QUALITATIVE PROPERTIES OF SOLUTIONS OF SOME KELLER - SEGEL TYPE SYSTEMS

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ABSTRACT

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The main objective of this thesis is to summarize results related with solutions of some Keller - Segel type systems, which model chemotaxis. This work surveys mathematical studies starting with the work that first presented these systems in 1970. This study emphasizes the local and global existence of solutions of Keller - Segel type systems, in particular the boundedness and blow-up of solutions.

Keywords: *Chemotaxis, Local Existence, Global Existence, Boundedness, Blow-up, Keller - Segel Model*

ÖZET

KELLER - SEGEL TİPİNDEKİ BAZI SİSTEMLERİN ÇÖZÜMLERİNİN KALİTATİF ÖZELLİKLERİ

Bu tezde temel amaç, kemotaksisi modelleyen bazı Keller - Segel tipi sistemlerin çözümleri ile ilgili sonuçları özetlemektir. Bu çalışma, 1970 yılında bu sistemleri ilk kez sunan çalışma ile başlayan matematiksel çalışmaları inceler. Bu çalışma, Keller - Segel tipi sistemlerin çözümlerinin yerel ve global varlığını, özellikle çözümlerin sınırlılığını ve patlamasını vurgular.

Anahtar Kelimeler: *Kemotaksis, Yerel Varlık, Global Varlık, Sınırlılık, Patlama, Keller - Segel Modeli*

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

α	$(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}$
D^α	$\frac{\partial^{\lvert\alpha\rvert}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \ldots \partial^{\alpha_n} x_n}$
$\mathbb{R}^n$	n-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$
Ω	open subset of $\mathbb{R}^n$
$\overline{\Omega}$	closure of Ω in $\mathbb{R}^n$
$\partial\Omega$	boundary of Ω
$V \subset\subset \Omega$	V is compactly contained in Ω
$C(\Omega)$	continuous functions on Ω
$C(\overline{\Omega})$	uniformly continuous functions on Ω
$C^k(\Omega)$	continuous functions that k-times continuously differentiable on Ω
$C^k(\overline{\Omega})$	continuous functions that k-times uniformly continuously differentiable on Ω
$C^\infty(\Omega)$	continuous functions that infinitely continuously differentiable
C^k_c	functions in C^k with compact support
$L^p(\Omega)$	space of measurable functions on Ω whose p-norm is finite
$\lVert u\rVert_{L^p(\Omega)}$	$(\int_\Omega \lvert u\rvert^p)^{\frac{1}{p}}$, u in $L^p(\Omega)$, $1 \le p < \infty$
$L^\infty(\Omega)$	L^p space of measurable functions on Ω such that $\lvert u(x)\rvert < C$ for some constant C and $x \in \mathbb{R}^n$
$\lVert u\rVert_{L^\infty(\Omega)}$	$\inf\{C > 0 : \lvert u(x)\rvert < C\}$, u in $L^\infty(\Omega)$
$L^p_{loc}(\Omega)$	the space of measurable functions on Ω such that $v \in L^p(V)$ for each $V \subset\subset \Omega$
$W^{k,p}(\Omega)$	Sobolev space of functions whose weak derivatives up to order k are in L^p
$W^{s,p}(\Omega)$	Sobolev space of functions of real order with $s = k + \epsilon$, $0 < \epsilon < 1$ and $k \ge 0$ is an integer
$\lVert u\rVert_{W^{k,p}(\Omega)}$	$(\sum_{\lvert\alpha\rvert\le k} \lVert D^\alpha u\rVert^p_{L^p(\Omega)})^{\frac{1}{p}}$, u in $W^{k,p}(\Omega)$
H^k	L^2-based Sobolev spaces, $H^k = W^{k,2}$

$C^k([a,b];X)$	Banach space of k times continuously differentiable functions $u : [a,b] \to X$
$C^{k,\alpha}$	Hölder space
$BUC(\Omega)$	Bounded uniformly continuous functions
Δ	$= \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$
∇u	$= (\partial_{x_1} u, \partial_{x_2} u, \ldots, \partial_{x_n} u)$
u_t	$= \partial_t u = \frac{\partial u}{\partial t}$
T_{max}	Maximal existence time

CHAPTER 1

INTRODUCTION

One of the main characteristics of cells is their reaction to environmental variations. Cells externally detect changes and alter their behavior accordingly. In general, a reaction to an external signal is called taxis, and cells exhibit a variety of tactical movements. One of them, chemotaxis, is the directed movement of cells in response to external signals. The movement of cells from low to high chemical signal intensity is called positive chemotaxis, and the reverse is called negative chemotaxis. Chemotaxis is prevalent in many biological events and regulates biological processes such as wound healing or neuron movement.

Mathematics provides insight for understanding biological processes. Chemotaxis is one such biological model that has a precise mathematical description. Theoretical and mathematical modeling of chemotaxis first appeared with the study of Patlak in the 1950s (Patlak, 1953) and next appeared in the work of Keller and Segel (1970). Keller and Segel offered the model of equations that characterize movements induced by a chemical matter.

The original Keller - Segel model to be detailed in the third chapter contains four partial differential equations. It can be reduced to two partial differential equations under suitable assumptions. This simplified model will form the main subject of this thesis.

The general structure of the model is as follows:

$$\begin{aligned} \partial_t u &= \nabla \cdot (D(u,v)\nabla u - S(u,v)\nabla v) + f(u,v), \\ \partial_t v &= \Delta v + g(u,v), \end{aligned} \tag{1.1}$$

where u denotes the cell density and v denotes the chemoattractant density on a given domain $\Omega \subset \mathbb{R}^n$. $D(u,v)$ describes the diffusivity of the cells, while $S(u,v)$ is the chemotactic sensitivity. Both of them may depend on u and v. $f(u,v)$ describes cell growth and death, while $g(u,v)$ is a kinetic function that describes the production and degradation of the chemical substance.

Analysis of the mathematical modeling of chemotaxis provides valuable insight into the underlying biological processes, and studies on this subject continue to increase every year. These studies examine the local or global existence of solutions and also find conditions under which solutions may blow up.

Following Keller Segel's seminal work (1970), Nanjundiah examined the problem of the nonlinear stability of the classical Keller Segel type system (1973). Next, Nanjundiah, Childdress and Percus, (1981) studied the asymptotic behavior of the solutions of some types of the Keller Segel model. Herrero and Velazquez (1996) then found that solutions form δ-singularity in finite time at the center of the $\mathbb{R}^2$ disk by using the asymptotic expansion method. This was an important contribution to the results on the blow-up behavior of solutions. Soon after, Yagi (1997) demonstrated the local existence of some Keller - Segel type systems of equations. Then Biler (1998) examined some Keller - Segel models with different boundary conditions and proved their global existence for some classes of solutions. In the same year, Gajewski and Zacharias (1998) showed the global existence of solutions for some regular spaces using the Lyapunov functional. Osaki and Yagi (2001) proved the global existence of solutions for the classical model (4.1) in dimension $n = 1$.

The review article by Horstmann (2003) can be considered a comprehensive introduction to the mathematics of Keller - Segel type equations. The review surveys different approaches for modeling chemotaxis and different perspectives on the model of chemotaxis in detail.

A survey by Hillen and Painter (2009) provides significant references to studies on the variations of Keller - Segel models. They studied the system of equations biologically and summarized important results regarding its properties.

An article by Bellomo et al. (2015) summarizes the qualitative characteristics of solutions of some types of systems. The study aimed to contribute to new approaches to research in this area.

The aim of this thesis is to examine the qualitative properties of the solutions of Keller - Segel models consisting of parabolic equations in studies from 1970 to the present. This thesis focuses on both the classical Keller - Segel model and related models. The models examined in the thesis are listed below.

The classical model:

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - \nabla\cdot(u\nabla v),\\ \frac{\partial v}{\partial t} &= \Delta v - v + u.\end{aligned}\right\} \tag{1.2}$$

Signal density dependent sensitivity model:

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - \nabla\cdot(uS(v)\nabla v),\\ \frac{\partial v}{\partial t} &= \Delta v - v + u.\end{aligned}\right\} \tag{1.3}$$

Cell density dependent sensitivity model:

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - \nabla\cdot(uS(u)\nabla v),\\ \frac{\partial v}{\partial t} &= \Delta v - v + u.\end{aligned}\right\} \tag{1.4}$$

Nonlinear diffusion and cell density dependent sensitivity model:

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\nabla v),\\ \frac{\partial v}{\partial t} &= \Delta v - v + u.\end{aligned}\right\} \tag{1.5}$$

Quasilinear model:

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \nabla\cdot(D(u,v)\nabla u) - \nabla\cdot(S(u,v)\nabla v),\\ \frac{\partial v}{\partial t} &= \Delta v - v + u.\end{aligned}\right\} \tag{1.6}$$

The thesis is organized as follows.

In Chapter 2, we collect some important tools from partial differential equations, functional analysis, and semi-group theory. In Chapter 3, we introduce the Keller - Segel model for chemotaxis and analyze the mathematical formulation of the general movement. In Chapter 4, we consider the qualitative characteristic of solutions of some Keller - Segel type systems. In the conclusion, we mention some open questions relating with these systems.

CHAPTER 2

PRELIMINARIES

This chapter reviews some mathematical tools to make the main chapters easier to read.

2.1. Lebesque and Hölder Spaces

We refer to (Kreyzing, 1991) for this section, which includes basic definitions and theorems about Lebesque and Hölder spaces.

Definition 2.1 *Let Ω be measure space. If u is a measurable function on Ω and $0 < p < \infty$, we define*

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \right)^{1/p}$$

and

$$L^p(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} : u \ \ is \ \ Lebesgue \ \ measurable, \|u\|_{L^p(\Omega)} < \infty\}.$$

If we consider the limiting case $p = \infty$, the space L^∞ will be defined as all functions that are essentially bounded. If u is a measurable function on Ω, we define

$$\|u\|_{L^\infty(\Omega)} = inf\{C \geq 0 : |u(x)| \leq C for a.e. x\}.$$

Definition 2.2 (Hölder space) *The Hölder space $C^{k,\gamma}(\overline{\Omega})$ consists of all functions $u \in C^k(\overline{\Omega})$ with the norm*

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha \leq k|} \|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{|\alpha = k|} [D^\alpha u]_{C^{0,\gamma}(\overline{\Omega})}.$$

So, the space $C^{k,\gamma}(\overline{\Omega})$ consists of those functions u that are k-times continuously differentiable and whose k^{th}-partial derivatives are Hölder continuous with exponent γ.

Definition 2.3 *Assume $\Omega \subset \mathbb{R}^n$ is open and $0 < \gamma \leq 1$. A function u is said to be Hölder continuous if it satisfies the following inequality*

$$|u(x) - u(y)| \leq C|x - y|^{\gamma}$$

for some positive constant C.

