PQ-Calculus of Fibonacci Divisors and Method of Images in Planar Hydrodynamics

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Abstract

By introducing the hierarchy of Fibonacci divisors and corresponding quantum derivatives, we develop the golden calculus, hierarchy of golden binomials and related exponential functions, translation operator and infinite hierarchy of Golden analytic functions. The hierarchy of Golden periodic functions, appearing in this calculus we relate with the method of images in planar hydrodynamics for incompressible and irrotational flow in bounded domain. We show that the even hierarchy of these functions determine the flow in the annular domain, bounded by concentric circles with the ratio of radiuses in powers of the Golden ratio. As an example, complex potential and velocity field for the set of point vortices with Golden proportion of images are calculated explicitly.

Keywords:Fibonacci divisors, golden calculus, hydrodynamic images

1 Golden Ratio and Inversion in Circle

The usual definition of Golden proportion or the Golden ratio is related with division of interval $x+y$ in proportion

$$\frac{x+y}{x} = \frac{x}{y} \quad \Rightarrow \quad \varphi^2 = \varphi + 1,$$

were $x/y = \varphi = \frac{1+\sqrt{5}}{2} \approx 1.6$ - Golden Ratio. Here we propose new definition of Golden Ratio, connected with reflection in circle with radius R. Let a and b are symmetric points with respect to the circle at distance R between them, satisfying equations

$$a\, b = R^2, \quad b - a = R.$$

Then, distances to these points from origin are in Golden proportion of R,

$$a = \frac{1}{\varphi} R, \quad b = \varphi R.$$

As well known, symmetric points with respect to circle at origin in complex plain are z and $R^2/\bar{z}$. These points correspond to position of a vortex and its image in the circle, according to method of images in hydrodynamics. Then, due to above definition, if the distance between vortex and its image is R, then positions of vortex and the image are in Golden proportion. For unit circle with $R = 1$, these positions are $z = \varphi e^{i\theta}$ and $z^* = \frac{1}{\varphi} e^{i\theta}$.

If method of images is applied to problem with two circles, then an infinite set of images arises [1]. These images can be counted by q-periodic functions [2] and two circle theorem [3] in q-calculus with $q = r_2^2/r_1^2$. For annular domain with two concentric circles of radiuses r_1 and r_2, it can be reformulated in terms of PQ-calculus, with $P = r_1^2$ and $Q = r_2^2$. Then, the PQ number in this calculus

$$[n]_{PQ} = \frac{P^n - Q^n}{P - Q}$$

for $P = \varphi^k$ and $Q = \varphi'^k$ becomes Binet formula for Fibonacci divisors (1). This implies that calculus of Fibonacci divisors [4] can be applied to problem of hydrodynamic images in annular domain with two circles and the Golden ratio of images.

2 Calculus of Fibonacci divisors

The ratio of two Fibonacci numbers F_n/F_m is not in general integer number. However, surprising fact is that F_{kn}, where $k, n \in Z$ is dividable by F_k. The infinite sequence of integer numbers

$$\frac{F_{kn}}{F_k} \equiv F_n^{(k)}$$

we call Fibonacci divisors conjugate to F_k. The Binet formula for these numbers

$$F_n^{(k)} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k}, \tag{1}$$

leads to recursion relation

$$F_{n+1}^{(k)} = L_k F_n^{(k)} + (-1)^{k-1} F_{n-1}^{(k)},$$

where L_k are Lucas numbers. The first few sequences of Fibonacci divisors $F_n^{(k)}$ for $k = 1, 2, 3, 4, 5$ and $n = 1, 2, 3, 4, 5, ...$ are

$$\begin{aligned}
k &= 1; \ F_n^{(1)} = F_n = 1, 1, 2, 3, 5, ... \\
k &= 2; \ F_n^{(2)} = F_{2n} = 1, 3, 8, 21, 55, ... \\
k &= 3; \ F_n^{(3)} = \frac{1}{2} F_{3n} = 1, 4, 17, 72, 305, ... \\
k &= 4; \ F_n^{(4)} = \frac{1}{3} F_{4n} = 1, 7, 48, 329, 2255, ... \\
k &= 5; \ F_n^{(5)} = \frac{1}{5} F_{5n} = 1, 11, 122, 1353, 15005, ...
\end{aligned}$$

