REIDEMEISTER TORSION OF CLOSED π-MANIFOLDS

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This thesis is dedicated to all women who never had a chance of an education, especially my lovely grandmother Fatma Dirican. Her enthusiasm for education always inspired me. I am glad to walk half of this long path with her. May she rest in peace.

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ABSTRACT

REIDEMEISTER TORSION OF CLOSED π-MANIFOLDS

Let M be a closed orientable $2n$-dimensional π-manifold such that $n \neq 2$ and M is either $(n-2)$-connected or $(n-1)$-connected. Such a manifold M can be decomposed as a connected sum of certain simpler manifolds. In this thesis, by using such connected sum decompositions, we develop multiplicative gluing formulas that express the Reidemeister torsion of M with untwisted $\mathbb{R}$-coefficients in terms of Reidemeister torsions of its building blocks in the decomposition. Then we apply these results to handlebodies, compact orientable smooth $(2n+1)$-dimensional manifolds whose boundary is a $(n-2)$-connected $2n$-dimensional closed π-manifold, and product manifolds.

ÖZET

KAPALI π-MANİFOLDLARIN REIDEMEISTER TORSİYONU

Kabul edelim ki M yönlendirilebilir kapalı $2n$-boyutlu bir π-manifold olsun öyle ki $n \neq 2$ ve M ya $(n-2)$-bağlantılıdır yada $(n-1)$-bağlantılıdır. Böyle manifoldlar, daha basit manifoldların bağlantılı toplamı olarak ifade edilebilir. Bu tezde, bağlantılı toplamlar parçalanışı kullanılarak M manifoldunun $\mathbb{R}$-değerli Reidemeister torsiyonunu bağlantılı toplamı oluşturan manifoldların Reidemeister torsiyonları cinsinden ifade eden çarpımsal yapıştırma formülleri geliştirilmiştir. Daha sonra bu sonuçlar tutamaçlara, sınırı $(n-2)$-bağlantılı $2n$-boyutlu kapalı π-manifold olan kompakt yönlendirilebilir $(2n+1)$-boyutlu manifoldlara ve son olarak çarpım manifoldlarına uygulanmıştır.

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LIST OF ABBREVIATIONS

$\mathbb{R}$	The set of real numbers
$\mathbb{Z}$	The set of integers
$C^\infty(M)$	Space of smooth functions
$\pi_n(M)$	The n-th homotopy group of M
$\chi(M)$	The Euler characteristic of M
$k(M)$	The Kervaire semi-characteristic of M
$\kappa(M)$	The Arf-Kervaire invariant of M
$d(M)$	Double of M
#	Connected sum operation
$\oplus$	Direct sum
ξ	Vector bundle
$\varepsilon^n(M)$	The n-dimensional trivial vector bundle over M
$\varepsilon^1(M)$	Line bundle over M
T_xM	Tangent space to M at a point x
$\tau(M)$	Tangent bundle of M
$V_k(\mathbb{F}^n)$	Steifel manifold
M_K^{4k+2}	Kervaire manifold
$\Sigma_{g,n}$	The n-holed genus g surface
$\Sigma_{g,0}$	The closed genus g surface
M^{2n}	The $2n$-dimensional closed π-manifold
$\widetilde{\mathbb{S}^{2n}}$	The $2n$-dimensional homotopy sphere
$\mathbb{D}^n$	Open unit ball in $\mathbb{R}^n$
$\overline{\mathbb{D}^n}$	Closed unit ball in $\mathbb{R}^n$.

CHAPTER 1

INTRODUCTION

Reidemeister torsion is a topological invariant and an invariant of the basis of the homology of a manifold. It was first introduced by Reidemeister (1935) to give a homeomorphism classification of 3-dimensional lens spaces (up to PL equivalence). Franz (1935) classified higher dimensional lens spaces by extending the notion of Reidmeister torsion. The results of Reidemeister and Franz were extended to the spaces of constant curvature +1 by de Rham (1964). Kirby and Siebenmann (1969) showed that Reidemeister torsion is a topological invariant for manifolds. Then Chapman (1974) proved the invariance for arbitrary simplicial complexes and thus the classification of lens spaces of Reidemeister and Franz was shown to be topological.

Using Reidemeister torsion, the Hauptvermutung was disproved by Milnor in 1961. He constructed two homeomorphic but combinatorially distinct finite simplicial complexes. Then he described Reidemeister torsion with the Alexander polynomial which plays an important role in knot theory and links (Milnor, 1962, 1966, 1968).

Interesting applications of this invariant occur in several branches of mathematics and theoretical physics, such as topology (Franz, 1935; Milnor, 1961, 1962; Reidemeister, 1935), differential geometry (Müller, 1978; Cheeger, 1979; Ray and Singer, 1971), representation spaces (Sözen, 2008, 2012a; Witten, 1989), Chern-Simon theory (Witten, 1991), algebraic K-theory (Milnor, 1966), dynamical systems (Hutchings and Lee, 1999), theoretical physics and quantum field theory (Witten, 1989, 1991).

We briefly explain some of these applications. Ray and Singer (1971) defined an analytical torsion, called Ray-Singer analytic torsion, corresponding to the de Rham complex of the straight beam coefficient forms on a compact Riemann manifold. They also conjectured that Ray-Singer analytic torsion is equal to the Reidemeister torsion obtained by using the action of the fundamental group on the universal cover of a Riemannian manifold and the representation corresponding to the flat beam. This conjecture was proved independently by Müller (1978) and Cheeger (1979). The Ray-Singer analytic torsion can be considered as a "de Rham counterpart" of the Reidemeister torsion (cf. de Rham cohomology vs. cellular cohomology).Witten (1989) investigated the non-abelian Chern-Simons gauge field theory with the help of the Ray-Singer analytical torsion.

The Reidemeister torsion has proven its utility in a number of topics in three-dimen- sional topology. For instance, the Casson-Walker-Lescop invariants are defined by using Reidemeister torsion (see, (Lescop, 1996)). Meng and Taubes (1996) proved that the Turaev's maximal abelian torsion coincided with the Seiberg-Witten invariant on 3-dimensional manifolds if the first Betti number is not zero.

The Reidemeister torsion resembles to the Euler characteristic in many respects. Thanks to the classical Poincaré-Hopf theorem, Euler counts the stationary points of the characteristic smooth vector fields. Similarly, Fried (1983) proved that Reidemeister torsion counts closed orbits of Morse-Smale vector fields that are not zero anywhere on the smooth manifolds. Thus, the Euler characteristic counts points while the torsion counts circles. Hutchings and Lee (1999) generalized Fried's conclusion. Schwarz (1977) showed that the quantum field theory can be established on a manifold such that the partition function will be a power of analytical torsion.

Symplectic chain complex is an algebraic topological tool and it was introduced by Witten (1991). By using Reidemeister torsion and symplectic chain complex, he computed the volume of several moduli spaces of representations from a Riemann surface to a compact gauge group.

The Arf invariant of a non-singular quadratic form over a field of characteristic 2 was defined by Arf (1941) when he started the systematic study of quadratic forms over arbitrary fields of characteristic 2. The Arf invariant has an application in geometric topology. In particular, Kervaire (1960) used the Arf invariant and defined a $\mathbb{Z}_2$-valued invariant, called Arf-Kervaire invariant, for $(4k + 2)$-dimensional compact manifolds. After that, Browder (1969) extended this definition to framed closed $(4k - 2)$-dimensional manifolds.

A manifold is called a π-manifold if the direct sum of its stable tangent bundle is trivial. Such manifolds are defined by Whitehead (1940) as combinatorial manifolds that have product regular neighbourhoods when imbedded in a Euclidean space of sufficiently high dimension. In his work, he showed that a triangulated smooth combinatorial π-manifold has a trivial normal bundle.

This thesis is organized as follows. In Chapter 2 we give the some basic concepts and results from algebraic topology which are required for notational convenience in the rest of the thesis.

In Chapter 3 we recall some essential results on smooth manifolds. We first give the connected sum operation on these manifolds. Then we give a summary on vector bundles. Lastly, we give the definition of a π-manifold and discuss some important results on π-manifolds.

In Chapter 4 we give preliminaries on Reidemeister torsion. We first recall the definition of the Reidemeister torsion of a general chain complex. After giving some properties of Reidemeister torsion that are essential for this thesis, we mention the notion of a symplectic chain complex. Then we give the definition of Reidemeister torsion for manifolds.

In Chapter 5 we state and prove the main results of this thesis, which are computations of the Reidemeister torsions of certain π-manifolds with untwisted $\mathbb{R}$-coefficients. We start with closed orientable surfaces. Note that every such surface is a π-manifold that admits a connected sum decomposition. For this purpose, we first obtain a formula for the Reidemeister torsion of one-holed-torus $\Sigma_{1,1}$ by using the notion of a symplectic chain complex and homological algebra techniques. Applying this result and considering the connected sum decomposition $\underset{j=1}{\overset{g}{\#}}(\Sigma_{1,0})$ of $\Sigma_{g,0}$, we compute the Reidemeister torsion of $\Sigma_{g,0}$ in terms of the Reidemeister torsion of $\Sigma_{1,1}$. Then, for $n \geq 3$, we compute the Reidemeister torsion of $(n-2)$-connected $2n$-dimensional closed π-manifold M^{2n} in terms of the Reidemeister torsions of its building blocks in the decomposition $M^{2n} = \underset{j=1}{\overset{p}{\#}}(\mathbb{S}^n \times \mathbb{S}^n)\# M_1^{2n}$ due to (Ishimoto, 1969). Lastly, we consider $(n-1)$-connected $2n$-dimensional closed π-manifold M^{2n} for $(n \geq 3)$ with the decomposition $M^{2n} = \underset{j=1}{\overset{p}{\#}}(\mathbb{S}^n \times \mathbb{S}^n)\#\widetilde{\mathbb{S}^{2n}}$ as given by (Ishimoto, 1969) and we compute the Reidemeister torsion of such manifolds in terms of the Reidemeister torsions of its building blocks.

In Chapter 6 we apply our main results to the manifolds such as handlebodies; compact, orientable, smooth $(2n+1)$-dimensional manifolds whose boundary is $(n-2)$-connected $2n$-dimensional closed π-manifold, and product manifolds.

CHAPTER 2

BASIC NOTIONS IN ALGEBRAIC TOPOLOGY

In this chapter, we recall the essential background in algebraic topology used in this thesis. We refer the reader to (Hatcher, 2002) for details of the definitions, theorems and their proofs given in this chapter.

2.1. Chain Complexes and Homology Groups

In this section, we give definitions and properties of chain complexes and homology groups. For more details, see (Hatcher, 2002, Chapter 2).

