation and Figures

In the following subsections, some figures and equations that are repetitively required in different problems are presented.

3.1.1. CCA

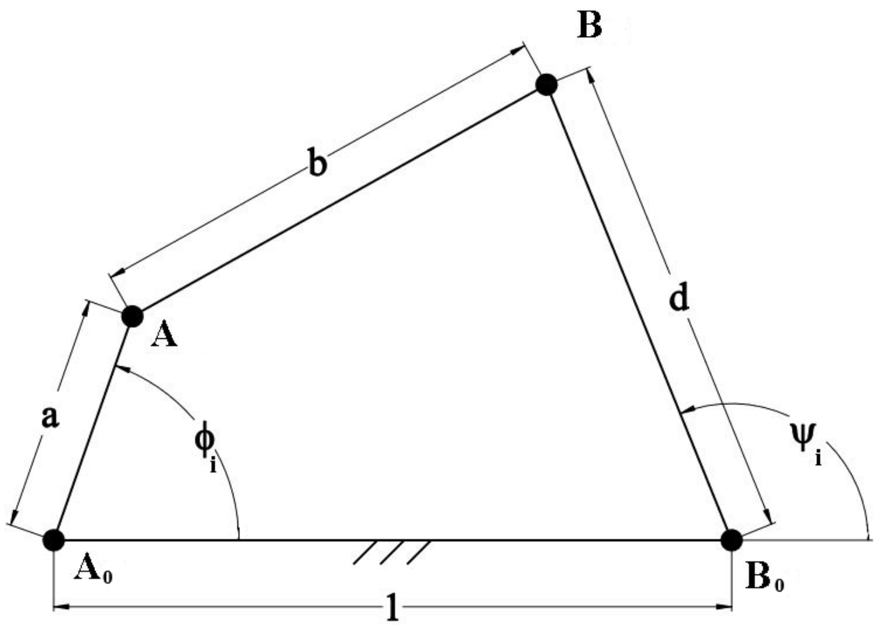


Figure 3.1. Notation for CCA

Note that, for all problems, the link length dimensions can all be scaled with the same ratio without affecting the I/O relationship between input angle ϕ and output angle ψ, so without loss of generality we may assume $|A_0B_0| = 1$.

For a planar four-bar mechanism (Figure 3.1), the loop closure equations are written as follows:

$$ac\phi + bc\gamma = 1 + dc\psi \;\; ; \;\; as\phi + bs\gamma = ds\psi \tag{1}$$

where c and s stand for cosine and sine, respectively. The interrelation of the mechanism input and output is given by the I/O of the four-bar mechanism which can be obtained by eliminating the coupler angle γ from the loop closure Eq. (1):

$$(bc\gamma)^2 + (bs\gamma)^2 = (1 + dc\psi - ac\phi)^2 + (ds\psi - as\phi)^2 \tag{2}$$

Expanding Eq. (2):

$$1 + a^2 - b^2 + d^2 - 2ac\phi + 2dc\psi - 2adc(\phi - \psi) = 0 \tag{3}$$

Dividing all terms in Eq. (3) to 2ad:

$$\frac{1 + a^2 - b^2 + d^2}{2ad} - \frac{1}{d}c\phi + \frac{1}{a}c\psi = c(\phi - \psi) \tag{4}$$

Let

$$P_1 = \frac{1 + a^2 - b^2 + d^2}{2ad}, \;\; P_2 = \frac{1}{d} \;\; \text{and} \;\; P_3 = \frac{1}{a} \tag{5}$$

3.1.2. DCP where input angle is given

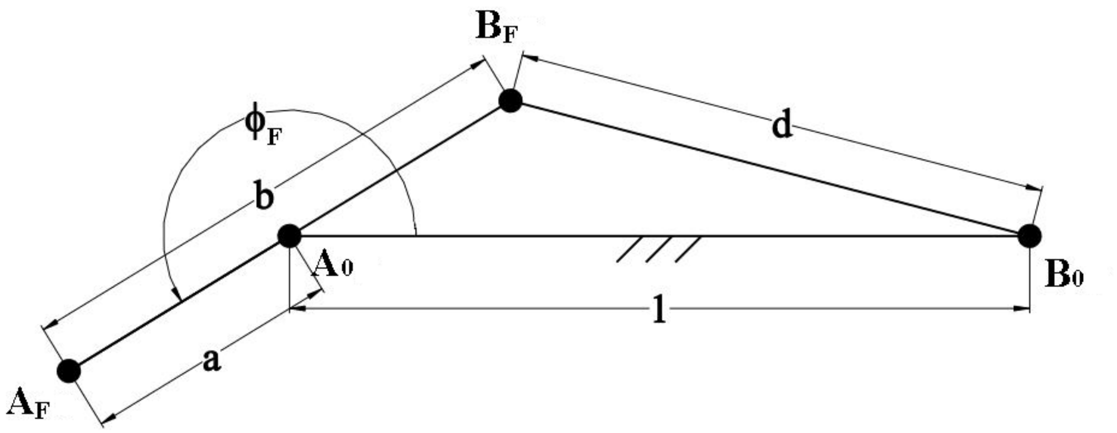


Figure 3.2. Notation for FDCP (input angle is given)

By using cosine theorem at FDCP (Figure 3.2):

$$d^2 = (b-a)^2 + 1 - 2(b-a)c(\phi_F - \pi) = (b-a)^2 + 1 + 2(b-a)c\phi_F \tag{6}$$

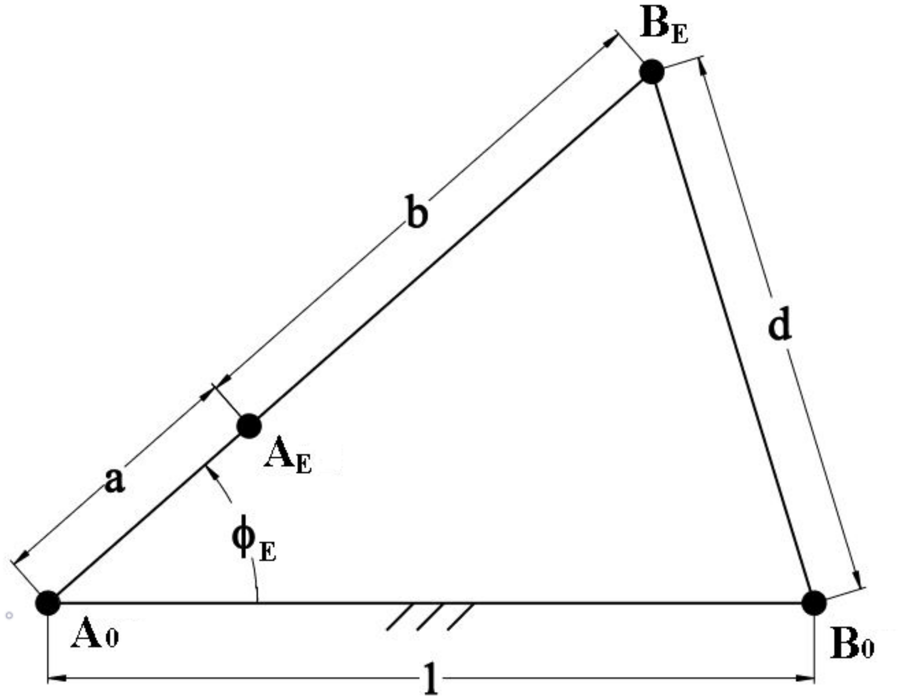


Figure 3.3. Notation for EDCP (input angle is given)

By using cosine theorem at EDCP (Figure 3.3):

$$d^2 = (a+b)^2 + 1 - 2(b+a)c\phi_E \tag{7}$$

3.1.3. DCP where output angle is given

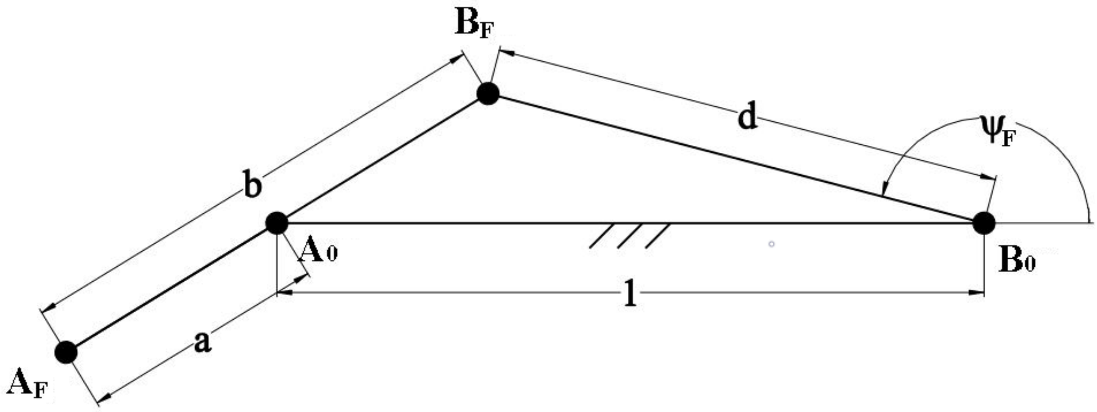


Figure 3.4. Notation for FDCP (output angle is given)

By using cosine theorem at FDCP (Figure 3.4):

$$(a-b)^2 = 1 + d^2 + 2dc\psi_F \tag{8}$$

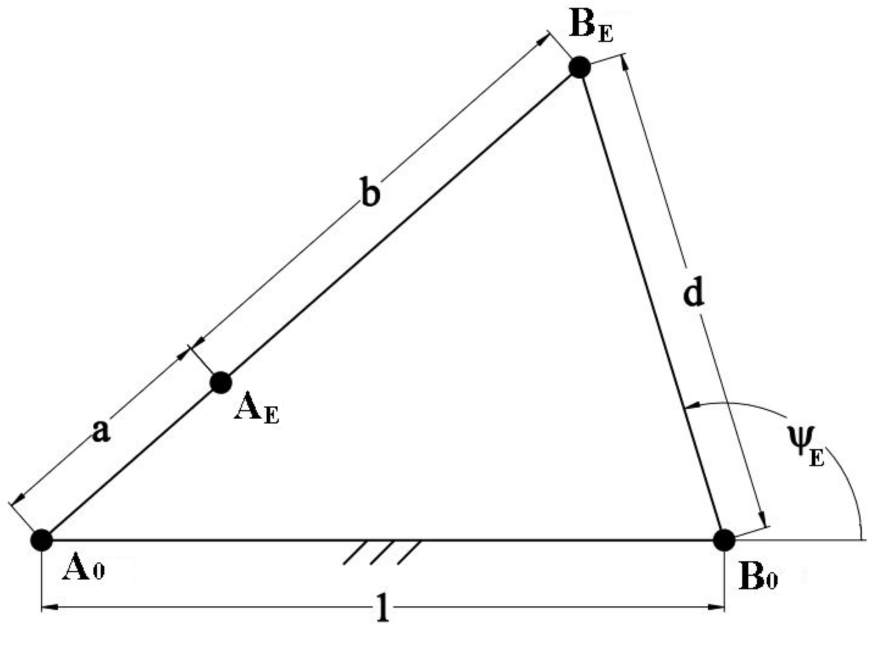


Figure 3.5. Notation for EDCP (output angle is given)

By using cosine theorem at EDCP (Figure 3.5):

$$(a+b)^2 = 1 + d^2 + 2dc\psi_E \tag{9}$$

3.1.4. OPT

When the transmission angle, μ is 90° for a four-bar mechanism where dynamic, gravitational and frictional effects are neglected, all the force transmitted from the coupler link to the output link is used to rotate the output link. If the transmission angle is 0°, no matter how much torque is applied to the input link, the output link cannot be rotated.

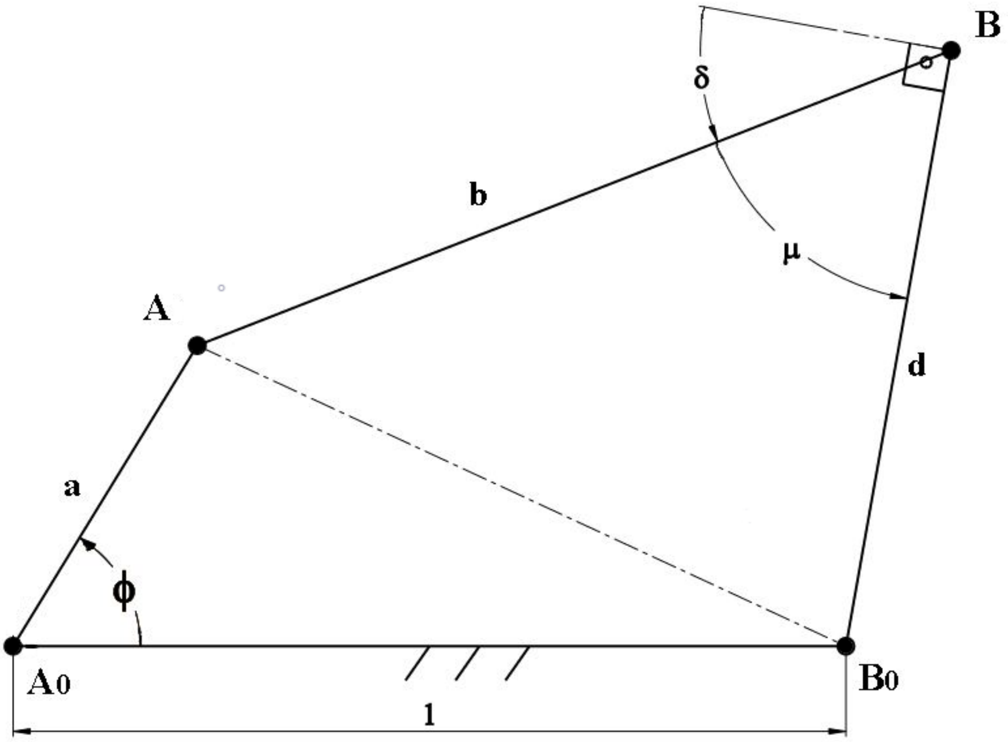


Figure 3.6. The transmission angle μ and the pressure angle δ

The transmission angle can be expressed in terms of the input angle and link lengths by writing the cosine theorem for AB_0 using the triangles A_0AB_0 and ABB_0 and equating them (Figure 3.6):

$$c\mu = \frac{d^2 + b^2 - 1 - a^2}{2bd} + \frac{a}{bd} c\phi \tag{10}$$

In order to find the minimum and maximum value of the transmission angle, the derivative of Eq. (10) with respect to ϕ is taken and equated to zero:

$$s\mu \frac{d\mu}{d\phi} = \frac{a}{bd} s\phi = 0 \tag{11}$$

Extremum values of transmission angle occur when $\sin\phi = 0$, that is when $\phi = 0$ or π (Figure 3.7). The extremum values of the transmission angle for the four-bar mechanism are as follows:

$$c\mu_{\substack{min \\ max}} = \frac{d^2 + b^2 - 1 - a^2}{2bd} \pm \frac{a}{bd} \tag{12}$$

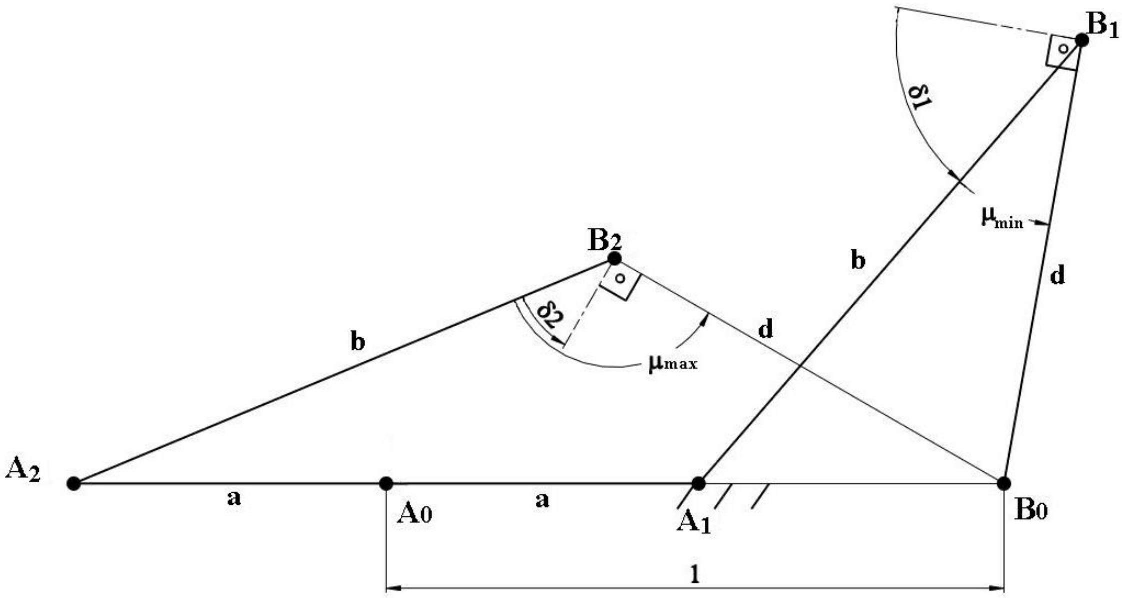


Figure 3.7. Maximum and minimum transmission angle of four-bar mechanism

After finding the maximum and minimum values of the transmission angle, the one that deviates more from 90° is the critical one. Cases where the minimum and maximum values are equal to each other are optimal if there are no other conditions. If Eq. (12) is examined, equality of the minimum and maximum transmission angle results in

$$1 + a^2 = d^2 + b^2 \tag{13}$$

The four-bar mechanisms satisfying Eq. (13) are called centric four-bar mechanisms. For a centric four-bar mechanism, the amount of crank rotation between the DCPs is 180°.

In order to optimize the transmission angle, the mechanism with the least deviation of the transmission angle (DTA) from 90° should be selected. In doing so, several different methods have been used. To equate the minimum and maximum DTA, a centric four-bar can be used. Centric four-bar mechanisms have a better force transmission characteristic compared with the other crank-rocker proportions (Söylemez, 2018).

3.2. Three Positions Synthesis for Four-Bar Mechanisms

Matlab and Mathematica have been used to for the symbolic solutions of some problems. Parametric solutions of all problems were applied using Excel. The correctness of the solutions has been verified using Solidworks.

