

# Existence and Uniqueness of Solution for Discontinuous Conewise Linear Systems

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**Abstract:** In this study, we give necessary and sufficient conditions for well posedness of Conewise Linear Systems in 3-dimensional space where the vector field is allowed to be discontinuous. The conditions are stated using the subspaces derived from subsystem matrices and the results are compared with the existing conditions given in the literature. We show that even we don't have a fixed structure on system matrices as in bimodal systems, similar subspaces and numbers again determines well posedness.

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## 1. INTRODUCTION

Conewise linear systems (CLS) are simply piecewise linear systems where the entire space is divided into convex cones. Partition of the space can be done either using bounding matrices  $C_i$  or via the vectors  $\{v_1, v_2, \dots, v_l\}$ , as shown by Farkas-Minkowski-Weyl Theorem – (Schrijver A., 1986). Each subsystem of CLS is a linear time invariant (LTI) system and the trajectory moves in the cone  $\chi_i$  as  $C_i x \geq 0$ . As the transition between the modes depends on the the states, CLS can also be regarded as a state depended switched system or alternatively linear hybrid automata, (Lygeros et al., 2003), (Shen et al., 2009). Although all the submodes are linear, behaviour of CLS is quite complex. Existence and uniqueness of the solution, in other words well posedness (WP), on the bounds of subsystems of discontinuous CLS has been considered by many authors, (Imura and Van der Schaft, 2000), (Xia, 2002), (Thuan and Çamlıbel, 2014), (Şahan and Eldem, 2015 and 2019) and (Ozguler and Zakwan, 2019). A detailed investigation for WP of discontinuous dynamical systems see also (Georgescu et al, 2012). The stability of CLS is also considered by (Pachter and Jacobson, 1981), (Araposthatis and Broucke, 2007), (Shen et al., 2009), (Zhendong and Shuzhi, 2011), (Eldem and Şahan, 2014, 2016) and (Eldem and Oner, 2015). However, obtained results are generally valid for some special subclasses and thus, the issue of WP is still being investigated by many researchers.

For the solution of WP problem, there exist some special solution structures in the literature such as Caratheodory, Filippov, Krasovski, Euler, (Cortes, 2008), (Filippov, 1998). In this work only Caratheodory Solutions (CS) are considered. WP is resolved in planar case by introducing the “flow continuation condition” (Pachter and Jacobson, 1981). On the other hand, (Imura and van der Schaft, 2000) and (Xia, 2002) used CS and smooth continuation sets  $S_i$ 's for WP of bimodal and multimodal CLS in  $\mathbb{R}^n$ . The structure of the system matrices of a well posed CLS is given for planar and multimodal case by (Şahan and Eldem, 2019) and for bimodal case in  $\mathbb{R}^n$  by (Şahan and Eldem, 2015), respectively. Submodes are classified as transitive and nontransitive for a

class of CLS by (Ozguler and Zakwan, 2019). Roughly speaking, if all the trajectory starting from a cone changes mode then it is transitive. Categorization of trajectories in this way is closely related with WP and can be regarded as a first step to explore stability.

In this work, we consider a multimodal CLS in  $\mathbb{R}^3$  and give necessary and sufficient conditions for WP. We state how the conditions given in (Imura and van der Schaft, 2000) and (Xia, 2002) are possible. We also relate our results to system matrices and so generalize (Şahan and Eldem, 2015, 2019) to 3-dimensional space and multimodal cases respectively.

**Nomenclature:** Throughout this work, we use  $\mathbb{R}^n$  for n-dimensional real vector space,  $\chi_i^0$  for the interior of a cone,  $I_n$  for  $n \times n$  identity matrix,  $sgn(k)$  to define the sign of  $k \in \mathbb{R}$ ,  $|A|$  for the cardinality of the set  $A$ .

## 2. PRELIMINARIES

**Definition 1 :** [Zhendong S., Shuzhi S.G., 2011] A CLS is a differential or difference equation

$$\dot{x} = f(x) = A_i x, \quad x \in \chi_i \quad (1)$$

where  $A_1, A_2, \dots, A_l$  are real  $n \times n$  constant matrices, and  $\{\chi_1, \chi_2, \dots, \chi_l\}$  is a set of convex polyhedral cones with  $\bigcup_{i=1}^l \chi_i = \mathbb{R}^n$  and  $\chi_i^0 \cap \chi_j^0 = \emptyset$  for  $i \neq j$ .

Using the partition above, the family of the polyhedral cones forms a conic subdivision for  $\mathbb{R}^n$ . The polyhedral cones and bounding matrices can be defined precisely as follows, (Scholtes, 2012), (Shen, 2010).

$$\chi_i = \{x : C_i x \geq 0\}, \quad i = 1, 2, \dots, l \text{ and } C_i \in \mathbb{R}^{l_i \times n}, \quad (2)$$

$$C_i = \begin{bmatrix} n_{i1} \\ \vdots \\ n_{il_i} \end{bmatrix}, \quad n_{ij} \in \mathbb{R}^{1 \times n}, \quad 1 \leq j \leq l_i, \quad C_i \in \mathbb{R}^{l_i \times n}. \quad (3)$$

Conventionally, it is assumed that  $l_i$  is minimum, in other words, the description is not redundant.