2.0.1 Golden Derivatives

The Golden derivative operator ${}_{(k)}D_F^x$ corresponding to Fibonacci divisors, conjugate to F_k, $k \in Z$ acts on arbitrary function $f(x)$ as

$${}_{(k)}D_F^x[f(x)] = \frac{f(\varphi^k x) - f(\varphi'^k x)}{(\varphi^k - \varphi'^k)\, x}. \tag{2}$$

For even k in the limit $k \to 0$ it gives usual derivative

$$\lim_{k \to 0} {}_{(k)}D_F^x f(x) = f'(x) \tag{3}$$

and

$${}_{(k)}D_F^x\, x^n = F_n^{(k)} x^{n-1}. \tag{4}$$

The Leibnitz and the quotient rules for this derivative are

$${}_{(k)}D_F^x(f(x)g(x)) = {}_{(k)}D_F^x(f(x))\, g(\varphi^k x) + f\left(\varphi'^k x\right)\, {}_{(k)}D_F^x(g(x)),$$

$${}_{(k)}D_F^x\left(\frac{f(x)}{g(x)}\right) = \frac{{}_{(k)}D_F^x(f(x))\, g(\varphi^k x) - f(\varphi^k x)\, {}_{(k)}D_F^x(g(x))}{g(\varphi^k x)\, g\left(\varphi'^k x\right)}.$$

2.0.2 Fibonacci Divisors and Fibonomials

The product of Fibonacci divisors,

$$F_1^{(k)} F_2^{(k)} \dots F_n^{(k)} = \prod_{i=1}^{n} F_i^{(k)} \equiv F_n^{(k)}! \tag{5}$$

- the Fibonacci divisors factorial, can be considered as $k-th$ Fibonorial or generalized Fibonorial. The Fibonomial coefficients for Fibonacci divisors are

$$_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_1^{(k)} F_2^{(k)} \dots F_{n-m+1}^{(k)}}{F_1^{(k)} F_2^{(k)} \dots F_m^{(k)}} = \frac{F_n^{(k)}!}{F_m^{(k)}! F_{n-m}^{(k)}!}.$$

2.0.3 Hierarchy of Golden Binomials

The $k-th$ Golden binomial is defined by polynomial,

$$_{(k)}(x-a)_F^n = \prod_{s=1}^{n} \left(x - \varphi^{k(n-s)} \varphi'^{k(s-1)} a \right)$$

with following factorization rule

$$\begin{aligned} _{(k)}(x-a)_F^{n+m} &= {}_{(k)}\left(x - \varphi^{km} a \right)_F^n {}_{(k)}\left(x - \varphi'^{kn} a \right)_F^m \\ &= {}_{(k)}\left(x - \varphi'^{km} a \right)_F^n {}_{(k)}\left(x - \varphi^{kn} a \right)_F^m . \end{aligned}$$

It can be expanded in powers of x:

$$_{(k)}(x+y)_F^n = \sum_{m=0}^{n} {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m} y^m.$$

The $k-th$ Golden derivative acts on this binomial as

$$\begin{aligned} _{(k)}D_F^x \; {}_{(k)}(x+y)_F^n &= F_n^{(k)} \; {}_{(k)}(x+y)_F^{n-1}, \\ _{(k)}D_F^y \; {}_{(k)}(x+y)_F^n &= F_n^{(k)} \; {}_{(k)}\left(x + (-1)^k y \right)_F^{n-1}, \\ _{(k)}D_F^y \; {}_{(k)}(x-y)_F^n &= -F_n^{(k)} \; {}_{(k)}\left(x - (-1)^k y \right)_F^{n-1}. \end{aligned}$$