3.2.1. 3-CCA

A practical solution to this problem was presented by Freudenstein (Freudenstein, 1954). 3 CCA positions are given (ϕ_1, ϕ_2, ϕ_3, ψ_1, ψ_2, ψ_3) and link lengths are desired. From Eqs. (4) and (5),

$$P_1 - P_2 c\phi_i + P_3 c\psi_i = c\left(\phi_i - \psi_i\right) \text{ for } i = 1, 2, 3 \tag{14}$$

For given 3 precision points, Freudenstein's coefficients (P_1 P_2 and P_3) can be found by inverting the coefficient matrix:

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & -c\phi_1 & c\psi_1 \\ 1 & -c\phi_2 & c\psi_2 \\ 1 & -c\phi_3 & c\psi_3 \end{bmatrix}^{-1} \begin{bmatrix} c\left(\phi_1 - \psi_1\right) \\ c\left(\phi_2 - \psi_2\right) \\ c\left(\phi_3 - \psi_3\right) \end{bmatrix}$$

After P_1, P_2 and P_3 are found, a, b and d can be computed using Eq. (5):

$$a = \frac{1}{P_3}, \quad d = \frac{1}{P_2} \text{ and } b = \sqrt{1 + a^2 + d^2 - 2adP_1} \tag{15}$$

3.2.2. 2-CCA with OPT

In this problem, 2 CCA are given and OPT is required. This problem can be solved in 2 ways:

- Equating the DTA from 90° at given positions
- Equating the maximum and minimum DTA from 90°

3.2.2.1. Equating the DTA from 90° at given positions

2 CCA positions are given (ϕ_1, ϕ_2, ψ_1, ψ_2) and link lengths are desired for OPT. Söylemez was solved this problem by equating DTAs at given two positions (δ_1 and δ_2 in Figure 3.8) (Söylemez, 2022).

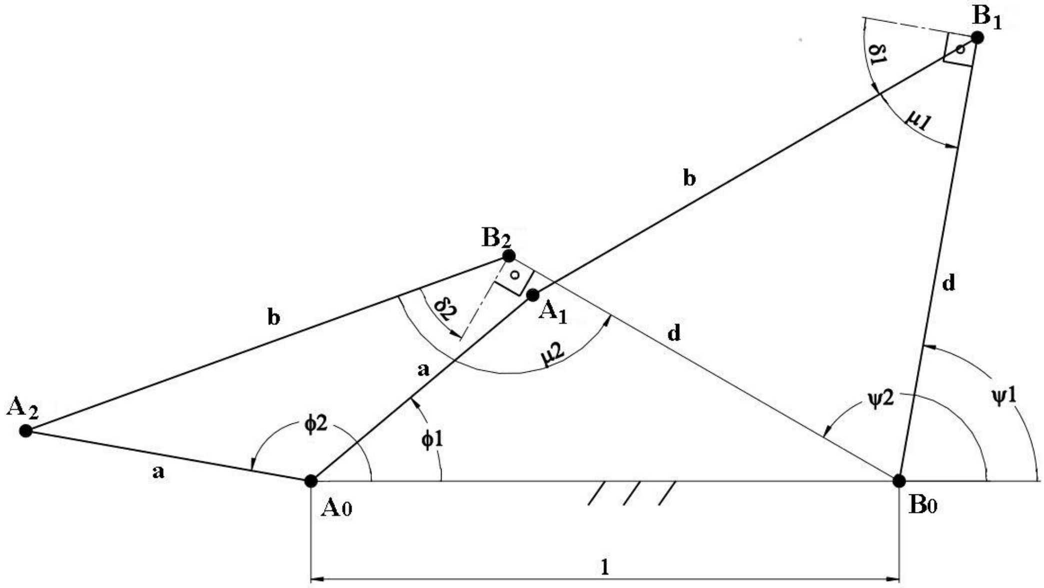


Figure 3.8. 2 CCA with OPT

The I/O function is written for given 2 CCA positions as follows:

$$P_1 - P_2 c\phi_i + P_3 c\psi_i = c(\phi_i - \psi_i) \text{ for } i = 1, 2 \tag{16}$$

where P_1, P_2, P_3 are as in Eq. (5). Eq. (16) gives 2 equations for 3 unknowns. So, the number of unknowns can be reduced to one. Solving P_2 and P_3 in terms of P_1:

$$P_2 = \frac{1}{\Delta}(T_2 P_1 + Q_2) \text{ and } P_3 = \frac{1}{\Delta}(T_1 P_1 + Q_1) \tag{17}$$

for $\Delta = c\phi_1 c\psi_2 - c\phi_2 c\psi_1$, $T_1 = c\phi_2 - c\phi_1$, $Q_1 = c\phi_1 c(\psi_2 - \phi_2) - c\phi_2 c(\psi_1 - \phi_1)$, $T_2 = c\psi_2 - c\psi_1, Q_2 = c\psi_1 c(\psi_2 - \phi_2) - c\psi_2 c(\psi_1 - \phi_1)$. Multiplying Eq. (10) with b/a:

$$\frac{b}{a} c\mu = -\frac{1 + a^2 - b^2 + d^2}{2ad} + \frac{d}{a} + \frac{1}{d} c\phi \tag{18}$$

When $\phi = \phi_1$, $\mu = \mu_1 - \delta$ and when $\phi = \phi_2$, $\mu = \mu_2 = \pi/2 + \delta$ (or $\mu_1 = \pi/2 + \delta$, $\mu_2 = \pi/2 - \delta$). Then for the two positions:

$$\frac{b}{a} s\delta = -P_1 + \frac{P_3}{P_2} + P_2 c\phi_1 \text{ and } \frac{b}{a} s\delta = P_1 - \frac{P_3}{P_2} - P_2 c\phi_2 \tag{19}$$

Equating the two equations in Eq. (19):

$$2P_1 - \frac{2P_3}{P_2} - P_2\left(c\phi_1 + c\phi_2\right) = 0 \tag{20}$$

So, a degree 2 univariate polynomial equation of P_1 is obtained:

$$AP_1^2 + BP_1 + C = 0 \tag{21}$$

where $A = T_2\left(\Delta - \frac{1}{2}T_2\left(c\phi_2 + c\phi_2\right)\right)$, $B = \left(Q_2 - P_1\right)\Delta - P_2Q_2\left(c\phi_1 + c\phi_2\right)$, $C = -Q_1\Delta - \frac{Q_2^{\,2}}{2}\left(c\phi_2 + c\phi_1\right)$.

The two roots of Eq. (21) can be found analytically. Then P_2 and P_3 can be found by using Eq. (17). Then a, b and d can be found using Eq. (15). So, we can synthesize a mechanism with input/output angles (ϕ_1, ψ_1) and (ϕ_2, ψ_2) also its transmission angle is optimized in between these two positions.

3.2.2.2. Equating the maximum and minimum DTA from 90°

2 CCA positions are given (ϕ_1, ϕ_2, ψ_1, ψ_2) and link lengths are desired for OPT. If the motion range of the mechanism is not limited with the two specified CCA positions, the maximum and minimum transmission angle for the whole range of motion can be equated. In this case, a centric-four bar mechanism is used, where the maximum and minimum transmission angle are equal. The I/O function is written for given 2 CCA positions as follows:

$$P_1 - P_2c\phi_i + P_3c\psi_i = c\left(\phi_i - \psi_i\right) \text{ for } i = 1,2 \tag{22}$$

where P_1, P_2, P_3 are as in Eq. (5). Solving P_1 from Eq. (22) in terms of P_2 and P_3 gives Eq. (17). Note that the DTA at the two extreme positions (δ_{max} and δ_{min} in Figure 3.7) will be equal if Eq. (13) is satisfied. Using Eq. (13) and rewriting Eq. (13) in terms of P_1, P_2 and P_3:

$$P_3 - P_1P_2 = 0 \tag{23}$$

Substituting Eq. (17) in Eq. (23):

$$P_1^2 + \left(Q_2 - T_1\right)P_1 - Q_1 = 0 \tag{24}$$

Eq. (24) yields 2 solutions. P_2 and P_3 can be found from Eq. (17). Then a, b and d can be found using Eq. (15).

3.2.3. 2-CCA and 1-DCP

In this problem, 2 CCA and a DCP are given. This problem can be formulated in 2 ways:

- Input angle ϕ is given for DCP
- Output angle ψ is given for DCP

3.2.3.1. Given input angle ϕ for DCP

This problem was solved by Kiper and Erez (2019). 2 CCA positions (ϕ_1, ϕ_2, ψ_1, ψ_2) and input angle for FDCP or EDCP (ϕ_F or ϕ_E) are given. Link lengths are desired. The I/O function is written for given 2 CCA positions as follows:

$$P_1 - P_2 c\phi_i + P_3 c\psi_i = c(\phi_i - \psi_i) \text{ for } i = 1, 2 \tag{25}$$

where P_1, P_2, P_3 are as in Eq. (5). Eq. (25) gives 2 equations for 3 unknowns. So, the number of unknowns can be reduced to one. Solving P_2 and P_3 in terms of P_1:

$$P_2 = \frac{1}{\Delta}(T_2 P_1 + Q_2) \text{ and } P_3 = \frac{1}{\Delta}(T_1 P_1 + Q_1) \tag{26}$$

for $\Delta = c\phi_1 c\psi_2 - c\phi_2 c\psi_1$, $T_1 = c\phi_2 - c\phi_1$, $Q_1 = c\phi_1 c(\psi_2 - \phi_2) - c\phi_2 c(\psi_1 - \phi_1)$, $T_2 = c\psi_2 - c\psi_1$, $Q_2 = c\psi_1 c(\psi_2 - \phi_2) - c\psi_2 c(\psi_1 - \phi_1)$. By using cosine theorem at FDCP and EDCP:

$$\begin{aligned} d^2 &= (b-a)^2 + 1 - 2(b-a)c(\phi_F - \pi) \\ d^2 &= (b+a)^2 + 1 - 2(b+a)c\phi_E \end{aligned} \tag{27}$$

Eq. (27) is simplified, and terms with sign differences are added up on the same side:

$$\begin{aligned} b &= \frac{1 + a^2 + b^2 - d^2 - 2ac\phi_F}{2a - 2c\phi_F} \\ -b &= \frac{1 + a^2 + b^2 - d^2 - 2ac\phi_E}{2a - 2c\phi_E} \end{aligned} \tag{28}$$

The two equations in Eq. (28) are equated to each other by taking the square of both sides. Thus, a single equation can be used for both EDCP and FDCP:

$$b^2 = \left(\frac{1 + a^2 + b^2 - d^2 - 2ac\phi_D}{2a - 2c\phi_D} \right)^2 \tag{29}$$

Eq. (29) is written in terms of P_1, P_2 and P_3 by using Eq. (15):

$$\left(\frac{1}{P_3}+\frac{P_3-P_1/P_2}{1-P_3\,c\phi_D}\right)^2=1+\frac{1}{P_3{}^2}+\frac{1}{P_2{}^2}-\frac{2P_1}{P_2P_3} \tag{30}$$

Then, Eq. (30) is written in terms of only P_1 by using Eq. (17). So, a degree 4 univariate polynomial equation of P_1 is obtained

$$AP_1{}^4+BP_1{}^3+CP_1{}^2+DP_1+E=0 \tag{31}$$

where

$$
\begin{aligned}
A &= T_1^2T_2{}^2s^2\phi_D\\
B &= 2T_1T_2s^2\phi_D\left(Q_1T_2+Q_2T_1-\Delta^2\right)\\
C &= s^2\phi_D\begin{bmatrix}\left(Q_1T_2+Q_2T_1\right)^2+2Q_1Q_2T_1T_2\\ -2Q_1T_2\Delta^2-2Q_2T_1\Delta^2\end{bmatrix}+\Delta^2\left(T_2{}^2-2T_2\Delta c\phi_D+\Delta^2-T_1{}^2c^2\phi_D\right)\\
D &= -2\begin{pmatrix}Q_1c^2\phi_D\left(Q_1Q_2T_2+Q_2{}^2T_1-\left(Q_2-T_1\right)\Delta^2\right)+\\ \Delta^3c\phi_D\left(Q_2-T_1\right)-Q_2\left(Q_1{}^2T_2+Q_1Q_2T_1-\left(Q_1-T_2\right)\Delta^2\right)\end{pmatrix}\\
E &= -Q_1{}^2c^2\theta_D\left(Q_2{}^2+\Delta^2\right)+2Q_1\Delta^3c\theta_D+Q_1{}^2Q_2{}^2+Q_2{}^2\Delta^2-\Delta^4
\end{aligned}
$$

Eq. (31) yields 4 solutions. P_2 and P_3 can be found from Eq. (17). Then a, b and d can be found using Eq. (15).

3.2.3.2. Given output angle ψ for DCP

2 CCA positions (ϕ_1, ϕ_2, ψ_1, ψ_2) and output angle for FDCP or EDCP (ψ_F or ψ_E) are given. The I/O function is written for given 2 CCA positions as follows:

$$P_1-P_2c\phi_i+P_3c\psi_i=c\left(\phi_i-\psi_i\right)\ \text{ for } i=1,2 \tag{32}$$

where P_1, P_2, P_3 are as in Eq. (5). Eq. (32) gives 2 equations for 3 unknowns. So, the number of unknowns can be reduced to one. Solving P_2 and P_3 in terms of P_1:

$$P_2=\frac{1}{\Delta}\left(T_2P_1+Q_2\right)\ \text{ and }\ P_3=\frac{1}{\Delta}\left(T_1P_1+Q_1\right) \tag{33}$$

for $\Delta=c\phi_1c\psi_2-c\phi_2c\psi_1$, $T_1=c\phi_2-c\phi_1$, $Q_1=c\phi_1c\left(\psi_2-\phi_2\right)-c\phi_2c\left(\psi_1-\phi_1\right)$, $T_2=c\psi_2-c\psi_1$, $Q_2=c\psi_1c\left(\psi_2-\phi_2\right)-c\psi_2c\left(\psi_1-\phi_1\right)$. By using cosine theorem at FDCP and EDCP:

$$
\begin{aligned}
\left(b-a\right)^2 &= 1+d^2+2dc\psi_F\\
\left(b+a\right)^2 &= 1+d^2+2dc\psi_E
\end{aligned} \tag{34}
$$

Eq. (34) is simplified, and terms with sign differences are added up on the same side:

$$-b=\frac{1-a^2-b^2+d^2+2d\,c\,\psi_F}{2a}$$
$$b=\frac{1-a^2-b^2+d^2+2d\,c\,\psi_E}{2a} \tag{35}$$

The two equations in Eq. (35) are equated to each other by taking the square of both sides. Thus, a single equation can be used for both EDCP and FDCP.

$$b^2=\left(\frac{1-a^2-b^2+d^2+2dc\psi_D}{2a}\right)^2 \tag{36}$$

Eq. (36) is written in terms of P_1, P_2 and P_3 by using Eq. (15):

$$1+\frac{1}{P_3{}^2}+\frac{1}{P_2{}^2}-\frac{2P_1}{P_2P_3}=\left(\frac{P_1+P_3\,c\,\psi_D}{P_2}-\frac{1}{P_3}\right)^2 \tag{37}$$

Then, Eq. (37) is written in terms of only P_1 by using Eq. (17). So, a degree 2 univariate polynomial equation of P_1 is obtained:

$$AP_1{}^2+BP_1+C=0 \tag{38}$$

where $A=T_2{}^2-\left(T_1\,c\,\psi_D+\Delta\right)^2$, $B=2\left(T_2Q_2-T_1Q_1\,c^2\,\psi_D-Q_1\Delta\,c\,\psi_D+T_2\Delta\,c\,\psi_D\right)$ and $C=\Delta^2+Q_2{}^2-Q_1{}^2\,c^2\,\psi_D+2Q_2\Delta\,c\,\psi_D$. Eq. (38) yields 2 solutions. P_2 and P_3 can be found from Eq. (17). Then a, b and d can be found using Eq. (15).

3.2.4. 1-CCA and 1-DCP with OPT

In this problem a CCA and a DCP are given and OPT is required. This problem can be solved in 2 ways:

- Equating the DTA from 90° at given positions
- Equating the maximum and minimum DTA from 90°

3.2.4.1. Equating the DTA from 90° at given positions

In this problem a CCA and a DCP are given and OPT is required. To optimize the transmission angle, DTA from 90° at given positions are equated. This problem can be formulated in 2 ways:

- Input angle ϕ is given for DCP
- Output angle ψ is given for DCP

3.2.4.1.1. Given input angle ϕ for DCP

A CCA position (ϕ_1, ψ_1) and an input angle for FDCP or EDCP (ϕ_F or ϕ_E) are given. Also, OPT is requested. To optimize transmission angle, the DTAs of given positions are equated. The I/O function is written for given a CCA position as follows:

$$P_1 - P_2 c\phi_1 + P_3 c\psi_1 = c(\phi_1 - \psi_1) \tag{39}$$

where P_1, P_2, P_3 are as in Eq. (5). By using cosine theorem at FDCP and EDCP:

$$\begin{gathered} d^2 = (b-a)^2 + 1 - 2(b-a)c(\phi_F - \pi) = (b-a)^2 + 1 + 2(b-a)c\phi_F \\ d^2 = (b+a)^2 + 1 - 2(b+a)c\phi_E \end{gathered} \tag{40}$$

Eq. (40) is simplified, and terms with sign differences are added up on the same side:

$$\begin{gathered} b = \frac{1 + a^2 + b^2 - d^2 - 2ac\phi_F}{2a - 2c\phi_F} \\ -b = \frac{1 + a^2 + b^2 - d^2 - 2ac\phi_E}{2a - 2c\phi_E} \end{gathered} \tag{41}$$

The two equations are equated to each other by taking the square of both sides. Thus, a single equation can be used for both EDCP and FDCP.

$$b^2 = \left(\frac{1 + a^2 + b^2 - d^2 - 2ac\phi_D}{2a - 2c\phi_D} \right)^2 \tag{42}$$

Eq. (42) is written in terms of P_1, P_2 and P_3 by using Eq. (15):

$$\left(\frac{1}{P_3} + \frac{P_3 - P_1/P_2}{1 - P_3 c\phi_D} \right)^2 = 1 + \frac{1}{P_3{}^2} + \frac{1}{P_2{}^2} - \frac{2P_1}{P_2 P_3} \tag{43}$$

For the two positions, DTA can be found by using Eq. (10) as follows:

$$\frac{b}{a} s\delta_1 = -P_1 + \frac{P_3}{P_2} + P_2 c\phi_1 \quad \text{and} \quad \frac{b}{a} s\delta_E = P_1 - \frac{P_3}{P_2} - P_2 c\phi_D \tag{44}$$

Equating the two equations in Eq. (44).

$$2P_1 - \frac{2P_3}{P_2} - P_2 (c\phi_1 + c\phi_D) = 0 \tag{45}$$

By using Eqs. (39) and (45) P_1 and P_3 can be found in terms of P_2.

$$\begin{gathered} P_1 = \frac{P_2{}^2 (c\psi_1 (c\phi_D + c\phi_1)) + 2P_2 c\phi_1 + 2c(\phi_1 - \psi_1)}{2(1 + P_2 c\psi_1)} = \frac{u_1}{u_3} \\ P_3 = \frac{P_2{}^2 (c\phi_1 - c\phi_D) + 2P_2 c(\phi_1 - \psi_1)}{2(1 + P_2 c\psi_1)} = \frac{u_2}{u_3} \end{gathered} \tag{46}$$

Substituting P_1 and P_2 into Eq. (43) a degree 6 univariate polynomial equation in P_2 is obtained:

$$u_1^2 + P_2^2\left(u_2^2 s^2\phi_D + u_3^2\right) = \left(u_2 c\phi_D - u_3\right)^2 + 2P_2 u_1\left(u_3 c\phi_D + u_2 s^2\phi_D\right) \tag{47}$$

There is no analytical solution for a degree 6 polynomial equation. But it can be solved numerically. There are most 6 real solutions. The root of equation, P_2 can be found by using "goal seek" in Excel. P_1 and P_3 can be found by using Eqs. (46) and (47). Then a, b and d can be found using Eq. (15).