The borders of the polyhedral cones for 2-dimensional spaces are just two lines. However, the borders of a cone in  $\mathbb{R}^3$  are planes and lines passing through the origin. Separators

of polyhedral cones may be more than two and may have more different formations. This is a well known and a widely studied issue in Geometric Combinatorics, (Scholtes, 2012), (Schrijver, 1986) and (Ziegler, 1998).

We assume for a uniform partition of CLS that the system is memoryless, (Iwatani Y. and Hara S., 2006). So the interior of each pairwise intersection is empty, i.e.  $\chi_i^0 \cap \chi_k = \emptyset$  for  $i \neq k$  and that  $\chi_1 \cup \chi_2 \cup \dots \cup \chi_l = \mathbb{R}^n$ . Since the existence and uniqueness of the solutions are to be considered especially at the boundaries of the cones, we first define and classify the borders formally and give more detailed definitions about geometric structure of the polyhedra.

**Definition 2:** (Schrijver, 1986), (Bremner et al. 2009)

Consider the CLS defined by equations (1)-(3) in  $\mathbb{R}^3$  and its conic subdivisions.

- $\mathcal{F}_{ij}^k$  is a  $k$ -dimensional **face** of  $\chi_i$  if and only if  $\mathcal{F}_{ij}^k := \{x \in \chi_i | n_{ij}x = 0\} \subset \text{Ker}(n_{ij})$ ,  $k = 1, 2$ . (4)
- A **facet** of  $\chi_i$  is a maximal face of dimension  $k = 2$ . We do not mark dimension if it is a facet.
- Faces of dimension 0 are called **vertices** and faces of dimension 1 are called **edges (or extreme rays)**.

Two modes are said to be **adjacent** if they share a common face. We also define  $\ell_S(X)$  as the set of the indices which is active on the set  $X \subset \mathcal{F}_{ij}$ , (Shen, 2010).

A trajectory may start on a face and stay on it. In order to eliminate these kind of trajectories, i.e. sliding modes, we assume throughout paper that “**All the  $(n_{ij}, A_i)$  pairs are observable**”. So all the trajectories that we consider are crossing-boundary or invariant types. That is, when a trajectory touches a face either it changes mode or turn back its own mode. Otherwise, the system may stay infinitely long in a single co-dimension-one part of the boundary between two modes.

**Example 1:** Consider 8 octants of  $\mathbb{R}^3$ . Each octant is bounded with 3 facets which lie in  $\text{Ker}(e_i)$  planes for standard basis  $\{e_i\}$  in  $\mathbb{R}^3$ . So we have  $\mathcal{F}_{11}, \mathcal{F}_{12}$  and  $\mathcal{F}_{13}$  facets for the 1st octant for  $C_1 = I_3$ . Also edges for each octant, i.e.  $\mathcal{F}_{ij}^1$ 's, are half of coordinate axis  $x, y$  and  $z$  and origin is the only vertice. Thus, for a CLS in  $\mathbb{R}^3$ ,  $\mathcal{F}_{ij}$ 's (facets) and  $\mathcal{F}_{ij}^1$ 's (edges) are a portion of a plane and a line through origin.

In order to explain the problems which may arise for a CLS in  $\mathbb{R}^3$ , let us consider the simplest case, which is Bimodal System, (Şahan and Eldem 2015), (Eldem and Şahan, 2016).

**Remark 1:** Consider the bimodal CLS

$$\dot{x} = \begin{cases} A_1x, & \text{if } c^T x \geq 0 \\ A_2x, & \text{if } c^T x \leq 0 \end{cases}, A_1, A_2 \in \mathbb{R}^{3 \times 3}, c^T = [0 \ 0 \ 1]. \quad (5)$$

We only have one face  $\mathcal{F}_{11} = \text{Ker}(c^T)$ . Note that  $\text{Ker}(c^T) \cap \text{Ker}(c^T A_1)$  and  $\text{Ker}(c^T) \cap \text{Ker}(c^T A_2)$  are

separating lines for 1<sup>th</sup> order derivatives. For WP, these lines must coincide on a line  $\mathcal{L}$ . If these lines do not coincide, they may induce some ill posed (IP) regions on the face  $\mathcal{F}_{11} = \mathcal{F}_{21} = \text{Ker}(c^T)$  (Fig (1)).

The first order derivatives are equal to zero for the initial conditions on the line  $\mathcal{L}$ . It also separates the trajectories starting from  $\text{Ker}(c^T)$  that have positive ( $\mathcal{P}^+$ ) and negative ( $\mathcal{P}^-$ ) first order derivatives at  $t = 0$ , i.e.  $c^T A_i x_0 > 0$  for  $x_0 \in \mathcal{P}^+$  and  $c^T A_i x_0 < 0$  for  $x_0 \in \mathcal{P}^-$ .