Definition 2.4

(i) If $\Omega \longrightarrow \mathbb{R}$ is bounded and continuous, we write

$$||u||_{C(\overline{\Omega})} := \sup_{x \in \Omega} |u(x)|.$$

(ii) The γ^{th}-Hölder seminorm of $u : \Omega \longrightarrow \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{x,y \in \Omega, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \right\}$$

and γ^{th}-Hölder norm is

$$||u||_{C^{0,\gamma}(\overline{\Omega})} := ||u||_{C(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}.$$

2.2. Sobolev Spaces

We refer to (Evans, 2010) for this section, which includes basic definitions and theorems about Sobolev spaces.

Definition 2.5 *Let us suppose that $u, v \in L^1_{loc}(\Omega)$ and α is a multiindex. We say that v is the α^{th}-weak derivative of u, written as*

$$D^{\alpha} u = v,$$

provided

$$\int_\Omega uD^\alpha \phi dx = (-1)^{|\alpha|}\int_\Omega v\phi dx$$

for all test functions $\phi \in C_c^\infty(\Omega)$.

Definition 2.6 (Sobolev Space) *The Sobolev space* $W^{p,k}(\Omega)$ *consists of all locally summable functions* $u : \Omega \longrightarrow \mathbb{R}$ *such that for each multiindex* α *with* $|\alpha| \leq k$, $D^\alpha u$ *exists in the weak sense and belogs to* $L^p(\Omega)$.
If $u \in W^{p,k}(\Omega)$, *we define its norm as*

$$||u||_{W^{k,p}(\Omega)} := \begin{cases} (\sum_{|\alpha|\leq k}\int_\Omega |D^\alpha u|^p)^{1/p}, & 1 \leq p < \infty, \\ \sum_{|\alpha|\leq k} esssup_\Omega |D^\alpha u|, & p = \infty. \end{cases}$$

Definition 2.7 *Let* $\{u_m\}_{m=1}^\infty$, $u \in W^{p,k}(\Omega)$. *We say* u_m *converges to* u *in* $W^{p,k}(\Omega)$, *written*

$$u_m \longrightarrow u, \quad in \ W^{p,k}(\Omega)$$

provided

$$\lim_{m\to\infty} ||u_m - u||_{W^{p,k}(\Omega)} = 0.$$

Definition 2.8 *Let* $s = k + \epsilon$ *with* $k \geq 0$ *being an integer and* $0 < \epsilon < 1$. *Then the fractional Sobolev space is defined by*

$$W^{s,p}(\Omega) = \left\{u \in W^{k,p} : \frac{|D^\alpha u(x) - D^\alpha u(y)|}{||x-y||^{\epsilon+\frac{d}{p}}} \in L^p(\Omega\times\Omega), \forall\alpha : |\alpha| = k\right\}$$

with the norm

$$||u||_{W^{s,p}(\Omega)} = \left(||u||^p_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k}\int_{\Omega\times\Omega}\frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{||x-y||^{\epsilon p+d}}dxdy\right)^{\frac{1}{p}}.$$

When $p = 2$, $H^s(\Omega) = W^{s,2}(\Omega)$.

2.3. Semigroup Theory

We refer to (Pazy, 2012) and (Kesavan, 1994) for this section, which includes basic definitions and theorems about semigroup theory.

Definition 2.9 *Let X be a Banach space. A one parameter family $T(t), 0 \leq t \leq \infty$ of bounded linear operators from X into X is called a semigroup of bounded linear operators on X if*

(i) $T(0) = I$*, (I is the identity operator),*

(ii) $T(s+t) = T(s)T(t)$ *for every* $t, s \geq 0$.

A linear operator A is the infinitesimal generator of semigroup $T(t)$ if

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t} = \left.\frac{d^+T(t)x}{dt}\right|_{t=0},$$

for every $x \in D(A)$, where

$$D(A) := \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \ exists \right\}.$$

A semigroup of the bounded linear operators $T(t)$ is called uniformly continuous if

$$\lim_{t\to 0} ||T(t) - I|| = 0.$$

Definition 2.10 *A semigroup $T(t), 0 \leq t \leq \infty$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t\to 0} T(t)x = x$$

for every $x \in X$. A strongly continuous semigroup of bounded linear operators on X is called a C_0 semigroup.

Definition 2.11 *A semigroup $T(t)$ will be called analytic if it is analytic in some sector containing the nonnegative real axis.*

On bounded domains, some L^p - L^q estimates have been proven for the Neumann heat semigroup.

Lemma 2.1 *(Cao, 2015)*
Assume $(e^{t\Delta})$ is the Neumann heat semigroup in Ω, and let λ_1 be the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist positive constants c_1, c_2, c_3 and c_4 which depend only on domain and which have the following properties:

(i) If $1 \leq q \leq p \leq \infty$, then

$$||e^{t\Delta}w||_{L^p(\Omega)} \leq c_1(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}||w||_{L^q(\Omega)}, \forall t \geq 0$$

holds for all $w \in L^q(\Omega)$ with $\int_\Omega w = 0$.

(ii) If $1 \leq q \leq p \leq \infty$, then

$$||\nabla e^{t\Delta}w||_{L^p(\Omega)} \leq c_2(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}||w||_{L^q(\Omega)}, \forall t \geq 0$$

holds for all $w \in L^q(\Omega)$.

(iii) If $2 \leq q \leq p < \infty$, then

$$||\nabla e^{t\Delta}w||_{L^p(\Omega)} \leq c_3(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}||\nabla w||_{L^q(\Omega)}, \forall t \geq 0$$

holds for all $w \in W^{1,p}(\Omega)$.

(iv) If $1 \leq q \leq p \leq \infty$, then

$$||e^{t\Delta}\nabla.w||_{L^p(\Omega)} \leq c_4(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}||\nabla w||_{L^q(\Omega)}, \forall t \geq 0$$

holds for all $w \in (W^{1,p}(\Omega))^n$.

2.4. Definitions of Some Keller - Segel Model

We refer to (Horstmann, 2003) for this section, which includes definitions of the solutions of some Keller - Segel type systems.

Definition 2.12 *Let T_{max} be the maximal existence time of the solution (u, v) of (4.1). A point $x_0 \in \overline{\Omega}$ is said to be a blow-up point of u if there exist $\{t_k\}_{k=1}^{\infty} \subset (0, T_{max})$ and $\{x_k\}_{k=1}^{\infty} \subset \overline{\Omega}$ satisfying*

$$u(t_k, x_k) \to \infty, t_k \to T_{max}, x_k \to x_0 \ \ \textit{when} \ \ k \to \infty.$$

Definition 2.13 *We say that the solution of (4.1) blows up provided there is a time $T_{max} \leq \infty$ such that*

$$\limsup_{t \to T_{max}} \|u(t, x)\|_{L^{\infty}(\Omega)} = \infty$$

or

$$\limsup_{t \to T_{max}} \|v(t, x)\|_{L^{\infty}(\Omega)} = \infty$$

If $T_{max} < \infty$ we say that the solution of (4.1) blows up in finite time, and if $T_{max} = \infty$ we say that the solution of (4.1) blows up in infinite time.

Definition 2.14 *Let $\chi > 0$, $u_0 \in L^1(\Omega)$, $v_0 \in L^1(\Omega)$ and $T > 0$. A weak solution of (4.6) in $(0, T) \times \Omega$ is a pair (u, v) of nonnegative functions*

$$u \in L^1_{loc}((0, T) \times \overline{\Omega}), \quad v \in L^1_{loc}((0, T) \times \overline{\Omega})$$

with the properties

$$\frac{u}{v} \nabla v \in L^1_{loc}((0, T) \times \overline{\Omega}),$$

$$-\int_0^T \int_{\Omega} u\psi_t - \int_0^T \int_{\Omega} u\Delta\psi - \chi - \int_0^T \int_{\Omega} \frac{u}{v} \nabla v \cdot \nabla\psi = \int_{\Omega} u_0 \psi(0, \cdot),$$

and

$$-\int_0^T \int_{\Omega} v\psi_t - \int_0^T \int_{\Omega} v\Delta\psi + \int_0^T \int_{\Omega} v\psi - \int_0^T \int_{\Omega} u\psi = \int_{\Omega} v_0 \psi(0, \cdot)$$

for all $\psi \in C_0^{\infty}([0, T) \times \overline{\Omega})$ with $\frac{\partial\psi}{\partial\nu} = 0$ on $(0, T) \times \partial\Omega$.

Definition 2.15 *(Stinner and Winkler, 2011)*

Let $p \in (0,1)$ *and* $(u_0, v_0) \in L^p(\Omega) \times L^p(\Omega)$. *Then the pair of non-negative functions* (u, v) *is called a weak power-λ solution of (4.6) if* $u \in L^p_{loc}([0,T) \times \overline{\Omega})$ *and* $v \in L^p_{loc}([0,T) \times \overline{\Omega})$ *with* $T > 0$ *such that*

$$\begin{cases} (u+1)^{p-2}|\nabla u|^2 \in L^p_{loc}([0,T)\times\overline{\Omega}) \text{ and } (v+1)^{p-2}|\nabla v|^2 \in L^p_{loc}([0,T)\times\overline{\Omega}) \\ u^p v^{-2}|\nabla v|^2 \in L^p_{loc}([0,T)\times\overline{\Omega}) \text{ and } u(v+1)^{p-1} \in L^p_{loc}([0,T)\times\overline{\Omega}) \end{cases} \tag{2.1}$$

that satisfy the identities

$$-\tfrac{1}{p}\int_0^T\int_\Omega (u+1)^p\psi_t + (p-1)\int_0^T\int_\Omega (u+1)^{p-2}|\nabla u|^2\psi + \int_0^T\int_\Omega (u+1)^{p-1}\nabla u\cdot\nabla\psi$$

$$-\chi(p-1)\int_0^T\int_\Omega (u+1)^{p-2}\frac{u}{v}\nabla u\cdot\nabla v\psi - \chi\int_0^T\int_\Omega (u+1)^{p-1}\frac{u}{v}\nabla v\cdot\nabla\psi$$

$$= \tfrac{1}{p}\int_\Omega (u_0+1)^p\psi(\cdot,0)$$

and

$$-\tfrac{1}{p}\int_0^T\int_\Omega (v+1)^p\psi_t + (p-1)\int_0^T\int_\Omega (v+1)^{p-2}|\nabla v|^2\psi + \int_0^T\int_\Omega (v+1)^{p-1}\nabla v\cdot\nabla\psi$$

$$+\int_0^T\int_\Omega (v+1)^{p-1}v\psi - \int_0^T\int_\Omega u(v+1)^{p-1}\psi = \frac{1}{p}\int_\Omega (v_0+1)^p\psi(\cdot,0)$$

for all $\psi \in C_0^\infty([0,T)\times\overline{\Omega})$.

CHAPTER 3

KELLER - SEGEL MODEL FOR CHEMOTAXIS

In this chapter, we introduce the well-known Keller - Segel models for chemotaxis. The first mathematical modeling of chemotaxis was by Keller and Segel (1970). In their paper, they presented a mathematical formulation that analyzed the movement of the amoeba and the chemical described by continuous functions. It should also be pointed out that the model they obtained builds on previous studies by Patlak (1953).