3.2.4.1.2. Given output angle ψ for DCP

A CCA position (ϕ_1, ψ_1) and an output angle for FDCP and EDCP (ψ_F, ψ_E) are given. The I/O function is written for given a CCA position as follows:

$$1 + a^2 - b^2 + d^2 - 2ac\phi_1 + 2dc\psi_1 = 2adc\left(\phi_1 - \psi_1\right) \tag{48}$$

By using cosine theorem at FDCP and EDCP:

$$\begin{aligned} (b-a)^2 &= 1 + d^2 + 2dc\psi_F \\ (b+a)^2 &= 1 + d^2 + 2dc\psi_E \end{aligned} \tag{49}$$

Eq. (49) is simplified, and terms with sign differences are added up on the same side:

$$\begin{aligned} -b &= \frac{1 - a^2 - b^2 + d^2 + 2dc\psi_F}{2a} \\ b &= \frac{1 - a^2 - b^2 + d^2 + 2dc\psi_E}{2a} \end{aligned} \tag{50}$$

The two equations are equated to each other by taking the square of both sides. Thus, a single equation can be used for both EDCP and FDCP.

$$b^2 = \left(\frac{1 - a^2 - b^2 + d^2 + 2dc\psi_D}{2a}\right)^2 \tag{51}$$

For two position DTA can be found by using Eq. (10) as follows:

$$s\delta_1 = \frac{-1 - a^2 + b^2 + d^2 + 2ac\phi_1}{2bd} \quad \text{and} \quad c^2\delta_D = \frac{s^2\psi_D}{1 + d^2 + 2dc\psi_D} \tag{52}$$

DTA in two positions should be equal. So, by using equality $s^2\delta_1 + c^2\delta_D = 1$, we have:

$$\left(\frac{-1 - a^2 + b^2 + d^2 + 2ac\phi_1}{2bd}\right)^2 + \frac{s^2\psi_D}{1 + d^2 + 2dc\psi_D} = 1 \tag{53}$$

There are have 3 equations (Eqs. (48), (51) and (53)) and 3 unknowns (a, b and d). If a univariate equation is obtained using Eqs. (48), (51) and (53), it will be seen that this equation has degree 8. Therefore, it is recommended to make a completely numerical solution using these equations. The values of a, b and d that satisfy these three equations are sought. The solution can be found by using the "Solver" in Excel.

3.2.4.2. Equating the maximum and minimum DTA from 90°

In this problem a CCA and a DCP are given and OPT is required. To optimize the transmission angle, maximum and minimum DTA from 90° are equated. This problem can be formulated in 2 ways

- Input angle ϕ is given for DCP
- Output angle ψ is given for DCP

3.2.4.2.1. Given input angle ϕ for DCP

A CCA position (ϕ_1, ψ_1) and input angle for FDCP or EDCP (ϕ_F or ϕ_E) are given. Also, OPT is requested. To optimize the transmission angle, a centric four-bar mechanism is used. The I/O function is written for given a CCA position as follows:

$$P_1 - P_2 c\phi + P_3 c\psi = c(\phi - \psi) \tag{54}$$

where P_1, P_2, P_3 are as in Eq. (5). Note that the DTA at the two extreme positions will be equal if:

$$d^2 + b^2 = 1 + a^2 \tag{55}$$

Eq. (55) can be written in terms of P_1 P_2 and P_3 as follows:

$$P_3 = P_1 P_2 \tag{56}$$

Substitute P_3 in Eq. (56) into Eq. (54) solve for P_1:

$$P_1 = \frac{c(\phi - \psi) + P_2 c\phi}{1 + P_2 c\psi} \tag{57}$$

By using cosine theorem at FDCP and EDCP:

$$\begin{gathered} d^2 = (b-a)^2 + 1 - 2(b-a)c(\phi_F - \pi) = (b-a)^2 + 1 + 2(b-a)c\phi_F \\ d^2 = (b+a)^2 + 1 - 2(b+a)c\phi_E \end{gathered} \tag{58}$$

Eq. (58) is simplified, and terms with sign differences are added up on the same side.

$$\begin{aligned} b &= \frac{1+a^2+b^2-d^2-2ac\phi_F}{2a-2c\phi_F} \\ -b &= \frac{1+a^2+b^2-d^2-2ac\phi_E}{2a-2c\phi_E} \end{aligned} \tag{59}$$

The two equations are equated to each other by taking the square of both sides. Thus, a single equation can be used for both EDCP and FDCP.

$$b^2 = \left(\frac{1+a^2+b^2-d^2-2ac\phi_D}{2a-2c\phi_D}\right)^2 \tag{60}$$

Eq. (60) is written in terms of P_1, P_2 and P_3 by using Eq. (15):

$$\left(\frac{1}{P_3}+\frac{P_3-P_1/P_2}{1-P_3c\phi_D}\right)^2 = 1+\frac{1}{P_3^{\,2}}+\frac{1}{P_2^{\,2}}-\frac{2P_1}{P_2P_3} \tag{61}$$

Substituting P_3 and P_1 in Eqs. (56) and (57) into Eq. (61) we obtain a degree 6 univariate polynomial equation in terms of P_2. But this degree 6 polynomial equation can be written as multiplication of two equations, one of which has degree 2 and the other has degree 4:

$$\left(P_2^{\,2}-1\right)\left(AP_2^{\,4}+BP_2^{\,3}+CP_2^{\,2}+DP+E=0\right)=0 \tag{62}$$

where $A=-c^2\phi s^2\phi_D$, $B=-2c\phi c(\phi-\psi)s^2\phi_D$, $C=c^2\phi-c^2\psi-c^2(\phi-\psi)s^2\phi_D$, $D=-2s\phi s(\phi-\psi)$ and $E=-s^2(\phi-\psi)$. From the first equation, $P_2 = 1$ or -1. If $P_2 = 1$, $P_3 = P_2P_1 = P_1$ can be found and also it means that $a_2 = a_4$ and $a_3 = a_1$. If $P_2 = -1$, $P_3 = -P_1$ which means that $a = -d$ and $f = -b$. These solutions give us kite mechanisms. In FDCP of kite mechanisms, input and coupler links are collinear, and fixed and output links are also collinear. So, the $|P_2| = 1$ case is not feasible. The second equation is a degree 4 univariate polynomial equation, so there are at most 4 real solutions. The roots of equation can be found analytically. P_1 and P_3 can be found using Eqs. (56) and (57). Then a, b and d can be found using Eq. (15). Finally, there are maximum at most 4 real solutions.

3.2.4.2.2. Given output angle ψ for DCP

A CCA position (ϕ_1, ψ_1) and output angle for FDCP or EDCP (ψ_F or ψ_E) are given. Also, OPT is requested. To optimize the transmission angle, a centric four-bar mechanism is used. The I/O function is written for given a CCA position as follows:

$$P_1 - P_2 c\phi + P_3 c\psi = c(\phi - \psi) \tag{63}$$

where P_1, P_2, P_3 are as in Eq. (5). Note that the DTA at the two extreme positions will be equal if:

$$d^2 + b^2 = 1 + a^2 \tag{64}$$

Eq. (64) can be written in terms of P_1 P_2 and P_3 as follows:

$$P_3 = P_1 P_2 \tag{65}$$

Substitute P_3 in Eq. (65) into Eq. (63) solve for P_1.

$$P_1 = \frac{c(\phi - \psi) + P_2\, c\phi}{1 + P_2\, c\psi} \tag{66}$$

By using cosine theorem at FDCP and EDCP:

$$\begin{aligned} (b-a)^2 &= 1 + d^2 + 2dc\psi_F \\ (b+a)^2 &= 1 + d^2 + 2dc\psi_E \end{aligned} \tag{67}$$

Eq. (67) is simplified, and terms with sign differences are added up on the same side.

$$\begin{aligned} -b &= \frac{1 - a^2 - b^2 + d^2 + 2d\, c\psi_F}{2a} \\ b &= \frac{1 - a^2 - b^2 + d^2 + 2d\, c\psi_E}{2a} \end{aligned} \tag{68}$$

The two equations are equated to each other by taking the square of both sides. Thus, a single equation can be used for both EDCP and FDCP.

$$b^2 = \left(\frac{1 - a^2 - b^2 + d^2 + 2d\, c\psi_D}{2a} \right)^2 \tag{69}$$

Eq. (69) is written in terms of P_1, P_2 and P_3 by using Eq. (15):

$$1 + \frac{1}{P_3^{\,2}} + \frac{1}{P_2^{\,2}} - \frac{2P_1}{P_2 P_3} = \left(\frac{P_1 + P_3\, c\psi_D}{P_2} - \frac{1}{P_3} \right)^2 \tag{70}$$

Substituting P_3 and P_1 in Eqs. (65) and (66) into Eq. (70) we obtain a degree 4 univariate polynomial equation of P_2:

$$AP_2^{\,4} + BP_2^{\,3} + CP_2^{\,2} + DP + E = 0 \tag{71}$$

where $A = c^2\psi - c^2\phi\, c^2\psi_D$, $B = 2\left(c\psi + c\psi_D\left(c^2\psi - c^2\phi\right) - c\phi\, c(\phi-\psi)\, c^2\psi_D\right)$,

$C = 1 - c^2\phi + c^2\psi - 4c\phi\, c(\phi - \psi)\, c\psi_D + 4c\psi\, c\psi_D - c^2(\phi-\psi)\, c^2\psi_D$,

$D = 2\left(c\psi_D\, s^2(\phi - \psi) + c\psi - c\phi\, c(\phi - \psi)\right)$ and $E = s^2(\phi - \psi)$. Eq. (71) yields 4 solutions.

P_1 and P_3 can be found from Eqs. (66) and (65). Then a, b and d can be found using Eq. (15).

3.2.5. 1-CCA and 2 Different Type DCPs

In this problem a CCA and 2 different type DCPs are given. This problem can be formulated in 3 ways:

- Input angle ϕ is given for DCPs
- Output angle ψ is given for DCPs
- Input angle ϕ is given for one DCP and output angle ψ is given for the other DCP

3.2.5.1. Given input angles ϕ for both DCPs

A CCA position (ϕ_1, ψ_1) and input angles for FDCP and EDCP (ϕ_F, ϕ_E) are given. The I/O function is written for the given CCA position as follows:

$$1 + a^2 - b^2 + d^2 - 2a\,c\phi_1 + 2d\,c\psi_1 = 2ad\,c\left(\phi_1 - \psi_1\right) \tag{72}$$

By using cosine theorem at FDCP and EDCP:

$$d^2 = \left(b + a\right)^2 + 1 - 2\left(b + a\right)c\,\phi_E \tag{73}$$

$$d^2 = \left(b - a\right)^2 + 1 + 2\left(b - a\right)c\,\phi_F \tag{74}$$

Subtracting Eq. (74) from Eq. (73) and solving for b:

$$2ab = c\,\phi_E\left(b + a\right) + c\,\phi_F\left(b - a\right) \Rightarrow b = \frac{a\left(c\,\phi_E - c\,\phi_F\right)}{2a - \left(c\,\phi_E + c\,\phi_F\right)} \tag{75}$$

Substituting d^2 in Eq. (73) into Eq. (72) and solving for d using Eq. (75):

$$d = \frac{1 + a^2 + ab - ac\phi_1 - \left(b + a\right)c\phi_E}{ac\left(\phi_1 - \psi_1\right) - c\psi_1} = \frac{2a\left(a - c\phi_E\right)\left(a - c\phi_F\right) + \left(1 - ac\phi_1\right)\left(2a - c\phi_E - c\phi_F\right)}{\left(ac\left(\phi_1 - \psi_1\right) - c\psi_1\right)\left(2a - c\phi_E - c\phi_F\right)} \tag{76}$$

Substituting Eq. (76) in Eq. (74) we obtain a degree 6 univariate polynomial equation in terms of a:

$$\begin{pmatrix} 2a\left(a - c\phi_E\right)\left(a - c\phi_F\right) \\ +\left(1 - ac\phi_1\right)\left(2a - c\phi_E - c\phi_F\right) \end{pmatrix}^2 = \begin{pmatrix} 4a^2\left(c\phi_E - a\right)^2 + \left(2a - c\phi_E - c\phi_F\right)^2 \\ +4a\left(c\phi_E - a\right)\left(2a - c\phi_E - c\phi_F\right)c\phi_F \end{pmatrix}\begin{pmatrix} ac\left(\phi_1 - \psi_1\right) \\ -c\psi_1 \end{pmatrix}^2 \tag{77}$$

This is a degree 6 polynomial equation, so there are most 6 real solutions. The roots of the equation can be found numerically by using "goal seek" in Excel. b and d can be found by using Eqs. (75) and (76).

3.2.5.2. Given output angles ψ for both DCPs

A CCA position (ϕ_1, ψ_1) and output angles for FDCP and EDCP (ψ_F, ψ_E) are given. The I/O function is written for the given CCA position as follows:

$$1+a^2-b^2+d^2-2a\,c\phi_1+2d\,c\psi_1=2ad\,c(\phi_1-\psi_1) \tag{78}$$

By using cosine theorem at FDCP and EDCP:

$$(b+a)^2=1+d^2+2d\,c\psi_E \tag{79}$$

$$(b-a)^2=1+d^2+2d\,c\psi_F \tag{80}$$

Subtracting Eq. (80) from Eq. (79) and solving for b:

$$b=\frac{d}{2a}(c\psi_E-c\psi_F) \tag{81}$$

Substituting b in Eq. (81) into Eq. (79) and substituting b^2 in Eq. (79) into Eq. (78) and solving for d:

$$d=\frac{a(a-c\phi_1)}{a\,c(\phi_1-\psi_1)+c\psi_F-c\psi_1+\frac{c\psi_E-c\psi_F}{2}} \tag{82}$$

Substituting b and d in Eqs. (81) and (82) into Eq. (80) a degree 4 univariate polynomial equation in terms of a is obtained:

$$Aa^4+Ba^3+Ca^2+Da+E=0 \tag{83}$$

where

$$A=4s^2(\phi_1-\psi_1)\;\;,\;\;B=8s\psi_1\,s(\phi_1-\psi_1)$$
$$C=4\big(c\psi_E\,c\psi_F-c\phi_1(c\psi_F+c\psi_E)c(\phi_1-\psi_1)-c^2\psi_1+c^2(\phi_1-\psi_1)+c^2\phi_1\big)$$
$$D=4\big((c\psi_F-2c\psi_1+c\psi_E)c(\phi_1-\psi_1)+c\phi_1(c\psi_1(c\psi_F+c\psi_E)-2c\psi_E\,c\psi_F)\big)$$
$$E=-4c\psi_1(c\psi_F+c\psi_E)+c\psi_E(c2\phi_1+3)c\psi_F+s^2\phi_1\,c^2\psi_F+4c^2\psi_1+c^2\psi_E\,s^2\phi_1$$

The roots of the equation can be found analytically. b and d can be found using Eqs. (81) and (82).