A *chain complex of abelian groups of length m* is a sequence

$$C_* = (C_*, \partial_*) = (0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0)$$

of abelian groups and homomorphisms such that $\mathrm{Im}(\partial_p) \subset \mathrm{Ker}(\partial_p)$ or equivalently $\partial_p \circ \partial_{p+1} = 0$ for all $p \in \{1, \ldots, m\}$. The map ∂_p is called a *boundary homomorphism*. For each $p \in \{1, \ldots, m\}$, define

$$B_p(C_*) = \mathrm{Im}\{\partial_{p+1} : C_{p+1} \to C_p\},$$
$$Z_p(C_*) = \mathrm{Ker}\{\partial_p : C_p \to C_{p-1}\}.$$

The *p−th homology* of the chain complex C_* is defined by the quotient

$$H_p(C_*) = Z_p(C_*)/B_p(C_*).$$

An element in $\mathrm{Ker}\{\partial_p\}$ is called a *cycle* and an element in $\mathrm{Im}\{\partial_{p+1}\}$ is called a *boundary*. Elements in $H_p(C_*)$ are called *homology classes*.

Definition 2.1 *Let (C_*, ∂_*) be a chain complex of length m. If for all $p \in \{1, \ldots, m\}$, $Z_p(C_*) = B_p(C_*)$, or equivalently the homology $H_p(C_*) = 0$, then (C_*, ∂_*) is called exact (or acyclic). If $m = 2$, then (C_*, ∂_*) is called a short exact sequence and if $m \geq 3$, then it is called a long exact sequence.*

Now, we recall the following well-known result.

Lemma 2.1 (Splitting Lemma) *For a short exact sequence of abelian groups*

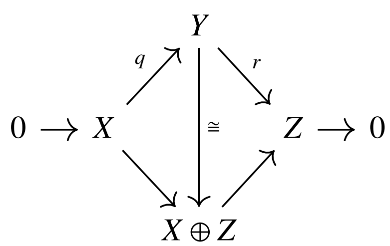
$$0 \to X \xrightarrow{q} Y \xrightarrow{r} Z \to 0,$$

the following statements are equivalent:

(i) There is a homomorphism $t : Y \to X$ such that $t \circ q = \mathrm{id}_X : X \to X$.

(ii) There is a homomorphism $u : Z \to Y$ such that $r \circ u = \mathrm{id}_Z : Z \to Z$.

(iii) There is an isomorphism $Y \cong X \oplus Z$ making a commutative diagram as at the below, where the maps in the lower row are the obvious ones, $x \mapsto (x, 0)$ and $(x, z) \mapsto z$.



If the above conditions are satisfied, the exact sequence is said to be *split*. Therefore, we can conclude that $Y = X \oplus u(Z)$.

Definition 2.2 *Let (C_*, ∂_*) and (C'_*, ∂'_*) be chain complexes. A chain map ϕ from (C_*, ∂_*) to (C'_*, ∂'_*) is a family of homomorphisms $\phi = \{\phi_p : C_p \to C'_p\}_{p \geq 0}$ such that the following diagram commutes for each p*

$$\begin{array}{ccccccc}
\ldots \to & C_p & \xrightarrow{\partial_p} & C_{p-1} & \xrightarrow{\partial_{p-1}} & C_{p-2} & \to \ldots \\
& \downarrow\phi_p & & \downarrow\phi_{p-1} & & \downarrow\phi_{p-2} & \\
\ldots \to & C'_p & \xrightarrow{\partial'_p} & C'_{p-1} & \xrightarrow{\partial'_{p-1}} & C'_{p-2} & \to \ldots
\end{array}$$

Remark 2.1 *If $\phi : C_* \to C'_*$ is a chain map between chain complexes (C_*, ∂_*) and (C'_*, ∂'_*), then for each p there is an induced homomorphism $\phi_* : H_p(C_*) \to H_p(C'_*)$.*

Definition 2.3 *A homotopy between given continuous functions $f, g : X \to Y$ is a continuous function $H : X \times [0, 1] \to Y$ such that*

$$\begin{cases} H(x, 0) = f(x), & \forall x \in X \\ H(x, 1) = g(x), & \forall x \in X. \end{cases}$$

If there is a homotopy between f and g, then we say f and g are homotopic and we write $f \simeq g$.

Let X and Y be two topological spaces and let $f : X \to Y$ be continuous function. If there is a continuous function $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, then f is called a *homotopy equivalence* and the spaces X and Y are called *homotopy equivalent* which is denoted by $X \simeq Y$. Here, id_X is the identity map of X and id_Y is the identity map of Y.

The following theorem shows that the singular homology is a homotopy invariant.

Theorem 2.1 (Homotopy Invariance Theorem) *Let $f, g : X \to Y$ be homotopic maps. Then they induce the same homomorphism on homology; that is,*

$$f_* = g_* : H_p(X) \to H_p(Y).$$

The following corollary is obtained from the Homotopy Invariance Theorem.

Corollary 2.1 *If $f : X \to Y$ is a homotopy equivalence, then the induced homomorphisms $f_* : H_p(X) \to H_p(Y)$ are isomorphisms for all p.*

Definition 2.4 *A sequence of the chain maps $0 \to A_* \xrightarrow{\alpha} D_* \xrightarrow{\beta} C_* \to 0$ is called a short exact sequence of chain complexes if for every $p \in \mathbb{N}$, $0 \to A_p \xrightarrow{\alpha_p} D_p \xrightarrow{\beta_p} C_p \to 0$ is a short exact sequence.*

In homological algebra, the Zig-Zag Lemma is a key lemma showing that a short exact sequence of chain complexes induces a natural long exact sequence in homology.

Lemma 2.2 (Zig-Zag Lemma) *For a short exact sequence of chain complexes*

$$0 \to A_* \xrightarrow{\alpha} D_* \xrightarrow{\beta} C_* \to 0,$$

there are connecting homomorphisms

$$\partial_p : H_p(C_*) \to H_{p-1}(A_*)$$

such that the following sequence is long exact

$$\cdots \xrightarrow{\partial} H_p(A_*) \xrightarrow{\alpha_*} H_p(D_*) \xrightarrow{\beta_*} H_p(C_*) \xrightarrow{\partial} H_{p-1}(A_*) \to \cdots$$

2.2. Mayer-Vietoris Sequence

The Mayer-Vietoris sequence provides an easy computation tool for homology of a topological space in terms of homologies of its subspaces. Let X be a topological space and $V, W \subseteq X$ such that

$$X = \overset{\circ}{V} \cup \overset{\circ}{W}.$$

Assume that $C_p(V+W)$ is the subgroup of $C_p(X)$ whose elements are sums of chains in either V or W. Restricting the usual boundary map $\partial : C_p(X) \to C_{p-1}(X)$ on $C_p(V+W)$, $C_*(V+W)$ becomes a chain complex. The inclusions

$$C_p(V+W) \hookrightarrow C_p(X)$$

induce isomorphisms on homology groups

$$H_p(V+W) \cong H_p(X).$$

There is a natural short exact sequence of chain complexes

$$0 \to C_*(V \cap W) \xrightarrow{\varphi} C_*(V) \oplus C_*(W) \xrightarrow{\phi} C_*(V + W) \to 0, \tag{2.1}$$

where $\varphi(x) = (x, x)$ and $\phi(x, y) = x + y$. The Mayer-Vietoris sequence is the long exact sequence of homology groups obtained by using the Zig-Zag Lemma for the short exact sequence (2.1)

$$\begin{aligned}
\mathcal{H}_* : \quad \cdots \longrightarrow H_p(V \cap W) &\xrightarrow{\alpha_p} H_p(V) \oplus H_p(W) \xrightarrow{\beta_p} H_p(X) \xrightarrow{\partial_p} \\
H_{p-1}(V \cap W) &\xrightarrow{\alpha_{p-1}} H_{p-1}(V) \oplus H_{p-1}(W) \xrightarrow{\beta_{p-1}} H_{p-1}(X) \xrightarrow{\partial_{p-1}} \\
H_{p-2}(V \cap W) &\xrightarrow{\alpha_{p-2}} \cdots
\end{aligned}$$

Here, $\partial_p : H_p(X) \to H_{p-1}(V \cap W)$ is the boundary map.

2.3. Homotopy Groups

In this section, we recall the definition of homotopy groups and the results. For more details, we refer to (Hatcher, 2002, Chapter 4).

Let $(\mathbb{S}^n, s_0)$ denote the n-sphere with base point $s_0 \in \mathbb{S}^n$. The *n-th homotopy group* $\pi_n(X, x_0)$ of a topological space X with base point x_0 is the set of homotopy classes of maps from $(\mathbb{S}^n, s_0)$ into (X, x_0); that is,

$$\pi_n(X, x_0) = \{[f], f : (\mathbb{S}^n, s_0) \to (X, x_0)\},$$

where homotopies are required to satisfy $H(s_0, t) = x_0$ for all t. The sum $f + g$ is the composition

$$\mathbb{S}^n \xrightarrow{\mathbf{c}} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{\mathbf{f}\vee\mathbf{g}} X$$

where **c** collapses the equator $\mathbb{S}^{n-1}$ in $\mathbb{S}^n$ to a point and we choose the base point s_0 to lie in $\mathbb{S}^{n-1}$ and $\vee$ denotes the wedge sum.

Definition 2.5 *A topological space X with a base point x_0 is called n-connected if $\pi_i(X, x_0) = 0$ whenever $i \leq n$, where 0 denotes the trivial group.*

It is evident from Definition 2.5 that

- -1-connectedness coincides with non-emptiness,
- 0-connectedness coincides with non-emptiness and path-connectedness,
- 1-connectedness coincides with simply connectedness,
- The n-sphere $\mathbb{S}^n$ is $(n-1)$-connected.

Theorem 2.2 *A topological space X is n-connected if and only if one of the following holds for $i \leq n$.*

(i) Every map $\mathbb{S}^i \to X$ is homotopic to a constant map.

(ii) Every map $\mathbb{S}^i \to X$ extends to a map $\mathbb{D}^{i+1} \to X$.

(iii) $\pi_i(X, x_0) = 0$ for every $x_0 \in X$.

Let X be a path-connected topological space. The *Hurewicz map* is a group homomorphism $h : \pi_n(X) \to H_n(X)$ defined by $h([f]) = f_*(\alpha)$, where α is the fixed generator of $H_n(S^n)$ and $f_* : H_n(\mathbb{S}^n) \to H_n(X)$ is induced by $f : \mathbb{S}^n \to X$ for $n > 0$. The Hurewicz theorem establishes the connection between homotopy groups and homology groups of a topological space.