2.0.4 Hierarchy of Golden functions

Let, entire complex valued function of complex variable z is

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}. \tag{6}$$

Then, for any integer k exists entire complex function

$${}_{(k)}\ f_F(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{F_n^{(k)}!}. \tag{7}$$

2.0.5 Hierarchy of Golden exponential functions

We introduce entire exponential functions

$$\begin{aligned} {}_{(k)}e_F^x &\equiv \sum_{n=0}^{\infty} \frac{x^n}{F_n^{(k)}!}, \\ {}_{(k)}E_F^x &\equiv \sum_{n=0}^{\infty} (-1)^{k\frac{n(n-1)}{2}} \frac{x^n}{F_n^{(k)}!}. \end{aligned}$$

The $k-th$ Golden derivative acts on these functions as

$$\begin{aligned} {}_{(k)}D_F^x \left({}_{(k)}e_F^{\lambda x}\right) &= \lambda\ {}_{(k)}e_F^{\lambda x}, \\ {}_{(k)}D_F^x \left({}_{(k)}E_F^{\lambda x}\right) &= \lambda\ {}_{(k)}E_F^{(-1)^k \lambda x}. \end{aligned}$$

Two exponential functions are related by formula

$${}_{(k)}E_F^x =_{(-k)} e_F^x. \tag{8}$$

The product of the exponentials is represented by series in powers of $k-$th Golden binomial

$${}_{(k)}E_F^x \cdot_{(k)} e_F^y = \sum_{n=0}^{\infty} \frac{{}_{(k)}(x+y)_F^n}{F_n^{(k)}!} \equiv_{(k)} e_F^{{}_{(k)}(x+y)_F}. \tag{9}$$

2.0.6 Translation operator

${}_{(k)}E_F^{y_{(k)}D_F^x}$ generates these binomials and $k-$th Golden functions

$${}_{(k)}E_F^{y_{(k)}D_F^x} x^n =_{(k)} (x+y)_F^n, \tag{10}$$

$${}_{(k)}E_F^{y_{(k)}D_F^x} f(x) =_{(k)} E_F^{y_{(k)}D_F^x} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \cdot {}_{(k)}(x+y)_F^n. \tag{11}$$

2.0.7 Hierarchy of Golden Analytic Functions

By translation operator we introduce complex valued $k-$th Golden analytic binomials

$$ {}_{(k)}E_F^{iy_{(k)}D_F^x} x^n =_{(k)} (x+iy)_F^n \tag{12}$$

and the hierarchy of $k-$th Golden analytic functions

$$ {}_{(k)}E_F^{iy_{(k)}D_F^x} f(x) = \sum_{n=0}^{\infty} a_n \cdot {}_{(k)}(x+iy)_F^n \equiv f\left({}_{(k)}(x+iy)_F\right), \tag{13}$$

for every integer k satisfying the $\bar{\partial}$-equation

$$ \frac{1}{2}\left({}_{(k)}D_F^x + i_{(-k)}D_F^y\right) f\left({}_{(k)}(x+iy)_F\right) = 0. \tag{14}$$

For the real and imaginary parts of these functions

$$ u(x,y) =_{(-k)} \cos_F\left(y_{(k)}D_F^x\right) f(x), \quad v(x,y) =_{(-k)} \sin_F\left(y_{(k)}D_F^x\right) f(x), \tag{15}$$

we have the Cauchy-Riemann equations

$$ {}_{(k)}D_F^x u(x,y) =_{(-k)} D_F^y v(x,y), \quad {}_{(-k)}D_F^y u(x,y) = -{}_{(k)}D_F^x v(x,y), \tag{16}$$

and functions are solutions of the hierarchy of Golden Laplace equations

$$ \left({}_{(k)}D_F^x\right)^2 \phi(x,y) + \left({}_{(-k)}D_F^y\right)^2 \phi(x,y) = 0. \tag{17}$$