3.2.5.3. Given input angle ϕ for one DCP and output angle ψ for other DCP

A CCA position (ϕ_1, ψ_1) and input angle for one of DCPs and output angle are given for other DCP (ϕ_E, ψ_F or ϕ_F, ψ_E). There are two possible cases:

i) Input angle ϕ is given for EDCP and output angle ψ is given for FDCP

ii) Input angle ϕ is given for FDCP and output angle ψ is given for EDCP

The solution of both cases turns out to be identical. The I/O function is written for the given CCA position as follows:

$$1 + a^2 - b^2 + d^2 - 2ac\phi_1 + 2dc\psi_1 = 2adc(\phi_1 - \psi_1) \tag{84}$$

For given input angle ϕ in EDCP and output angle ψ in FDCP, using cosine theorem:

$$(b - a)^2 = 1 + d^2 + 2dc\psi_F \tag{85}$$

$$d^2 = (b + a)^2 + 1 - 2(b + a)c\phi_E \tag{86}$$

Subtracting Eq. (85) from Eq. (86) and solving for b:

$$b = -\frac{dc\psi_F + 1 - ac\phi_E}{2a - c\phi_E} \tag{87}$$

Substituting b^2 in Eq. (84) into Eq. (85) and solving for d by using Eq. (87).

$$\begin{aligned} d &= \frac{a^2 - ac\phi_1 - ba}{ac(\phi_1 - \psi_1) + c\psi_F - c\psi_1} \\ &= \frac{a\left(2a^2 - 2ac\phi_D - 2ac\phi_1 + c\phi_1 c\phi_D + 1\right)}{2a^2c(\phi_1 - \psi_1) - 2ac\psi_1 - ac(\phi_1 - \psi_1)c\phi_D - ac\psi_D + c\phi_D(c\psi_1 - c\psi_D)} \end{aligned} \tag{88}$$

For given input angle ϕ in FDCP and output angle ψ in EDCP, using cosine theorem:

$$(b + a)^2 = 1 + d^2 + 2dc\psi_E \tag{89}$$

$$d^2 = (b - a)^2 + 1 + 2(b - a)c\phi_F \tag{90}$$

Subtracting Eq. (89) from Eq. (90) and solving for b:

$$b = \frac{dc\psi_E + 1 - ac\phi_F}{2a - c\phi_F} \tag{91}$$

Substituting b^2 in Eq. (84) into Eq. (89) and solving for d by using Eq. (91).

$$
\begin{aligned}
d &= \frac{a^2 - ac\phi_1 + ba}{ac(\phi_1 - \psi_1) + c\psi_F - c\psi_1} \\
&= \frac{a\left(2a^2 - 2ac\phi_D - 2ac\phi_1 + c\phi_1 c\phi_D + 1\right)}{2a^2c(\phi_1 - \psi_1) - 2ac\psi_1 - ac(\phi_1 - \psi_1)c\phi_D - ac\psi_D + c\phi_D(c\psi_1 - c\psi_D)}
\end{aligned} \tag{92}
$$

Substituting b and d in Eqs. (87) and (88) into Eq. (86) or substituting b and d in Eqs. (91) and (92) into Eq. (90), the solution is unique so, D can be used instead of F or E. We obtain a degree 6 univariate polynomial equation in terms of a:

$$
ak_2\left[2k_1\left(c\psi_1 - ac(\phi_1 - \psi_1)\right) + ak_2\right] + k_1^2\left(-2ac\phi_1 + a^2 + 1\right) = \left(\frac{ak_2c\psi_D + k_1(1 - ac\phi_D)}{c\phi_D - 2a}\right)^2 \tag{93}
$$

where $k_1 = 2a^2c(\phi_1 - \psi_1) + (c\psi_1 - c\psi_D)c\phi_D + a\left[c\psi_D - c\phi_D c(\phi_1 - \psi_1) - 2c\psi_1\right]$ and $k_2 = 2a^2 - 2a(c\phi_D + c\phi_1) + c\phi_1 c\phi_D + 1$. The roots of the equation can be found numerically by using "goal seek" in Excel. b and d can be found by using Eqs. (91) and (92). a and d are the same for the two cases and b has only sign difference. b can never be less than zero, so for the common solution the absolute value of b can be used:

$$
b = \left|\frac{d\,c\psi_D + 1 - a\,c\phi_D}{(2a - c\phi_D)}\right| \tag{94}
$$

3.2.6. 1-CCA and 2 Same Type DCPs

In this problem a CCA and 2 same type DCPs are given. This problem can be formulated in 2 ways:

- two input or two outputs angles are given for both DCPs
- input angle is given for a DCP and output angle is given for other DCP

3.2.6.1. Given two output or two input angles for both DCPs

For a CCA and 2-FDCP the followings are given: a CCA position (ϕ_1, ψ_1) and output angles for each FDCPs (ψ_{F1}, ψ_{F2}). ψ_{F1} is the output angle when input and coupler links are collinear in folded configuration and ψ_{F2} is the output angle when output and coupler links are collinear in folded configuration (Figure 3.9).

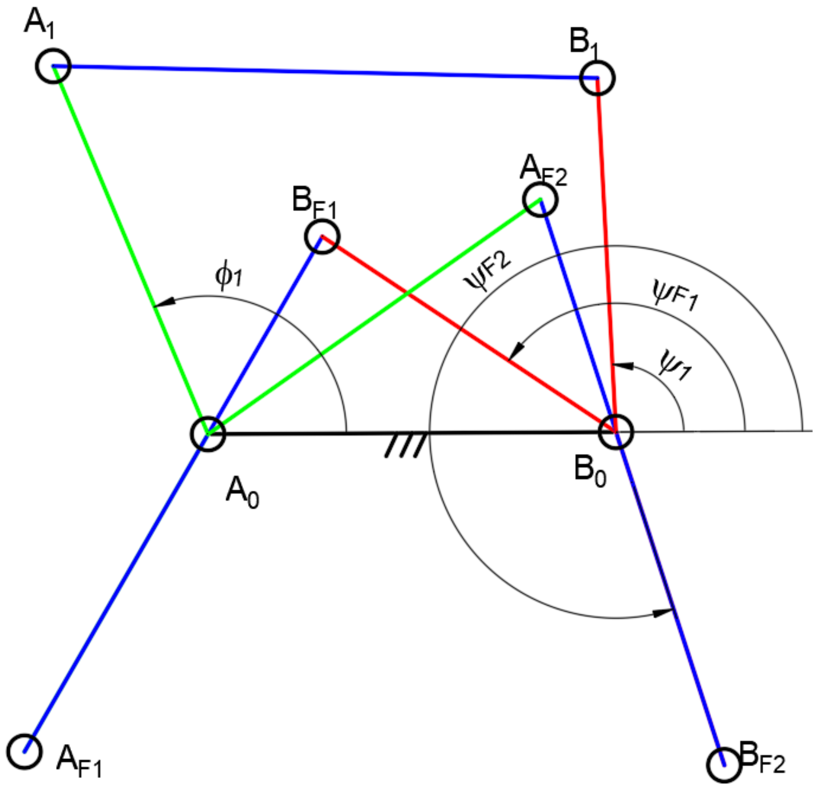


Figure 3.9. A CCA and 2-FDCP are given when output angles are given for both DCPs

The I/O function is written for given a CCA position as follows:

$$1 + a^2 - b^2 + d^2 - 2a\,c\phi_1 + 2d\,c\psi_1 = 2ad\,c(\phi_1 - \psi_1) \tag{95}$$

By using cosine theorem at each EDCP:

$$(b-a)^2 = 1 + d^2 + 2d\,c\psi_{F1} \tag{96}$$

$$a^2 = 1 + (b-d)^2 - 2(b-d)c\psi_{F2} \tag{97}$$

For a CCA and 2-EDCP the followings are given: a CCA position (ϕ_1, ψ_1) and output angles for each FDCPs (ψ_{E1}, ψ_{E2}). ψ_{E1} is the output angle when input and coupler links are collinear in extended configuration and ψ_{E2} is the output angle when output and coupler links are collinear in extended configuration (Figure 3.10).

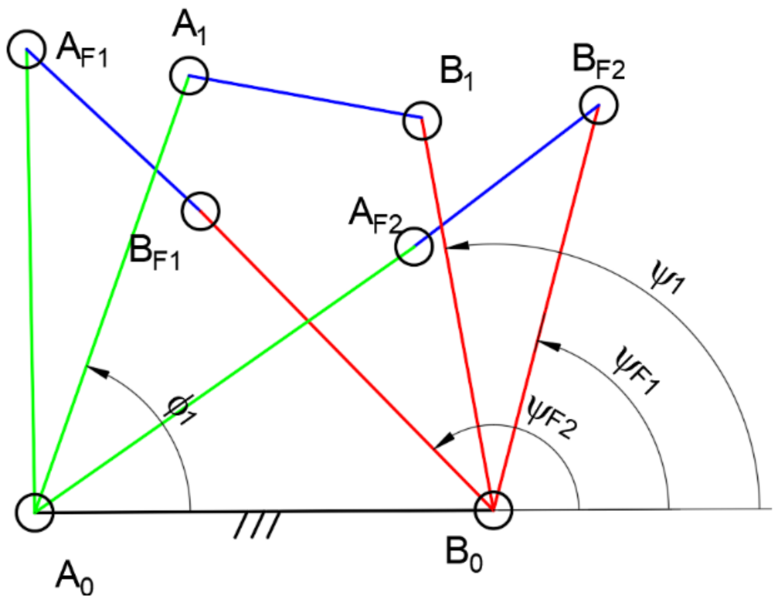


Figure 3.10. A CCA and 2-EDCP are given when output angles are given for both DCPs

The I/O function is written for given a CCA position as follows:

$$1+a^2-b^2+d^2-2a\cos\phi_1+2d\,c\,\psi_1=2ad\,c\left(\phi_1-\psi_1\right) \tag{98}$$

By using cosine theorem at each EDCP:

$$\left(b+a\right)^2=1+d^2+2d\,c\,\psi_{E1} \tag{99}$$

$$a^2=1+\left(b+d\right)^2+2\left(b+d\right)c\,\psi_{E2} \tag{100}$$

By using Eqs. (95), (96) and (97) or Eqs. (98), (99) and (100), d can be found as the root of a degree 4 univariate polynomial equation:

$$Ad^4+Bd^3+Cd^2+Dd+E=0 \tag{101}$$

where

$$A=\left[c\left(\phi_1-\psi_1\right)-1\right]\left[c\left(\phi_1-\psi_1\right)\left(1-2c\psi_{D1}c\psi_{D2}+c^2\psi_{F1}\right)+2c\psi_1\left(c\psi_{D1}-c\psi_{D2}\right)+s^2\psi_{D1}\right]$$

$$B=2\begin{pmatrix} c\phi_1\left[c\left(\phi_1-\psi_1\right)\left(-2c\psi_{D1}c\psi_{D2}+c^2\psi_{D1}+1\right)+\left(c\psi_1-c\psi_{D1}\right)\left(c\psi_{D1}-c\psi_{D2}\right)\right] \\ +c\psi_1\left[c\psi_{D2}c\left(\phi_1-\psi_1\right)\left(c\psi_{D1}-c\psi_{D2}\right)-s^2\psi_{D2}\right]+c^2\psi_1\left(c\psi_{D2}-c\psi_{D1}\right) \\ +c\psi_{D1}\left[c\left(\phi_1-\psi_1\right)-1\right]\left[s^2\psi_{D2}c\left(\phi_1-\psi_1\right)-c\psi_{D1}c\psi_{D2}+1\right] \end{pmatrix}$$

$$C=\begin{pmatrix} 2c\phi_1\left\{\left[c\psi_1-c\psi_{D1}\right]c\psi_{D2}\left[c\psi_{D1}-c\psi_{D2}\right]+2c\psi_{D1}s^2\psi_{D2}c\left(\phi_1-\psi_1\right)\right\} \\ +c^2\psi_{D2}\left[c^2\psi_{D1}+1\right]+2c\psi_1\left[c\psi_{D1}\left(c^2\psi_{D2}-2\right)+c\psi_{D2}\right] \\ +c^2\phi_1\left[1-2c\psi_{D1}c\psi_{D2}+c^2\psi_{D1}\right]-s\phi_1s^2\psi_{D2}s\left(\phi_1-2\psi_1\right)-2 \end{pmatrix}$$

$$D=-2s\phi_1s^2\psi_{D2}\left[s\left(\phi_1-\psi_1\right)+s\phi_1c\psi_{D1}\right]$$

$$E=-s^2\phi_1s^2\psi_{D2}$$

After finding d, a can be found as follows:

$$a = \frac{d\left(c\psi_1 - c\psi_{D1}\right)\left(\pm\sqrt{\rho} + d\,c\left(\phi_1 - \psi_1\right) + c\phi_1\right)}{d^2 c^2\left(\phi_1 - \psi_1\right) - \rho + 2d\,c\phi_1\,c\left(\phi_1 - \psi_1\right) + c^2\phi_1} \tag{102}$$

where $\rho = d^2 + 2d\,c\psi_{D1} + 1$. As a common solution for a CCA and 2-FDCP or 2-EDCP.

$$b = \left|\frac{d\left(c\psi_{D1} - c\psi_{D2}\right)\left(\mp\sqrt{\rho} + d + c\psi_{D2}\right)}{2d\left(c\psi_{D1} - c\psi_{D2}\right) - c^2\psi_{D2} + 1}\right| \tag{103}$$

Eq. (101) is a degree 4 polynomial equation in terms of d and it can be solved analytically. In Eq. (102), there is a term that has $\pm$ sign and in Eq. (103), there is a term that has $\mp$ sign. This means that, in this solution, for each d value, there are two solutions for a and b. Actually, for each d value, there is only one solution of a and b but the solution is too long to write here. For finding the proper a and b values a verification is needed. Two solutions of a and b should be found and substituted into Eq. (95). Then one of the solutions satisfies Eq. (95).

The same formulation can be used in the case where two input angles are given for both DCPs. The symmetry of the mechanism with respect to the y-axis can be considered. Thus, d is used instead of a, and a is used instead of d in the formulation. $\pi - \phi_1$ is used instead of ψ_1, $\pi - \psi_1$ is used instead of ϕ_1, $\pi - \phi_{D2}$ is used instead of ψ_{D1}, $\pi - \phi_{D1}$ is used instead of ψ_{D2}.

3.2.6.2. Given input angle for a DCP and output angle for other DCP

In this problem there are two possible cases:

- Input angle is given for DCP where input and coupler links are collinear and output angle is given for the other DCP where output and coupler links are collinear.
- Input angle is given for DCP where output and coupler links are collinear and output angle is given for the other DCP where input and coupler links are collinear.

Solutions and formulation of these two cases are different.

3.2.6.2.1. Given input angle for DCP where input and coupler links are collinear and output angle for the other DCP where output and coupler links are collinear

For a CCA and 2-FDCP the following are given: a CCA position (ϕ_1, ψ_1), the input angle for the FDCP where input and coupler links are collinear (ϕ_{F1}) and output angle for the other FDCP where output and coupler links are collinear (ψ_{F2}) (Figure 3.11).

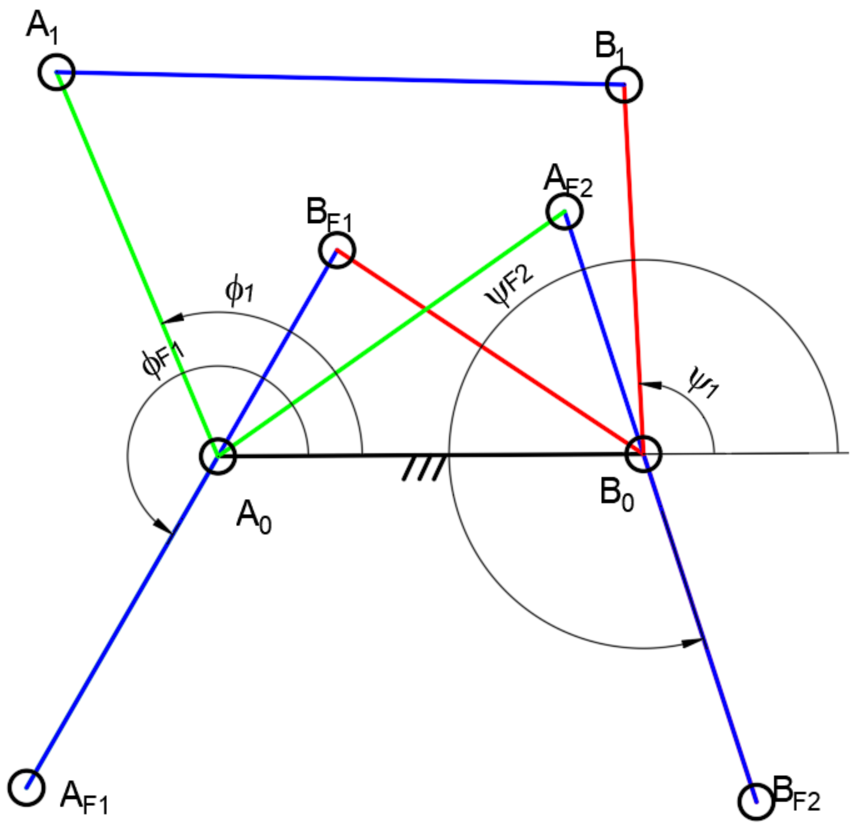


Figure 3.11. Given input angle for FDCP where input and coupler links are collinear and output angle for the other FDCP where output and coupler links are collinear

The I/O function is written for given CCA as follows:

$$1+a^2-b^2+d^2-2a\cos\phi_1+2d\cos\psi_1=2ad\cos\left(\phi_1-\psi_1\right) \tag{104}$$

By using cosine theorem at each folded DCPs:

$$d^2=\left(b-a\right)^2+1-2\left(b-a\right)\cos\left(\phi_{F1}-180\right) \tag{105}$$

$$a^2 = 1 + (b-d)^2 - 2(b-d)\cos\psi_{F2} \qquad (106)$$

For a CCA and 2-EDCP, the following are given: a CCA position (ϕ_1, ψ_1), the input angle for the EDCP where input and coupler links are collinear (ϕ_{E1}) and output angle for the other EDCP where output and coupler links are collinear (ψ_{E2}) (Figure 3.12).