3.1. Formulation of the model for chemotaxis

Keller and Segel (1970)'s model includes a system of four strongly coupled parabolic partial differential equations. To describe chemotaxis, they assumed some of the events that took place during chemotaxis. The hypotheses and their notations are as follows:

- External signal is produced at a rate $c(v)$.
- There exists an enzyme that destroys the external signal. The concentration of the enzyme is denoted by p. The enzyme is produced at a rate g.
- The enzyme and chemoattractant react and form products. The concentration is given as η.

$$v + p \rightleftharpoons \eta \rightarrow p + product$$

Let Ω be the domain in the $\mathbb{R}^n$ in which the cells are located. Then by conservation of mass,

$$\frac{\partial}{\partial t}\int_\Omega u dx = \int_\Omega Q^{(u)} dx - \int_{\partial\Omega} J^{(u)} \cdot n ds \tag{3.1}$$

where $Q^{(u)}(x,t)$ denotes the amoeba mass, $J^{(u)}(x,t)$ is the flux of amoeba mass and n is the unit normal vector to the $\partial\Omega$. Reproduction was ignored, so $Q^{(u)} \equiv 0$. (3.1) implies

$$\frac{\partial u}{\partial t} = -\nabla \cdot J^{(u)} = -\nabla \cdot (f_2 \nabla v - f_1 \nabla u).$$

The flux terms are the following :

$$J^{(v)} = -k_v \nabla v, \quad J^{(\eta)} = -k_\eta \nabla \eta, \quad J^{(p)} = -k_p \nabla p$$

where k_v , k_η and k_p are taken as constants. The following system was obtained:

$$\left.\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla \cdot (f_1(u,v)\nabla u) - \nabla \cdot (f_2(u,v)\nabla v), \\
\frac{\partial v}{\partial t} &= k_v \Delta v - w_1 v p + w_{-1}\eta + uc(v), \\
\frac{\partial p}{\partial t} &= k_p \Delta p - w_1 v p + (w_{-1} + r_2)\eta + ug(v,p), \\
\frac{\partial \eta}{\partial t} &= k_\eta \Delta \eta + w_1 v p - (w_{-1} + w_2)\eta
\end{aligned}\right\} \tag{3.2}$$

where $u(t,x)$ indicates the amoebae density of the cells and $v(t,x)$ indicates a chemoattractant concentration. w_{-1}, w_1 and w_2 are constants that show reaction rates.

Keller and Segel (1970) devoted their main attention to the aggregation process. They made Haldane's assumption and assumed that the total concentration of the enzyme is constant. Then they simplified the problem into two equations that involve only u and v:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla \cdot (k_1(u,v)\nabla u) - \nabla \cdot (k_2(u,v)\nabla v), \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} &= k_v \Delta v - k_3(v)v + uf(v), \quad x \in \Omega, t > 0,
\end{aligned} \tag{3.3}$$

Keller and Segel (1970) presented an instability condition given by

$$\frac{k_2 v_0}{k_1 u_0} + \frac{u_0 f'(v_0)}{\overline{k_3}} > 1.$$

where it is understood that k_1 and k_2 are evaluated at $u = u_0$, $v = v_0$ and

$$\overline{k_3} \equiv k_3(v_0) + v_0 k'(v_0).$$

To find this condition, they assumed that the right-hand sides of equations in (3.3) were replaced by Taylor expansions in u and v around the equilibrium point (u_0, v_0) with small perturbations.

Patlak (1953) considered the situation in which only the directions in which the particle travels are correlated and there are only one-step correlations. He also assumed that there are external forces and anisotropy of the surroundings. He derived the partial differential equation of the random walk problem with the persistence of direction and external bias.

Patlak (1953) assumed the following:

- The particles interact with each other, but this interaction can be neglected.
- Each time the particle begins its motion, the data on the previous c and τ is deleted, where τ denotes a certain length of time and c denotes an average speed of particles.
- The time the particle spends while moving is insignificant.
- The number of particles and the values c and τ remain approximately the same per unit time.

In its original form, this model consists of four coupled equations. These equations can be written as a model containing two unknowns u and v. Assumptions that do not contradict biological facts can be made. The review of the models with only these unknown functions will be the focus of this thesis.

The simplified model takes the following form:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \nabla(k_1(u,v)\nabla u) - \nabla(k_2(u,v)u\nabla v) + k_3(u,v), \\ \frac{\partial v}{\partial t} &= k_c \Delta v + k_4(u,v), \end{aligned}\right\} \tag{3.4}$$

where u denotes the cell density on a given domain $\Omega \subset \mathbb{R}^n$ and v denotes the concentration of the chemical signal. $k_1(u,v)$ describes the diffusivity of cells, while $k_2(u,v)$ is the chemotactic sensitivity, and both functions may depend on u and v. $k_3(u,v)$ describes cell growth and death while $k_4(u,v)$ is the kinetic function that describes degradation and production of the chemical signal.

CHAPTER 4

QUALITATIVE PROPERTIES OF SOLUTIONS OF SOME KELLER - SEGEL TYPE SYSTEMS

Many researchers have studied mathematical properties of solutions of Keller - Segel type systems and found conditions for existence of global or blow-up solutions. This section will review qualitative properties of solutions of Keller - Segel type systems.

In Subsection 4.1, we give some results on local existence, global existence, boundedness, and blow-up of solutions of the classical Keller - Segel model.

In Subsection 4.2, we give the results for the signal density-dependent Keller - Segel model.

In Subsection 4.3, we summarize some results for the system with cell density-dependent chemotactic sensitivity function.

In Subsection 4.4, we consider a more general type of classical model which includes nonlinear diffusion and cell density-dependent sensitivity functions.

In Subsection 4.5, we give the results for a general quasilinear Keller - Segel system.

In Subsection 4.6, we give important results for a relatively more general form of the Keller - Segel system.

4.1. Classical Keller - Segel Model

In this subsection we consider the classical Keller - Segel equations:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u - \nabla(u \nabla v), \\ \frac{\partial v}{\partial t} &= \Delta v - v + u. \end{aligned}\right\} \tag{4.1}$$

where $x \in \Omega \subset \mathbb{R}^n$ and $t > 0$. In this model, the function $u(t, x)$ is the density of the particle at time t and position x; the function $v(t, x)$ describes the external signal density. Furthermore, the normal derivative of the density of particle and external signal are assumed to be zero on $\partial\Omega$. The initial datas are u_0 and v_0 when $x \in \Omega$ and $t = 0$.

A system of this type describes the evolution of cell populations and their movement affected by the gradient of a chemical signal produced by the cells themselves. This section is devoted to a review of classical Keller - Segel system. We will focus on local and global existence, boundedness, and blow-up of solutions in finite time or infinite time.

For the classical Keller - Segel model (4.1), it is well-known that the qualitative properties of solutions are related to the space dimension. Many researchers explored whether solutions exist globally or blow up in finite time. One interesting problem is to find ways to prevent the blowing-up of solutions, because assumptions should be supported by biological facts.

4.1.1. Existence of Solutions of Classical Keller - Segel Model

In this subsection, we recall the results on the local and global existence of solutions of the classical Keller - Segel model. It is well known that (4.1) is well-posed, meaning that for any smooth initial data we can find a unique classical solution to the classical Keller - Segel type model (Bellomo et al., 2015). In addition, according to N. Bellomo et al. (2015), the solution (u, v) of (4.1) in $(0, T_{max}) \times \Omega$ satisfies the mass identity

$$\int_\Omega u(t)dx = \int_\Omega u_0 dx, \ \ for\ all\ 0 < t < T_{max}. \tag{4.2}$$

In general, two main methods have been used to prove the existence of solutions. One of them is to find a L^∞ bound for the function $-k_2 u \nabla v$ in (3.4). Finding a Lyapunov function is another method (Hillen and Painter, 2009). One has the following lemma:

Lemma 4.1 *(Hillen and Painter, 2009)*
Let the components of the vector field $\psi : (0, \infty) \times \Omega \longrightarrow \mathbb{R}^n$ be uniformly bounded and let $u_0 \in L^\infty(\Omega) \cap L^1(\Omega)$ be nonnegative. Then the solution of the initial boundary value problem

$$u_t = \nabla \cdot (\nabla u - u\psi), \quad u(0, x) = u_0$$

satisfying the zero-flux boundary condition on $\partial\Omega$ *satisfies* $u \in L^\infty((0,\infty) \times \Omega)$ *and*

$$\sup_{t\in(0,\infty)} \|u\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^1(\Omega)}, \|u_0\|_{L^\infty(\Omega)}, \sup_t \|\psi\|_{L^\infty(\Omega)}, n)$$

for some positive constant C.

Finding a Lyapunov functional is important for studying qualitative properties of solutions. The Lyapunov functional associated with Keller - Segel model is below:

$$\mathbf{F}(u,v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega v^2 + \int_\Omega u \ln u - \int_\Omega uv \tag{4.3}$$

Bellomo et al. (2015) calls this Lyapunov functional the energy functional of the Keller - Segel system, and they used it to study the qualitative properties of the model. They argued that this functional plays a crucial role in deriving various results related with global solutions or for solutions which cease to exist globally (Bellomo et al., 2015).

The local solutions of (4.1) have been studied by Yagi, (1997), who suggested that the Keller - Segel system possesses a global solution in one dimension. The global existence of solutions to (4.1) in a bounded domain in $\mathbb{R}$ was derived in (Osaki and Yagi, 2001). The authors supposed that the initial data u_0 and v_0 are non-negative functions for model (4.1). They summarized these results in the following theorem :

Theorem 4.1 *(Osaki and Yagi, 2001)*
Let Ω *be bounded in* $\mathbb{R}$. *Assume* $u_0 \in C^0(\overline{\Omega})$ *and* $v_0 \in \cup_{q>n} W^{1,q}(\Omega)$ *are non-negative in* Ω. *Let* T_{max} *be the maximal existence time for the classical solution of (4.1) and* (u,v) *denote the corresponding maximally extended classical solution of (4.1) in* $(0,T_{max}) \times \Omega$. *Then* (u,v) *is global and bounded. This means that there exists* $K > 0$ *such that*

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} + \|v(t,\cdot)\|_{L^\infty(\Omega)} \leq K$$

for all $t > 0$.
Moreover, the solution converges to a stationary solution as $t \to \infty$.

Osaki and Yagi (2001) succeeded in proving global existence for (4.1) in the case $n = 1$. They have studied the more general form of the Keller - Segel model (4.1). The authors studied (1.1) with $D(u, v) = \text{constant}$, $S(u, v) = u\phi(v)$, $g(u, v) = -\beta v + \alpha u$ and $f \equiv 0$. In order to prove existence of local solutions, Osaki and Yagi used the Galerkin method. Then they obtained the existence of global solutions by constructing a priori estimates for local solutions (Osaki and Yagi, 2001).