Theorem 2.3 (Hurewicz Theorem) *Let X be an $(n-1)$-connected topological space $n \geq 2$. Then the Hurewicz map $h : \pi_n(X) \to H_n(X)$ becomes an isomorphism. Moreover, $H_i(X) = 0$, $1 \leq i < n$.*

CHAPTER 3

SOME ESSENTIAL BACKGROUND ON SMOOTH MANIFOLDS

Throughout this chapter, we follow the definitions and notations of (Lee, 2013). We start with the definition of a manifold.

An n-dimensional *topological manifold* M is a topological space with the following properties:

- M is a Hausdorff space: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$.
- M is second-countable: there exists a countable basis for the topology of M.
- M is locally Euclidean of dimension n: for each point p of M there is an open subset $p \in U \subset M$ which is homeomorphic to an open subset of $\mathbb{R}^n$.

An n-dimensional *topological manifold with boundary* is a second-countable Hausdorff space in which every point has a neighbourhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of closed n-dimensional upper half-space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. A manifold is called compact if the underlying the topological space is compact. A *closed manifold* is a compact manifold without boundary.

Let M be an n-dimensional topological manifold. A coordinate chart (shortly a chart) on M is a pair (U, ψ), where U is an open subset of M and $\psi : U \to \widetilde{U}$ is a homeomorphism from U to an open subset $\widetilde{U} = \psi(U)$ of $\mathbb{R}^n$. If (U, ψ) and (V, ϕ) are two charts such that $U \cap V \neq \emptyset$, the composite map $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is called the *transition map* from ψ to ϕ.

Two charts (U, ψ) and (V, ϕ) are said to be *smoothly compatible* if either $U \cap V = \emptyset$ or the transition map

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$$

is a diffeomorphism. An *atlas* for M to be a collection of charts whose domains cover M. If any two charts in $\mathcal{A}$ are smoothly compatible with each other, then $\mathcal{A}$ is called a smooth atlas. A smooth atlas $\mathcal{A}$ on M is maximal if it is not properly contained in any larger smooth atlas.

Definition 3.1 *A smooth structure on a topological manifold M is a maximal smooth atlas on M. A smooth manifold is a pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.*

A real-valued function $f : M \to \mathbb{R}$ is called smooth if for every $p \in M$; there is a smooth chart (U, ψ) for M with $p \in U$ such that $f \circ \psi^{-1}$ is smooth on $\psi(U)$. The set of all smooth functions from M to $\mathbb{R}$ is denoted by $C^\infty(M)$. The set $C^\infty(M)$ is a vector space over $\mathbb{R}$ with point-wise addition and scalar multiplication.

In generally, a finite product of smooth manifolds with boundary is not a smooth manifold with boundary. But a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary. More precisely,

Proposition 3.1 *Let $M_1, \ldots, M_k$ be smooth manifolds without boundary and N a smooth manifold with boundary. Then $M_1 \times \cdots \times M_k \times N$ is a smooth manifold with boundary $\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial(N)$.*

For a finite CW-complex X, the *Euler characteristic* is defined to be the alternating sum

$$\chi(X) = \sum_n (-1)^n t_n,$$

where t_n is the number of n-cells of X. Equivalently, $\chi(X)$ can be defined purely in terms of homology as follows

$$\chi(X) = \sum_n (-1)^n \operatorname{rank}(H_n(X)).$$

Hence, $\chi(X)$ depends only on the homotopy type of X, that is, it is independent of the choice of CW-structure on X. Here, the rank of a finitely generated abelian group is the number of $\mathbb{Z}$ summands when the group is expressed as a direct sum of cyclic groups.

3.1. Connected Sum of Manifolds

In this section, we recall the definition and some important properties of the connected sum operation on smooth manifolds. For more details, we refer the reader to (Lee, 2013) and (Kervaire and Milnor, 1963).

For connected n-dimensional manifolds M_1 and M_2, choose smooth imbeddings

$$i_1 : \mathbb{D}^n \to M_1, \quad i_2 : \mathbb{D}^n \to M_2$$

such that i_1 preserves orientation and i_2 reverses orientation, where $\mathbb{D}^n$ denotes the open unit ball in $\mathbb{R}^n$.

Definition 3.2 *The connected sum $M_1\#M_2$ is the space of the disjoint union*

$$M_1 - i_1(0) \sqcup M_2 - i_2(0)$$

by identifying $i_1(tv)$ with $i_2((1-t)v)$ for each unit vector $v \in \mathbb{S}^{n-1}$ and each $0 < t < 1$. Since the correspondence $i_1(tv) \to i_2((1-t)v)$ preserves orientation, the orientation on $M_1\#M_2$ can be chosen to be compatible with the orientation of M_1 and M_2.

By the Invariance of Domain, the projections $M_k - i_k(0) \to M_1\#M_2,\ k = 1,2$ are open maps, so $M_1\#M_2$ is second countable and Hausdorff. Moreover, the orientation preserving diffeomorphism $i_2 \circ i_1^{-1}$ together with these open maps imply that the smooth structures on $M_1 - i_1(0)$ and $M_2 - i_2(0)$ are compatible. Hence, there is a smooth structure on $M_1\#M_2$. The following theorem is due to (Kervaire and Milnor, 1963).

Theorem 3.1 *The connceted sum $M_1\#M_2$ is a closed, oriented, smooth n-dimensional manifold and it is independent of the choice of the imbeddings $i_k : \mathbb{D}^n \to M_k$, $k = 1,2$.*

It is well-known that many classical invariants are well-behaved under the connected sum operation, in particular, homology groups. Let M_1 and M_2 be connected, closed, oriented n-dimensional manifolds and N_i the image in $M_1\#M_2$ of $M_k - i_k(0),\ k = 1,2$. Note that $N_1 \cap N_2$ has the homotopy type of $\mathbb{S}^{n-1}$. The Mayer-Vietoris sequence for the pair (N_1, N_2) gives the following isomorphism for $0 < i < n$

$$H_i(M_1\#M_2) \cong H_i(M_1) \oplus H_i(M_2).$$

Two orientation preserving embeddings $\psi_1, \psi_2 : \mathbb{D}^n \to M$ are called *isotopic* if there exist some diffeomorphism φ of M such that $\psi_2 = \varphi \circ \psi_1$. Palais (1959) and Cerf (1961) showed that any two orientation preserving smooth embeddings of $\mathbb{D}^n$ into a connected, compact, oriented, smooth n-dimensional manifold are isotopic. By using this result, Kervaire and Milnor (1963) proved the following theorem.

Theorem 3.2 *Let $\mathcal{M}$ be the set of connected, compact, oriented, smooth n-dimensional manifolds. Then $(\mathcal{M}, \#)$ is an associative and commutative monoid (up to the orientation preserving diffeomorphism), where the identity element is the n-sphere $\mathbb{S}^n$.*

The above theorem can be restated as follows: For any $M_1, M_2, M_3 \in \mathcal{M}$ the following axioms hold

(i) $M_1 \# M_2 \in \mathcal{M}$,

(ii) $M_1 \# \mathbb{S}^n \cong M_1$,

(iii) $(M_1 \# M_2) \# M_3 \cong M_1 \# (M_2 \# M_3)$,

(iv) $M_1 \# M_2 \cong M_2 \# M_1$.

Here, $\cong$ indicates that two manifolds are diffeomorphic.

Theorem 3.3 *Let M and N be oriented, smooth n-dimensional manifolds with non-empty boundaries and $f : \partial M \to \partial N$ a diffeomorphism between the boundaries. Then the adjunction space*

$$M \cup_f N,$$

formed by identifying each $x \in \partial M$ with $f(x) \in \partial N$, is a closed, smooth n-dimensional manifold. If M and N are both compact, then $M \cup_f N$ is compact, and if they are both connected, then $M \cup_f N$ is connected.

In the above theorem, if $N = M$ and $f = \mathrm{id}_{\partial M}$, then the corresponding adjunction space $M \cup_{\mathrm{id}_{\partial M}} M$ is called the *double of M* and denoted by $d(M)$.

3.2. Vector Bundles

In this section, we recall the definition of a vector bundle and state some results on tangent bundles. We refer to (Hatcher, 2003) and (Lee, 2013) and references therein for details. Throughout this section, $\mathbb{F}$ denotes the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

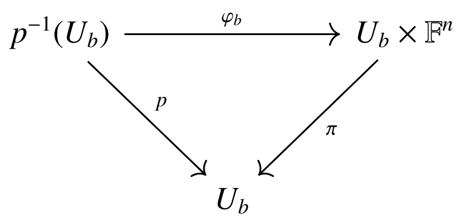
Definition 3.3 *An n-dimensional vector bundle over $\mathbb{F}$ is a triple $\xi = (E, p, B)$, where E and B are topological spaces and $p : E \to B$ is a continuous surjection satisfying the following conditions:*

(i) For each $b \in B$, the fiber $E_b = p^{-1}(b)$ over b is an n-dimensional vector space.

(ii) For each $b \in B$, there is a neighbourhood U_b of b and a homeomorphism

$$\varphi_b : p^{-1}(U_b) \to U_b \times \mathbb{F}^n,$$

called a local trivialization of the vector bundle ξ over U_b, such that the restriction of φ_b to E_b is a vector space isomorphism from $E_b = p^{-1}(b)$ to $\{b\} \times \mathbb{F}^n \cong \mathbb{F}^n$ and the below diagram is commutative



Here, $\pi : U_b \times \mathbb{F}^n \to U_b$ is the natural projection. The space B is called the base space, E is the total space of the bundle.

An $\mathbb{F}-$vector bundle is called a real vector bundle if $\mathbb{F} = \mathbb{R}$, a complex vector bundle if $\mathbb{F} = \mathbb{C}$.

Definition 3.4 *An isomorphism between vector bundles $\xi = (E, p_1, B)$ and $\xi' = (E', p_2, B)$ over the same base space B is a homeomorphism*

$$h : E \to E'$$

taking each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b')$ by a linear isomorphism. Thus, *an isomorphism preserves all the structure of a vector bundle. If* ξ *and* ξ' *are isomorphic, then we write* $\xi \cong \xi'$.

Example 3.1 *Some examples of vector bundles are given as follows.*

(i) The n-dimensional product or trivial vector bundle $\varepsilon^n(M)$ *over a topological space* M *is the bundle*

$$(M \times \mathbb{F}^n, p, M)$$

with the vector space structure of $\mathbb{F}^n$ *defining the vector space structure on* $b \times \mathbb{F}^n = p^{-1}(b)$ *for* $b \in B$. *The local triviality condition holds by letting* $U_b = B$ *and* $\varphi_b = \mathrm{id}_{p^{-1}(U_b)}$. *Here,* $p : B \times \mathbb{F}^n \to B$ *is the projection onto the first factor.*

(ii) The line bundle is the one-dimensional vector bundle. Thus, the trivial line bundle over a manifold M *is isomorphic to the product bundle* $\varepsilon^1(M) = (M \times \mathbb{R}, p, M)$.