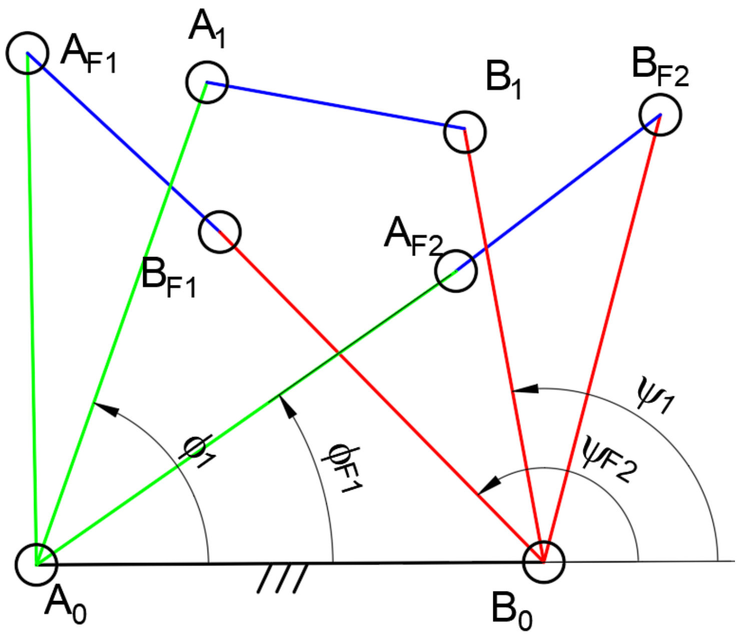


Figure 3.12. Given input angle for EDCP where input and coupler links are collinear and output angle for the other EDCP where output and coupler links are collinear

The I/O function is written for given crank angle correlations as follows:

$$1 + a^2 - b^2 + d^2 - 2a\cos\phi_1 + 2d\cos\psi_1 = 2ad\cos(\phi_1 - \psi_1) \qquad (107)$$

By using cosine theorem at each extended DCPs:

$$d^2 = (b+a)^2 + 1 - 2(b+a)\cos\phi_{E1} \qquad (108)$$

$$a^2 = 1 + (b+d)^2 - 2(b+d)\cos(\psi_{E2} - 180) \qquad (109)$$

By using Eqs. (104), (105) and (106) or Eqs. (107), (108) and (109), d can be found as the root of a degree 4 univariate polynomial equation:

$$Ad^4 + Bd^3 + Cd^2 + Dd + E = 0 \qquad (110)$$

where

$$A = \left[c(\phi_1-\psi_1)-1\right]\left[c(\phi_1-\psi_1)\left(1-2c\phi_{D1}+c^2\phi_{D1}\right)-2c\phi_{D1}(c\psi_1-c\psi_{D2})-2c\psi_1 c\psi_{D2}+s^2\phi_{D1}\right]$$

$$B = 2\begin{pmatrix} c^2\phi_{D1}\left[c(\phi_1-\psi_1)-1\right]\left[c\psi_{D2}\left(c(\phi_1-\psi_1)-1\right)+c\phi_1-c\psi_1\right]+ \\ c\phi_{D1}\begin{bmatrix} c(\phi_1-\psi_1)\left(c\psi_{D2}(2c\phi_1-3c\psi_1+c\psi_{D2})-2\right)-c\phi_1 c\psi_{D2} \\ +c^2(\phi_1-\psi_1)\left(c^2\psi_{D2}+1\right)+c\psi_1(+2c\psi_{D2}-c\phi_1+c\psi_1)+1 \end{bmatrix} \\ +\left[c^2\psi_1 c\psi_{D2}-c(\phi_1-\psi_1)\right]\left[c\psi_1-c\psi_{D2}\left(c(\phi_1-\psi_1)-1\right)-c\phi_1\right] \end{pmatrix}$$

$$C = \begin{pmatrix} -2c\phi_1\begin{pmatrix} c\phi_{D1}\left(2-\left(c(2\psi_{D2})+3\right)c(\phi_1-\psi_1)+3c\psi_1 c\psi_{D2}+c^2\psi_{D2}\right)-2c\psi_{D2}c(\phi_1-\psi_1) \\ +c\psi_1 c^2\psi_{D2}+\cos(y)+c\psi_{D2}+c^2\phi_{D1}\left(c\psi_1-2c\psi_{D2}\left(c(\phi_1-\psi_1)-1\right)\right) \end{pmatrix}- \\ 2c\phi_{D1}\begin{pmatrix} c\psi_1 c^2\psi_{D2}\left(c(\phi_1-\psi_1)-1\right)-c\psi_{D2}\left(c^2\psi_1+2\right)-2c\psi_1 \\ +c(\phi_1-\psi_1)\left(c\psi_{D2}\left(4-c(\phi_1-\psi_1)\right)+c\psi_1\right) \end{pmatrix}-2c\psi_1 c\psi_{D2}\left(c(\phi_1-\psi_1)-1\right) \\ -c^2\phi_{D1}\left(2c(\phi_1-\psi_1)\left(c\psi_{D2}(c\psi_1+c\psi_{D2})+1\right)-c^2(\phi_1-\psi_1)-c\psi_{D2}(4c\psi_1+c\psi_{D2})-1\right) \\ +c^2\psi_{D2}\left(\left(c(\phi_1-\psi_1)-1\right)^2+c^2\psi_1\right)-2c(\phi_1-\psi_1)+c^2\psi_1+1+c^2\phi_1\left(2c\phi_{D1}c\psi_{D2}+c^2\phi_{D1}+1\right) \end{pmatrix}$$

$$D = 2\begin{bmatrix} c\phi_1\left(c\phi_{D1}[c\psi_1+c\psi_{D2}]+c\psi_1 c\psi_{D2}+1\right)-c\psi_{D2} \\ +s\phi_1 s\psi_1[c\phi_{D1}+c\psi_{D2}]-c\phi_{D1}[c\psi_1 c\psi_{D2}+1]-c\psi_1 \end{bmatrix}\left[c\phi_1(c\phi_{D1}+c\psi_{D2})-c\phi_{D1}c\psi_{D2}-1\right]$$

$$E = \left(1-c\phi_1[c\phi_{D1}+c\psi_{D2}]+c\phi_{D1}c\psi_{D2}\right)^2$$

After finding d, a can be found as follows:

$$a = \frac{[d+c\psi_{D2}]\left[1+c^2\phi_{D1}+\left(d\pm\sqrt{\rho}\right)(c\phi_{D1}+c\psi_{D2})\right]+c\phi_{D1}\left(1+c\psi_{D2}[2d+c\psi_{D2}]\right)}{2\left(c\phi_{D1}(c\psi_{D2}+d)+dc\psi_{D2}\right)+1+c^2\psi_{D2}}$$

(111)

where $\rho = d^2+c^2\phi_{D1}-1$. As a common solution for a CCA and 2-FDCP or 2-EDCP:

$$b = \left|\frac{\begin{pmatrix} d\left(1+c\phi_{D1}(d+c\psi_{D2})-c^2\phi_{D1}\right)+c\psi_{D2}\left(1-c^2\phi_{D1}+d^2+dc\psi_{D2}\right) \\ \mp\sqrt{\rho}\left((c\phi_{D1}+c\psi_{D2})(d+1)+1\right) \end{pmatrix}}{2\left(c\phi_{D1}(c\psi_{D2}+d)+dc\psi_{D2}\right)+1+c^2\psi_{D2}}\right| \qquad (112)$$

Eq. (110) is a degree 4 polynomial equation in terms of d and it can be solved analytically. In Eq. (111), there is a term that has $\pm$ sign and in Eq. (112), there is a term that has $\mp$ sign. This means that, in this solution, for each d value, there are two solutions for a and b. Actually, for each d value, there is only one solution of a and b but the solution is too long to write here. For finding the proper a and b values a verification is needed. Two solutions of a and b should be found and substituted into Eq. (104). Then one of the solutions satisfy Eq. (104).

3.2.6.2.2. Given input angle for DCP where output and coupler links are collinear and output angle for the other DCP where input and coupler links are collinear

For a CCA and 2-FDCP, the following are given: a CCA position (ϕ_1, ψ_1), the input angle for the FDCP where output and coupler links are collinear (ϕ_{F1}) and output angle for the other FDCP where input and coupler links are collinear (ψ_{F2}) (Figure 3.13).

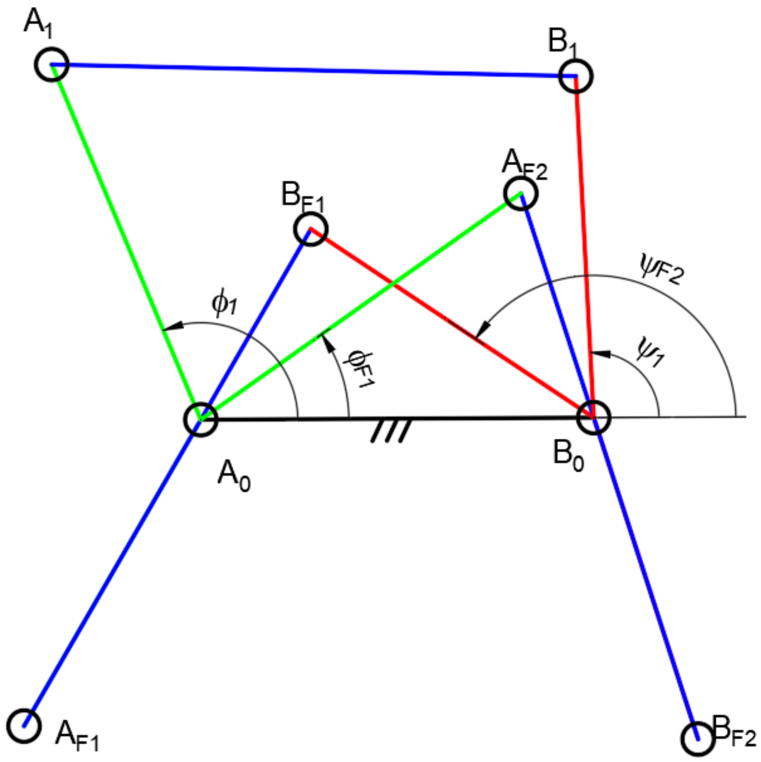


Figure 3.13. Given input angle for FDCP where output and coupler links are collinear and output angle for the other FDCP where input and coupler links are collinear

The I/O function is written for given crank angle correlations as follows:

$$1+a^2-b^2+d^2-2a\cos\phi_1+2d\cos\psi_1=2ad\cos\left(\phi_1-\psi_1\right) \tag{113}$$

By using cosine theorem at each folded DCPs:

$$\left(b-d\right)^2=a^2+1-2a\cos\phi_{F1} \tag{114}$$

$$\left(b-a\right)^2=1+d^2-2d\cos\left(\psi_{F2}-180\right) \tag{115}$$

For a CCA and 2-EDCP, the following are given: a CCA position (ϕ_1, ψ_1), the input angle for the EDCP where output and coupler links are collinear (ϕ_{E1}) and output angle for the other EDCP where input and coupler links are collinear (ψ_{E2}) (Figure 3.14).

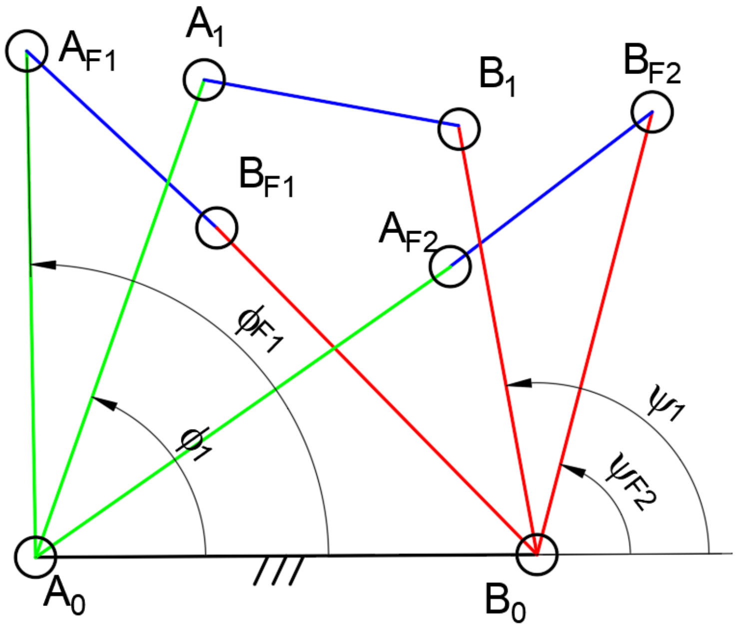


Figure 3.14. Given input angle for EDCP where output and coupler links are collinear and output angle for the other EDCP where input and coupler links are collinear

The I/O function is written for given crank angle correlations as follows:

$$1 + a^2 - b^2 + d^2 - 2a\cos\phi_1 + 2d\cos\psi_1 = 2ad\cos(\phi_1 - \psi_1) \qquad (116)$$

By using cosine theorem at each extended DCPs:

$$(d + b)^2 = a^2 + 1 - 2a\cos\phi_{E1} \qquad (117)$$

$$(a + b)^2 = 1 + d^2 - 2d\cos(\psi_{E2} - 180) \qquad (118)$$

By using Eqs. (113), (114) and (115) or Eqs. (116), (117) and (118), d can be found as the root of a degree 2 univariate polynomial equation:

$$Ad^2 + Bd + C = 0 \qquad (119)$$

where

$$A=\left[c\left(\phi_1-\psi_1\right)-1\right]\left[2\left(c\psi_{D1}\left[c\phi_{D2}-c\psi_1\right]-c\psi_1 c\phi_{D2}\right)+c^2\phi_{D2}+s^2\phi_{D2}\left(c\phi_1 c\psi_1+s\phi_1 s\psi_1\right)+1\right]$$

$$B=2\begin{pmatrix} c^2\phi_{D2}\left(c\phi_1\left(c\psi_1\left(c\psi_{D1}+c\phi_1\right)-s\phi_1 s\psi_1+1\right)+c\psi_{D1}\left(c\psi_{D1} si\phi_1 s\psi_1-1\right)\right)+ \\ c^2\phi_1 c\psi_1-c\phi_{D2}+s\phi_1 c\phi_{D2} s\psi_1+c\psi_1\left(c\psi_1 c\psi_{D1}+c\psi_1 c\phi_{D2}-s\phi_1 s\psi_1\left(1+c\psi_{D1} c\phi_{D2}\right)\right) \\ +c\phi_1\left[-c^2\psi_1\left(1+c\psi_{D1} c\phi_{D2}\right)+c\psi_{D1}\left(c\phi_{D2}-c\psi_1\right)+s\phi_1 s\psi_1\right] \end{pmatrix}$$

$$C=\begin{pmatrix} 2\left\{\left(c\psi_{D1}-c\phi_1\right)\left(c\psi_1-c\phi_{D2}\right)+c\psi_{D1} c\phi_{D2}\left[c\phi_1\left(c\phi_{D2}-c\psi_1\right)+c\psi_{D1} c\psi_1\right]\right\}+ \\ c^2\phi_1 s^2\phi_{D2}+c^2\psi_1\left(s^2\psi_{D1}-c^2\phi_{D2}\right)-1 \end{pmatrix}$$

After finding d, a can be found as follows:

$$a=\frac{d\left(c\phi_{D2}\left(d^2+dc\psi_{D1}+dc\phi_{D2}+1\right)\pm\left(\sqrt{\rho}+1\right)\left(c\psi_{D1}+c\phi_{D2}\right)\right)}{2dc\psi_{D1}-c^2\psi_{D1}+2dc\phi_{D2}+1} \tag{120}$$

where $\rho=d^2+2c^2\phi_{D2}+1$. As a common solution for a CCA and 2-FDCP or 2-EDCP:

$$b=\frac{d\left[1+dc\phi_{D2}\left(1+c\psi_{D1}\right)\right]\mp\sqrt{\rho}\left[1-c^2\psi_{D1}\left(1+d^2\right)+d\left(c\psi_{D1}+c\phi_{D2}\right)\right]}{2d\left[c\psi_{D1}+c\phi_{D2}\right]-c^2\psi_{D1}+1} \tag{121}$$

Eq. (119) is a degree 2 polynomial equation in terms of d and it can be solved analytically. In Eq. (120), there is a term that has $\pm$ sign and in Eq. (121), there is a term that has $\mp$ sign. This means that, in this solution, for each d value, there are two solutions for a and b. Actually, for each d value, there is only one solution of a and b but the solution is too long to write here. For finding the proper a and b values a verification is needed. Two solutions of a and b should be found and substituted into Eq. (113). Then one of the solutions satisfy Eq. (113).

3.2.7. 2 Different Type DCPs with OPT

In this problem 2 different type DCPs are given and OPT required. This problem can be formulated in 3 ways:

- Input angle ϕ is given for DCP
- Output angle ψ is given for DCP
- Input angle ϕ is given for one DCP and output angle ψ is given for the other DCP

3.2.7.1. Given input angles ϕ for both DCPs

Two input angles of both FDCP and EDCP (ϕ_E, ϕ_F) are given and OPT is required (Figure 3.15). The same problem can also be described given the amount of input rotation and input angle in DCP. But here the formulation is formed over the input angles of the two DCPs.

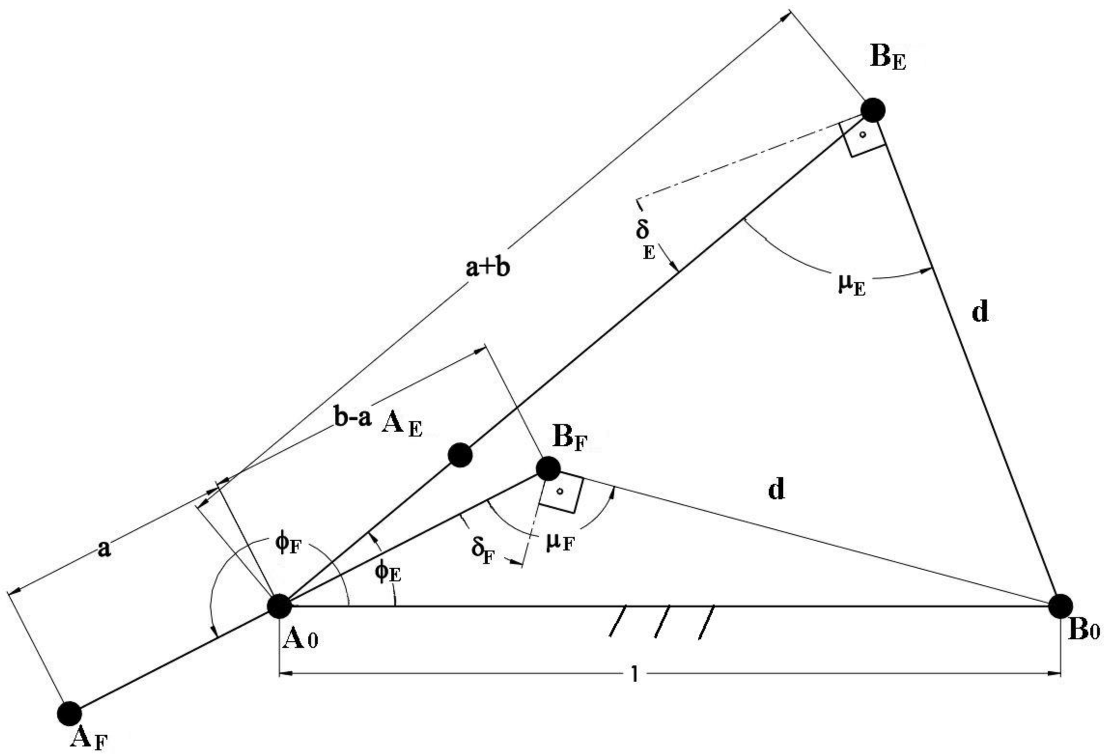


Figure 3.15. Two different type DCPs are given with OPT and input angle ϕ is given for both DCPs