Now, we will give the results of the existence of solutions for the case $n = 2$. For the case $n = 2$, the global existence of solutions is given depending on a threshold value of L^1 norm of initial data. If the initial data is below the threshold, then the global solutions of the classical Keller - Segel model exist. If the initial data is above the threshold, then the solutions of the system blow up in finite time or infinite time. Yagi (1997) suggested that the Keller - Segel system possesses a global solution for any sufficiently small initial data in two-dimensional spaces. His results can be summarized as follows :

Theorem 4.2 *(Yagi, 1997)*
Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Assume $u_0, v_0 \in H^{1+\epsilon_0}(\Omega)$ for some $0 < \epsilon_0 \leq 1$ and u_0 and v_0 are nonnegative in Ω. Let T_{max} be the maximal existence time of the solution (u, v) of (4.1).

(i) Then (4.1) has a nonnegative solution (u, v) satisfying

$$u, v \in C([0, T_{max}); H^{1+\epsilon_1}(\Omega)) \cap C^1((0, T_{max}); L^2(\Omega)) \cap C((0, T_{max}); H^2(\Omega))$$

for any positive $\epsilon_1 < min\{\epsilon_0, \frac{1}{2}\}$.

(ii) If $T_{max} < \infty$, then

$$\lim_{t \to T_{max}} (\|u(t, \cdot)\|_{H^{1+\epsilon_0}(\Omega)} + \|v(t, \cdot)\|_{H^{1+\epsilon_0}(\Omega)}) = \infty.$$

Nagai, Senba, and Yoshida (1997) proved that a global solution exists if the L^1 norm of u_0 is smaller than a certain positive number. Their results can be summarized as the follows :

Theorem 4.3 *(Nagai and Senba and Yoshida, 1997)*
Let Ω be bounded in $\mathbb{R}^2$ with smooth boundary. Assume $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in \cup_{q>n} W^{1,q}(\Omega)$ are non-negative in Ω. Let T_{max} be the maximal existence time of classical solution (u, v) of (4.1) and (u, v) denote the corresponding maximally extended classical solution of (4.1) in $(0, T_{max}) \times \Omega$. . If $\int_\Omega u_0 < 4\pi$, then (u, v) exists globally in time, and its L^∞ norm is uniformly bounded for all times. That is, the solution fulfills

$$||u(t, \cdot)||_{L^\infty(\Omega)} + ||v(t, \cdot)||_{L^\infty(\Omega)} \leq C.$$

If moreover Ω is a disk and (u_0, v_0) is radially symmetric, then the same result is obtained under the assumption that $\int_\Omega u_0 < 8\pi$.

The summarized results were obtained by using semigroup theory. Nagai et al. (1997) used the Lyapunov functional to find a bound to the function $\nabla v(t, \cdot)$ for all $t \geq 0$ in the L^∞ norm.

Winkler (2010) realized that there were no conclusions concerning the existence of global solutions for the classical Keller - Segel model in higher space dimension. Winkler (2010) considered the classical Keller - Segel model (4.1) under Neumann boundary conditions in $\mathbb{R}^n$ with $n \geq 3$. His main result stated the following:

Theorem 4.4 *(Winkler, 2010)*
Let Ω be smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$. Assume $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in \cup_{q>n} W^{1,q}(\Omega)$ are non-negative in Ω. If $p > n$, $q > \frac{n}{2}$ and initial data satisfy

$$||u_0||_{L^q(\Omega)} < \epsilon \text{ and } ||\nabla v_0||_{L^p(\Omega)} < \epsilon$$

for some sufficiently small positive ϵ, then the solution (u, v) of (4.1) is global in time and is bounded.

Thanks to the properties of the Neumann heat semigroup, Winkler proved that the global solution exists and is bounded. Furthermore, he described the behavior of solutions in large time and showed that the solution (u, v) approaches the steady-state (m, m) as $t \to \infty$, where m is the total mass of the population. Cao (2015) extended this result in the corresponding critical case, that is, for $q = \frac{n}{2}$ and $p = n$. He proved the global existence and boundedness under the assumption that initial data is small.

Theorem 4.5 *(Cao, 2015)*
Let $n \geq 2$ and $0 < \lambda < \lambda_1$, where λ_1 denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Then there exists $\epsilon_0 > 0$ depending on λ_1 and Ω with the following property: If $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in \cup_{q>n} W^{1,q}(\Omega)$ with $q > n$ are non-negative in Ω and satisfy

$$||u_0||_{L^{n/2}(\Omega)} \leq \epsilon \ \ and \ \ ||\nabla v_0||_{L^n(\Omega)} \leq \epsilon.$$

for some $\epsilon < \epsilon_0$, then (4.1) possess a global solution (u, v) which is bounded and satisfies

$$||u(t,\cdot) - \overline{u_0}||_{L^\infty(\Omega)} \leq C.e^{-\lambda t} \ \ and \ \ ||v(t,\cdot) - \overline{u_0}||_{L^\infty(\Omega)} \leq C.e^{-min\{\lambda,1\}t} \ \ for \ all \, t > 0$$

where $C > 0$ depends on λ_1 and Ω and $\overline{u_0} = \frac{1}{|\Omega|}\int_\Omega u_0$.

His proof is based on a priori estimates of the total mass of cells and the chemical gradient.

Hieber et al. (2021) addressed the classical Keller - Segel model (4.1) with convex bounded domains of $\mathbb{R}^3$. However, they did not assume that the boundary of the domain is smooth. The authors established the local and strong solutions of the classical Keller - Segel model. This solution can be extended globally if initial data are small. This result is one of the most recent results regarding the classical Keller - Segel model.

4.1.2. Blow-up of Solutions of Classical Keller - Segel Model

After the main article written by Keller and Segel (1970), Nanjundiah's article (1973) was a landmark in examining the qualitative characteristics of solutions of the classical Keller - Segel model (Horstmann, 2003). His paper studied nonlinear stability for the classical model of Keller - Segel equations in dimension $n = 2$. He predicted finite time blow-up for some solutions. Furthermore, he revealed that the blow-up of solutions can only be in the form of a δ function type.

Subsequently, two articles (Childress and Percus, 1981; Childress, 1983) developed conjectures for the asymptotic behavior of the solution of the Keller - Segel type system (Horstmann, 2003). Childress and Percus (1981) focused on the relation between the dimension of space and aggregation. They proved that the singular behavior of the solution is dependent on the dimension of space. They also refer that aggregation leads to the formation of δ function type in the cell density as chemotactic collapse. Studies of the existence of such blow-up solutions increased in the 1990s.

The first result was given by Herrero and Velazquez (1997). The authors were able to describe the blow-up profile of the system (4.1) using asymptotic expansion theory, and they showed that there exist initial data such that the corresponding solution of the Keller - Segel model blows up either in finite or in infinite time under the condition $\int_\Omega u_0(x)dx > 8\pi$ (Herrero and Velazquez, 1997). Moreover, they constructed some radially symmetric initial data such that the solution forms a singularity in finite time in the center of a disc Ω.

Horstmann and Wang (2001) considered the nonradial case in $\mathbb{R}^2$ and showed that there exist some unbounded solutions provided the domain $\Omega \subset \mathbb{R}^2$ is simply-connected. They proved that the corresponding solution of (4.1) blows up in the finite or infinite time.

Nagai, Senba and Suzuki (2000) showed the following results that gives information about the general blow - up behavior of the classical Keller - Segel model (4.1) in their article:

Theorem 4.6 *(Nagai and Senba and Suzuki, 2000)*
Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. Let (u, v) be a classical solution of (4.1) in $(0, T_{max}) \times \Omega$ with $T_{max} \in (0, \infty]$ satisfying $\|u\|_{L^\infty((0,T_{max})\times\Omega)} = \infty$. Moreover, let the set B denote all blow-up points in $\overline{\Omega}$ and the set $B_I \subset B$ denote all isolated blow-up points.

(i) If $x_0 \in B_I$, then we have $R > 0$ and the nonnegative function

$$f \in L^1(B(x_0, R) \cap \Omega) \cap C(\overline{B(x_0, R) \cap \Omega} \setminus \{x_0\})$$

such that $u(t, \cdot)$ converges to $m\delta_{x_0} + f$ as $t \to T_{max}$ in the sense of radon measures on

$$\overline{B(x_0, R) \cap \Omega} \text{ where } \begin{cases} m \geq 8\pi \;\; if \;\; x_0 \in \Omega \\ m \geq 4\pi \;\; if \;\; x_0 \in \partial\Omega \end{cases} \quad \text{and } B(x_0, R) := \{x \in \mathbb{R}^2 : |x - x_0| < R\}.$$

(ii) If $\inf_{0 \leq t < T_{max}} F(u(t,\cdot), v(t,\cdot)) > 0$ *or* $\lim_{t \to T_{max}} F(u(t,\cdot), v(t,\cdot)) = -\infty$*, then* $B_I = B$.

(iii) If the solution of (4.1) is radially symmetric and $T_{max} < \infty$ *, then* $B = \{0\}$.

Theorem 4.6 implies that if there is a solution that blows up in finite time, then the blow-up must occur at the boundary of the domain. However, the results do not show whether the blow-up time is finite or infinite.

Results on the blow-up of solutions of the classical Keller - Segel model are limited in the literature. Results of the classical Keller - Segel model depending on the particular conditions associated with the blow-up of this model are summarized below.

Winkler (2010) showed that there exist unbounded solutions emanating from initial data (u_0, v_0) having total mass $\int_\Omega u_0 = m$ in three or higher dimensional balls.

Theorem 4.7 *(Winkler, 2010)*
Assume that $\Omega \subset \mathbb{R}^n$ *is a ball with* $n \geq 3$. *For each small mass, there is an initial data* u_0 *and* v_0 *such that the solution of (4.1) blows up either in finite or infinite time.*

The above result obtained by Winkler does not answer the question of which solutions blow up in finite time and which solutions blow up in infinite time.

The following blow-up result of the classical Keller - Segel model (4.1) for two and higher dimensions was obtained by Bellomo et al. (Bellomo et al., 2015).

Theorem 4.8 *(Bellomo et al., 2015)*
Assume that $\Omega \subset \mathbb{R}^n$ *is a ball with* $n \geq 2$. *Let* $\overline{u_0} > 0$ *and* $\overline{v_0} > 0$. *Then there is* $T(\overline{u_0}, \overline{v_0}) > 0$ *and* $C(\overline{u_0}, \overline{v_0}) > 0$ *such that for any initial data* (u_0, v_0) *from the given set*

$$\begin{aligned} S(\overline{u_0}, \overline{v_0}) \;\; := \;\; & \Big\{(u_0, v_0) \in C(\overline{\Omega}) \times W^{1,\infty}(\Omega) : 0 < u_0 \textit{ and } 0 < v_0 \textit{ are radially symmetric in } \overline{\Omega} \\ & \textit{with } \overline{u_0} = \int_\Omega u_0, \;\; \overline{v_0} \geq \|v_0\|_{W^{1,2}(\Omega)} \textit{ and } \mathbf{F}(u_0, v_0) \leq -C\Big\} \end{aligned}$$

the corresponding solution (u, v) *of (4.1) blows up before or at time* $T(\overline{u_0}, \overline{v_0})$.