2.0.8 Golden periodic functions

The set of Golden derivatives, determines hierarchy of Golden periodic functions for every natural k. If function $f(x)$ is Golden periodic ($k=1$),

$$ D_F^x(f(x)) = 0 \Longleftrightarrow f(\varphi x) = f(\varphi' x), \tag{18}$$

then, it is also periodic for arbitrary $k-th$ order Golden derivative,

$$\begin{array}{ccc} D_F^x(f(x)) = 0 & \Rightarrow & {}_{(k)}D_F^x(f(x)) = 0, \\ f(\varphi x) = f(\varphi' x) & \Rightarrow & f(\varphi^k x) = f(\varphi'^k x), \end{array}$$

for $k = 2, 3, \ldots$ But the opposite is not in general true. Indeed, function

$$ f(x) = \sin\left(\frac{\pi}{\ln \varphi^2} \ln|x|\right)$$

is Golden periodic function with $k=2$, but it is not Golden periodic, since

$$ D_F^x(f(x)) = 2\,\frac{\cos\left(\frac{\pi}{\ln(\varphi^2)}\ln|x|\right)}{(\varphi-\varphi')x} \neq 0. \tag{19}$$

3 Hydrodynamic Images and Golden Periodic Functions

3.1 Two dimensional stationary flow

We consider incompressible and irrotational planar flow,

$$div\,\vec{u} = 0 \Rightarrow u_1 = \frac{\partial\psi}{\partial y}, \quad u_2 = -\frac{\partial\psi}{\partial x}, \tag{20}$$

$$rot\,\vec{u} = 0 \Rightarrow u_1 = \frac{\partial\varphi}{\partial x}, \quad u_2 = \frac{\partial\varphi}{\partial y}, \tag{21}$$

where real functions $\varphi(x,y)$ and $\psi(x,y)$ are velocity potential and stream function, correspondingly. These functions are harmonically conjugate and satisfy Cauchy-Riemann equations,

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}.$$

Combined together, they determine complex potential $f(z) = \varphi + i\psi$, as analytic function $\frac{\partial}{\partial\bar{z}}f(z) = 0$, of $z = x + iy$. Corresponding complex velocity $\bar{V}(z) = \frac{\partial}{\partial z}f(z)$ is anti-analytic function of z.

For hydrodynamic flow in bounded domain, the problem is for given C-the boundary curve, find analytic function (complex potential) $F(z)$, with boundary condition

$$\Im F|_C = \psi|_C = 0.$$

This equation determines the stream lines of the flow, such that normal velocity to the curve $v_n|_C = 0$.

3.1.1 Two Circle Theorem

Applying two circles theorem [3] for flow $f(z)$, restricted to annular domain: $1 < |z| < \sqrt{\varphi}$ between two concentric circles $C_1 : |z| = 1$, $C_2 : |z| = \sqrt{\varphi}$, we get complex potential

$$F_\varphi(z) = f_\varphi(z) + \bar{f}_\varphi\left(\frac{1}{z}\right),$$

where $q = \frac{r_2^2}{r_1^2} = \varphi$, flow in even annulus -

$$f_\varphi(z) = \sum_{n=-\infty}^{\infty} f(\varphi^n z),$$

and flow in odd annulus -

$$\bar{f}_\varphi\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \bar{f}\left(\varphi^n \frac{1}{z}\right).$$

3.1.2 Golden φ-periodicity of flow

The Golden periodicity

$$f_\varphi(\varphi z) = f_\varphi(z) \Rightarrow F_\varphi(\varphi z) = F_\varphi(z)$$

implies that complex potential of the flow is invariant under Golden Ratio rescaling and as follows it is Golden φ-periodic function,

$$D_z f_\varphi(z) = \frac{f(\varphi z) - f(z)}{(\varphi - 1)z} = 0.$$

Corresponding complex velocity

$$\bar{V}(z) = \sum_{n=-\infty}^{\infty} \varphi^n \bar{v}(\varphi^n z) - \frac{1}{z^2} \sum_{n=-\infty}^{\infty} \varphi^n v\left(\varphi^n \frac{1}{z}\right)$$