Let the transmission angle at EDCP be μ_E and at FDCP be μ_F. Writing sine theorem in triangle $A_0B_EB_0$ at EDCP and triangle $A_0B_FB_0$ at FDCP:

$$\frac{s\mu_E}{1} = \frac{s\phi_E}{d} \quad \text{and} \quad \frac{s\mu_F}{1} = \frac{s\left(\phi_F - \pi\right)}{d} \tag{122}$$

Dividing the two equations to each other:

$$\frac{s\mu_E}{s\mu_F} = -\frac{s\phi_E}{s\phi_F} \tag{123}$$

ϕ_E and ϕ_F are given parameters so, $\sin\phi_E / \sin\phi_F$ is constant then we can say there is a linear proportion between $s\mu_E$ and $s\mu_F$. To optimize DTA, μ should be equal to $\pi/2$ at EDCP or FDCP, in which case the other μ will have a minimum value. In other problems which have OPT requirement, a centric four-bar is used or DTAs in given positions are equated to each other. But in this problem, it's impossible, because amount of input rotation is a constant parameter. There are 3 possible cases about amount of input rotation:

i) If $\phi_F - \phi_E > 180°$, then μ_E should be 90°

Writing sine theorem in triangle $A_0B_EB_0$ at EDCP:

$$\frac{s\,\phi_E}{d} = \frac{s\,\mu_E}{1} = \frac{s(\pi - \mu_E - \phi_E)}{b+a} \Rightarrow b + a = c\,\phi_E \text{ and } d = s\,\phi_E \tag{124}$$

Writing sine theorem in triangle $A_0B_FB_0$ at FDCP:

$$\frac{s\,\mu_F}{1} = \frac{s(\phi_F - \pi)}{d} = \frac{s(\pi - \mu_F - (\phi_F - \pi))}{b-a} \Rightarrow b - a = -c\phi_F - \sqrt{s^2\,\phi_E - s^2\,\phi_F} \tag{125}$$

So,

$$a = \frac{c\phi_E}{2} + \frac{c\phi_F + \sqrt{s^2\phi_E - s^2\phi_F}}{2},\ \ b = \frac{c\phi_E}{2} - \frac{c\phi_F + \sqrt{s^2\phi_E - s^2\phi_F}}{2} \text{ and } d = s\phi_E \tag{126}$$

ii) If $\phi_F - \phi_E < 180°$, then μ_F should be 90°

Writing sine theorem in triangle $A_0B_FB_0$ at FDCP:

$$\frac{s\mu_F}{1} = \frac{s(\phi_F - \pi)}{d} = \frac{s(\pi - \mu_F - (\phi_F - \pi))}{b-a} \Rightarrow d = -s\phi_F \text{ and } b - a = -c\phi_F \tag{127}$$

Writing sine theorem in triangle $A_0B_EB_0$ at EDCP:

$$\frac{s\phi_E}{d} = \frac{s\mu_E}{1} = \frac{s(\pi - \mu_E - \phi_E)}{b+a} \Rightarrow b + a = \frac{s(\mu_E + \phi_E)}{s\mu_E} = c\phi_E + \sqrt{s^2\phi_F - s^2\phi_E} \tag{128}$$

So,

$$a = \frac{c\phi_F}{2} + \frac{c\phi_E + \sqrt{s^2\phi_F - s^2\phi_E}}{2},\ \ b = -\frac{c\phi_F}{2} + \frac{c\phi_E + \sqrt{s^2\phi_F - s^2\phi_E}}{2} \text{ and } d = -s\phi_F \tag{129}$$

iii) If $\phi_F - \phi_E = 180°$, it means that the mechanism is a centric four-bar. In this situation, minimum feasible input link length gives the minimum DTA.

As seen above, the solutions in cases 1 and 2 are very similar to each other. Only b and d have a sign difference.

3.2.7.2. Given output angles ψ for both DCPs

Two output angles of DCP (ψ_E, ψ_F) are given and OPT is required (Figure 3.16). The same problem can also be defined for given amount of output rotation and output angle in a DCP. But here the formulation is formed over the output angles of the two DCPs.

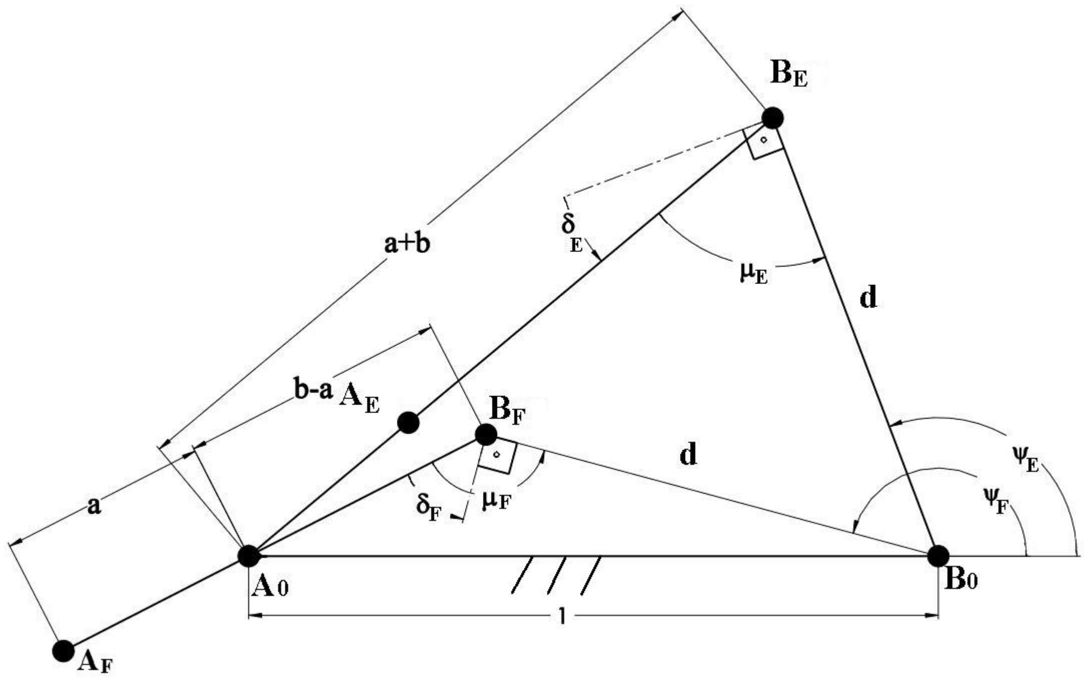


Figure 3.16. Two different type DCPs are given with OPT and output angle ψ is given for both DCPs

To optimize the transmission angle, two different methods were applied above. One is to equate the DTA in the given positions to each other, the other is to use a centric four-bar. In this problem, two methods give the same solution. Because the DTAs in the DCPs of the centric four-bar are equal to each other.

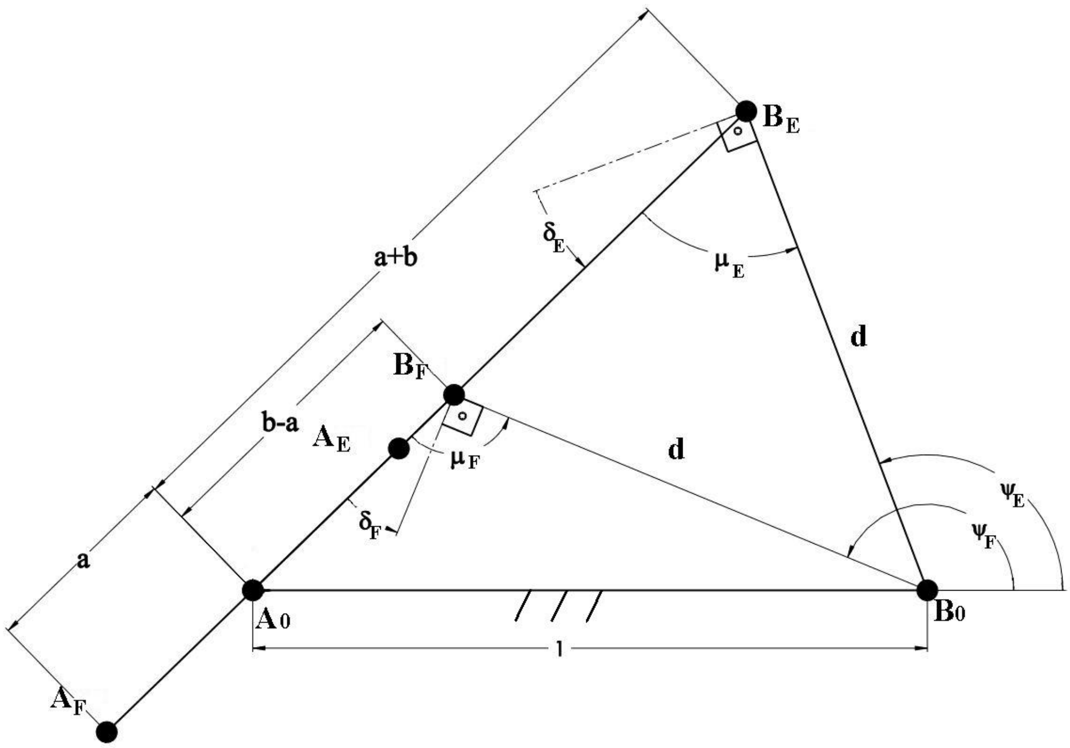


Figure 3.17. Two different type DCPs are given with OPT and output angle ψ is given for both DCPs when $\phi_F - \phi_E = 180°$

The sine theorem can be written for triangle $A_0B_EB_0$ (Figure 3.17):

$$\frac{s(\pi-\psi_E)}{b+a}=\frac{s\mu_E}{1}=\frac{s(\pi-(\pi-\psi_E+\mu_E))}{d} \tag{130}$$

The sine theorem can be written for triangle $A_0B_FB_0$:

$$\frac{s(\pi-\psi_F)}{b-a}=\frac{s\mu_F}{1}=\frac{s(\pi-(\pi-\psi_F+\mu_F))}{d} \tag{131}$$

There is an isosceles triangle ($B_FB_EB_0$) so, $\mu_E = \pi - \mu_F$ and $|B_EB_F| = b + a - (b - a) = 2a$. By using this triangle μ_E can be found as follows:

$$\mu_E=\frac{\pi-\psi_E+\psi_F}{2} \tag{132}$$

By using Eqs. (130), (131) and (132), link lengths can be found as follows:

$$a=\frac{s\psi_E-s\psi_F}{2c\left(\frac{\psi_E-\psi_F}{2}\right)},\quad b=s\left(\frac{\psi_E+\psi_F}{2}\right)\quad\text{and}\quad d=-\frac{c\left(\frac{\psi_E+\psi_F}{2}\right)}{c\left(\frac{\psi_E-\psi_F}{2}\right)} \tag{133}$$

3.2.7.3. Given input angle ϕ for one DCP and output angle ψ for other DCP

Input angle for one of DCP and output angle for other DCP are given (ϕ_E, ψ_F or ϕ_F, ψ_E). To optimize the transmission angle, two different methods were applied above. One is to equate the DTA in the given positions to each other, the other is to use a centric four-bar. In this problem, two methods give the same solution. Because the DTAs in the DCPs of the centric four-bar are equal to each other. There are two possible ways to formulate this problem:

i) Input angle ϕ is given for FDCP and output angle ψ is given for EDCP

ii) Input angle ϕ is given for EDCP and output angle ψ is given for FDCP

The solution of the two problems turns out to be identical. Let's consider case (i) first.

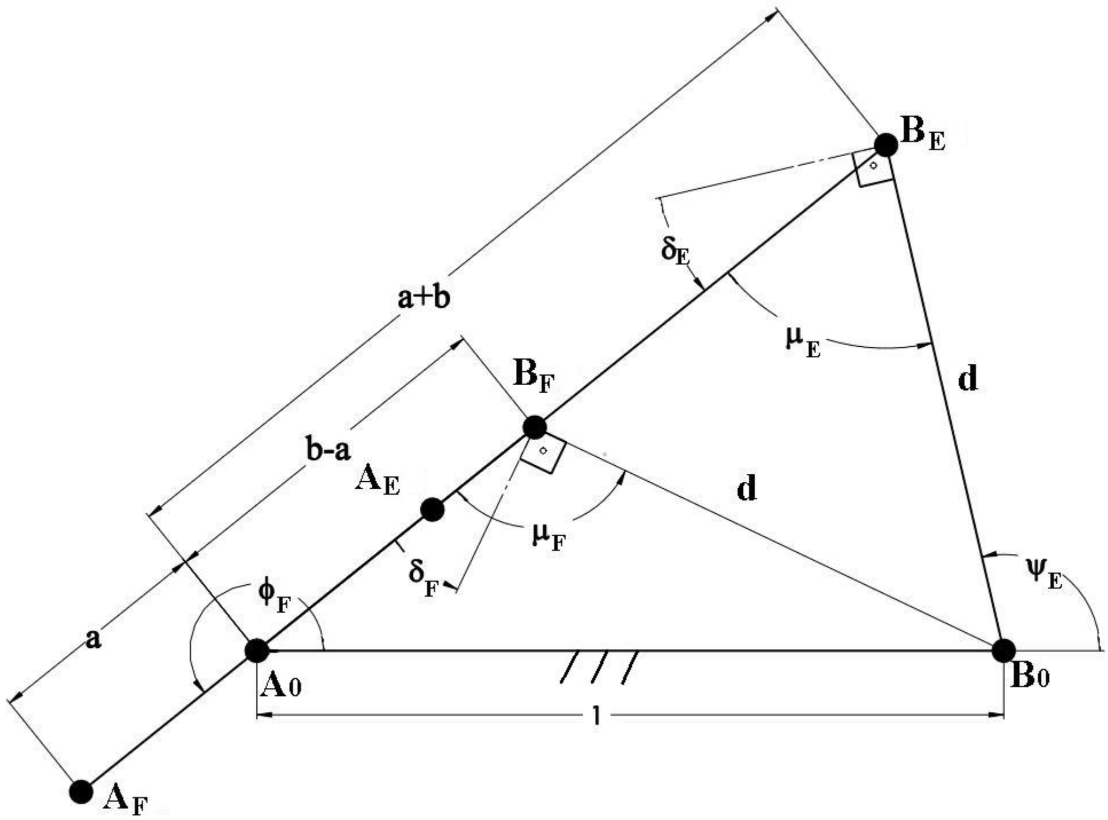


Figure 3.18. Input angle ϕ is given for FDCP and output angle ψ is given for EDCP

There is an isosceles triangle as seen in Figure 3.18 ($B_FB_EB_0$) so, $\mu_E = \pi - \mu_F$ and $|B_EB_F| = b + a - (b - a) = 2a$. By using this triangle μ_E can be found as follows:

$$\mu_E = \frac{\pi}{2} - \phi_F - \psi_E \qquad (134)$$

Using sine theorem for triangle $A_0B_EB_0$:

$$\frac{s(\pi-\psi_E)}{a+b}=\frac{s\mu_E}{1}=\frac{s(\phi_F-\pi)}{d} \tag{135}$$

Using sine theorem for triangle $A_0B_FB_0$:

$$\frac{s(\pi-(\mu_F+\phi_F-\pi))}{b-a}=\frac{s\mu_F}{1}=\frac{s(\phi_F-\pi)}{d} \tag{136}$$

By using Eqs. (134), (135) and (136) link lengths can be found as follows:

$$a=\frac{s\phi_F}{\tan(\phi_F-\psi_E)},\quad b=-c\phi_F \ \text{ and } \ d=-\frac{s\phi_F}{s(\phi_F-\psi_E)} \tag{137}$$

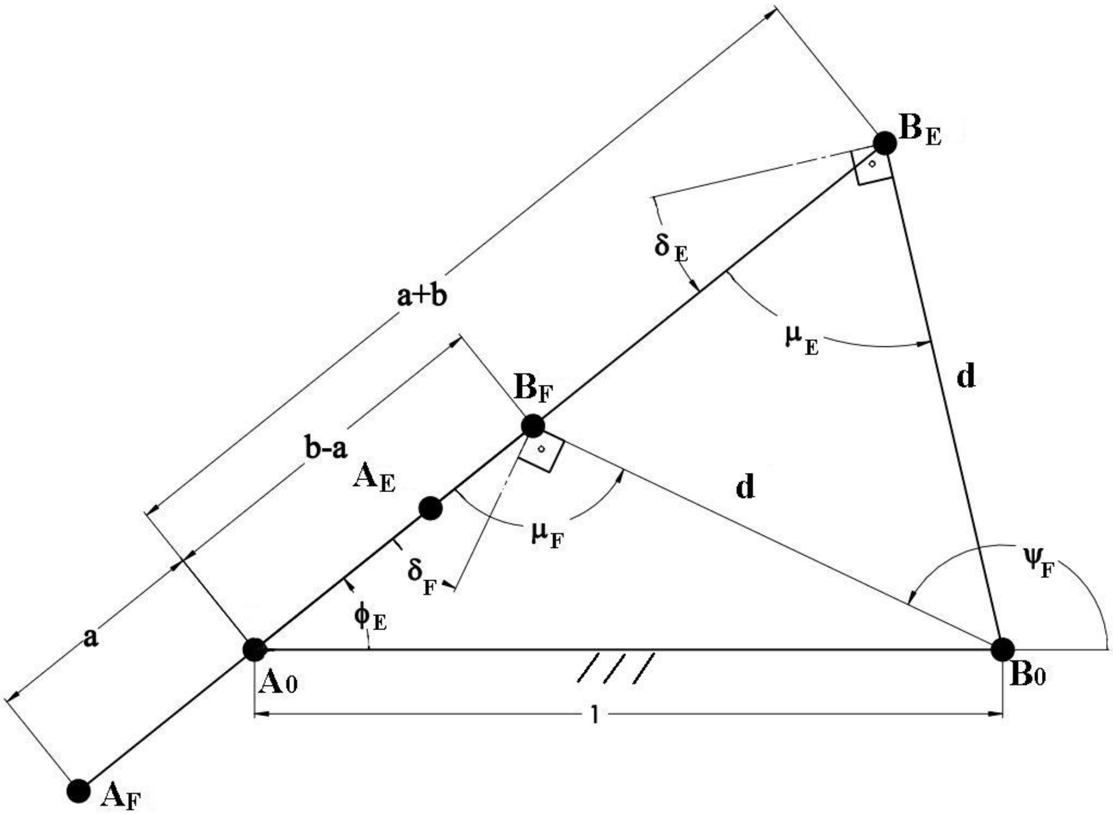


Figure 3.19. Input angle ϕ is given for EDCP and output angle ψ is given for FDCP

For case (ii) there is an isosceles triangle as seen in Figure 3.19 ($B_FB_EB_0$) so, $\mu_E = \pi - \mu_F$ and $|B_EB_F| = b + a - (b - a) = 2a$. By using this triangle μ_E can be found as follows:

$$\mu_E=\frac{\pi}{2}-\phi_E-\psi_F \tag{138}$$

Using sine theorem at triangle $A_0B_EB_0$:

$$\frac{s(\pi-(\phi_F+\mu_E))}{a+b}=\frac{s\mu_E}{1}=\frac{s\phi_F}{d} \tag{139}$$

Using sine theorem at triangle $A_0B_FB_0$:

$$\frac{s(\pi-\psi_E)}{b-a}=\frac{s\mu_F}{1}=\frac{s\phi_F}{d} \tag{140}$$

By using Eqs. (138), (139) and (140) link lengths can be found as follows:

$$a=\frac{s\phi_E}{\tan(\phi_E-\psi_F)},\quad b=c\phi_E \quad \text{and} \quad d=-\frac{s\phi_E}{s(\phi_E-\psi_F)} \tag{141}$$

As seen in two cases, all link lengths are similar, but have sign differences.