Definition 3.5 *Let* M *be a smooth manifold with or without boundary, and let* x *be a point of* M. *A derivation at* x *is a linear map*

$$\mathcal{D} : C^\infty(M) \to \mathbb{R}$$

that satisfies the Leibniz rule

$$\mathcal{D}(fg) = f(x)\mathcal{D}(g) + g(x)\mathcal{D}(g)$$

for all $f, g \in C^\infty(M)$. *The tangent space* T_xM *to* M *at* x *is the set of all derivations of* $C^\infty(M)$. *Moreover,* T_xM *is a vector space and its elements are called tangent vectors at* x.

Definition 3.6 *The tangent bundle of a smooth manifold* M, *with or without boundary, is the triple*

$$\tau(M) = (TM, p, M),$$

where TM *is the disjoint union of the tangent spaces at all points of* M

$$TM = \coprod_{x \in M} T_xM = \{(x, v) \mid x \in M, v \in T_xM\},$$

and $p : TM \to M$ is the projection map which sends each vector v in T_xM to the point x at which it is tangent: $p(x, v) = x$.

Example 3.2 *The tangent bundle of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ is a vector bundle*

$$\tau(\mathbb{S}^n) = (E, p, \mathbb{S}^n),$$

where $E = \{(x, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \mid x \perp v\}$ and v is the tangent vector to $\mathbb{S}^n$ by translating it so that its tail is at the head of x, on $\mathbb{S}^n$. The map

$$p : E \to \mathbb{S}^n$$

sends (x, v) to x. Choose any point $x \in \mathbb{S}^n$ and let $U_x \subset \mathbb{S}^n$ be the open hemisphere containing x and bounded by the hyperplane through the origin orthogonal to x. The local trivialization

$$\varphi_x : p^{-1}(U_x) \to U_x \times p^{-1}(x) \cong U_x \times \mathbb{R}^n$$

given by

$$\varphi_x(y, v) = (y, \pi_x(v)),$$

where π_x is orthogonal projection onto the hyperplane $p^{-1}(x)$ and it restricts to an isomorphism of $p^{-1}(y)$ onto $p^{-1}(x)$ for each $y \in U_x$.

Definition 3.7 *A compact smooth manifold M is called parallelizable if its tangent bundle $\tau(M)$ is trivial. If the tangent bundle of $M - \{x\}$ is trivial for some $x \in M$, then M is called almost parallelizable.*

Example 3.3 *Here we list some important examples of parallelizable manifolds.*

(i) The n-torus for all n.

(ii) Lie groups such as Euclidean spaces $\mathbb{R}^n$, orthogonal groups $O(n)$, and unitary groups $U(n)$, etc.

(iii) *The tangent bundle* $\tau(\mathbb{S}^1)$ *of* $\mathbb{S}^1$ *is trivial since there is an isomorphism*

$$h : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$$
$$(e^{i\theta}, ite^{i\theta}) \mapsto (e^{i\theta}, t)$$

for $e^{i\theta} \in \mathbb{S}^1$ *and* $t \in \mathbb{R}$. *Moreover, Bott and Milnor (1958) and independently Kervaire (1958) showed that the only parallelizable spheres are* $\mathbb{S}^1, \mathbb{S}^3, \mathbb{S}^7$.

(iv) *All compact, connected, orientable* 3*-dimensional manifold is parallelizable. The details of the proof can be found in (Milnor and Stasheff, 2016).*

(v) *The Stiefel manifold* $V_k(\mathbb{R}^n)$ *is a subspace of the product of* n *copies of the unit sphere* $\mathbb{S}^{k-1}$, *namely, the subspace of orthogonal* n *tuples. Since the product of spheres is compact,* $V_k(\mathbb{R}^n)$ *is also a compact manifold. The complex Stiefel manifold* $V_k(\mathbb{C}^n)$ *and quaternionic Stiefel manifold* $V_k(\mathbb{H}^n)$ *are defined analogously using the standard Hermitian product on* $\mathbb{C}^n$ *and the standard quaternionic product* $\mathbb{H}^n$ *defined as*

$$p \cdot p' = \sum_{1 \le i \le n} \bar{p}_i p'_i$$

for $p, p' \in \mathbb{H}^n$. *The Stiefel manifolds* $V_k(\mathbb{R}^n)$, $V_k(\mathbb{C}^n)$ *and* $V_k(\mathbb{H}^n)$ *are parallelizable if* $k > 2$ *by (Sutherland, 1964; Handel, 1965; Lam, 1975).*

Proposition 3.2 *Every parallelizable smooth manifold is orientable.*

Definition 3.8 *Let* $\xi = (E, p_1, B)$ *and* $\xi' = (E', p_2, B')$ *be two vector bundles. The Cartesian product of* ξ *and* ξ' *is the bundle*

$$\xi \times \xi' = (E \times E', p_1 \times p_2, B \times B')$$

with fibers the products $p_1^{-1}(b) \times p_2^{-1}(b') = E_b \times E'_{b'}$. *If we have local trivializations*

$$\varphi_b : p_1^{-1}(U_b) \to U_b \times \mathbb{R}^n,\ \varphi_{b'} : p_2^{-1}(U_{b'}) \to U_{b'} \times \mathbb{R}^m$$

for E and E', then $\varphi_b \times \varphi_{b'}$ is a local trivialization for $E \times E'$.

Given two vector bundles $\xi_1 = (E, p_1, B)$ and $\xi_2 = (E', p_2, B)$ over the same base space B, a third vector bundle over B can be created such that its fiber over each point of B is the direct sum of the fibers of E and E' over this point which is called the Whitney Sum of the bundles ξ_1 and ξ_2. Its formal definition is given as follows:

Definition 3.9 *The Whitney Sum of two vector bundles ξ_1 and ξ_2 over B is the bundle*

$$\xi_1 \oplus \xi_2 = (E \oplus E', p, B),$$

where the total space $E \oplus E'$ is defined as

$$\{(e_1, e_2) \in E \times E' \mid p_1(e_1) = p_2(e_2)\}$$

and the projection $p : E \oplus E' \to B$ sending (e_1, e_2) to the point $p_1(e_1) = p_2(e_2)$. The fiber at each $b \in B$ is the direct sum $E_b \oplus E'_b$.

Definition 3.10 *Given a vector bundle $\xi = (E, p, B)$ and a subspace $A \subset B$, the triple*

$$\xi_{|_A} = (p^{-1}(A), p, A)$$

is a vector bundle, called the restriction of E over A.

For $\xi_1 = (E, p_1, B)$ and $\xi_2 = (E', p_2, B)$, the restriction of the product $E \times E'$ over the diagonal $B = \{(b, b) \in B \times B\}$ is exactly $\xi_1 \oplus \xi_2$.

Proposition 3.3 *Given a map $f : A \to B$ and a vector bundle (E, p, B), there exists a vector bundle (E', p', A) with a map $f' : E' \to E$ taking the fiber of E' over each point $a \in A$ isomorphically onto the fiber of E over $f(a)$, and such a vector bundle E' is unique up to isomorphism*

$$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

From the uniqueness statement it follows that the isomorphism type of E' depends only on the isomorphism type of E since we can compose the map f' with an isomorphism of E with another vector bundle over B. Thus, there is a function

$$f^* : Vect(B) \to Vect(A)$$

taking the isomorphism class of E to the isomorphism class of E'. Often the vector bundle E' is written as $f^*(E)$ and called the *pull back* of E by f.

Note that the Whitney sum $\xi_1 \oplus \xi_2$ can be considered as the pull-back bundle of the diagonal map from B to $B \times B$, where the bundle over $B \times B$ is $E \times E'$.

3.3. π-Manifolds

The purpose of this section is to give the basic definitions and theorems for the class of manifolds that can be imbedded in a Euclidean space of a sufficiently high dimension with a trivial normal bundle. This class of manifolds can be characterized by a condition on the tangent bundle. The discussions and results of this section appear in (Kervaire and Milnor, 1963) and (Whitehead, 1940). For the sake of the completeness, we also provide some of the proofs.

Let M be a compact, oriented, smooth manifold with tangent bundle $\tau(M)$, and let $\varepsilon^1(M)$ denote a trivial line bundle. The Whitney sum $\tau(M) \oplus \varepsilon^1(M)$ is called the *stable tangent bundle* of M.

Definition 3.11 *A manifold is said to be a π-manifold or stably parallelizable if its stable tangent bundle is trivial.*

For brevity, we call such manifolds π-manifolds.

Theorem 3.4 *(Whitehead, 1940) Every parallelizable manifold is a π-manifold. Every π-manifold is almost parallelizable.*

The following lemmas give the necessary and sufficient conditions for being a π-manifold.

Lemma 3.1 *(Kervaire and Milnor, 1963) Let ξ be a k-dimensional vector bundle over an n-dimensional complex, $k > n$. If the Whitney sum of ξ with a trivial bundle ε^r is trivial, then ξ itself is trivial.*

Proof Suppose that $r = 1$, and ξ is oriented. From the following isomorphism

$$\xi \oplus \varepsilon^1 \cong \varepsilon^{k+1}$$

it follows that there is a bundle map f from ξ to φ^k of oriented k-planes in $(k + 1)$-space. The dimension of the base space of ξ is n, and also the base space of φ^k is the k-sphere $\mathbb{S}^k$, $k > n$. Thus, f is null-homotopic; and hence ξ is trivial. □

Lemma 3.2 *(Whitehead, 1940) Let M be an n-dimensional submanifold of $\mathbb{S}^{n+k}$, $n < k$. Then M is a π-manifold if and only if its normal bundle is trivial.*

Proof Since the Whitney sum $\tau(M) \oplus \nu(M)$ is trivial, $(\tau(M) \oplus \varepsilon^1(M)) \oplus \nu(M)$ is trivial. By Lemma 3.1, the conclusion follows. □

Lemma 3.3 *(Kervaire and Milnor, 1963) A connected manifold with non-empty boundary is a π-manifold if and only if it is parallelizable.*

Proof This follows by a similar argument. By the hypothesis on the manifold, every map into a sphere of the same dimension is null-homotopic. □

The class of π-manifolds behaves nicely under the product operation. So, the following two propositions are straightforward from Lemma 3.2 and Lemma 3.3.

Proposition 3.4 *Let M_1, M_2 be closed smooth manifolds.*

(i) The product manifold $M_1 \times M_2$ is a π-manifold if and only if M_1 and M_2 are both π-manifolds.

(ii) The product manifold $M_1 \times M_2$ is parallelizable provided that M_1 is a π-manifold and M_2 is parallelizable.