is Golden φ-scale invariant function

$$\bar{V}(\varphi z) = \varphi^{-1} \bar{V}(z).$$

3.1.3 Golden φ scale-invariant analytic fractal

For scale invariant function $f(\varphi z) = \varphi^d f(z) \rightarrow$

$$D_z f(z) = \frac{f(\varphi z) - f(z)}{(\varphi - 1)z} = \frac{(\varphi^d - 1)}{(\varphi - 1)z} f(z),$$

and the φ-difference equation is

$$z D_z f(z) = [d]_\varphi f(z).$$

Solution of this equation can be represented as

$$f(z) = z^d A_\varphi(z),$$

where

$$A_\varphi(\varphi z) = A_\varphi(z)$$

is arbitrary φ-periodic function, playing the role of φ-periodic modulation.

3.1.4 Golden Weierstrass-Mandelbrot fractal

As an example we consider

$$W(t) = \sum_{n=-\infty}^{\infty} \frac{1 - \cos \varphi^n t}{\varphi^{nd}}, \quad 0 < d < 1,\, \varphi > 1,$$

- continuous but nowhere differentiable function, representing fractal with dimension $2 - d$. It is Golden self-similar function

$$W(\varphi t) = \varphi^d W(t),$$

satisfying φ-difference equation

$$t D_t W(t) = [d]_\varphi W(t).$$

By decomposing it as

$$W(t) = t^d A_\varphi(t)$$

we extract the Golden φ-scale periodic part $A_\varphi(\varphi t) = A_\varphi(t)$, where

$$A_\varphi(t) = \sum_{n=-\infty}^{\infty} \frac{1 - e^{i\varphi^n t}}{(\varphi^d)^n t^d},$$

or in terms of Fibonacci numbers

$$A_\varphi(t) = \sum_{n=-\infty}^{\infty} \frac{1 - \cos(\varphi F_n + F_{n-1})t - i \sin(\varphi F_n + F_{n-1})t}{(\varphi^d)^n t^d}.$$

3.1.5 Elliptic Function Form

Let complex potential is Golden periodic analytic function $F(\varphi z) = F(z)$. The Golden ratio can be represented

$$\varphi = e^{2\pi i \frac{\omega'}{\omega}}$$

by arbitrary real ω and pure imaginary $\omega' = -i\frac{\omega}{2\pi} \ln \varphi$. Function

$$F(z) \equiv \Phi\left(\frac{\omega}{i\pi} \ln z\right) = \Phi(u)$$

is double periodic function

$$\Phi(u + 2\omega') = \Phi(u), \quad \Phi(u + 2\omega) = \Phi(u).$$

It is elliptic function on Golden torus, determined by its singular points.

3.1.6 Golden φ periodic flow

Simplest example of Golden φ periodic function (as principal branch) is

$$F(z) = z^{\frac{2\pi i}{\ln \varphi}} = e^{\frac{2\pi i}{\ln \varphi} \ln z} = F(\varphi z).$$

Rewritten in polar coordinates $z = re^{i\theta}$,

$$F(z) = e^{-\frac{2\pi}{\ln \varphi}\theta} \left(\cos(2\pi \log_\varphi r) + i \sin(2\pi \log_\varphi r)\right)$$

it gives stream function

$$\psi(r, \theta) = e^{-\frac{2\pi}{\ln \varphi}\theta} \sin(2\pi \log_\varphi r)$$

and complex velocity

$$\bar{V}(z) = \frac{dF}{dz} = \frac{2\pi i}{\ln \varphi} \frac{1}{z} e^{\frac{2\pi i}{\ln \varphi} \log_\varphi z} = \frac{\Gamma}{2\pi i} \frac{1}{z} A_\varphi(z).$$