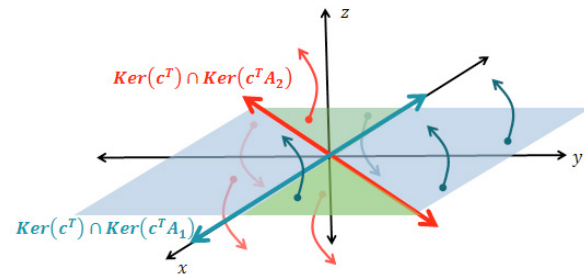


Fig.1

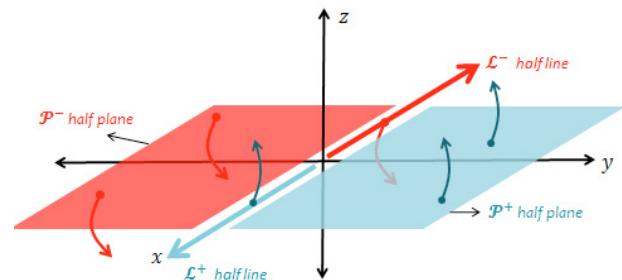


Fig.2

In addition, since the first order derivatives are zero for  $x_0 \in \mathcal{L}$ , we need to check second order derivatives. For one part of the line  $\mathcal{L}$ , the second order derivatives are positive ( $\mathcal{L}^+$ ), for one part they are negative ( $\mathcal{L}^-$ ), i.e.  $c^T A_i^2 x_0 > 0$  for  $x_0 \in \mathcal{L}^+$  and  $c^T A_i^2 x_0 < 0$  for  $x_0 \in \mathcal{L}^-$ . So the smooth continuation sets for each mode are disjoint, (Fig (2)) and their union gives  $\mathbb{R}^3$  which is equivalent to WP conditions given by (Imura and van der Schaft, 2000), (Imura, 2002).

**Definition 3:** Consider mode  $i$  and its  $j^{\text{th}}$  face  $\mathcal{F}_{ij}$ . We define

$$\mathcal{L}_{ij} := \text{ker}(n_{ij}) \cap \text{ker}(n_{ij} A_i). \quad (6)$$

Note that different  $\mathcal{L}_{ij}$  spaces (lines through origin) can be defined for different active modes and corresponding  $A_k$ 's on  $\mathcal{F}_{ij}$ . We'll simply use the terminology “ $\mathcal{L}_{ij}$  spaces on  $\mathcal{F}_{ij}$ ” for them.

**Definition 4:** Consider mode  $i$  and its  $j^{\text{th}}$  face  $\mathcal{F}_{ij}$ . We define

$$\alpha_{ij(k)}^1(x_0) := n_{ij} A_k x_0 \quad \text{and} \quad \alpha_{ij(k)}^2(x_0) := n_{ij} A_k^2 x_0$$

for points  $x_0 \in \mathcal{F}_{ij}$ .

Thus, the numbers  $\alpha_{ij(k)}^1(x_0)$  and  $\alpha_{ij(k)}^2(x_0)$  are 1<sup>th</sup> and 2<sup>nd</sup> order derivatives determined by  $k^{\text{th}}$  system matrix and  $j^{\text{th}}$  row vector of  $C_i$ . We do not mark order if it is 1. We also generalize the regions  $\mathcal{P}^+$  and  $\mathcal{P}^-$  on the faces. We do this to separate the regions on a face as the ones where  $m^{\text{th}}$  order derivative is positive or negative.

**Definition 5:** Consider mode  $i$ , and its  $j^{\text{th}}$  face  $\mathcal{F}_{ij}$  and assume that  $k \in \ell_S(\mathcal{F}_{ij})$ .

$$\mathcal{P}_{ij(k)}^{m,+} := \{x_0 \in \mathcal{F}_{ij}: \alpha_{ij(k)}^m(x_0) = n_{ij} A_k^m x_0 > 0\}, m = 1, 2.$$

$$\mathcal{P}_{ij(k)}^{m,-} := \{x_0 \in \mathcal{F}_{ij}: \alpha_{ij(k)}^m(x_0) = n_{ij} A_k^m x_0 < 0\}, m = 1, 2.$$

### 3. WELL POSEDNESS OF CLS

Now let us consider the main result of (Şahan and Eldem, 2015) and adapt it to CLS. (We do not use affine terms in CLS)

**Theorem 1:** Consider the following bimodal CLS

$$\dot{x} = \begin{cases} A_1 x, & \text{if } c^T x \geq 0 \\ A_2 x, & \text{if } c^T x \leq 0 \end{cases}, A_1, A_2 \in \mathbb{R}^{n \times n}, c^T = [0 \quad \dots \quad 0 \quad 1]$$

where  $A_1$  and  $A_2$  are as in equation (2) of (Şahan and Eldem, 2015). The system is well posed iff the following hold.

1) The structure of  $A_1$  is such that  $a_{ij} = 0$  if  $i = 3, 4, \dots, n$  and  $j = 1, 2, \dots, i - 2$ , or equivalently the following hold  $\bigcap_{r=0}^s \ker(c^T A_1^r) = \bigcap_{r=0}^s \ker(c^T A_2^r)$  for  $s = 1, 2, \dots, n - 1$ .

2) The entries of  $A_1$  which are below diagonal are positive, i.e.  $a_{i+1,i} > 0$  for  $i = 1, 2, \dots, n - 1$ .