Proposition 3.5 *(Kervaire and Milnor, 1963) The connected sum of two π-manifolds is a π-manifold.*

Example 3.4 *Here we give some important examples of π-manifolds.*

(i) *The n-sphere* $\mathbb{S}^n$ *is a* π*-manifold. For any* $n \in \mathbb{N}$, *the Whitney sum* $\tau(\mathbb{S}^n) \oplus \varepsilon^1(\mathbb{S}^n)$ *is isomorphic to the trivial bundle* $\mathbb{S}^n \times \mathbb{R}^{n+1}$ *since elements of the direct sum are triples* $(x, v, t) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \times \mathbb{R}$ *with* $x \perp v$ *and the map*

$$(x, v, t) \mapsto (x, (v, t))$$

gives an isomorphism of the direct sum bundle $\tau(\mathbb{S}^n) \oplus \varepsilon^1(\mathbb{S}^n)$ *with* $\mathbb{S}^n \times \mathbb{R}^{n+1}$.

(ii) *Let* $\Sigma_{g,0}$ *be a closed, orientable, connected surface of genus* g. *The Euler characteristic of* $\Sigma_{g,0}$ *is* $\chi(\Sigma_{g,0}) = 2 - 2g$.

- *If* $g = 0$, *then* $\Sigma_{g,0}$ *is* 2*-sphere* $\mathbb{S}^2$ *and it is a* π*-manifold,*
- *If* $g = 1$, *then* $\Sigma_{g,0}$ *equals the torus* $\mathbb{S}^1 \times \mathbb{S}^1$ *which is parallelizable,*
- *If* $g > 1$, *then* $\Sigma_{g,0}$ *is the connected sum of* g*-copies of torus which is a* π*-manifold by Proposition 3.5.*

(iii) *Let* W *be a parallelizable manifold with boundary* $M = \partial(W)$. *Then* W *is necessarily orientable and the normal bundle* ν *to the inclusion* $M \hookrightarrow W$ *is a trivial line bundle. Thus,* $\tau(W)_{|_M} \cong \tau(M) \oplus \varepsilon^1(M)$. *The triviality of the bundle* $\tau(W)_{|_M}$ *implies that* $\tau(M) \oplus \varepsilon^1(M)$ *is trivial; that is,* M *is a* π*-manifold. Moreover, the boundary of a* π*-manifold is also a* π*-manifold.*

One of the main themes in geometric topology is the study of smooth manifolds and their piecewise linear (PL) triangulations. Shortly after Milnor (1956)'s discovery of exotic smooth spheres in seven dimensions, Kervaire (1960) constructed the first example (in dimension 10) of a PL-manifold with no differentiable structure, and a new exotic smooth 9-sphere.

The *Kervaire manifold* M_K^{4k+2} is a closed, almost parallelizable, PL-manifold with the same homology as the product $\mathbb{S}^{2k+1} \times \mathbb{S}^{2k+1}$ of spheres. It is simply-connected when $k > 0$. The Kervaire manifold can be constructed as follows: for $2k + 1 \neq 1, 3, 7$, let

$$p : T \to \mathbb{S}^{2k+1}$$

be the tangent disc bundle of $\mathbb{S}^{2k+1}$ and $\overline{\mathbb{D}^{2k+1}}$ the closed $(2k+1)$-ball,

$$h : \overline{\mathbb{D}^{2k+1}} \to \mathbb{S}^{2k+1}$$

an embedding,

$$k : \overline{\mathbb{D}^{2k+1}} \times \overline{\mathbb{D}^{2k+1}} \to T$$

a bundle map covering h. Assume that $\widetilde{T}$ is a copy of T and

$$M^{4k+2} = T \cup \widetilde{T}$$

with $h(x, y)$ identified to $h(y, x)$ for each $(x, y) \in \overline{\mathbb{D}^{2k+1}} \times \overline{\mathbb{D}^{2k+1}}$. Thus, M^{4k+2} is a manifold with boundary and the boundary $\partial(M^{4k+2}) = \Sigma^{4k+1}$ is a smooth homotopy sphere, called the Kervaire sphere. By Smale (1961), Σ^{4k+1} is always PL-homeomorphic to the standard sphere $\mathbb{S}^{4k+1}$. Let $f : \partial(M^{4k+2}) \to \mathbb{S}^{4k+1}$ be a homeomorphism. Then the *Kervaire manifold* M_K^{4k+2} is the adjunction space

$$M^{4k+2} \cup_f \overline{\mathbb{D}^{4k+2}}$$

formed by identifying each $x \in \partial(M^{4k+2})$ with $f(x) \in \mathbb{S}^{4k+1}$.

Theorem 3.5 *(Brown Jr and Peterson, 1965) If M is a smooth manifold with the same homotopy type as Kervaire manifold M_K^{2n}, then M is a π-manifold.*

The Kervaire semi-characteristic is an invariant of closed $(4n + 1)$-dimensional manifolds and it is introduced by Kervaire (1956).

Definition 3.12 *Let M be a closed, oriented, smooth $(4n + 1)$-dimensional manifold. The Kervaire semi-characteristic $k(M)$ of M is a mod 2 invariant defined by*

$$\sum_{i=0}^{2n} \operatorname{rank}(H^{2i}(M; \mathbb{R})) \bmod 2.$$

In the following theorem, Sutherland (1964) answered the questions: How many parallelizable manifolds are and what style they have.

Theorem 3.6 *(Sutherland, 1964) Let M^n be a closed n-dimensional π-manifold. Then M^n is parallelizable if and only if*

(i) *n is even and the Euler characteristic $\chi(M^n)$ of M^n is zero, or*

(ii) *n is odd, $n \neq 1,3,7$, and the Kervaire semi-characteristic $k(M^n)$ of M^n is zero* mod 2*, or*

(iii) $n = 1,3,7$.

By Theorem 3.6, it is concluded that $\mathbb{S}^2$ is not parallelizable since its Euler characteristic $\chi(\mathbb{S}^2)$ is 2. But it is a closed π-manifold.

A *homotopy n-sphere* $\widetilde{\mathbb{S}^n}$ is a closed n-dimensional manifold with the homotopy type of n-sphere $\mathbb{S}^n$. That is, it has the same homotopy groups and the same homology groups as the n-sphere. Namely,

$$H_i(\widetilde{\mathbb{S}^n};\mathbb{Z}) = \begin{cases} \mathbb{Z}\ , & i = 0,n \\ 0\ , & i \neq 0,n. \end{cases}$$

Using the results of Adams and Walker (1965), Kervaire and Milnor (1963) proved the following theorem.

Theorem 3.7 *Kervaire and Milnor (1963) Homotopy spheres are π-manifolds.*

The proof of Theorem 3.7 based on the obstruction to the triviality of $\tau(\widetilde{\mathbb{S}^n}) \oplus \varepsilon^1(\widetilde{\mathbb{S}^n})$ is a well-defined cohomology class

$$\mathfrak{o}_n(\widetilde{\mathbb{S}^n}) \in H^n(\widetilde{\mathbb{S}^n}, \pi_{n-1}(SO_{n+1})) = \pi_{n-1}(SO_{n+1}).$$

The coefficient group can be identified with the stable group $\pi_{n-1}(SO_{n+1})$. Bott (1959) computed these stable groups for $n > 2$, as follows

$$\pi_{n-1}(SO_{n+1}) = \begin{cases} \mathbb{Z}\ , & n \equiv 0,4 \pmod 8 \\ \mathbb{Z}_2\ , & n \equiv 1,2 \pmod 8 \\ 0\ , & n \equiv 3,5,6,7 \pmod 8. \end{cases}$$

Here $\mathbb{Z}$, $\mathbb{Z}_2$, and 0 denote the cyclic groups of order ∞, 2, and 1, respectively.

If $n \equiv 3, 5, 6$, or 7 (mod 8), then $\pi_{n-1}(SO_{n+1}) = 0$. Hence, $\mathfrak{o}_n(\Sigma)$ is trivially zero. For the case $n \equiv 0$ or 4 (mod 8), let $n = 4k$. By (Kervaire, 1959; Milnor and Kervaire, 1960), every homotopy $4k$-sphere is a π-manifold. When $n \equiv 1$ or 2 (mod 8), the Hopf-Whitehead homomorphism $J_{n-1} : \pi_{n-1}(SO_{n+1}) \to \pi_{n+k-1}(\mathbb{S}^k)$, in the stable range $k > n$, is injective by Adams and Walker (1965). Moreover, an argument of Rohlin implies that $J_{n-1}(\mathfrak{o}_n(\widetilde{\mathbb{S}^n})) = 0$, so $\mathfrak{o}_n(\widetilde{\mathbb{S}^n}) = 0$.

CHAPTER 4

REIDEMEISTER TORSION

In this chapter, we first give the basic definitions and facts about Reidemeister torsion. We then present the notion of a symplectic chain complex. This chapter are mainly based on the results in (Özel and Sözen, 2012; Porti, 1997; Sözen, 2008, 2012a,b; Turaev, 2002; Witten, 1991).

4.1. Reidemeister Torsion of Chain Complexes

First, we give the notations. Let $\mathbb{F}$ be a field and let V be a finite dimensional vector space over $\mathbb{F}$. Suppose that $\dim(V) = k$ and all bases of V are ordered. For any bases $\mathbf{b} = (b_1, \ldots, b_k)$ and $\mathbf{c} = (c_1, \ldots, c_k)$ of the space V, the following equality holds

$$b_i = \sum_{j=1}^{k} a_{ij} c_j, \quad i = 1, \ldots, k,$$

where the *transition matrix* (a_{ij}) is a non-singular $(k{\times}k)$-matrix over $\mathbb{F}$. For the determinant of a matrix, the following notation will be used

$$[\mathbf{b}, \mathbf{c}] = \det(a_{ij}) \in \mathbb{F}^*(= \mathbb{F} - \{0\}).$$

Clearly, the determinant of the transition matrix satisfies the following properties :

- $[\mathbf{b}, \mathbf{b}] = 1$,
- if $\mathbf{d}$ is a third basis of V, then $[\mathbf{b}, \mathbf{d}] = [\mathbf{b}, \mathbf{c}] \cdot [\mathbf{c}, \mathbf{d}]$,
- For the trivial vector space $V = \{0\}$, $[\mathbf{h}, \mathbf{h}] = 1$ by using the convention $1 \cdot 0 = 0$.