In the last form it represents Golden modulated point vortex at origin with strength $\Gamma = -\frac{4\pi^2}{\ln \varphi}$ and stream lines $\psi|_C = 0$ at $\sin(2\pi \log_\varphi r) = 0$ or $2\pi \log_\varphi r = \pi n$, $n = 0, \pm 1, \pm 2, \ldots$. These lines represent an infinite set of concentric circles with radiuses

$$r_n = \varphi^{\frac{n}{2}}.$$

The ratio of two successive radiuses is the Golden Ratio

$$q = \frac{r_{n+1}^2}{r_n^2} = \varphi.$$

For the flow in Golden annulus $r_0 = 1$ and $r_1 = \sqrt{\varphi}$ we have $D_\varphi F(z) = 0$, and in k-th Golden annulus $r_0 = 1$ and $r_k = \varphi^{\frac{k}{2}}$ it gives $D_{\varphi^k} F(z) = 0$.

Superposition

$$F_k(z) = \sum_{N=-\infty}^{+\infty} a_N z^{\frac{2\pi i}{k \ln \varphi} N}$$

describes flow in circular annulus with radius $r = 1$ and $R = \varphi^{\frac{k}{2}}$, so that

$$F_k(\varphi^k z) = F_k(z),$$

and the flow is φ^k- periodic.

3.2 Vortex in Golden annular domain

For point vortex at position z_0 in Golden annular domain, $1 < |z_0| < \varphi^{\frac{k}{2}}$, by Two Circle Theorem

$$F_k(z) = \frac{\Gamma}{2\pi i} \sum_{n=-\infty}^{\infty} \ln \frac{z - z_0 \varphi^{kn}}{z - \frac{1}{\bar{z}_0} \varphi^{kn}}$$

and

$$\bar{V}(z) = \frac{\Gamma}{2\pi i} \sum_{n=-\infty}^{\infty} \left[\frac{1}{z - z_0 \varphi^{kn}} - \frac{1}{z - \frac{1}{\bar{z}_0} \varphi^{kn}} \right].$$

The flow is Golden φ^k periodic

$$F_k(\varphi^k z) = F_k(z),$$

with self-similar complex velocity

$$\bar{V}_k(\varphi^k z) = \frac{1}{\varphi^k} \bar{V}(z).$$

It represents modulation of point vortex by Golden periodic function

$$\bar{V}(z) = \frac{\Gamma}{2\pi i\, z} A_k(z).$$

3.2.1 Golden Ratio of pole singularities

Pole singularities are located at positions

$$z_n = z_0 \varphi^{kn}$$

and at symmetric points

$$z_n^* = \frac{1}{\bar{z}_0} \varphi^{kn},$$

where $n = 0, \pm 1, \pm 2, ... \pm \infty$. The ratio of two image positions is power of Golden ratio

$$\frac{|z_{n+1}|}{|z_n|} = \varphi^k.$$

The distance between symmetric points is growing in geometric progression

$$|z_n - z_n^*| = |z_0 - z_0^*| \left(\varphi^k \right)^n.$$

3.2.2 Hierarchy of Golden Logarithmic Functions

The set of vortex images is determined completely by singularities of the φ-Logarithmic function,

$$Ln_\varphi(1-z) \equiv -\sum_{n=1}^{\infty} \frac{z^n}{[n]_\varphi}.$$

It converges for $|z| < \varphi$, where φ - number

$$[n]_\varphi \equiv 1 + \varphi + \varphi^2 + ... + \varphi^{n-1} = \frac{\varphi^n - 1}{\varphi - 1}$$

expressed by Fibonacci numbers is $[n]_\varphi = (F_{n+1} - 1)\varphi + F_n$. More general function, φ^k-logarithm $(0 < |z| < \varphi^k)$:

$$Ln_{\varphi^k}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{[n]_{\varphi^k}} = \frac{1}{\varphi^k} \sum_{n=1}^{\infty} \frac{z}{\varphi^{kn} + z}$$

is expressible by Fibonacci divisors $\varphi^{kn} = \varphi^k F_n^{(k)} + (-1)^{k+1} F_{n-1}^{(k)}$ and

$$[n]_{\varphi^k} = \frac{\varphi^k F_n^{(k)} + (-1)^{k+1} F_{n-1}^{(k)} - 1}{\varphi^k - 1}.$$

It has an infinite number of simple pole singularities at $z = -\varphi^{kn}$.