3.3. Four Positions Synthesis for Four-Bar Mechanisms

Problems including 4 positions for four-bar mechanism are tabulated by adding one more CCA position to problems including 3 positions for the four-bar mechanism. So, the number of possible problems is the same. Only a problem including 4 positions is solved and presented in this section.

3.3.1. 3 CCA and 1 DCP

There are 4 ways to formulate this:

- Offset is at input angle and input angle is given at the DCP
- Offset is at input angle and output angle is given at the DCP
- Offset is at output angle and input angle is given at the DCP
- Offset is at output angle and output angle is given at the DCP

But only the problem in which the offset is at output and input angle is given the DCP has been solved.

3.3.1.1. Given input angle ϕ for DCP and offset at output angle

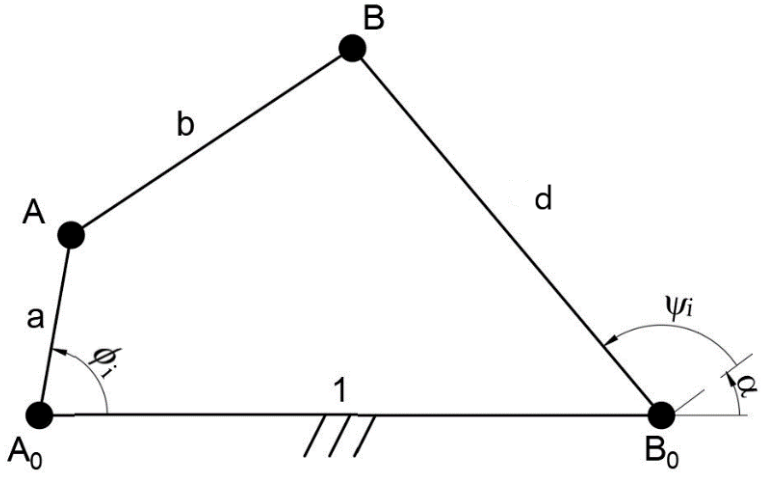


Figure 3.20. Illustration of offset angle at output angle (Source: Kadak & Kiper, 2021)

This problem is solved by Kadak and Kiper (2021). 3 CCA positions (ϕ_1, ϕ_2, ϕ_3, ψ_1, ψ_2, ψ_3) and input angle for FDCP or EDCP (ϕ_F or ϕ_E) are given (Figure 3.20). Link lengths are desired. The I/O function is written for given 3 CCA positions as follows:

$$P_1 - P_2\, c\phi_i + P_3\, c(\psi_i + \alpha) = c(\psi_i - \phi_i + \alpha) \quad \text{for } i = 1, 2, 3 \tag{142}$$

where P_1, P_2, P_3 are as in Eq. (5). Equations for three positions are written in matrix format and can be solved as:

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & -c\phi_1 & c(\psi_1 + \alpha) \\ 1 & -c\phi_2 & c(\psi_2 + \alpha) \\ 1 & -c\phi_3 & c(\psi_3 + \alpha) \end{bmatrix}^{-1} \begin{bmatrix} c(\psi_1 - \phi_1 + \alpha) \\ c(\psi_2 - \phi_2 + \alpha) \\ c(\psi_3 - \phi_3 + \alpha) \end{bmatrix} \tag{143}$$

The P_1, P_2, and P_3 solved here are in terms of the α angle. For FDCP and EDCP, the cosine theorem for triangles A_0BB_0 is written.

$$\begin{gathered} d^2 = (b-a)^2 + 1 - 2(b-a)c(\phi_F - \pi) \\ d^2 = (b+a)^2 + 1 - 2(b+a)c\phi_E \end{gathered} \tag{144}$$

Eq. (144) is simplified, and terms with sign differences are added up on the same side:

$$\begin{gathered} b = \frac{1 + a^2 + b^2 - d^2 - 2a\, c\phi_F}{2a - 2\, c\phi_F} \\ -b = \frac{1 + a^2 + b^2 - d^2 - 2a\, c\phi_E}{2a - 2\, c\phi_E} \end{gathered} \tag{145}$$

The two equations are equated to each other by taking the square of both sides. Thus, a single equation can be used for both EDCP and FDCP.

$$b^2 = \left(\frac{1 + a^2 + b^2 - d^2 - 2a\,c\,\phi_D}{2a - 2c\,\phi_D} \right)^2 \tag{146}$$

Eq. (146) is written in terms of P_1, P_2 and P_3 by using Eq. (15):

$$\left(\frac{1}{P_3} + \frac{P_3 - P_1/P_2}{1 - P_3\,c\,\phi_D} \right)^2 = 1 + \frac{1}{P_3^{\,2}} + \frac{1}{P_2^{\,2}} - \frac{2P_1}{P_2P_3} \tag{147}$$

Eq. (147) can be written in terms of P_1, P_2 and P_3 after some simplifications as follows:

$$\left[\left(1+P_2^{\,2}\right) P_3^{\,2} - 2P_1P_2P_3 \right] s^2\phi_D + 2\left(P_3 - P_1P_2\right) c\phi_D + P_1^{\,2} + P_2^{\,2} - P_3^{\,2} - 1 = 0$$

P_1, P_2 and P_3 are expressed in terms of α. So, Eq. (147) is a univariate equation in terms of α. In order to be able to determine the maximum number of solutions for this equation, the tangent of the half-angle can be used.

$$c\left(\psi_i + \alpha\right) = \frac{1-t^2}{1+t^2}\,c\,\psi_i - \frac{2t}{1+t^2}\,s\,\psi_i$$

$$c\left(\psi_i - \theta_i + \alpha\right) = \frac{1-t^2}{1+t^2}\,c\left(\psi_i - \theta_i\right) - \frac{2t}{1+t^2}\,s\left(\psi_i - \theta_i\right)$$

for $t = \tan\dfrac{\alpha}{2}$ and $i = 1,\ 2,\ 3$. If P_1, P_2 and P_3 are written in terms of t, then a degree 12 univariate polynomial equation in terms of t is obtained. So, there are at most 12 real solutions. It is possible that some solutions give the same mechanism solution.

In all numerical examples, it has been seen that if the value of α is a solution, the value of $\alpha + 180$ is also a solution. These 180° different solutions of the equation coincide with the same solution of the mechanism, since in one solution the rocker length d is positive, in the other it is negative. That is, 12 roots correspond to a maximum of 6 mechanism solutions.

3.4. Three Position Synthesis for Slider-Crank Mechanisms

All problems formulated for four-bar mechanisms can also be used for slider-crank mechanisms, with the exception of 1-CCA and two same type DCPs. Because it is not possible to obtain two same types of DCP for the same slider-crank mechanism. A solution of some of the problems are given in this section.

3.4.1. 2-CCA and 1-DCP

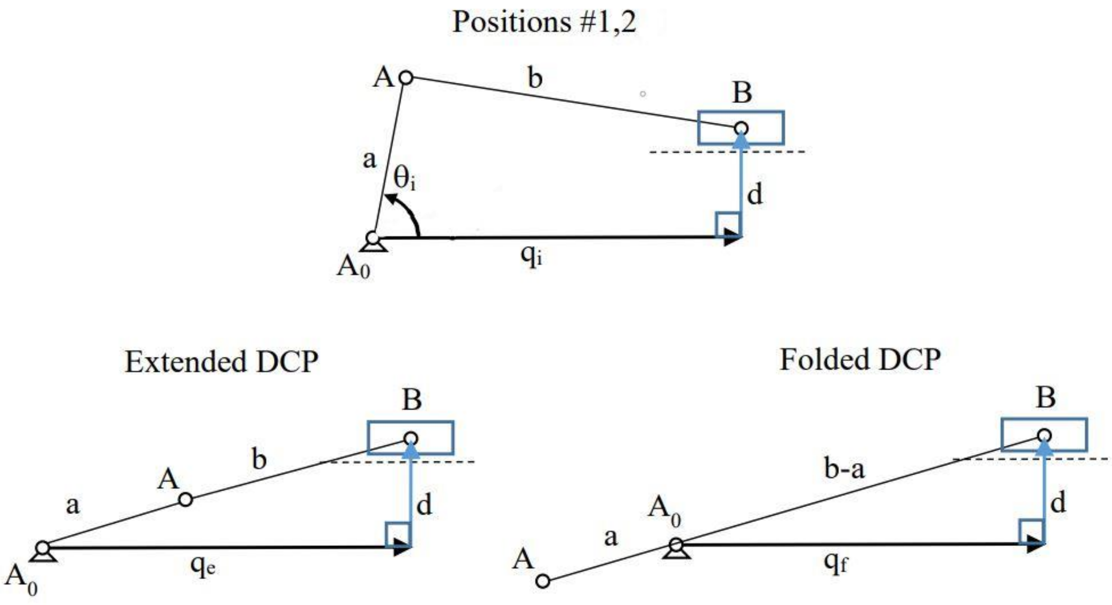


Figure 3.21. Two CCA and a DCP (EDCP or FDCP) are given for a slider-crank linkage (Source: Kiper et al., 2020)

Figure 3.21 shows an arbitrary position, EDCP and FDCP of a slider-crank mechanism. Crank length is a, coupler length is b, eccentricity is d. These three unknowns (a, b, d) are to be determined for two given general positions expressed in terms of pairs (θ_1, q_1), (θ_2, q_2), and an EDCP or FDCP, where only the output slider displacement, q_E or q_F is specified. For a planar slider-crank mechanism, the loop closure equations are written as follows:

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{A_0B} - \overrightarrow{A_0A}\right| \Rightarrow b^2 = \left(q_i - ac\theta_i\right)^2 + \left(d - as\theta_i\right)^2$$
$$\Rightarrow b^2 - a^2 - d^2 + 2aq_ic\theta_i + 2ads\theta_i = q_i^{\,2} \tag{148}$$

Let $P_1 = b^2 - a^2 - d^2$, $P_2 = 2a$ and $P_3 = 2ad$. Then:

$$P_1 + P_2 q_i\, c\theta_i + P_3\, s\theta_i = q_i^{\,2} \quad \text{for} \quad i = 1,\ 2 \tag{149}$$

P_1 and P_3 can be written in terms of P_2:

$$P_1 = \frac{q_2^2\, s\theta_1 - q_1^2\, s\theta_2 + P_2\left(q_1\, s\theta_2\, c\theta_1 - q_2\, s\theta_1\, c\theta_2\right)}{s\theta_1 - s\theta_2}$$
$$P_3 = \frac{q_1^2 - q_2^2 + P_2\left(q_2\, c\theta_2 - q_1\, c\theta_1\right)}{s\theta_1 - s\theta_2} \tag{150}$$

Solving for a, d, b^2:

$$a = \frac{P_2}{2},\ d = \frac{P_3}{P_2},\ b^2 = P_1 + a^2 + d^2 \text{ or } b = \sqrt{P_1 + a^2 + d^2} \tag{151}$$

For the EDCP and FDCP:

$$\begin{aligned} (a+b)^2 &= q_E{}^2 + d^2 \Rightarrow a^2 + b^2 - d^2 - q_E{}^2 = -2ab \\ (b-a)^2 &= q_D{}^2 + d^2 \Rightarrow a^2 + b^2 - d^2 - q_F{}^2 = 2ab \end{aligned} \tag{152}$$

In order not to have a square root expression, taking square of Eq. (152):

$$\left(a^2 + b^2 - d^2 - q_D{}^2\right)^2 = 4a^2b^2 \tag{153}$$

By taking square of Eq. (152), the solution formulation for the EDCP and FDCP problems becomes identical. Writing Eq. (153) in terms of P_1, P_2, P_3 and rearranging:

$$\left(P_1 - q_D{}^2\right)^2 - P_3^2 - q_D{}^2 P_2^2 = 0 \tag{154}$$

Substituting P_1 and P_3 in Eq. (150) into Eq. (154) and:

$$AP_2{}^2 + BP_2 + C = 0 \tag{155}$$

where

$$\begin{aligned} A &= \left(q_1 s\theta_2 c\theta_1 - q_2 s\theta_1 c\theta_2\right)^2 - q_D^2\left(s\theta_1 - s\theta_2\right)^2 - \left(q_1 c\theta_1 - q_2 c\theta_2\right)^2 \\ B &= 2\begin{pmatrix} q_2 c\theta_2\left[q_D^2 s\theta_1\left(s\theta_1 - s\theta_2\right) + q_2^2 c^2\theta_1\right] + q_2 q_1^2\left(s\theta_1 s\theta_2 - 1\right)c\theta_2 + \\ q_1 c\theta_1\left[-q_D^2 s\theta_2\left(s\theta_1 - s\theta_2\right) + q_2^2\left(s\theta_1 s\theta_2 - 1\right)\right] + q_1^3 c\theta_1 c^2\theta_2 \end{pmatrix} \\ C &= \begin{pmatrix} q_D^4\left(s\theta_1 - s\theta_2\right)^2 - q_1^4 c^2\theta_2 - 2q_2^2 q_D^2 s\theta_1\left(s\theta_1 - s\theta_2\right) + \\ 2q_1^2\left[q_D^2 s\theta_2\left(s\theta_1 - s\theta_2\right) - q_2^2\left(s\theta_1 s\theta_2 - 1\right)\right] - q_2^4 c^2\theta_1 \end{pmatrix} \end{aligned}$$

Eq. (155) is a degree 2 polynomial equation in P_2. The two roots of the equation can be found analytically. a, b and d can be found by using Eq. (151).

3.5. Four Positions Synthesis for Slider-Crank Mechanisms

Problems including 4 positions for slider-crank mechanism are tabulated by adding one more CCA position to problems including 3 positions for slider-crank mechanism. Only a problem including 4 positions is solved and presented in this section.

3.5.1. 3-CCA and 1-DCP

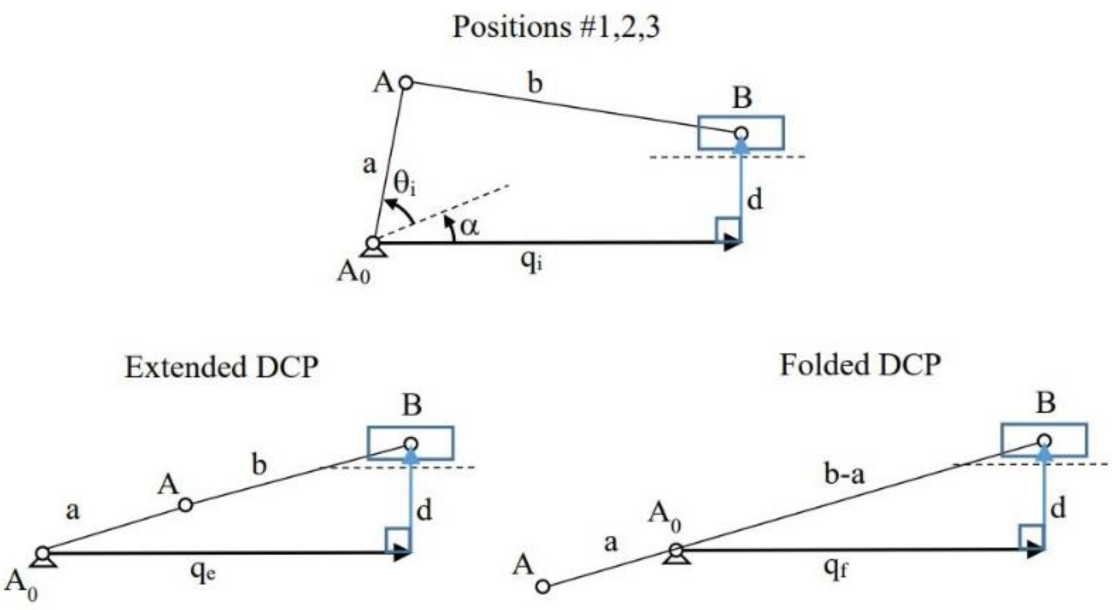


Figure 3.22. 3 CCA and a DCP (EDCP or FDCP) are given for a slider-crank linkage (Source: Kiper et al., 2020)

This problem was solved by Kiper et al (2020). Figure 3.22 shows an arbitrary position, EDCP and FDCP of a slider-crank mechanism. Crank length is a, coupler length is b, eccentricity is d and offset angle is α. These four unknowns (a, b, d and α) are to be determined for 3 given general positions expressed in terms of pairs (θ_1, q_1), (θ_2, q_2) and (θ_3, q_3), and an EDCP or FDCP, where only the output slider displacement, q_E or q_F is specified. For a planar slider-crank mechanism, the loop closure equations are written as follows:

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{A_0B} - \overrightarrow{A_0A}\right| \Rightarrow b^2 = \left[q_i - a\,c\left(\alpha + \theta_i\right)\right]^2 + \left[d - a\,s\left(a + \theta_i\right)\right]^2 \tag{156}$$