Let $\mathbb{F}$ denote the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, and let C_* be a chain complex of finite dimensional vector spaces over $\mathbb{F}$

$$C_* = (0 \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0).$$

For $p = 0, \ldots, n$, let

$$H_p(C_*) = Z_p(C_*)/B_p(C_*)$$

be the p-th homology space of the chain complex C_*, where

$$B_p(C_*) = \operatorname{Im}\{\partial_{p+1} : C_{p+1} \to C_p\},$$

$$Z_p(C_*) = \operatorname{Ker}\{\partial_p : C_p \to C_{p-1}\}.$$

Considering the First Isomorphism Theorem for the sequence (4.1) and the definition of $H_p(C_*)$ for the sequence (4.2), it is easily shown that the following sequences are short-exact

$$0 \longrightarrow Z_p(C_*) \overset{\imath}{\hookrightarrow} C_p \xrightarrow{\partial_p} B_{p-1}(C_*) \longrightarrow 0, \tag{4.1}$$

$$0 \longrightarrow B_p(C_*) \overset{\imath}{\hookrightarrow} Z_p(C_*) \overset{\varphi_p}{\twoheadrightarrow} H_p(C_*) \longrightarrow 0. \tag{4.2}$$

Here, $\imath$ and φ_p are the inclusion and the natural projection, respectively.

Suppose that $s_p : B_{p-1}(C_*) \to C_p$ and $\ell_p : H_p(C_*) \to Z_p(C_*)$ are sections of $\partial_p : C_p \to B_{p-1}(C_*)$ and $\varphi_p : Z_p(C_*) \to H_p(C_*)$, respectively. Then the short exact sequences (4.1) and (4.2) yield

$$C_p = B_p(C_*) \oplus \ell_p(H_p(C_*)) \oplus s_p(B_{p-1}(C_*)). \tag{4.3}$$

Let $\mathbf{c}_p = \{c_p^1, \ldots, c_p^{m_p}\}$, $\mathbf{b}_p = \{b_p^1, \ldots, b_p^{l_p}\}$, and $\mathbf{h}_p = \{h_p^1, \ldots, h_p^{n_p}\}$ be bases of C_p, $B_p(C_*)$, and $H_p(C_*)$, respectively. By equation (4.3), $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ becomes the new basis of C_p for $p \in \{0, \ldots, n\}$.

By using the above arguments, Milnor (1966) defined the Reidemeister torsion of a general chain complex as follows.

Definition 4.1 *Reidemeister torsion of a general chain complex C_* with respect to bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is defined as the alternating product*

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_{p=0}^n, \{\mathbf{h}_p\}_{p=0}^n\right) = \prod_{p=0}^{n} \left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p\right]^{(-1)^{(p+1)}} \in \mathbb{F}^*,$$

where $[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p]$ is the determinant of the transition matrix from the initial basis $\mathbf{c}_p$ to the obtained basis $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ of C_p.

Milnor (1966) proved that Reidemeister torsion is independent of the bases $\mathbf{b}_p$, and sections s_p, ℓ_p. More precisely, for a different choice of bases $\widetilde{\mathbf{b}}_p$ for $B_p(C_*)$, the following equality holds

$$\prod_{p=0}^{n} \left(\left[\widetilde{\mathbf{b}}_p, \mathbf{b}_p\right] \cdot \left[s_p(\widetilde{\mathbf{b}}_{p-1}), s_p(\mathbf{b}_{p-1})\right]\right)^{(-1)^{(p+1)}} = 1.$$

Then Reidemeister torsion becomes

$$\begin{aligned}
&\prod_{p=0}^{n} \left(\left[\widetilde{\mathbf{b}}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\widetilde{\mathbf{b}}_{p-1}), \mathbf{c}_p\right]\right)^{(-1)^{(p+1)}} \\
= &\prod_{p=0}^{n} \left(\left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p\right] \cdot \left[\widetilde{\mathbf{b}}_p, \mathbf{b}_p\right] \cdot \left[s_p(\widetilde{\mathbf{b}}_{p-1}), s_p(\mathbf{b}_{p-1})\right]\right)^{(p+1)} \\
= &\prod_{p=0}^{n} \left(\left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p\right]\right)^{(-1)^{(p+1)}}.
\end{aligned}$$

On the other hand, it depends on the bases $\mathbf{c}_p$ and $\mathbf{h}_p$. If one makes a change $\mathbf{c}_p \mapsto \widetilde{\mathbf{c}}_p$ and $\mathbf{h}_p \mapsto \widetilde{\mathbf{h}}_p$, then Reidemeister torsion changes as follows

$$\mathbb{T}\left(C_*, \{\widetilde{\mathbf{c}}_p\}_{p=0}^n, \{\widetilde{\mathbf{h}}_p\}_{p=0}^n\right) = \prod_{p=0}^{n} \left(\left[\widetilde{\mathbf{c}}_p, \mathbf{c}_p\right]\left[\widetilde{\mathbf{h}}_p, \mathbf{h}_p\right]^{-1}\right)^{(-1)^p} \mathbb{T}\left(C_*, \{\mathbf{c}_p\}_{p=0}^n, \{\mathbf{h}_p\}_{p=0}^n\right).$$

Applying the Zig-zag Lemma to the short exact sequence of chain complexes

$$0 \to A_* \overset{\imath}{\hookrightarrow} B_* \overset{\pi}{\to} D_* \twoheadrightarrow 0,$$

one can obtain the long exact sequence $\mathcal{H}_*$ of vector spaces as follows

$$\cdots \longrightarrow H_p(A_*) \xrightarrow{\iota_p} H_p(B_*) \xrightarrow{\pi_p} H_p(D) \xrightarrow{\delta_p} H_{p-1}(A_*) \xrightarrow{\iota_{p-1}} H_{p-1}(B_*) \xrightarrow{\pi_{p-1}} H_{p-1}(D) \xrightarrow{\delta_{p-1}} H_{p-2}(A_*) \xrightarrow{\iota_{p-2}} \cdots$$

Namely, $\mathcal{H}_*$ is an exact (or acyclic) complex C_* of length $3n+2$ with $C_{3p}(\mathcal{H}_*) = H_p(D_*)$, $C_{3p+1}(\mathcal{H}_*) = H_p(A_*)$, and $C_{3p+2}(\mathcal{H}_*) = H_p(B_*)$. Clearly, the bases $\mathbf{h}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^B$ are considered as bases for $C_{3p}(\mathcal{H}_*)$, $C_{3p+1}(\mathcal{H}_*)$, and $C_{3p+2}(\mathcal{H}_*)$, respectively.

By using the above set-up, Milnor (1966) showed that Reidemeister torsion has a multiplicativity property. More precisely,

Theorem 4.1 *(Milnor, 1966) Suppose that* $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, *and* $\mathbf{h}_p^D$ *are bases of* A_p, B_p, D_p, $H_p(A_*)$, $H_p(B_*)$, *and* $H_p(D_*)$, *respectively. Suppose also that* $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, *and* $\mathbf{c}_p^D$ *are compatible in the sense that*

$$\left[\mathbf{c}_p^B, \mathbf{c}_p^A \sqcup \widetilde{\mathbf{c}_p^D}\right] = \pm 1,$$

where $\pi_p\left(\widetilde{\mathbf{c}_p^D}\right) = \mathbf{c}_p^D$. *Then the following formula holds*

$$\mathbb{T}\left(B_*, \{\mathbf{c}_p^B\}_{p=0}^n, \{\mathbf{h}_p^B\}_{p=0}^n\right) = \mathbb{T}\left(A_*, \{\mathbf{c}_p^A\}_{p=0}^n, \{\mathbf{h}_p^A\}_{p=0}^n\right)\ \mathbb{T}\left(D_*, \{\mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^D\}_{p=0}^n\right) \times \mathbb{T}\left(\mathcal{H}_*, \{\mathbf{c}_{3p}\}_{p=0}^{3n+2}, \{0\}_{p=0}^{3n+2}\right).$$

Definition 4.2 *(Borghini, 2015) The Reidemeister torsion of* $\mathcal{H}_*$, $\mathbb{T}(\mathcal{H}_*, \{\mathbf{c}_{3p}\}_{p=0}^{3n+2}, \{0\}_{p=0}^{3n+2})$, *stated in Theorem 4.1 is called the corrective term.*

It is clear from Theorem 4.1 that

Lemma 4.1 *(Milnor, 1966) If* A_*, D_* *are two chain complexes, and if* $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, *and* $\mathbf{h}_p^D$

are respectively bases of A_p, D_p, $H_p(A_*)$, *and* $H_p(D_*)$, *then*

$$T\left(A_* \oplus D_*, \{\mathbf{c}_p^A \sqcup \mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^A \sqcup \mathbf{h}_p^D\}_{p=0}^n\right) = \mathbb{T}\left(A_*, \{\mathbf{c}_p^A\}_{p=0}^n, \{\mathbf{h}_p^A\}_{p=0}^n\right)\ \mathbb{T}\left(D_*, \{\mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^D\}_{p=0}^n\right).$$

4.2. Symplectic Chain Complex

A *symplectic chain complex* of vector spaces over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is a chain complex of length q

$$(C_*, \partial_*, \{\omega_{*,q-*}\}) : 0 \to C_q \stackrel{\partial_q}{\to} C_{q-1} \to \cdots \to C_{q/2} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0$$

with the following properties:

(i) $q \equiv 2 \pmod 4$,

(ii) There is a non-degenerate bilinear form

$$\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{R}$$

for $p = 0, \ldots, q/2$ such that

- $\partial-$compatible: $\omega_{p,q-p}(\partial_{p+1}a, b) = (-1)^{p+1}\omega_{p+1,q-(p+1)}(a, \partial_{q-p}b)$,
- anti-symmetric: $\omega_{p,q-p}(a, b) = (-1)^{p(q-p)}\omega_{q-p,p}(b, a)$.

From the fact that $q \equiv 2 \pmod 4$ it follows $\omega_{p,q-p}(a, b) = (-1)^p\omega_{q-p,p}(b, a)$. Using the $\partial-$compatibility of $\omega_{p,q-p}$, they can be extended to homologies

$$[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathbb{R},$$

where $[\omega_{p,q-p}]([x], [y]) = \omega_{p,q-p}(x, y)$ is an anti-symmetric, non-degenerate, bilinear form. For details, see (Sözen, 2008).

Definition 4.3 *Let $(C_*, \partial_*, \{\omega_{*,q-*}\})$ be a symplectic chain complex. The bases $\mathbf{c}_p$ and $\mathbf{c}_{q-p}$ of C_p and C_{q-p} are ω-*compatible *if the matrix of $\omega_{p,q-p}$ in bases $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ is equal to*

$$\begin{cases} I_{k\times k} & , p \neq n/2, \\ \begin{pmatrix} 0_{l\times l} & I_{l\times l} \\ I_{l\times l} & 0_{l\times l} \end{pmatrix} & , p = n/2. \end{cases}$$

Here, $k = \dim(C_p) = \dim(C_{q-p})$, and $2l = \dim(C_{q/2})$.