The logarithm function is related to entire exponential functions

$$e_\varphi(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_\varphi!}, \quad E_\varphi(z) = \sum_{n=0}^{\infty} \varphi^{n(n-1)/2} \frac{z^n}{[n]_\varphi!},$$

which by Euler identities for φ-binomial can be written as infinite product

$$e_\varphi\left(z\right) = E_{\frac{1}{\varphi}}\left(z\right) = \prod_{n=0}^{\infty} \left(1 + \frac{z}{\varphi^{n+2}}\right).$$

Zeroes of φ - exp function

$$\varphi Ln_\varphi(1-\alpha z) = z\frac{d}{dz} \ln e_\varphi(-\varphi\alpha z)$$

contribute to complex potential

$$F(z) = \sum_{s=1}^{N} i\kappa_s \ln(z - z_s) + \sum_{s=1}^{N} i\kappa_s \ln \frac{e_\varphi\left(-\varphi\frac{z}{z_s}\right) e_\varphi\left(-\varphi\frac{z_s}{z}\right)}{e_\varphi\left(-\varphi z\bar{z}_s\right) e_\varphi\left(-\frac{\varphi^2}{z\bar{z}_s}\right)},$$

so that all images in the second sum are determined by zeros of these functions. Then, complex velocity is expressible as

$$\bar{V}(z) = \sum_{s=1}^{N} \frac{i\kappa_s}{z - z_s} +$$

$$\frac{i\varphi}{z} \sum_{k=s}^{N} \kappa_s \left(Ln_\varphi \left(1 - \frac{z}{z_s}\right) - Ln_\varphi \left(1 - z\bar{z}_s\right) + Ln_\varphi \left(1 - \frac{\varphi}{z\bar{z}_s}\right) - Ln_\varphi \left(1 - \frac{z_s}{z}\right)\right)$$

3.3 Hydrodynamic Images and k-th Golden Derivatives

For even $k = 2l$, the Fibonacci divisor derivative is determined by finite difference

$$z\,{}_{(k)}D_F^z[f(z)] = \frac{f(\varphi^k z) - f(\frac{1}{\varphi^k} z)}{\left(\varphi^k - \frac{1}{\varphi^k}\right)},$$

vanishing for Golden periodic function

$$_{(k)}D_F^z F(z) = 0. \tag{22}$$

In annular domain, bounded by circles $1 < |z| < \varphi^{\frac{k}{2}}$ the flow is k-th Golden periodic $F_k(\varphi^k z) = F_k(z)$, so that

$$_{(k)}D_F^z F_k(z) = 0. \tag{23}$$

3.4 Single Vortex Motion

For single vortex motion, subject to equation

$$\dot{z}_0 = \varphi \frac{i\kappa}{\bar{z}_0} \left[Ln_\varphi \left(1 - |z_0|^2\right) - Ln_\varphi \left(1 - \frac{\varphi}{|z_0|^2}\right)\right],$$

the solution is described by uniform rotation $z_0(t) = z_0(0)e^{i\omega t}$, with angular velocity

$$\omega = \frac{\varphi\kappa}{|z_0|^2} \left(Ln_\varphi \left(1 - |z_0|^2\right) - Ln_\varphi \left(1 - \frac{\varphi}{|z_0|^2}\right)\right).$$