Let $c_i = \cos\left(\alpha + \theta_i\right)$ and $s_i = \sin\left(\alpha + \theta_i\right)$. Expanding and rearranging Eq. (156):

$$b^2 - a^2 - d^2 + 2aq_ic_i + 2ads_i = q_i^{\ 2} \tag{157}$$

Let $P_1 = b^2 - a^2 - d^2$, $P_2 = 2a$ and $P_3 = 2ad$. For given (θ_1, q_1), (θ_2, q_2) and (θ_3, q_3), P_1, P_2 and P_3 can be linearly solved from Eq. (157) as a function of α:

$$\begin{aligned} P_1 + q_1c_1P_2 + s_1P_3 &= q_1^{\ 2} \\ P_1 + q_2c_2P_2 + s_2P_3 &= q_2^{\ 2} \\ P_1 + q_3c_3P_2 + s_3P_3 &= q_3^{\ 2} \end{aligned} \Rightarrow \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & q_1c_1 & s_1 \\ 1 & q_2c_2 & s_2 \\ 1 & q_3c_3 & s_3 \end{bmatrix}^{-1} \begin{bmatrix} q_1^{\ 2} \\ q_2^{\ 2} \\ q_3^{\ 2} \end{bmatrix} \tag{158}$$

Using Cramer's rule:

$$
\begin{aligned}
P_1 &= \frac{q_1^2\left(q_2c_2s_3 - q_3c_3s_2\right) - q_2^2\left(q_1c_1s_3 - q_3c_3s_1\right) + q_3^2\left(q_1c_1s_2 - q_2c_2s_1\right)}{q_2c_2s_3 - q_3c_3s_2 - q_1c_1s_3 + q_3c_3s_1 + q_1c_1s_2 - q_2c_2s_1} \\
P_2 &= \frac{\left(q_3^2 - q_2^2\right)s_1 + \left(q_1^2 - q_3^2\right)s_2 + \left(q_2^2 - q_1^2\right)s_3}{q_2c_2s_3 - q_3c_3s_2 - q_1c_1s_3 + q_3c_3s_1 + q_1c_1s_2 - q_2c_2s_1} \\
P_3 &= \frac{\left(q_2^2 - q_3^2\right)q_1c_1 + \left(q_3^2 - q_1^2\right)q_2c_2 + \left(q_1^2 - q_2^2\right)q_3c_3}{q_2c_2s_3 - q_3c_3s_2 - q_1c_1s_3 + q_3c_3s_1 + q_1c_1s_2 - q_2c_2s_1}
\end{aligned}
\tag{159}
$$

Solving for a, d, b^2 (expressed as a function of α):

$$
a = \frac{P_2}{2},\ d = \frac{P_3}{P_2},\ b^2 = P_1 + a^2 + d^2 \text{ or } b = \sqrt{P_1 + a^2 + d^2} \tag{160}
$$

For the EDCP or FDCP:

$$
\begin{aligned}
(a+b)^2 &= q_E^2 + d^2 \Rightarrow a^2 + b^2 - d^2 - q_E^2 = -2ab \\
(b-a)^2 &= q_D^2 + d^2 \Rightarrow a^2 + b^2 - d^2 - q_F^2 = 2ab
\end{aligned}
\tag{161}
$$

In order not to have a square root expression, taking square of Eq. (161):

$$
\left(a^2 + b^2 - d^2 - q_D^2\right)^2 = 4a^2b^2 \tag{162}
$$

By taking square of Eq. (161), the solution formulation for the EDCP and FDCP problems becomes identical. Writing Eq. (162) terms of P_1, P_2, P_3 and rearranging:

$$
\left(P_1 - q_D^2\right)^2 - P_3^2 - q_D^2 P_2^2 = 0 \tag{163}
$$

Eq. (163) is an equation in terms of α and can be numerically solved using a root finding algorithm. Then the corresponding a, b and d values can be determined using Eq. (160).

In order to find an upper bound for the number of roots let $t = \tan\frac{\alpha}{2}$ so that:

$$
c_i = \frac{1-t^2}{1+t^2} c\theta_i - \frac{2t}{1+t^2}\sin\theta_i \quad \text{and} \quad s_i = \frac{1-t^2}{1+t^2} s\theta_i + \frac{2t}{1+t^2}\cos\theta_i \tag{164}
$$

Substituting Eq. (164) into Eq. (159) and substituting Eq. (159) into Eq. (164) results in a degree 8 polynomial in t. So, there are at most 8 real solutions.

CHAPTER 4

NUMERICAL EXAMPLES

All problems in Chapter 3 were implemented in Excel. Some macros used for mechanism analysis and a macro which solves degree 4 polynomial equations have been used. For given parameters, spin buttons are assigned. In problems requiring numerical solutions, the "goal seek" macro that use the Newton-Rapshon method was used. This macro is assigned to all spin buttons. Thus, even in problems that require a numerical solution, a simultaneous solution can be obtained as the spin buttons are used. Finally, a simulation has been prepared to visualize the mechanism.

Being able to obtain the solution instantaneously provides many advantages. While obtaining the solution, the ratio of the link lengths of the mechanism or the transmission angle can be monitored. As mentioned in the introduction, the solutions to these problems can also be a loop of a multi-loop mechanism. In such cases, it is necessary to solve these problems instantly for changing problem definitions.

The numerical solutions of two problems that can be solved analytically and solved numerically have been examined in the next sections.

Table 4.1 lists all numerical examples of problems in this thesis. The column DoE indicates the degree of polynomial equation. So, this column is also the number of maximum solutions. However only one of the solutions is shown in the table as a sample. If the equation has 4 or less roots, there are examples in which all the roots are real in all these problems. But some of the mechanisms synthesized with these real roots may not be feasible. In some cases, link lengths may be too large or too small. In some cases, the mechanism does not provide the given positions in the same configuration. In some equations with more than 4 roots, no example has been found where all roots are real. However, since the examples are worked out with trial and error, there may be an example where all roots are real.

The given parameters of the problems are indicated by yellow cells and desired parameters (link lengths and offset angle in radians) are indicated by green cells.

Table 4.1. All numerical examples

Section	DoE	Given parameters are yellow, link lengths are green										
3.2.1	1	ϕ_1	ϕ_2	ϕ_3	ψ_1	ψ_2	ψ_3	a	b	d		
		25	50	80	60	70	90	0.3586	0.9449	0.4696		
□	2	ϕ_1	ϕ_2	ψ_1	ψ_2	a	b	d				
		40	170	80	150	0.4923	0.8736	0.7660				
3.2.2.2	2	ϕ_1	ϕ_2	ψ_1	ψ_2	a	b	d				
		40	170	100	150	0.3457	0.7807	0.7142				
3.2.3.1	4	ϕ_1	ϕ_2	ϕ_D	ψ_1	ψ_2	a	b	d			
		140	100	200	90	40	0.4884	1.3745	0.3462			
3.2.3.2	2	ϕ_1	ϕ_2	ψ_1	ψ_2	ψ_D	a	b	d			
		40	120	10	70	100	0.3471	1.5023	0.7774			
3.2.4.1.1	6	ϕ_1	ϕ_D	ψ_1	a	b	d					
		60	210	100	0.2275	0.8752	0.5987					
3.2.4.1.2	8	ϕ_1	ψ_1	ψ_D	a	b	d					
		60	110	80	0.2823	0.7814	0.2282					
3.2.4.2.1	2+4	ϕ_1	ϕ_D	ψ_1	a	b	d					
		60	230	120	0.2857	0.6428	0.8176					
3.2.4.2.2	4	ϕ_1	ψ_1	ψ_D	a	b	d					
		45	60	50	0.5836	0.9407	0.6750					
3.2.5.1	6	ϕ_1	ϕ_E	ϕ_F	ψ_1	a	b	d				
		80	35	195	120	0.2700	0.7018	0.5935				
3.2.5.2	4	ϕ_1	ψ_1	ψ_E	ψ_F	a	b	d				
		80	120	110	170	0.2726	0.7003	0.5940				
3.2.5.3	6	ϕ_1	ϕ_{D1}	ψ_1	ψ_{D2}	a	b	d				
		80	35	120	170	0.2725	0.7003	0.5938				
3.2.6.1	4	ϕ_1	ψ_1	ψ_{F1}	ψ_{F2}	a	b	d				
		110	60	140	280	1.2839	1.9545	0.9571				
3.2.6.2.1	4	ϕ_1	ψ_1	ϕ_{F1}	ψ_{F2}	a	b	d				
		110	60	245	280	1.3448	1.9635	0.8741				
3.2.6.2.2	2	ϕ_1	ψ_1	ψ_{F1}	ϕ_{F2}	a	b	d				
		110	60	143	54	1.3643	1.9680	0.8466				
3.2.7.1	1	ϕ_E	ϕ_F	a	b	d						
		60	210	0.1705	0.3295	0.8660						
3.2.7.2	1	ψ_E	ψ_F	a	b	d						
		140	170	0.2428	0.4226	0.9383						
3.2.7.3	1	ϕ_{D2}	ψ_{D1}	a	b	d						
		60	170	0.3152	0.5000	0.9216						
3.3.1.1	12	ϕ_1	ϕ_2	ϕ_3	ϕ_D	ψ_1	ψ_2	ψ_3	a	b	d	α
		70	135	260	200	115	140	160	0.3199	0.6415	0.7064	0.0581
3.4.1	2	θ_1	θ_2	q_1	q_2	q_D	a	b	d			
		35	45	1	0.9	1.1	0.6371	0.4913	0.2520			
3.5.1	8	θ_1	θ_2	θ_3	q_1	q_2	q_3	q_D	a	b	d	α
		110	60	40	0.5	1	1.2	1.45	0.5944	1.0159	0.7008	0.5410

4.1. Numerical Example of 1-CCA and 2 Different Type DCPs when Input Angles ϕ Are Given for Both DCPs for Four-Bar Mechanism

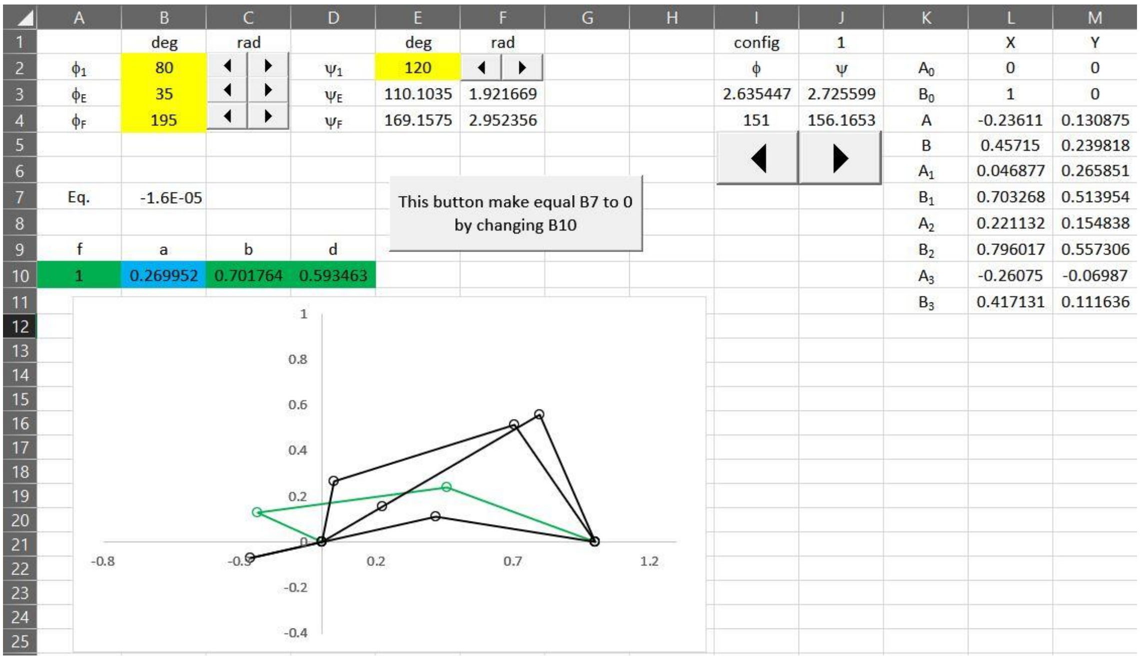
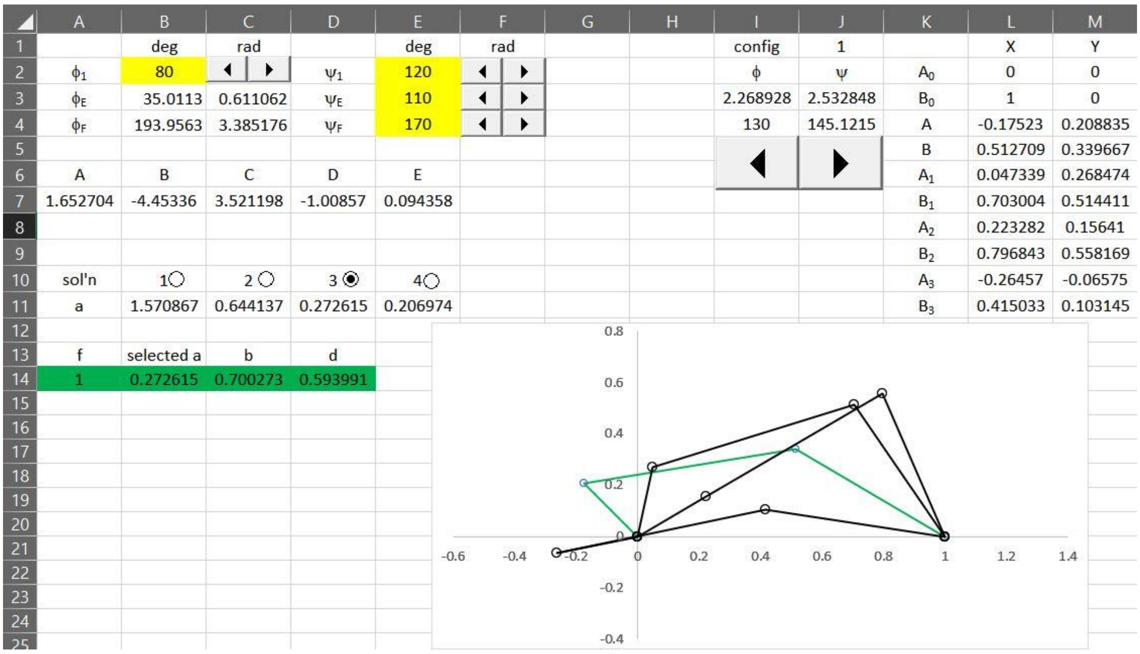


Figure 4.1. The numerical solution of 1-CCA and 2 different type DCPs when input angles ϕ are given for both DCPs for a four-bar mechanism

Given parameters (ϕ_1, ϕ_2, ϕ_F, ϕ_E) are shown by yellow cells and spin buttons change them in Figure 4.1. The computed link lengths are shown by green cells. The cell B7 contains Eq. (77). A macro has been written that equates the equation in cell B7 to zero by changing the value of a in cell B10 using "goal seek". This macro is assigned to all spin buttons. If the macro is run again by changing the initial value of a, different values of a can be found. By using selected a value, b and d can be calculated using Eqs. (75) and (76). Then, the mechanism is simulated. The black lines in the graph indicate the given positions of the mechanism. The green lines are animated using the spin button in cell I5.

4.2. Numerical Example of 1-CCA and 2 Different Type DCPs when Output Angles ψ Are Given for Both DCPs For Four-Bar Mechanism



	A	B	C	D	E	F	G	H	I	J	K	L	M
1		deg	rad		deg	rad			config	1		X	Y
2	ϕ_1	80		ψ_1	120				ϕ	ψ	A_0	0	0
3	ϕ_E	35.0113	0.611062	ψ_E	110				2.268928	2.532848	B_0	1	0
4	ϕ_F	193.9563	3.385176	ψ_F	170				130	145.1215	A	-0.17523	0.208835
5											B	0.512709	0.339667
6	A	B	C	D	E						A_1	0.047339	0.268474
7	1.652704	-4.45336	3.521198	-1.00857	0.094358						B_1	0.703004	0.514411
8											A_2	0.223282	0.15641
9											B_2	0.796843	0.558169
10	sol'n	1	2	3	4						A_3	-0.26457	-0.06575
11	a	1.570867	0.644137	0.272615	0.206974						B_3	0.415033	0.103145
12													
13	f	selected a	b	d									
14	1	0.272615	0.700273	0.593991									

Figure 4.2. The numerical solution of 1-CCA and 2 different type DCPs when output angles ψ are given for both DCPs for a four-bar mechanism

Given parameters (ϕ_1, ϕ_2, ψ_F, ψ_E) are shown by yellow cells and spin buttons change them in Figure 4.2. The computed link lengths are shown by green cells. A, B, C, D and E are the coefficients of Eq. (83) which is a degree 4 polynomial equation in a. Therefore, there are at most 4 possible real values of a. All solutions can be found by using a macro. By using buttons on cells B10, C10, D10 and E10, a value of a is selected. By using selected value of a, b and d can be calculated with Eqs. (81) and (82). Then the mechanism is simulated. The black lines in the graph indicate the given positions of the mechanism. The green lines are animated using the spin button in cell I5.

CHAPTER 5

CONCLUSIONS

In this thesis, an overview of mixed function generation synthesis problems for planar mechanisms is presented, the problems that can be formulated are listed, and analytical or semi-analytical solutions of some of the problems are presented.

All function generation problems including 3 positions for planar four-bar mechanisms as mixed problems of CCA and dead-center design is presented in Section 3.2. One of the problems for four-bar mechanisms including 4 positions is presented in Section 3.3. Mixed function generation problems including 3 positions for planar slider-crank mechanisms are identified and one of them is solved in Section 3.4. One of the problems including 4 positions for slider-crank mechanisms is solved in Section 3.5.

Solutions are solved and simulated by either analytical or semi-analytical methods. In CHAPTER 4, numerical examples are examined. The numerical solutions are obtained for all problems. The numerical examples of Sections 3.2.5.1 and 3.2.5.2 are examined in detail in Sections 4.1 and 4.2. Thus, function generation synthesis of planar mechanisms as a mixed problem of CCA and dead-center design is studied.

Any problems including 5 positions are not addressed in this thesis. It may be the subject of future studies. By taking derivative of the I/O equation with respect to time, velocity and acceleration can also be used in the problem definitions. Also, mixed function generation problems for other planar crank mechanisms can be formulated. Problem definitions and solutions of other crank mechanisms may be the subject of future studies.

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