Note that every symplectic chain complex has $\omega-$compatible bases. Using the existence of $\omega-$compatible bases, the following result gives the Reidemeister torsion of an $\mathbb{F}-$symplectic chain complex.

Theorem 4.2 *Let C_* be an $\mathbb{F}-$symplectic chain complex of length $2n$. Suppose that $\mathbf{c}_p$ is an $\omega-$compatible basis of C_p and $\mathbf{h}_p$ is a basis of $H_p(C_*)$ for $p = 0, \dots, 2n$.*

(i) If C_* is an $\mathbb{R}-$symplectic chain complex, then

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_{p=0}^{2n}, \{\mathbf{h}_p\}_{p=0}^{2n}\right) = \prod_{p=0}^{n-1} \Delta_{p,2n-p}(\mathbf{h}_p, \mathbf{h}_{2n-p})^{(-1)^p} \cdot \sqrt{\Delta_{n,n}(\mathbf{h}_n, \mathbf{h}_n)}^{\,(-1)^n}.$$

(ii) If C_* is a $\mathbb{C}-$symplectic chain complex, then

$$\left|\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_{p=0}^{2n}, \{\mathbf{h}_p\}_{p=0}^{2n}\right)\right| = \prod_{p=0}^{n-1} \left|\Delta_{p,2n-p}(\mathbf{h}_p, \mathbf{h}_{2n-p})\right|^{(-1)^p} \cdot \sqrt{\left|\Delta_{n,n}(\mathbf{h}_n, \mathbf{h}_n)\right|}^{\,(-1)^n}.$$

Here, $\Delta_{p,2n-p}(\mathbf{h}_p, \mathbf{h}_{2n-p})$ is the determinant of the matrix of the non-degenerate pairing $[\omega_{p,2n-p}] : H_p(C_*) \times H_{2n-p}(C_*) \to \mathbb{F}$ in the bases $\mathbf{h}_p, \mathbf{h}_{2n-p}$.

The details of the proof of Theorem 4.2 can be found in (Sözen, 2008; Sözen, 2014).

4.3. Reidemeister Torsion of Manifolds

Let M be an n-dimensional manifold and let K be a cell decomposition of M. Denote the set of p-cells by $C_p(K)$. The cell decomposition K of M canonically defines a chain complex $C_*(K)$ of free abelian groups as follows

$$C_*(K) = (0 \to C_n(K) \stackrel{\partial_n}{\to} C_{n-1}(K) \to \cdots \to C_1(K) \stackrel{\partial_1}{\to} C_0(K) \to 0),$$

where ∂_p is the boundary operator for $p \in \{1, \ldots, n\}$. By orienting the p-cells and ordering $C_p(K)$, this chain complex has a *geometric basis* $\mathbf{c}_p = \{c_p^1, \cdots, c_p^{m_p}\}$ for $C_p(K)$.

Definition 4.4 *(Milnor, 1966) Let M be an n-dimensional manifold with a cell decomposition K. Let $\mathbf{c}_p$ and $\mathbf{h}_p$ be bases of $C_p(K)$ and $H_p(M)$, respectively. Reidemeister torsion of M is defined as follows*

$$\mathbb{T}\left(C_*(K), \{\mathbf{c}_p\}_{p=0}^n, \{\mathbf{h}_p\}_{p=0}^n\right).$$

Following the arguments introduced in (Sözen, 2008, Lemma 2.0.5), one can obtain the following lemma.

Lemma 4.2 *Reidemeister torsion of M does not depend on the cell decomposition.*

From the lemma above, we can conclude that the Reidemeister torsion of M is a well-defined invariant. Thus, we denote by $\mathbb{T}(M, \{\mathbf{h}_p\}_{p=0}^n)$ the Reidemeister torsion of M in the basis $\mathbf{h}_p$ of $H_p(M)$, $p = 0, \ldots, m$.

Let M be a compact, orientable, $2n$-dimensional manifold and let K' be the dual cell decomposition of M corresponding to the cell decomposition K. Without loss of generality, assume that cells $\sigma \in K$ and $\sigma' \in K'$ do not meet more than once. We can do this since Reidemeister torsion is invariant under subdivision. Recall that the dual cell decomposition K' is obtained as follows: Let $K = \{\sigma_\alpha^p\}_{\alpha,p}$. Denote by $\{\tau_\alpha^p\}_{\alpha,p}$ the first barycentric subdivision of K. For each vertex $\sigma_\alpha^0 \in K$, associate the following $2n$-cell

$$(\sigma_\alpha^0)' = \bigcup_{\sigma_\alpha^0 \in \tau_\beta^{2n}} \tau_\beta^{2n}$$

which is the union of all $2n$-simplices τ_β^{2n} in the subdivision with σ_α^0 as a vertex. For every p-simplex in the cell decomposition K, let

$$(\sigma_\alpha^p)' = \bigcap_{\sigma_\beta^0 \in \sigma_\alpha^p} (\sigma_\beta^0)'$$

be the intersection of all $2n$-cells $(\sigma_\beta^0)'$ that are associated to the $p+1$ vertices of σ_α^p. Thus, the dual cell decomposition of M corresponding to K is given by

$$K' = \left\{ \triangle_\alpha^{2n-p} = (\sigma_\alpha^p)' \right\}_{\alpha,p} .$$

Note that $\triangle_\alpha^{2n-p} = (\sigma_\alpha^p)'$ and σ_α^p meet transversely. For a given orientation on σ_α^p, we can take the dual orientation on $\triangle_\alpha^{2n-p}$ as the one at $S \in \sigma_\alpha^p \cap (\sigma_\alpha^p)'$,

$$\imath_S(\sigma_\alpha^p, (\sigma_\alpha^p)') = 1,$$

where $\imath_S$ is the intersection number (index) at S.

Definition 4.5 *The intersection pairing*

$$(\cdot,\cdot)_{p,2n-p} : C_p(K;\mathbb{R}) \times C_{2n-p}(K';\mathbb{R}) \to \mathbb{R}$$

is defined by

$$(\alpha,\beta)_{p,2n-p} = \sum_{S \in \alpha \cap \beta} \imath_S(\alpha,\beta).$$

The intersection pairings $(\cdot,\cdot)_{p,2n-p} : C_p(K;\mathbb{R}) \times C_{2n-p}(K';\mathbb{R}) \to \mathbb{R}$ satisfy the following properties for all $\alpha \in C_p(K;\mathbb{R}),\ \beta \in C_{2n-p}(K';\mathbb{R})$

(i) $(\alpha,\beta)_{p,2n-p} = (-1)^{p(2n-p)}(\beta,\alpha)_{2n-p,p}$,

(ii) $(\alpha, \partial_{2n-p}\,\beta)_{(p+1),2n-(p+1)} = (-1)^{2n-p+1}(\partial_{p+1}\alpha,\beta)_{p,2n-p}$.

Here, ∂ denotes the boundary operator. Since the intersection number (index) is anti-symmetric, (i) is obtained. Moreover, (ii) follows from the fact that

$$\partial_{2n-p}(\triangle_{\alpha}^{2n-p}) = (-1)^{2n-p+1}(\partial_p(\alpha_{\alpha}^{p}))'.$$

For further information, we refer to (Griffiths and Harris, 1994). As a result, $(\cdot,\cdot)_{p,2n-p}$ are $\partial-$compatible and anti-symmetric.

Naturally, the intersection pairing for $2n$-dimensional manifold M can be extended to homologies for each $p \in \{0,\dots,2n\}$ as follows

$$(\cdot,\cdot)_{p,2n-p} : H_p(M)\times H_{2n-p}(M) \to \mathbb{R}.$$

In addition, Poincaré duality gives the following commutative diagram

$$\begin{array}{ccccc} H^{2n-p}(M) & \times & H^{p}(M) & \stackrel{\wedge_{p,2n-p}}{\longrightarrow} & H^{2n}(M) \\ \uparrow \mathrm{PD} & & \uparrow \mathrm{PD} & \circlearrowleft & \uparrow \\ H_p(M) & \times & H_{2n-p}(M) & \stackrel{(,)_{p,2n-p}}{\longrightarrow} & \mathbb{R}. \end{array}$$

Here, $\wedge_{k,2n-k}$ denotes the wedge product.

Let $D_p(K) = C_p(K;\mathbb{R})\oplus C_p(K';\mathbb{R})$ and define $(\cdot,\cdot)_{p,2n-p}$ as zero on

$$C_p(K;\mathbb{R})\times C_{2n-p}(K;\mathbb{R}),$$
$$C_p(K';\mathbb{R})\times C_{2n-p}(K';\mathbb{R}).$$

Then the following chain complex

$$0 \to D_n(K) \stackrel{\partial_n}{\to} D_{n-1}(K) \to \cdots \to D_1(K) \stackrel{\partial_1}{\to} D_0(K) \to 0 \tag{4.4}$$

becomes a symplectic chain complex, see (Sözen, 2012a).

Throughout this thesis, $\triangle^{M}_{p,2n-p}(\mathbf{h}_p,\mathbf{h}_{2n-p})$ denotes the determinant of the matrix of the intersection pairing $(\cdot,\cdot)_{p,2n-p} : H_p(M) \times H_{2n-p}(M) \to \mathbb{R}$ in he homology bases $\mathbf{h}_p, \mathbf{h}_{2n-p}$.

Using symplectic chain complex (4.4) and Poincaré duality, Sözen (2012b) proved the following results.

Theorem 4.3 *Let $\Sigma_{g,0}$ be a closed, orientable, genus $g \geq 1$ surface. Let $\mathbf{h}_p^{\Sigma_{g,0}}$ be basis of $H_p(\Sigma_{g,0})$ for $p = 0,1,2$. Assume that $\Gamma = \{\Gamma_i\}_{i=1}^{2g}$ is a canonical basis of $H_1(\Sigma_{g,0})$, i.e. Γ_i intersects Γ_{i+g} once positively and does not intersect others. Then*

$$\left|\mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_p^{\Sigma_{g,0}}\}_{p=0}^{2}\right)\right| = \left|\frac{\triangle_{0,2}^{\Sigma_{g,0}}\left(\mathbf{h}_0^{\Sigma_{g,0}}, \mathbf{h}_2^{\Sigma_{g,0}}\right)}{\det \wp\left(\mathbf{h}_{\Sigma_{g,0}}^1, \Gamma\right)}\right|.$$

Here, $\mathbf{h}_{\Sigma_{g,0}}^1 = \{\omega_i\}_1^{2g}$ is the Poincaré dual basis of $H^1(\Sigma_{g,0})$ corresponding to the basis $\mathbf{h}_1^{\Sigma_{g,0}}$ of $H_1(\Sigma_{g,0})$, where $\wp(\mathbf{h}^1, \Gamma) = \left[\int_{\Gamma_i} \omega_j\right]$ is the period matrix of $\mathbf{h}_{\Sigma_{g,0}}^1$ with respect to Γ.