If we state the Theorem 1 in  $\mathbb{R}^3$  and in terms of the terminology we used in Definitions 3-5, the first item is equivalent to say  $\mathcal{L}_{11} = \mathcal{L}_{21}$  where  $n_{11} = n_{21} = [0 \ 0 \ 1]$ .  $\ker(n_{ij}) \cap \ker(n_{ij} A_i) \cap \ker(n_{ij} A_i^2)$  is the origin and there is no more  $\bigcap_{r=0}^s \ker(c^T A_i^r)$  subspaces. For  $\mathbb{R}^3$ , the line  $\mathcal{L}_{11} = \mathcal{L}_{21}$  divides the face  $\mathcal{F}_{11} = \mathcal{F}_{21} = \text{Ker}(c^T)$  into two regions  $\mathcal{P}_{11(k)}^+$  and  $\mathcal{P}_{11(k)}^-$ . Secondly, the origin divides  $\mathcal{L}_{11} = \mathcal{L}_{21}$  into two parts  $\mathcal{P}_{11(k)}^{2,+}$  and  $\mathcal{P}_{11(k)}^{2,-}$  for  $k = 1, 2$ . The second item states that  $\mathcal{L}_{11}$  and origin divides  $\mathcal{F}_{11}$  into such regions that direction of the solutions are the same, as well. Consequently, two items state that  $\left\{ \begin{matrix} \mathcal{P}_{11(1)}^{1,+} = \mathcal{P}_{11(2)}^{1,+} \\ \mathcal{P}_{11(1)}^{1,-} = \mathcal{P}_{11(2)}^{1,-} \end{matrix} \right\}$ , and  $\left\{ \begin{matrix} \mathcal{P}_{11(1)}^{2,+} = \mathcal{P}_{11(2)}^{2,+} \\ \mathcal{P}_{11(1)}^{2,-} = \mathcal{P}_{11(2)}^{2,-} \end{matrix} \right\}$ . So the WP is resolved. For simplicity, we give a special name for this situation as following.

**Definition 6:** Consider a CLS in  $\mathbb{R}^3$  defined by (1)-(3), the mode  $i$ , its  $j^{\text{th}}$  face  $\mathcal{F}_{ij}$  and  $\emptyset \neq X \subset \mathcal{F}_{ij}$ . If we have either

$$\left\{ \begin{matrix} \mathcal{P}_{ij(i)}^{1,+} = \mathcal{P}_{ij(k)}^{1,+} \\ \mathcal{P}_{ij(i)}^{1,-} = \mathcal{P}_{ij(k)}^{1,-} \end{matrix} \right\} \text{ on } X \subset \mathcal{F}_{ij} \text{ or}$$

$$\alpha_{ij(k)}(X) = 0 \implies \left\{ \begin{matrix} \mathcal{P}_{ij(i)}^{2,+} = \mathcal{P}_{ij(k)}^{2,+} \\ \mathcal{P}_{ij(i)}^{2,-} = \mathcal{P}_{ij(k)}^{2,-} \end{matrix} \right\}, \forall k \in \ell_S(X),$$

and disjoint union of  $\mathcal{P}_{ij(k)}^{m,+/-}$  regions give  $X$ , then we simply state that “ $\mathcal{P}_{ij(k)}^{m,+/-}$  regions match on  $X$ ”.

Depending on the Definitions 3,4 and 5, we state that  $\mathcal{L}_{ij}$  subspaces (lines) may divide the initial conditions on a face  $\mathcal{F}_{ij}$  into two parts:  $\mathcal{P}_{ij(k)}^{m,+}$  and  $\mathcal{P}_{ij(k)}^{m,-}$ . But,  $\mathcal{L}_{ij}$  subspaces may also lie out of  $\mathcal{F}_{ij}$ , i.e.  $\mathcal{L}_{ij} \in \text{Ker}(n_{ij}) - \mathcal{F}_{ij}$ . Here,  $\mathcal{P}_{ij(k)}^{m,+}$  (or  $\mathcal{P}_{ij(k)}^{m,-}$ ) defines the set of initial conditions on a face  $\mathcal{F}_{ij}$  such that  $\alpha_{ij(k)}^m(x_0)$  (dot product of  $n_{ij}$  and  $m^{\text{th}}$  order derivative vector with respect to mode  $k$ ) are positive (negative).

Consider a CLS in  $\mathbb{R}^3$ , its mode 1 and its face  $\mathcal{F}_{11}$ . (See Figs.3 and 4). Let the adjacent mode on  $\mathcal{F}_{11} - \mathcal{F}_{11}^1$  be mode 2. In view of Theorem 1,  $\mathcal{P}_{11(1)}^{+/-}$  and  $\mathcal{P}_{11(2)}^{+/-}$  regions must match on  $\mathcal{F}_{11} - \mathcal{F}_{11}^1$  for WP. This means that the trajectories starting from  $x_0 \in \mathcal{F}_{11} - \mathcal{F}_{11}^1$  have the same sign for both modes. This is depicted in the Fig.3 for the case  $\mathcal{L}_{11} \in \mathcal{F}_{11}$ . Here,  $\mathcal{L}_{11} = \mathcal{L}_{21}$  is  $\mathcal{P}_{11(1)}^{2,+} = \mathcal{P}_{11(2)}^{2,+}$ . If we have the same decomposition on each mode & face  $\mathcal{F}_{ij}$ , then CLS is well posed.

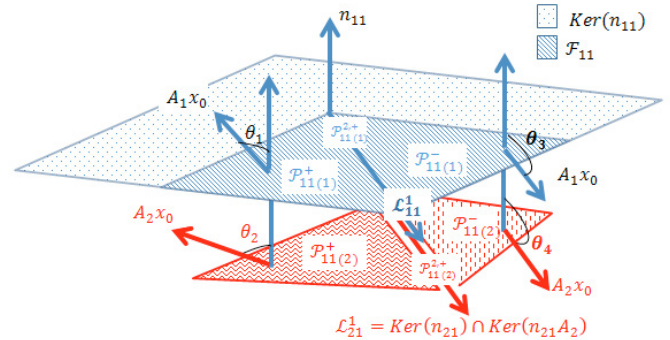


Fig.3

(Although the blue and red regions are actually same they are shown as separated just to show the related vectors and angles clearly)

Let us remember the smooth continuation sets for the mode  $i$  as  $S_i$ , [Imura and van der Schaft, 2000]. Now, we give the only WP condition for a CLS in  $\mathbb{R}^n$  that we know (Xia X., 2002).