Winkler (2020) considered (4.1) for dimensions $n \geq 3$. He used a basic theory of local existence and maximal extensibility of classical solutions. He proved the presence of blow-up solution in finite time with large sets of radially symmetric initial data, and also showed that blow-up occurs at the spatial origin.

Theorem 4.9 *(Winkler, 2020)*
Suppose $n \geq 3$ and $q > n$. Let $u_0 \in BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $v_0 \in W^{1,q}(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)$ be nonnegative and $u_0 \neq 0$.

(i) Then there is $T_{max} \in [0, \infty)$ and

$$\begin{cases} u \in C^0([0, T_{max}); BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \cap C^{1,2}((0, T_{max}) \times \mathbb{R}^n) \\ v \in C^0([0, T_{max}); W^{1,q}(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)) \cap C^{1,2}((0, T_{max}) \times \mathbb{R}^n) \end{cases}$$

which solve (4.1) in the classical sense and such that if $T_{max} < \infty$ then

$$\limsup_{t \to T_{max}} \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} = \infty.$$

In addition, if initial data are radially symmetric with respect to x_0, then solutions $u(t, \cdot)$ and $v(t, \cdot)$ are also radially symmetric with respect to $x = 0$ for all $0 < t < T_{max}$.

(ii) If initial data are radially symmetric, then for any choice of $p \in \left(1, \frac{2n}{n+2}\right)$ there are sequences (u_{0i}) and (v_{0i}) such that u_{0i} and v_{0i} are radially symmetric and positive for all $i \in \mathbb{N}$,

$$u_{0i} \to u_0 \quad \text{and} \quad v_{0i} \to v_0$$

when $i \to \infty$ and for all $i \in \mathbb{N}$ the classical solution of (4.1) from item(i) blows up in finite time at the spatial origin.

A more recent study by Winkler (2020) showed that any classical radially symmetric solution which blows up in finite time has a unique blow-up profile. He derived pointwise time-independent estimates for any radially symmetric solutions. Then these estimates were used to claim that any radial classical solution which blows up in finite time has a unique blow-up profile. Moreover, these estimates were used to ensure global extensibility for any such solution (Winkler, 2020). His results can be summarized as follows:

Theorem 4.10 *(Winkler, 2020)*
Let Ω be a ball in $\mathbb{R}^n$ for some $n \geq 2$ and assume $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are nonnegative and radially symmetric initial data.

(i) *Given any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that if $T_{max} \in (0, \infty]$ then the solution of the system (4.1) satisfies*

$$u(t, x) \leq C(\epsilon) \cdot |x|^{-n(n-1)-\epsilon}$$

$$|\nabla v(t, \cdot)| \leq C(\epsilon) \cdot |x|^{-(n-1)-\epsilon}$$

for all $x \in \Omega$ and $t \in (0, T_{max})$.

(ii) *If the corresponding solution of the system (4.1) blows up in finite time, then there is a nonnegative radially symmetric function $\overline{u} \in C^2(\overline{\Omega} - \{0\})$ such that*

$$u(t, \cdot) \to \overline{u}$$

in $C^2_{loc}(\overline{\Omega} - \{0\})$ when $t \to T_{max}$ and for all $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$\overline{u} \leq C(\epsilon) \cdot |x|^{-n(n-1)-\epsilon}$$

for all $x \in \Omega - \{0\}$.

4.2. Signal Dependent Sensitivity Keller - Segel Model

In this subsection we consider the following Keller - Segel equations:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u - \nabla\left(uS(v)\nabla v\right), \quad x \in \Omega, t \in (0,T), \\ \frac{\partial v}{\partial t} &= \Delta v - v + u, \quad x \in \Omega, t \in (0,T), \end{aligned}\right\} \tag{4.4}$$

where $S : (0,\infty) \to \mathbb{R}$. In this model, the function $u(t,x)$ is the density of the particle at time t and position x; the function $v(t,x)$ describes the external signal density. Furthermore, the normal derivative of the density of particle and external signal are zero on $\partial\Omega$.

This section is devoted to study of the system of equations referred to as the signal density dependent sensitivity Keller - Segel model, which focuses on local and global existence, boundedness, and blow-up in finite time or infinite time of solutions.

There are two versions of the above model. The first one is

$$\left.\begin{aligned} \partial_t u &= \Delta u - \nabla\left(\chi u \frac{1+\beta}{v+\beta}\nabla v\right), \\ \partial_t v &= \Delta v - v + u. \end{aligned}\right\} \tag{4.5}$$

where for $\beta \to \infty$ one gets the classical Keller - Segel model and for $\beta \to 0$ one gets a model with singular sensitivity :

$$\left.\begin{aligned} \partial_t u &= \Delta u - \nabla\left(\chi \frac{u}{v}\nabla v\right), \\ \partial_t v &= \Delta v - v + u. \end{aligned}\right\} \tag{4.6}$$

The second model is

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u - \nabla\left(\chi u \frac{1}{(1+\beta v)^2}\nabla v\right), \\ \frac{\partial v}{\partial t} &= \Delta v - v + u. \end{aligned}\right\} \tag{4.7}$$

where for $\beta \to 0$ we obtain the classical Keller - Segel model.

4.2.1. Existence of Solutions of Signal Dependent Sensitivity Keller - Segel Model

In this subsection, we recall the results on local and global existence of solutions of the signal-dependent Keller - Segel model and its existing variations.

The existence of local solutions of model (4.4) has been investigated by many authors. The general existence theorem which will be given in section (4.6) was demonstrated by Bellomo et al. (2015). However, this theorem does not hold for the model (4.6) since the model contains a singular sensitivity function. Although we do not consider the signal-dependent model with singular sensitivity (4.6) to be biologically relevant, it has been studied by many mathematicians.

The method used by Osaki and Yagi (2001) can also be applied to (4.6). This gives the global existence for the case $n = 1$. For all other dimensions in some special cases, some results show the global existence of solutions. As an example of this, Nagai, Senba and Yoshida (1997) considered a domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. The following theorem summarizes their results:

Theorem 4.11 *(Nagai and Senba and Yoshida, 1997)*
Assume that v_0 is positive in $\overline{\Omega} \subset \mathbb{R}^2$.

(i) If $\chi < 1$, then the solution of (4.6) globally exists in time. Moreover, $T_{max} = \infty$ and there is a constant C_T

$$sup_{t\in(0,T)} \left(\|u(t,\cdot)\|_{L^\infty(\Omega)}\right) + \|v(t,\cdot)\|_{L^\infty(\Omega)}\Big) = C_T < \infty$$

for $T > 0$.

(ii) Let $\Omega = \{x \in \mathbb{R}^2 : \|x\| < L\}$ be a disk and the initial data (u_0, v_0) be radial in x. If $\chi < \frac{5}{2}$, then the solution of (4.6) globally exists in time.

It was shown that the restriction $\chi < 1$ is not critical for the boundedness of the solutions. Lankeit (2015) introduced a new method to obtain the boundedness of solutions of the model (4.6). He considered the energy functional in order to prove global existence and boundedness of solutions for $0 < \chi < \chi^*$ for some $\chi^* > 1$ in convex, bounded and smooth domain in the space dimension $n = 2$. His theorem is given as follows:

Theorem 4.12 *(Lankeit, 2015)*
Let Ω be a subset of $\mathbb{R}^2$ and Ω be convex bounded domain with smooth boundary. Assume that $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in \cup_{q>2} W^{1,q}(\Omega)$ are nonnegative functions. Then there is $\chi^* > 1$ such that for any $0 < \chi < \chi^*$ the system (4.6) has a global classical solution which is bounded.

In higher dimensions, Winkler (2011) considered the Neumann boundary value problem for chemotaxis system (4.6) in a bounded domain Ω of $\mathbb{R}^n$. He proved global existence of classical solutions for reasonably smooth but arbitrarily large data under the assumption $\chi < \sqrt{\frac{2}{n}}$ and he showed that global existence of weak solutions provided $\chi < \sqrt{\frac{n+2}{3n-4}}$.

Theorem 4.13 *(Winkler, 2011)*
Consider the system (4.6).

(i) Suppose that $\chi < \sqrt{\frac{2}{n}}$. Then for all $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ satisfying $u_0 \geq 0$ and $v_0 > 0$ in $\overline{\Omega}$, the system (4.6) has a global classical solution.

(ii) Suppose that $n \geq 2$ and $\chi < \sqrt{\frac{(n+2)}{3n-4}}$. Then for all $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ satisfying $u_0 \geq 0$ and $v_0 > 0$ in $\overline{\Omega}$, there is a global weak solution (u, v) of the system (4.6).

However, the conclusions in his article did not give boundedness of solutions of (4.6). Recently, the uniform-in-time lower estimate for v was established by Fujie (2015), who showed global existence of classical solutions that are bounded solutions of (4.6) satisfying $0 < \chi < \sqrt{\frac{2}{n}}$.

Theorem 4.14 *(Fujie, 2015)*
Let Ω be a subset of $\mathbb{R}^n$ with $n \geq 2$. Suppose that $0 < \chi < \sqrt{\frac{2}{n}}$ and the non-negative initial data $(u_0, v_0) \in C(\overline{\Omega}) \times W^{1,\infty}(\Omega)$. Then the solution of (4.6) is bounded and one can find some positive constant C such that

$$||u(t, \cdot)||_{L^\infty(\Omega)} \leq C$$

for all $t > 0$.

Winkler and Stinner (2011) proved the global existence of weak power - λ solutions to (4.6). The authors showed that generalized solutions can be achieved without any restriction on the term χ. The main result is given in the following theorem :

Theorem 4.15 *(Stinner and Winkler, 2011)*
Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \geq 2$ and u_0 and v_0 be radially symmetric nonnegative initial data in $\overline{\Omega}$ and $\chi > 0$. Then there exists nonnegative (u, v) defined in $(0, \infty) \times \Omega$ so that the pair (u, v) is a global solution of (4.6) that is weak power - λ.

Stinner and Winkler studied radially symmetric nonnegative solutions of (4.6) in the case $\Omega \subset \mathbb{R}^n$ is ball , $n \geq 2$ and initial data $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are radially symmetric. They proved existence of global weak solutions for any $\chi > 0$. Also, to overcome the difficulties associated with the singularity, they described weak power-λ solutions. This result extends the previous result of Winkler for radially symmetric solutions.

Furthermore, Winkler (2010) proved that the model (4.6) has the global classical solution which is bounded in the case that

$$S(v) \leq \frac{\chi}{(1 + av)^k}$$

with $\chi > 0$, $k > 1$ and $a > 0$.

Fujie and Yokota (2014) established global existence and boundedness of classical solutions to the model (4.4) with $S(v) = \frac{\chi}{v^k}$ for $k > 1$ and $\chi > 0$.

Generally, it is not easy to demonstrate the global existence of bounded solutions for any positive value of χ. More recently, Mizukami and Yokota obtained global existence and boundedness under proper conditions for S in (4.4). They assumed that S satisfies

$$S \in C^{1+\lambda}(0, \infty) \quad \text{and} \quad 0 \leq S(v) \leq \frac{\chi}{(a + v)^k} \tag{4.8}$$

with $\lambda > 0$, $\chi > 0$, $k \geq 1$ and $a \geq 0$.