The vortex is stationary $\omega = 0$ at geometric mean distance $|z_0| = \varphi^{\frac{1}{4}}$ and ratio of frequencies at boundary circles is the Golden ratio

$$\frac{|\omega_1|}{|\omega_2|} = \varphi.$$

3.4.1 Semiclassical quantization of vortex motion

The Bohr-Zommerfeld quantization of single vortex motion gives discrete spectrum

$$E_n = \frac{\Gamma^2}{4\pi} \ln \left| e_\varphi \left(-\varphi \left(n + \frac{1}{2} \right) \right) e_\varphi \left(\frac{-\varphi^2}{(n + \frac{1}{2})} \right) \right|.$$

This expression never vanishes, since zeros of exponential functions in r.h.s. should satisfy following equations, $n + \frac{1}{2} = \varphi^{k+1}$ or $n + \frac{1}{2} = \varphi^{-k}$. But in both equations the l.h.s is rational number, while the r.h.s. is irrational.

3.5 N vortex dynamics

For N - point vortices with circulations Γ_1,...,Γ_N, at positions $z_1, ..., z_N$, equations of motion are

$$\dot{\bar{z}}_n = \frac{1}{2\pi i} \sum_{j=1(j\neq n)}^{N} \frac{\Gamma_j}{z_n - z_j} + \frac{1}{2\pi i} \sum_{j=1}^{N} \sum_{n=\pm 1}^{\pm\infty} \frac{\Gamma_j}{z_n - z_j \varphi^n} - \frac{1}{2\pi i} \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} \frac{\Gamma_j}{z_n - \frac{1}{\bar{z}_n} \varphi^n}.$$

This is Hamiltonian system with Hamiltonian function

$$H = -\frac{1}{4\pi} \sum_{i,j=1(i\neq j)}^{N} \Gamma_i \Gamma_j \ln |z_i - z_j| - \frac{1}{4\pi} \sum_{i,j=1}^{N} \Gamma_i \Gamma_j \ln \left| \frac{e_\varphi \left(-\varphi \frac{z_i}{z_j} \right) e_\varphi \left(-\varphi \frac{z_j}{z_i} \right)}{e_\varphi \left(-\varphi z_i \bar{z}_j \right) e_\varphi \left(-\frac{\varphi^2}{z_i \bar{z}_j} \right)} \right|,$$

where the second sum describes an infinite set of images with Golden proportion of positions. The Green function of the problem

$$G_I = -\frac{1}{2\pi} \ln |z - z_l| - \frac{1}{2\pi} \ln \left| \frac{e_\varphi \left(-\varphi \frac{z}{z_l} \right) e_\varphi \left(-\varphi \frac{z_l}{z} \right)}{e_\varphi \left(-\varphi z \bar{z}_l \right) e_\varphi \left(-\frac{\varphi^2}{z \bar{z}_l} \right)} \right| + \frac{1}{4\pi} \ln \varphi$$

satisfies following conditions:1. symmetry $G_I(z, z_l) = G_I(z_l, z)$; 2. boundary values, $G_I(z, z_l)|_{C_2} = 0$ - at the outer circle, $G_I(z, z_l)|_{C_1} = \frac{1}{2\pi} \ln \left| \frac{\sqrt{\varphi}}{z_l} \right|$ - at the inner circle.

Exact solution for N identical vortices $\Gamma_l = \Gamma, l = 1, ..., N$, located at the same distance $1 < r < \sqrt{\varphi}$ is

$$z_l(t) = r e^{i\omega t + i \frac{2\pi}{N} l},$$

where rotation frequency

$$\omega = \frac{\Gamma}{2\pi r^2}\left(\frac{N-1}{2} + \varphi\sum_{j=1}^{N}\left[Ln_{\varphi}\left(1-\frac{\varphi}{r^2}\,e^{i\frac{2\pi}{N}j}\right) - Ln_{\varphi}\left(1-r^2\,e^{-i\frac{2\pi}{N}j}\right)\right]\right).$$

At geometrical mean distance $r = \varphi^{1/4}$ the frequency is

$$\omega = \frac{\Gamma(N-1)}{4\pi\sqrt{\varphi}}.$$

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