**Theorem 2:** Consider the CLS defined by equation (1) and (3). The system is well posed if and only if

- $\bigcup_{i=1}^m S_i = \mathbb{R}^n$  and  $S_i \cap S_k = \{0\}$  for each different mode  $i, k$  where  $1 \leq i, k \leq m$

The condition of Theorem 2 uses the same idea in [Imura and van der Schaft, 2000] and try to divide the entire space into distinct smooth continuation sets. In what follows, we'll give the conditions under which this kind of a disjoint union is possible. Before that, let us see the WP problem in  $\mathbb{R}^3$ , with the following example.

**Example 2:** Consider the following CLS in  $\mathbb{R}^3$  where  $A_1 =$

$$\begin{bmatrix} k_1 & k_2 & k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } A_i = \begin{bmatrix} a_{i1}^i & a_{i2}^i & a_{i3}^i \\ a_{i21}^i & a_{i22}^i & a_{i23}^i \\ a_{i31}^i & a_{i32}^i & a_{i33}^i \end{bmatrix} \text{ for } i = 2,3,4,5.$$

$$C_1 = [0 \ 0 \ 1], \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$C_4 = -I_3 \text{ and } C_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \text{ Define the vectors } n_{ij} \text{ and}$$

the faces  $\mathcal{F}_{ij}$  as given in (3)-(4), respectively.

Note that 1<sup>th</sup> mode is upper side of the  $z = 0$  plane, the remaining 4 modes are the octants under  $z = 0$ . The face  $\mathcal{F}_{11}$  is  $\text{Ker}(n_{11})$  and it has intersections with faces  $\mathcal{F}_{i3} \subset \text{Ker}(n_{i3}) = \text{Ker}(n_{11})$ ,  $i = 2,3,4,5$ . (Fig.4)

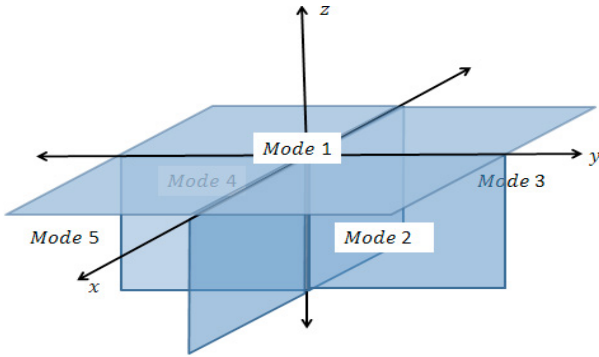


Fig.4

Now consider an initial condition  $x_0 = [\gamma_1 \ \gamma_2 \ 0]'$ ,  $\gamma_1, \gamma_2 \in \mathbb{R}$  and also vectors  $A_1 x_0 = [k_1 \gamma_1 + k_2 \gamma_2 \ \gamma_1 \ \gamma_2]'$ ,  $A_i x_0 = [a_{i1}^i \gamma_1 + a_{i2}^i \gamma_2 \ a_{i21}^i \gamma_1 + a_{i22}^i \gamma_2 \ a_{i31}^i \gamma_1 + a_{i32}^i \gamma_2]'$ ,  $i = 2,3,4,5$ . The solution with respect to 1<sup>st</sup> mode continues in mode 1 if  $\gamma_2 > 0$ , but continues in mode 4 and 5 for  $\gamma_2 < 0$  on corresponding regions. Because the line  $\mathcal{L}_{11} = \text{Ker}(n_{11}) \cap \text{Ker}(n_{11} A_1)$  is  $x$ -axes and it divides the face  $\mathcal{F}_{11}$  into two parts:  $\mathcal{P}_{11(1)}^+$  and  $\mathcal{P}_{11(1)}^-$ . (Fig.5).

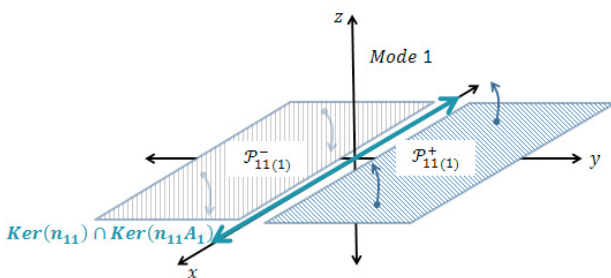


Fig.5

Thus, initial conditions on this face must point out the same direction for any active mode of it. According to mode 1, the directions of the solutions on the face  $\mathcal{F}_{11}$  are fixed because of the canonical form which  $A_1$  has. Hence, the directions of the solutions w.r.t. adjacent modes also must point out the same direction, too. As a result, we must have the following for the adjacent modes there.