Moreover, they assumed that the initial data u_0 and v_0 satisfy

$$0 \le u_0 \in C(\overline{\Omega}) - \{0\} \quad and \quad \begin{cases} 0 < v_0 \in W^{1,q}(\overline{\Omega}) \ (\exists q > n) \quad (a = 0) \\ 0 < v_0 \in W^{1,q}(\overline{\Omega}) - \{0\} \ (\exists q > n) \quad (a > 0). \end{cases} \tag{4.9}$$

Their main result is as follows :

Theorem 4.16 *(Mizukami and Yokota, 2017)*
Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose that the function S satisfies the condition in (4.8) and also $\chi < k(a+\eta)^{k-1}\sqrt{\frac{2}{n}}$ where $\eta = \sup_{\tau>0}(\min\{e^{-2\tau}\min_{x\in\overline{\Omega}} v_0(x), c_0 m(1-e^{-\tau})\})$. Then for any u_0, v_0 under the condition (4.9), there exists a unique pair (u, v) of functions

$$\begin{cases} u \in C([0, T_{max}) \times \overline{\Omega}) \cap C^{1,2}((0, T_{max}) \times \overline{\Omega}) \\ v \in C([0, T_{max}) \times \overline{\Omega}) \cap C^{1,2}((0, T_{max}) \times \overline{\Omega}) \end{cases}$$

which solves (4.4). Moreover, the solution (u, v) is bounded, which means there exists a constant $M > 0$ such that

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} + \|v(t,\cdot)\|_{W^{1,\infty}(\Omega)} \le M \quad \forall t \ge 0.$$

Black et al. (2020) noted that one should reconsider the above proof of boundedness as more quantitative information becomes necessary. Their main result is as follows:

Theorem 4.17 *(Black et al., 2020)*
Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Let $M := \int_\Omega u_0 > 0$, $v^* := min v_0 > 0$, $k > 1$ and $a \ge 0$. Then there exists $\delta = \delta(M, v^*, k, a)$ such that for all functions

$$S \in C^{1+\lambda}(0,\infty) \quad and \quad 0 \le S(v) \le \frac{1}{(a+v)^k}$$

for some $0 < \lambda < 1$*, all initial data*

$$\begin{cases} 0 \le u_0 \in C(\overline{\Omega}) - \{0\} \\ 0 \le v_0 \in W^{1,\infty}(\Omega) - \{0\} \end{cases}$$

and for all $\chi < \delta$*, the system (4.4) has a global classical solution. Moreover, positive constants* θ *and L can be found such that*

$$\|u(t,\cdot) - \frac{M}{|\Omega|}\|_{L^\infty(\Omega)} + \|v(t,\cdot) - \frac{M}{|\Omega|}\|_{L^\infty(\Omega)} \le Le^{-\theta t}$$

for all $t > 0$.

4.3. Cell Dependent Sensitivity Keller - Segel Model

In this subsection we consider the following Keller - Segel system :

$$\left.\begin{array}{rcl} \frac{\partial u}{\partial t} & = & \Delta u - \nabla(uS(u)\nabla v), \quad x \in \Omega, t \in (0,T), \\ \frac{\partial v}{\partial t} & = & \Delta v - v + u, \quad x \in \Omega, t \in (0,T), \\ \frac{\partial u}{\partial N} & = & \frac{\partial v}{\partial N} \quad = 0, \quad x \in \partial\Omega, t > 0, \\ u|_{t=0} & = & u_0, \quad v|_{t=0} = v_0, \quad x \in \Omega. \end{array}\right\} \tag{4.10}$$

where $S : (0,\infty) \to \mathbb{R}$.

In this model, the function $u(t,x)$ is the density of the particle at time t and position x; the function $v(t,x)$ describes the external signal density. This section is devoted to the study of the system of equations above referred to as the cell density dependent sensitivity Keller - Segel type equations. This section focuses on local and global existence, boundedness, and blow up in finite time or infinite time of solutions.

Two versions of the above model have been studied.

The first one is

$$\left.\begin{array}{rcl}\frac{\partial u}{\partial t} &=& \Delta u - \nabla(u(1-\frac{u}{\beta})\nabla v), \quad x \in \Omega, t \in (0,T), \\ \\ \frac{\partial v}{\partial t} &=& \Delta v - v + u, \quad x \in \Omega, t \in (0,T), \end{array}\right\} \tag{4.11}$$

where for $\beta \to \infty$, the classical Keller - Segel model is obtained. The second one is

$$\left.\begin{array}{rcl}\frac{\partial u}{\partial t} &=& \Delta u - \nabla(u(\frac{1}{1+\beta u})\nabla v), \quad x \in \Omega, t \in (0,T), \\ \\ \frac{\partial v}{\partial t} &=& \Delta v - v + u, \quad x \in \Omega, t \in (0,T), \end{array}\right\} \tag{4.12}$$

where for $\beta \to 0$, the classical Keller - Segel model is obtained.

Horstmann and Winkler (2005) explained the motivation for studying (4.10) in their article as follows :

"*. . . , to our knowledge it has never been analyzed whether the solution of system (4.10) might become unbounded if $uS(u)$ equals other powers of u, i.e. $uS(u) = u^\alpha$ with some $\alpha > 0$. Of course, this question is more motivated from the mathematical point of view than from the biological one, but it will help to get more insights into the understanding of the blow-up mechanism of the problem. . . .*"

4.3.1. Existence of Solutions of Cell Dependent Sensitivity Keller - Segel Model

Theorem 4.18 *(Horstmann and Winkler, 2005)*
Suppose $q > n$ and that $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ are nonnegative in Ω. Then there exists $T_{max} \leq \infty$ (depending on $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{W^{1,q}(\Omega)}$ only) and a unique nonnegative function pair (u, v) fulfilling

$$\begin{cases} u \in C^0([0,T_{max}); C^0(\overline{\Omega})) \cap C^{1,2}((0,T_{max}) \times \overline{\Omega}) \\ v \in C^0([0,T_{max}); C^0(\overline{\Omega})) \cap L^\infty_{loc}([0,T_{max}); W^{1,q}(\Omega)) \cap C^{1,2}((0,T_{max}) \times \overline{\Omega}) \end{cases}$$

that solves (4.10) in the classical sense. If $T_{max} < \infty$ then

$$\lim_{t \to T_{max}} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,q}(\Omega)}) = \infty.$$

Moreover, the solution (u, v) satisfies the mass identities

$$\int_\Omega u(t)dx = \int_\Omega u_0 dx$$

and

$$\int_\Omega v(t)dx = \int_\Omega u_0 dx + \left(\int_\Omega v_0 dx - \int_\Omega u_0 dx \right) e^{-t}$$

for all $t \in (0, T_{max})$.

The local existence of the solution in the above theorem has been achieved by using Banach's fixed point theorem and semigroup theory. Yagi (1997) also proved similar local existence results for more general forms of the system (4.10), but he did not consider the chemotactic sensitivity function that depends on the power of u.

Another important result is the boundedness of all solutions of (4.10). The following theorem gives this result :

Theorem 4.19 *(Horstmann and Winkler, 2005)*

(i) If $n \geq 1$ and $uS(u) \leq c_0 u^\alpha$ for all $u \geq 1$ some $c_0 > 0$ and some $0 < \alpha < \frac{2}{n}$, then all solutions of (4.10) are global in time and uniformly bounded.

Moreover, given $\kappa := max\{\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}\} > 0$ and $\zeta \in (0, 1)$ there exist $c(\kappa, \zeta) > 0$,$m > 0$ and $\nu > 0$ such that

$$\|u_0\|_{L^1(\Omega)} \leq \kappa \quad and \quad \|v_0\|_{L^1(\Omega)} \leq \kappa$$

implies

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} \leq c(\kappa, \zeta)(1 + \bar{K}^m(\zeta) e^{-\nu t})$$

for all $t \geq \zeta$ where $\bar{K}(\zeta) := \max_{t \in \frac{\zeta}{4}, \zeta} (\|u(t)\|_{L^\infty(\Omega)} + \|\nabla v(t)\|_{L^2(\Omega)})$

(ii) If $n \geq 2$ and there exists $c_0 > 0$ such that $uS(u) \geq c_0 u^{\alpha}$ for all $u \geq 1$ and some $0 < \alpha < \frac{2}{n}$ then there does not exist a priori estimate in the sense of (4.19)-(i).

This result of boundedness is obtained step by step starting from the $L^1(\Omega)$ bounds for u and v. Both equations in (4.10) are used to establish estimates in higher L^p spaces. This method is based on an iterative procedure.

4.3.2. Blow-up of Solutions of Cell Dependent Sensitivity Keller - Segel Model

Horstmann and Winkler (2005) studied blow-up of solutions of (4.10) and proved the existence of solutions that become unbounded in finite or infinite time.

Theorem 4.20 *(Horstmann and Winkler, 2005)*

(i) Suppose $\Omega = B_R$ is a ball in $\mathbb{R}^n$, $n \geq 2$ and $uS(u) \geq c_0 u^{\alpha}$ for all $u \geq 1$ with some c_0 and $\alpha > 2$. Then for any $\lambda > 0$ there are radially symmetric solutions (u, v) of the system (4.10) which blow up and have mass $\int_{\Omega} u(t) \equiv \lambda$.

(ii) Suppose that $\Omega = B_R$ is a ball in $\mathbb{R}^n$, where $n = 2$ or $n = 3$ and $c_0 u^{\alpha} \leq uS(u) \leq c_1 u^{\alpha_+}$ for all $u \geq 1$ and

$$\alpha \in \begin{cases} (1,2] & if \quad n = 2 \\ (1,2) & if \quad n = 3 \end{cases}$$

$\alpha_+ \in \left[\alpha, \frac{1}{2-\alpha}\right]$. Then for any $\lambda > 0$ there are radially symmetric solutions (u, v) of the system (4.10) which blow up and have mass $\int_{\Omega} u(t) \equiv \lambda$.

To prove the existence of unbounded solutions of (4.10), Horstmann and Winkler (2005) used an indirect method. At first, they found some initial data, for which the corresponding solution of (4.10) does not remain bounded, which means the solution will have to blow up either in finite time or in infinite time. They explained in their article the reasons for reaching these results: "*To become more concrete, the existence of Lyapunov* *functional encourages us to suspect a connection between the ω-limit set of a supposedly bounded solution of (4.10) and some kind of steady-state solutions of (4.10) . . .* "

Their results explained :

(i) for the space dimension $n = 1$, why there is no blow-up,

(ii) for the space dimension $n = 2$, if the initial data has sufficiently large mass, there is the possibility of unbounded solutions, where the mass is the L^1 norm of the initial data,

(iii) for the space dimension $n \geq 3$, why solutions blow up even though there are no constraints for the initial data.