$$\begin{aligned} x_0 \in \{\mathcal{F}_{11} \cap \mathcal{F}_{23}\} \\ x_0 \in \{\mathcal{F}_{11} \cap \mathcal{F}_{33}\} \end{aligned} \Rightarrow \alpha_{11(i)}(x_0) > 0, \quad i = 2,3$$

or if  $\alpha_{11(i)}(x_0) = 0$ , then  $\alpha_{11(i)}^2(x_0)$  must be positive. In a similar fashion, we also must have

$$\begin{aligned} x_0 \in \{\mathcal{F}_{11} \cap \mathcal{F}_{43}\} \\ x_0 \in \{\mathcal{F}_{11} \cap \mathcal{F}_{53}\} \end{aligned} \Rightarrow \alpha_{11(i)}(x_0) < 0, \quad i = 4,5$$

or if  $\alpha_{11(i)}(x_0) = 0$ , then  $\alpha_{11(i)}^2(x_0)$  must be negative. Let's investigate the sign of  $\alpha_{11(2)}(x_0)$  in detail. Depending on the entries of  $A_2$ , the line  $\mathcal{L}_{23}$  may change on  $\text{Ker}(n_{23})$ . Now assume that the entries are  $a_{31}^2 = -a_{32}^2 = 1$ . Then  $\mathcal{L}_{23}$  lies on  $\mathcal{F}_{23}$  and it has a positive slope, (It's actually  $y = x$  line in  $\mathbb{R}^2$ ). As a result, while  $\mathcal{P}_{11(1)}^+$  includes whole  $\{\mathcal{F}_{23} - \mathcal{F}_{23}^1\} \cup \{\mathcal{F}_{23} \cap \mathcal{F}_{33}\} \cup \{\mathcal{F}_{33} - \mathcal{F}_{33}^1\}$  for mode 1 (as depicted in Fig.5),  $\mathcal{L}_{23}$  divides  $\mathcal{F}_{23}$  into two parts  $\mathcal{P}_{11(2)}^+$  and  $\mathcal{P}_{11(2)}^-$ . This fact destroys WP on this face. For instance, consider  $x_0 = [3 \ 1 \ 0]'$  and assume again that  $a_{31}^2 = -a_{32}^2 = 1$ . Then  $A_1 x_0 = [\times \ \times \ 1]'$  and  $A_2 x_0 = [\times \ \times \ 2]'$ . Hence, the trajectories starting from this initial condition goes to mode 1, which is consistent with WP. However, when we consider  $x_0 = [1 \ 3 \ 0]'$ , then  $A_1 x_0 = [\times \ \times \ 3]'$ ,  $A_2 x_0 = [\times \ \times \ -2]'$ . Then the trajectories starting from  $x_0$  according to both mode 1 and mode 2 go to their own modes. Thus the solution is not unique, which again destroys WP. Hence, the subspace  $\mathcal{L}_{23}$  must be same with  $\mathcal{L}_{11}$ . As a result, we must have  $a_{31}^2 a_{32}^2 \geq 0$ . If one of the entries  $a_{31}^2, a_{32}^2$  is zero, then  $\mathcal{L}_{23}$  lies on  $x$ -axes or  $y$ -axes. Then, the remaining nonzero one of  $a_{31}^2, a_{32}^2$  must be positive.

Similar computations should be done for  $\mathcal{F}_{11} \cap \mathcal{F}_{33}$ .  $\mathcal{L}_{33}$  must be the same as  $\mathcal{L}_{11}$  or lie outside of  $\mathcal{F}_{23}$ . As a result, we must have  $a_{31}^3 a_{32}^3 \leq 0$  in order to get a nonnegative slope for  $\mathcal{L}_{33}$ . But in case of  $a_{32}^3 < 0$ ,  $\mathcal{F}_{11} \cap [\mathcal{F}_{33} - \mathcal{F}_{33}^1]$  is completely  $\mathcal{P}_{11(3)}^-$  for mode 2. Then  $a_{31}^3$  and  $a_{32}^3$  must be negative and positive respectively. But then  $\mathcal{P}_{11(3)}^+$  also covers  $\mathcal{F}_{11} \cap \mathcal{F}_{33} \cap \mathcal{F}_{43}$  and we have an interesting situation for this boundary of  $\mathcal{F}_{33}$ , i.e. for  $x_0 \in \mathcal{L}_{11}$  and  $\gamma_2 < 0$ . Here,  $\alpha_{11(1)}(x_0) = 0$ . Thus, we need  $\alpha_{11(1)}^2(x_0)$  again.

$$x_0 = [\gamma_1 \ 0 \ 0]' \text{ and } \gamma_1 > 0 \Rightarrow \alpha_{11(1)}^2(x_0) > 0,$$

$$x_0 = [\gamma_1 \ 0 \ 0]' \text{ and } \gamma_1 < 0 \Rightarrow \alpha_{11(1)}^2(x_0) < 0.$$

Consequently,  $\mathcal{L}_{11}$  is decomposed into 3 parts:  $\mathcal{P}_{11(1)}^{2,+} \cup \{0\} \cup \mathcal{P}_{11(1)}^{2,-}$ . Here,  $\mathcal{P}_{ij(k)}^{2,+/-}$  regions must match for active modes at these regions, i.e. for  $k \in \ell_5(\mathcal{L}_{11})$ . Therefore, we must have the followings:

\*  $x_0 = [\gamma_1 \ 0 \ 0]'$  and  $\gamma_1 > 0 \Rightarrow$  we must have  $x_0 \in \mathcal{P}_{11(2)}^+$  and  $x_0 \in \mathcal{P}_{11(5)}^+$ . Otherwise, if  $\alpha_{11(2)}(x_0) = 0$ , then we must have  $x_0 \in \mathcal{P}_{11(2)}^{2,+}$  and  $\alpha_{11(5)}(x_0) = 0 \Rightarrow x_0 \in \mathcal{P}_{11(5)}^{2,+}$



\*  $x_0 = [\gamma_1 \ 0 \ 0]'$  and  $\gamma_1 < 0 \Rightarrow$  we must have  $x_0 \in \mathcal{P}_{11(3)}^-$  and  $x_0 \in \mathcal{P}_{11(4)}^-$ . Otherwise, if  $\alpha_{11(3)}(x_0) = 0$ , then we must have  $x_0 \in \mathcal{P}_{11(3)}^{2,-}$  and  $\alpha_{11(4)}(x_0) = 0 \Rightarrow x_0 \in \mathcal{P}_{11(4)}^{2,-}$ .

Then  $a_{31}^2 = 0$ ,  $a_{32}^2 \neq 0 \Rightarrow$  we must have  $a_{21}^2 > 0$ . When we regard  $x_0 = [\gamma_1 \ 0 \ 0]'$  and  $\gamma_1 < 0$ , then  $\alpha_{11(3)}(x_0) = a_{31}^3 \gamma_1$ . We must have  $a_{31}^3 \geq 0$ . But remember that if  $a_{31}^3 > 0$  and  $x_0 \in \mathcal{F}_{11} \cap [\mathcal{F}_{33} - \mathcal{F}_{33}^1]$  then  $\alpha_{11(3)}(x_0) = a_{31}^3 \gamma_1 + a_{32}^3 \gamma_2$  may be negative for some  $x_0$ . Thus, it leads to have  $a_{31}^3 = 0$ .

Similar computations should be done for the rest of the modes taking into account the cautions mentioned above. For example, in Fig.6,  $\mathcal{L}_{21}$  and  $\mathcal{L}_{31}$  both lie in the same face but not equal. This causes IP again.

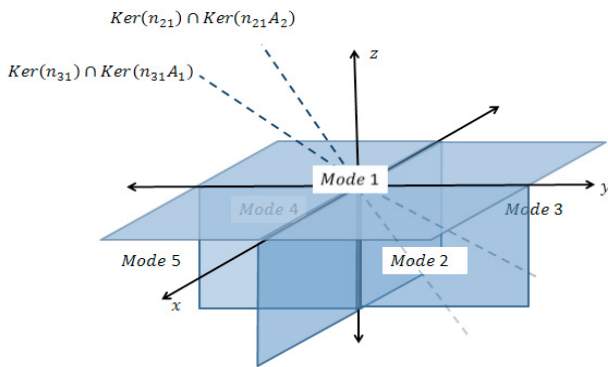


Fig.6

**Remark 2:** Note that for  $\mathbb{R}^n$ ,  $\mathcal{L}_{ij}^s := \bigcap_{r=0}^s \ker(c^T A_i^r)$  subspaces divide the previous space into two parts. For instance, in  $\mathbb{R}^3$ , the subspaces  $\mathcal{L}_{ij}^2 := \ker(n_{ij}) \cap \ker(n_{ij}A_i) \cap \ker(n_{ij}A_i^2) = \{0\}$  divides  $\mathcal{L}_{ij}$  into two parts. In  $\mathbb{R}^n$ , for each  $s$  we have 2 parts on  $\mathcal{L}_{ij}^{s-1}$  which have positive and negative signed  $(s-1)^{th}$  order derivatives. This idea should be used for generalization of WP of CLS to  $\mathbb{R}^n$ .

**Remark 3 :** We have seen above that WP for discontinuous CLS induces a fixed ratio between some entries, makes some entries zero and fixed sign. On the other hand, continuity causes much more restrictions. Now let us show this. If we had assumed that the system is continuous, then we would have the following.

$$A_2 x_0 = A_3 x_0 \text{ for any } x_0 = [0 \ \gamma_2 \ \gamma_3]' \in X = \mathcal{F}_{21} - \mathcal{F}_{21}^1$$

$$\Rightarrow a_{12}^2 \gamma_2 + a_{13}^2 \gamma_3 = a_{12}^3 \gamma_2 + a_{13}^3 \gamma_3$$

It leads to take the entries  $a_{12}^2 = a_{12}^3$  and  $a_{13}^2 = a_{13}^3$ . The only difference between system matrices remains  $a_{11}^2$  and  $a_{11}^3$ . Also note that we just regard one part of boundary. Thus, it requires to have more identical elements, even equal system matrices for some modes. This shows that **continuity for CLS is rather restrictive**.

Consequently, it is observed that if we apply a transformation and put one of the system matrices into some

special canonical forms as in the work [Şahan and Eldem, 2015] and as we considered in Example 1 above, then it forces its adjacent modes to have fixed proportional entries.

With the help of these notations and assumptions, now we give our main result.

**Theorem 3:** Consider the CLS defined by equation (1)-(3) in  $\mathbb{R}^3$  and assume that the vector field is not necessarily continuous. The system is well posed if and only if one of the following conditions hold.