4.4. Nonlinear Diffusion and Cell Dependent Sensitivity Keller -Segel Model

In this subsection we consider the following Keller - Segel system :

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \nabla(D(u)\nabla u) - \nabla(S(u)\nabla v), \quad x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} &= \Delta v - v + u, \quad x \in \Omega, t > 0, \end{aligned}\right\} \tag{4.13}$$

where $D, S : (0, \infty) \to \mathbb{R}$. In this model, the function $u(t, x)$ is the density of the particle at time t and position x; the function $v(t, x)$ describes the external signal density. Furthermore, the normal derivative of the density of particle and external signal are zero on $\partial\Omega$.

This model involves a nonlinear diffusion function and signal density chemotactic sensitivity function. This section is devoted to study of the system of equations referred to as the cell density dependent Keller - Segel type equations. We focus on local and global existence, boundedness, and blow-up in finite time or infinite time of solutions.

Winkler et al. (2012) indicated that (4.13) was first examined by Hillen and Painter (2002). Hillen and Painter considered approaches in which such equations could arise based on a biological mechanism. One of the approaches is the existence of the volume-filling effect. They stated that cell overcrowding is not biologically realistic due to the ignorance of the finite size of cells. The volume-filling effect includes the finite size of individual cells. They explained that what is expected from this approach is that the cells are collected until they reach maximum capacity, after which cells cannot move.

The mass conservation of cells holds in (4.13).

Lemma 4.2 *(Tao and Winkler, 2012)*
The first component u of the solution of (4.13) satisfies the mass conservation property

$$||u(t)||_{L^1(\Omega)} = ||u_0||_{L^1(\Omega)}, \forall t \in (0, T_{max}).$$

4.4.1. Existence of Solutions of Nonlinear Diffusion and Cell Density Dependent Sensitivity Keller - Segel Model

In this section, we review some results on the local and global existence of solutions of nonlinear diffusion and the signal-dependent sensitivity Keller - Segel model.

Hillen and Painter (2001) studied the particular choices for diffusion and cross-diffusion functions. They assumed that chemotactic movement stops at high cell density. They demonstrated the global existence of solutions in any space dimension for (4.13). Thus these assumptions avoid overcrowding.

Cieslak (2008) considered the case of absence of value at which chemotactic movement is terminated. His purpose was to find sufficient conditions on the functions $D(u)$ and $S(u)$ for (4.13) to have global solutions. He studied the problem with the following assumptions:

$$\begin{cases} D \in C^2([0,\infty)) \\ S \in C^2([0,\infty)) \end{cases} \tag{4.14}$$

are bounded and positive functions.

Cieslak's theorems give conditions on $D(u)$ and $S(u)$ for which (4.13) has a unique global solution. These conditions are dimension dependent. His results can be summarized as follows:

Theorem 4.21 *(Cieslak, 2008)*

(i) *Suppose* $n = 1$. *Assume that* $(u_0, v_0) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ *for* $p > 2$, *nonnegative, the assumption (4.14) holds and the ratio*

$$\frac{S(u)}{D(u)} \le Ku$$

is satisfied for some $K > 0$. *Then there exists a unique global classical solution that is uniformly bounded in time.*

(ii) *Suppose* $n = 2, 3$. *Assume that* $(u_0, v_0) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ *for* $p > n$, *nonnegative, the assumption (4.14) holds and*

$$\frac{S(u)}{D(u)} \le Ku^{\alpha}$$

is satisfied for some $K > 0$ *and* $\alpha < \frac{2}{n}$ *for every* $u > 0$. *Then there exists a unique global classical solution that is uniformly bounded in time.*

In the above theorem, (i) means that the solution of (4.13) will not blow up when $n = 1$ under given assumptions. (ii) specifies the blow-up preventing conditions for the system (4.13) in space dimensions 2 and 3. Cieslak (2008) extended the results obtained by Horstmann and Winkler (2005).

Global existence and boundedness of solutions of model (4.13) were studied by Tao and Winkler (2012). They explored this system of equations in more detail and allowed fairly general choices for functions D and S. They considered the nonlinear diffusion and signal-dependent sensitivity Keller - Segel model in a bounded, smooth, convex domain Ω in the space dimension $n \ge 1$. They assumed that

$$\begin{cases} D \in C^2([0,\infty)^2) \\ S \in C^2([0,\infty)^2) \;\; satisfies \;\; such \;\; that \;\; S(0) = 0, \end{cases} \tag{4.15}$$

and

$$\frac{S(u)}{D(u)} \leq C_1(u+1)^{\alpha_1} \quad \forall u \geq 0 \tag{4.16}$$

with some $C_1 > 0$ and $\alpha_1 > 0$. They also assumed that

$$\begin{cases} D(u) \geq C_2(u+1)^{\alpha_2 - 1} & \forall u \geq 0 \\ D(u) \leq C_3(u+1)^{\alpha_3 - 1} & \forall u \geq 0 \end{cases} \tag{4.17}$$

are satisfied for some $\alpha_2 \in \mathbb{R}$, $\alpha_3 \in \mathbb{R}$, $C_2 > 0$ and $C_3 > 0$.

Theorem 4.22 *(Tao and Winkler, 2012)*
Let Ω be a convex, bounded subset of $\mathbb{R}^n$, $n \geq 1$ with smooth boundary. Suppose that the functions $D(u)$ and $S(u)$ satisfy the conditions (4.15), (4.16) and (4.17) and $\alpha_1 < \frac{2}{n}$. For each nonnegative $(u_0, v_0) \in C(\overline{\Omega}) \times C^1(\overline{\Omega})$ there is a nonnegative pair $(u, v) \in C([0, \infty) \times \overline{\Omega}) \cap C^{1,2}((0, \infty) \times \overline{\Omega})$ that is a classical solution of (4.13).

The global existence and boundedness of solutions of (4.13) were studied in bounded non-convex domain by Ishida et al. (2014) and the convexity assumption was removed.

Bellomo et al. (2015) stated that the most complete picture for this model (4.13) would be obtained if the following functions were selected

$$D(u) = (u + \epsilon)^{-\gamma} \qquad \text{and} \qquad uS(u) = (u + \epsilon)^{-\beta}$$

for $u \geq 0$ with some $\gamma, \beta \in \mathbb{R}$ and $\epsilon \geq 0$ and gave the following theorem:

Theorem 4.23 *(Bellomo et al., 2015)*
Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in C^1(\overline{\Omega})$ be nonnegative initial data. Suppose that D and S satisfy above conditions and $\gamma + \beta < \frac{2}{n}$.

(i) If $\epsilon > 0$ then the (4.13) has a global classical solution

$$(u, v) \in (C^0([0, \infty) \times \overline{\Omega}) \cap C^{1,2}((0, \infty) \times \overline{\Omega}))^2$$

that is also bounded

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} + \|v(t,\cdot)\|_{L^\infty(\Omega)} \le L \qquad \forall t > 0$$

with some $L > 0$.

(ii) *If* $\epsilon = 0$ *and besides* $\gamma \ge 0$ *then the system (4.13) has a global bounded weak solution*

$$(u, v) \in L^\infty((0,\infty)\times\Omega) \times L^\infty((0,\infty); W^{1,\infty}(\Omega)).$$

Ding and Winkler (2021) considered the system (4.13) and showed that the system admits a bounded and global classical solution. They needed to assume that

$$\begin{cases} D \in C^2([0,R]) \;\; and \;\; D > 0 \;\; in \;\; [0,R] \\ S \in C^2([0,R]) \;\; and \;\; S(0) = 0 \end{cases} \tag{4.18}$$

with some $R > 0$. Their main theorem is the following:

Theorem 4.24 *(Ding and Winkler, 2021)*
Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$, *be a bounded domain with smooth boundary. Suppose also that* D *and* S *satisfy (4.18). Then for any* $\epsilon > 0$ *one can find* $\delta = \delta(\epsilon) \in (0,R)$, *and whenever* $0 \le u_0 \in W^{1,\infty}(\Omega)$ *and* $0 \le v_0 \in W^{1,\infty}(\Omega)$ *with*

$$\|v_0\|_{W^{1,\infty}(\Omega)} \le \epsilon \;\; and \;\; \|u_0\|_{W^{1,\infty}(\Omega)} \le \delta$$

there exist functions

$$\begin{cases} u \in C([0,\infty)\times\overline{\Omega}) \cap C^{1,2}((0,\infty)\times\overline{\Omega}) \\ v \in \cap_{q>n} C([0,\infty); W^{1,q}(\Omega)) \cap L^\infty((0,\infty); W^{1,\infty}(\Omega)) \cap C^{1,2}((0,\infty)\times\overline{\Omega}) \end{cases}$$

such that $v \ge 0$ *in* $[0,\infty)\times\overline{\Omega}$ *and* $0 \le u \le R$ *and that* (u, v) *solves (4.13) in the classical sense.*

Winkler (2017) studied the system (4.13) in bounded domain of $\mathbb{R}^n$, with smooth boundary. In his article, the self-diffusivity function D was allowed to decay exponentially. His work develops an alternative approach to obtain global existence of classical solutions for all reasonably regular initial data. His main result suggests a global and classical solution for (4.13) provided the ratio $\frac{S(u)}{D(u)} \leq Ke^{\alpha u}$ with $K > 0$ and $\alpha \in \mathbb{R}$.

4.4.2. Blow-up of Solutions of Nonlinear Diffusion and Cell Density Dependent Sensitivity Keller - Segel Model

(4.13) has solutions where the cell density u becomes unbounded in finite or infinite time. For the qualitative properties of solutions of (4.13), the behavior of the ratio $\frac{S(u)}{D(u)}$ at large value of cell density is very important.

The blow-up of solutions of (4.13) in finite or infinite time was studied in (Winkler, 2010). He stated that for sufficiently large u, $S(u)$ is identically zero, and he explored the extent to which solutions could prevent a blow-up. The main result is as follows:

Theorem 4.25 *(Winkler, 2010)*
Let $\Omega \subset \mathbb{R}^n$ be a ball with $n \geq 2$. Suppose

$$\frac{S(u)}{D(u)}$$

grows faster than $u^{\frac{2}{n}}$ as $u \to \infty$. Then the system (4.13) possesses unbounded solutions.

In the above theorem, it has been proved that if there exists $\epsilon > 0$ and $C > 0$ such that $\frac{S(u)}{D(u)} \geq Cu^{\frac{2}{n}+\epsilon}$ for all $u \geq 1$ then there is a smooth solution of (4.13) that blows up in finite or infinite time.

4.5. Quasilinear Keller - Segel Model

In this subsection we consider the following Keller - Segel equations:

$$\left.\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla(D(u,v)\nabla u) - \nabla(S(u,v)\nabla v), \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} &= \Delta v - v + u, \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial N} &= \frac{\partial v}{\partial N} = 0, \quad x \in \partial\Omega, t > 0, \\
u|_{t=0} &= u_0, \quad v|_{t=0} = v_0, \quad x \in \Omega.
\end{aligned}\right\} \tag{4.19}$$

where D is a nonnegative cell diffusion function and S is a nonnegative chemotactic sensitivity function.