- 1)  $\mathcal{P}_{ij(k)}^{m,+/-}$  regions match on  $X \subset \mathcal{F}_{ij}$ , for all modes  $i$  and their each indices  $j$  where  $1 \leq i \leq l$ ,  $1 \leq j \leq l_i$ ,  $k \in \ell_S(\mathcal{F}_{ij})$ .
- 2) Either  $\alpha_{ij(k)}(X) \neq 0$  for all  $k$  and  $\text{sgn}(\alpha_{ij(k)}(X))$  are all same or  $\alpha_{ij(k)}(X) = 0$  for all  $k$  and  $\text{sgn}(\alpha_{ij(k)}^2(X))$  are all same for  $k \in \ell_S(X)$ ,  $X \subset \mathcal{F}_{ij}$ .

**Proof :** Consider mode  $i$  and an initial condition  $x_0 \in C_i$ . If  $x_0 \notin \mathcal{F}_{ij}$ , then the WP is clear as it's already an LTI system. Hence we explore WP for  $x_0 \in \mathcal{F}_{ij}$ . Assume that the system is WP. There are two alternatives for  $x_0$ : Either  $x_0 \in \{\mathcal{F}_{ij} - \mathcal{F}_{ij}^1\}$  or  $x_0 \in \mathcal{F}_{ij}^1$ .

1) Let  $x_0 \in \{\mathcal{F}_{ij} - \mathcal{F}_{ij}^1\} =: X$ . For  $\mathbb{R}^3$ ,  $|\ell_S(X)| = 2$ . Let  $\ell_S(X) = \{i, k\}$ . We naturally two alternatives for two  $\mathcal{L}_{ij}$  lines on  $X$ , as well. They either both lie in  $X$  or both lie outside. Otherwise we have IP as in Example 2.

a) If they both lie inside, then they must coincide because of WP assumption. Otherwise some IP regions occur between them. Thus, they divide  $X$  into two sides and  $\mathcal{P}_{ij(i)}^{+/-}$  and  $\mathcal{P}_{ij(k)}^{+/-}$  regions must match on  $X - \mathcal{L}_{ij}$ , (see Fig.3). Here also  $\alpha_{ij(i)}(X - \mathcal{L}_{ij})$  and  $\alpha_{ij(k)}(X - \mathcal{L}_{ij})$  are nonzero and must be same sign. If they have different signs, we have IP here. In addition, for  $x_0 \in \mathcal{L}_{ij}$ , we have  $\alpha_{ij(i)}(\mathcal{L}_{ij}) = \alpha_{ij(k)}(\mathcal{L}_{ij}) = 0$ .

We thus check second order derivatives.  $\text{sgn}(\alpha_{ij(i)}^2(\mathcal{L}_{ij}))$  and  $\text{sgn}(\alpha_{ij(k)}^2(\mathcal{L}_{ij}))$  must be same for WP and so whole  $\mathcal{L}_{ij}$  is either  $\mathcal{P}_{ij(i)}^{2,+} = \mathcal{P}_{ij(k)}^{2,+}$  or  $\mathcal{P}_{ij(i)}^{2,-} = \mathcal{P}_{ij(k)}^{2,-}$ , (same with Fig.3).

b) If both  $\mathcal{L}_{ij}$  lines outside of  $X$ , then neither  $\alpha_{ij(i)}(X)$  nor  $\alpha_{ij(k)}(X)$  change sign throughout  $X$  and they must be same sign. So whole  $X$  is  $\mathcal{P}_{ij(i)}^+ = \mathcal{P}_{ij(k)}^+$  or  $\mathcal{P}_{ij(i)}^- = \mathcal{P}_{ij(k)}^-$ .

2) Now, let  $x_0 \in \mathcal{F}_{ij}^1$ . Note that  $\mathcal{F}_{ij}$  has actually two borders and assume without loss of generality that we choose one of them. This time  $X$  is an edge and so  $|\ell_S(X)| \geq 2$ . There exists another row, say  $p^{th}$ , of  $C_i$ . We define  $X := \mathcal{F}_{ij} \cap \mathcal{F}_{ip}$ . Because of WP assumption, smooth continuation is possible to any mode in  $\ell_S(X)$ . Let that mode be  $i^{th}$ . Then we have  $\alpha_{ij(i)}(X)$ ,  $\alpha_{ip(i)}(X) \geq 0$ . If one of  $\mathcal{L}_{ij}$  and  $\mathcal{L}_{ip}$  lies here, then that derivative is zero. It means  $\mathcal{P}_{ij(i)}^+ = \emptyset$  here. This requires to check corresponding second order derivative. But because of smooth continuation to this mode  $i$ , that must be positive. For a consensus on  $X$ , we must have  $\mathcal{P}_{ij(k)}^{2,+} = X$  for all

$k \in \ell_s(X)$ . This also implies that  $\alpha_{ij(k)}(X) = 0$  and  $\text{sgn}(\alpha_{ij(k)}^2(X))$  are all same.

The inverse implication should be easily proved in a same way.

## CONCLUSIONS

In this work, we investigated the WP conditions of CLS. It is shown that while WP forces some entries of subsystem coefficient matrices to have some fixed ratios continuity causes much more restrictions. But we do not conjecture that a similar structure with WP bimodal system for system matrices is possible also for CLS. But we showed that location of  $\mathcal{L}_{ij}^s := \cap_{r=0}^s \ker(c^T A_i^r)$  subspaces and sign of  $\alpha_{ij(k)}^1, \alpha_{ij(k)}^2, \dots$  again determines WP as we have in bimodal systems.

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