In this model, the function $u(t,x)$ is the density of the particle at time t and position x; the function $v(t,x)$ describes the external signal density. This model involves a nonlinear diffusion function and signal density chemotactic sensitivity function. Here, the model, which includes nonlinear cell diffusion and chemotactic sensitivity function depending on the density of cells and density of a chemical substance, known as the quasilinear chemotaxis problem. This section reviews this system from the points of local and global existence, boundedness, and blow up in finite time or infinite time of solutions.

A system of this type emerges in the diverse complex chemotaxis aggregation behavior of cells. Most studies focus on detecting or excluding the occurrence of unlimited cell densities (Winkler, 2019). For example, the details of the classical Keller-Segel model obtained by choosing the diffusion function $D(u,v) = 1$ and the chemotactic sensitivity function $S(u,v) = u$ were given in section 4.1.

The importance of nonlinear chemosensitivity is emphasized in (Hillen and Painter, 2002). In this article, the authors examined two different systems of chemotaxis arising from biological facts. One of them, called the volume filling chemotaxis model, in terms of the structure of the equations, has the same structure as the system presented by Keller and Segel (1970). The volume filling chemotaxis model includes a finite size of individual cells. In this model, the probability of cells bouncing depends on the area where they are able to move. Another model is called the quorum sensing chemotaxis model. However, this model is not the main subject of this thesis since it possesses a different structure than the Keller - Segel model.

4.5.1. Existence of Solutions of Quasilinear Keller - Segel Model

The local existence result and extensibility are based on the use of Schauder fixed point theorem along with standard parabolic regularity theory (Winkler, 2019). Winkler assumed in (4.19) that

$$\begin{cases} D \in C^2([0,\infty)^2) \;\; satisfies \;\; D > 0 \;\; in \;\; [0,\infty)^2 \\ S \in C^2([0,\infty)^2) \;\; satisfies \;\; S \geq 0 \;\; and \;\; such \;\; that \;\; S(0,v) = 0 \;\; for \;\; all \;\; v \geq 0. \end{cases} \tag{4.20}$$

Regarding initial data, he assumed that

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \;\; with \;\; u_0 > 0 \;\; in \;\; \overline{\Omega} \\ v_0 \in W^{1,\infty}(\Omega) \;\; with \;\; u_0 > 0 \;\; in \;\; \overline{\Omega}. \end{cases} \tag{4.21}$$

Theorem 4.26 *(Winkler, 2019)*
Assume that D and S satisfy (4.20) and that u_0 and v_0 comply with (4.21). Then there exists $T_{max} \in (0,\infty]$ and at least one pair (u,v) of functions

$$\begin{cases} u \in C^0([0,T_{max}) \times \overline{\Omega}) \cap C^{1,2}([0,T_{max}) \times \overline{\Omega}) \\ v \in \cap_{q>n} C^0([0,T_{max}); W^{1,q}(\Omega)) \cap C^{1,2}([0,T_{max}) \times \overline{\Omega}) \end{cases}$$

such that both u and v are positive in $(0,\infty) \times \overline{\Omega}$ that (u,v) solves (4.19) classically in $(0,T_{max}) \times \overline{\Omega}$ and that if $T_{max} < \infty$ then $\limsup_{t\to T_{max}} \|u(t,\cdot)\|_{L^\infty(\Omega)} + \|v(t,\cdot)\|_{W^{1,q}(\Omega)} = \infty$ for all $q > n$.

In the above theorem, (u,v) denotes the corresponding local solution of (4.19). Now we will give the result for its maximal existence time T_{max}. To this end, we assume

$$D(u,v) \geq k_D(v)(u+1)^{m-1} \tag{4.22}$$

for all $u \geq 0$ and $v \geq 0$, where k_D is some nonincreasing positive function on $[0, \infty)$ and some $m \in \mathbb{R}$ and

$$S(u, v) \leq k_S \tag{4.23}$$

for all $u \geq 0$ and $v \geq 0$ with number $k_S > 0$ and

$$\frac{\partial S(u, v)}{\partial v} \geq -k_s u^{-\lambda}(v + 1)^{-\mu} \tag{4.24}$$

for all $u \geq 0$ and $v \geq 0$ with constants $k_s > 0$ and $\mu \geq 0$.

Theorem 4.27 *(Winkler, 2019)*
Assume $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and suppose that D and S satisfy (4.20) as well as (4.22), (4.23) and (4.24) with

$$\lambda > \frac{n-2}{n}(1 - \mu)_+.$$

Then for any choice of u_0 and v_0 satisfying (4.21), the initial boundary value problem (4.19) possesses a global classical solution (u, v) such that

$$\begin{cases} u \in C^0([0, T_{max}) \times \overline{\Omega}) \cap C^{1,2}([0, T_{max}) \times \overline{\Omega}) \\ v \in \cap_{q>n} C^0([0, T_{max}); W^{1,q}(\Omega)) \cap C^{1,2}([0, T_{max}) \times \overline{\Omega}) \end{cases} \tag{4.25}$$

and both u and v are positive in $(0, \infty) \times \overline{\Omega}$.

In the case $S \equiv S(u)$ is independent of v (4.24) is fulfilled. On the basis of this theorem, if the positive smooth function D decays at most algebraically concerning u then for any smooth nonnegative and bounded S, the system possesses a globally defined classical solution.

4.6. General Form of Keller - Segel Model

The general form of Keller - Segel type system is the following :

$$\left.\begin{array}{rcl} \frac{\partial u}{\partial t} &=& \Delta u - \nabla(k_1(u,v)u\nabla v) + k_2(u,v), \quad x \in \Omega, t > 0, \\ \\ \frac{\partial v}{\partial t} &=& \Delta v - v + h(u,v), \quad x \in \Omega, t > 0, \\ \\ \frac{\partial u}{\partial N} &=& \frac{\partial v}{\partial N} \quad = 0, \quad x \in \partial\Omega, t > 0, \\ \\ u|_{t=0} &=& u_0, \qquad v|_{t=0} = v_0, \quad x \in \Omega. \end{array}\right\} \tag{4.26}$$

Here, we assume that for some $\alpha \in (0,1)$, we have $k_1 \in C^{1+\alpha}_{loc}(\mathbb{R}^2)$, $k_2 \in C^{1-}(\mathbb{R}^2)$ and $h \in C^{1-}_{loc}(\mathbb{R}^2)$. Moreover $k_2(0,v) \geq 0$ for all $v \in [0,\infty)$ and $h(u,0) \geq 0$ for all $u \in [0,\infty)$.

For all quite regular initial data, a local in time classical solution exists.

Theorem 4.28 *(Bellomo et al., 2015)*
Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $q > n$. Then for all nonnegative $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ there exists $T_{max} \in [0,\infty)$ and a uniquely determined pair of nonnegative functions

$$\left\{\begin{array}{lcl} u &\in& C^0([0,T_{max}) \times \overline{\Omega}) \cap C^{1,2}((0,T_{max}) \times \overline{\Omega}) \\ v &\in& C^0([0,T_{max}) \times \overline{\Omega}) \cap C^{1,2}((0,T_{max}) \times \overline{\Omega}) \cap L^\infty_{loc}([0,T_{max}); W^{1,q}(\Omega)) \end{array}\right. \tag{4.27}$$

such that (u,v) solves (4.26) classically in $(0,T_{max}) \times \Omega$ and such that

$$if \;\; T_{max} < \infty \;\; then \;\; \|u(t,\cdot)\|_{L^\infty(\Omega)} + \|v(t,\cdot)\|_{W^{1,q}(\Omega)} \to \infty \;\; as \;\; t \to T_{max}. \tag{4.28}$$

Most literature on the global existence of the chemotaxis of the system focused on proving estimates as in (4.28). The following theorem offers an approach to a significant quantitative relationship of criterion (4.28). In order to do this, let us assume that k_1, k_2 and h satisfy the followings

$$|k_1(u,v)| \le c_1(u+1)^{\alpha} \ \ for\ all\ (u,v) \in (0,\infty)\times(0,\infty) \tag{4.29}$$

and

$$|k_2(u,v)| \le c_2 \ \ for\ all\ (u,v) \in (0,\infty)\times(0,\infty) \tag{4.30}$$

and

$$|h(u,v)| \le c_3(u+1)^{\beta} \ \ for\ all\ (u,v) \in (0,\infty)\times(0,\infty) \tag{4.31}$$

where c_1 , c_2 and c_3 are positive constants , $\alpha \in \mathbb{R}$ and $\beta \in [0,1]$.

The following theorem shows global existence of solutions of the system (4.26).

Theorem 4.29 *(Bellomo et al., 2015)*
Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and assume that k_1, k_2 and h satisfy (4.29), (4.30) and (4.31). Suppose also that for some $q > n$ there is a nonnegative classical solution (u,v) of (4.26) in $(0,T)\times\Omega$ with initial data $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$. If there exists $M > 0$ and $p \ge 1$ with

$$p > max\left\{\frac{n(\alpha+\beta)}{2}, \frac{nq\alpha}{q-n}\right\}$$

such that

$$\|u(t,\cdot)\|_{L^p(\Omega)} \le M for\ \ all\ \ t \in (0,T)$$

then one can find $c > 0$

$$\|u(t,\cdot)\|_{L^{\infty}(\Omega)} + \|v(t,\cdot)\|_{W^{1,q}(\Omega)} \le c\ for\ all\ t \in (0,T).$$

CHAPTER 5

NOTES ON REFERENCES

For a completely remarkable well-written account of the formulation and perspective of the Keller - Segel type system we refer to (Keller and Segel, 1970). See also (Hillen and Painter, 2009) for derivation of chemotaxis.

Good sources for the concepts in Chapter 2 are (Evans, 2010), (Folland, 1999) and (Pazy, 2012). The definitions of the solutions of the Keller - Segel type system can be found in (Stinner and Winkler, 2011) and (Arumugam and Tyagi , 2021). See also (Kreyzing, 1991) and (Kesavan, 1994) for functional analysis tools.

Earlier reviews of the Keller Segel type system can be found in (Horstmann, 2003) and (Arumugam and Tyagi , 2021). See also the survey (Bellomo et al., 2015).

CHAPTER 6

CONCLUSIONS

This thesis aimed to collect the qualitative results of some Keller - Segel type systems. This thesis focused on systems of the parabolic-parabolic type Keller - Segel model. After introducing the model, we gave a survey and qualitative analysis of the classical Keller - Segel type system and its variations of this model to derive from the classical model.

We focused on the qualitative properties of solutions of some Keller - Segel models which describe the interplay between cells and external signals produced by cells themselves. However, changing the environment can have an important effect on the movement of cells. Recent studies have addressed the interaction between the movement of cell populations and the environment according to two different external signaling mechanisms.

According to our research so far, open problems related to Keller - Segel type systems and their variations are as follows.

(1) Although the threshold for blow-up and global existence of solutions exists for the classical Keller - Segel model, the threshold analysis for nonclassical models has not been done.

(2) Boundedness of solutions for model (4.13) is an open problem when the function $D(u)$ decays at exponential rates and $S(u)$ grows slowly with respect to $D(u)$.

(3) For the classical Keller - Segel model (4.1) the extensibility of the non-radial solutions beyond the blow-up time is an open problem.

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