Theorem 4.4 *(Sözen, 2012b) Let M be a closed, connected, orientable m-dimensional manifold and $\mathbf{h}_p$ a basis of $H_p(M)$ for $p = 0, ..., n$.*

(i) If $m = 2n$ $(n \geq 1)$, then

$$\left|\mathbb{T}\left(M, \{\mathbf{h}_p\}_{p=0}^{2n}\right)\right| = \prod_{p=0}^{n-1} \left|\triangle_{p,2n-p}^{M}\left(\mathbf{h}_p, \mathbf{h}_{2n-p}\right)\right|^{(-1)^p} \sqrt{\left|\triangle_{n,n}^{M}\left(\mathbf{h}_n, \mathbf{h}_n\right)\right|^{(-1)^n}}.$$

(ii) If $m = 2n+1$ $(n \geq 0)$, then

$$\left|\mathbb{T}\left(M, \{\mathbf{h}_p\}_{p=0}^{2n+1}\right)\right| = 1.$$

Theorem 4.4 yields the following result.

Remark 4.1 *For the unit spheres $\mathbb{S}^n$ with homology bases $\{\mathbf{h}_p^{\mathbb{S}^n}\}_{p=0}^{n}$,*

(i) if n is odd, then $|T(\mathbb{S}^n, \{\mathbf{h}_p^{\mathbb{S}^n}\}_{p=0}^{n})| = 1$,

(ii) if n is even, then $|T(\mathbb{S}^n, \{\mathbf{h}_p^{\mathbb{S}^n}\}_{p=0}^{n})| = |(\triangle_{0,n}^{\mathbb{S}^n}(\mathbf{h}_0^{\mathbb{S}^n}, \mathbf{h}_n^{\mathbb{S}^n}))|$.

Özel and Sözen (2012) established a formula to compute the Reidemeister torsion of the product manifolds in terms of the Reidemeister torsion of each factor and the corresponding Euler characteristic.

Theorem 4.5 *Assume that M_i is a closed, connected, orientable $2m_i$-dimensional manifold ($m_i \geq 1$) for each $i = 1, 2, \ldots, n$. Let $M = \bigtimes_{i=0}^{n} M_i$ be the product manifold and $\{\mathbf{h}_{p,i}^{M_i}\}_{p=0}^{2m_i}$ be the homology basis of $H_p(M_i)$ for each i. Then the Reidemeister torsion of M satisfies the following formula*

$$\left|\mathbb{T}\left(M, \{\bigoplus_{|\alpha|=p} \mathbf{h}_{\alpha_1,1} \otimes \cdots \otimes \mathbf{h}_{\alpha_n,n}\}_{p=0}^{2m}\right)\right| = \prod_{i=1}^{n} \left|\mathbb{T}\left(M_i, \{\mathbf{h}_{p,i}^{M_i}\}_{p=0}^{2m_i}\right)\right|^{\chi(M)/\chi(M_i)}.$$

Here, $m = \sum_{i=1}^{n} m_i$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$ is the length of the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$.

CHAPTER 5

REIDEMEISTER TORSION OF π-MANIFOLDS VIA CONNECTED SUM DECOMPOSITIONS

In this chapter, we give the main results of this thesis and their proofs. We consider the closed π-manifolds that admit a connected sum decomposition. More precisely, 0-connected 2-dimensional closed π-manifold $\Sigma_{g,0}$ with the connected sum decomposition $\mathop{\#}\limits_{j=1}^{g}(\Sigma_{1,0})$, and $(n-2)$-connected $2n$-dimensional closed π-manifolds ($n \geq 3$) which have the connected sum decomposition $\mathop{\#}\limits_{j=1}^{p}(\mathbb{S}^n \times \mathbb{S}^n)\# M_1^{2n}$ stated in (Ishimoto, 1969), and $(n-1)$-connected $2n$-dimensional closed π-manifolds ($n \geq 3$) with the connected sum decomposition $\mathop{\#}\limits_{j=1}^{p}(\mathbb{S}^n \times \mathbb{S}^n)\#\widetilde{\mathbb{S}^{2n}}$ given in (Ishimoto, 1969).

Throughout this thesis, we consider Reidemeister torsion with untwisted $\mathbb{R}$-coefficients. For a manifold M, we mean by $H_i(M)$ the homology space $H_i(M;\mathrm{R})$ with $\mathbb{R}$-coefficient.

5.1. Reidemeister Torsion of 0-Connected 2-Dimensional Closed π-Manifold

Let $\Sigma_{g,n}$ be a compact, connected, smooth, orientable surface of genus g with $n > 0$ disjoint open disks removed. We will refer to such a surface as n-holed genus g surface. Let $\Sigma_{g,0}$ be a closed, connected, smooth, orientable surface of genus g. This surface is 0-connected because it is path-connected, and it has also the following homology spaces

$$H_i(\Sigma_{g,0}) = \begin{cases} \mathbb{R}^{2g}\,, & i = 1, \\ \mathbb{R}\,, & i = 0, 2, \\ 0\,, & i \neq 0, 1, 2. \end{cases}$$

The genus one closed surface $\Sigma_{1,0}$, namely torus, is a closed π-manifold. It is well-known that $\Sigma_{1,0}$ is the building block of the surface $\Sigma_{g,0}$. That is, $\Sigma_{g,0}$ is expressed as a connected sum of g-copies of $\Sigma_{1,0}$. By Proposition 3.5, $\Sigma_{g,0}$ is a closed π-manifold.

The aim of this section is to describe the behaviour of Reidemeister torsion on the 0-connected 2-dimensional closed π-manifold $\Sigma_{g,0}$ ($g \geq 2$) with respect to gluings along a circle. Considering the surface $\Sigma_{g,0}$ as the following connected sum

$$\Sigma_{g,0} = \mathop{\#}_{j=1}^{g} (\Sigma_{1,0})$$

and using the fact that Reidemeister torsion acts multiplicatively with respect to gluings in the sense of Milnor (1966), we establish a formula to compute the Reidemeister torsion of $\Sigma_{g,0}$ in terms of the Reidemeister torsion of $\Sigma_{1,1}$. To obtain this formula, we first prove a formula for computing the Reidemeister torsion of $\Sigma_{1,1}$ (Theorem 5.1) through the determinant of the period matrix of the Poincaré dual basis of $H^1(\Sigma_{2,0})$. Then we establish a formula (Proposition 5.2) for the Reidemeister torsion of $\Sigma_{1,2}$ with regard to the Reidemeister torsion of $\Sigma_{1,1}$. By using these results, we obtain the formulas (Theorem 5.4-Theorem 5.5) that compute the Reidemeister torsion of $\Sigma_{g,0}$ in terms of the Reidemeister torsion of $\Sigma_{1,1}$. The results of this section appear in (Dirican and Sözen, 2016).

Let $\Sigma_{1,1}$ be a one-holed genus one surface, namely one-holed torus with boundary circle γ. Obviously, the double of $\Sigma_{1,1}$ is a closed, orientable surface $\Sigma_{2,0}$ of genus 2 (see, Figure 5.1).

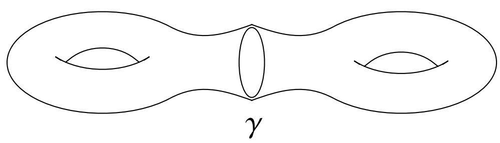


Figure 5.1. The double of $\Sigma_{1,1}$.

Then there is the following short exact sequence of the chain complexes

$$0 \to C_*(\gamma) \longrightarrow C_*(\Sigma_{1,1}) \oplus C_*(\Sigma_{1,1}) \longrightarrow C_*(\Sigma_{2,0}) \to 0. \tag{5.1}$$

Associated to the sequence (5.1), there exists the following Mayer-Vietoris sequence

$$\mathcal{H}_* : \quad 0 \longrightarrow H_2(\Sigma_{2,0}) \xrightarrow{f} H_1(\gamma) \xrightarrow{g} H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1}) \xrightarrow{h} H_1(\Sigma_{2,0})$$

$$\xrightarrow{i} H_0(\gamma) \xrightarrow{j} H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1}) \xrightarrow{k} H_0(\Sigma_{2,0}) \xrightarrow{\ell} 0.$$

By the exactness of $\mathcal{H}_*$ and the First Isomorphism Theorem, it is concluded that

$$\begin{aligned}
&\mathrm{Im}(g) = \mathrm{Im}(i) = \{0\}\\
&\mathrm{Im}(k) = H_0(\Sigma_{2,0})\\
&\mathrm{Im}(f) \cong H_2(\Sigma_{2,0})\\
&\mathrm{Im}(h) \cong H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})\\
&\mathrm{Im}(j) \cong H_0(\gamma).
\end{aligned}$$

Theorem 5.1 *Let $\Sigma_{1,1}$ be a one-holed torus with boundary circle γ and let $\Sigma_{2,0}$ be the double of $\Sigma_{1,1}$. For the given bases $\mathbf{h}_p^{\Sigma_{1,1}}$ and $\mathbf{h}_p^{\gamma}$ of $H_p(\Sigma_{1,1})$ and $H_p(\gamma)$, $p = 0, 1$, there exists a basis $\mathbf{h}_i^{\Sigma_{2,0}}$ of $H_i(\Sigma_{2,0})$, $i = 0, 1, 2$ such that the corrective term is 1 and the following formula holds*

$$\left|\mathbb{T}\left(\Sigma_{1,1}, \{\mathbf{h}_p^{\Sigma_{1,1}}\}_{p=0}^1\right)\right| = \left|\triangle_{0,2}^{\Sigma_{2,0}}\left(\mathbf{h}_0^{\Sigma_{2,0}}, \mathbf{h}_2^{\Sigma_{2,0}}\right)\right|^{1/2} \left|\triangle_{1,1}^{\Sigma_{2,0}}\left(\mathbf{h}_1^{\Sigma_{2,0}}, \mathbf{h}_1^{\Sigma_{2,0}}\right)\right|^{-1/4}.$$

Moreover, if $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ is a canonical basis of $H_1(\Sigma_{2,0})$, i.e. $i = 1, 2$, Γ_i intersects Γ_{i+2} once positively and does not intersect others, then

$$\left|\mathbb{T}\left(\Sigma_{1,1}, \{\mathbf{h}_p^{\Sigma_{1,1}}\}_{p=0}^1\right)\right| = \left|\frac{\triangle_{0,2}^{\Sigma_{2,0}}\left(\mathbf{h}_0^{\Sigma_{2,0}}, \mathbf{h}_2^{\Sigma_{2,0}}\right)}{\det \wp\left(\mathbf{h}^1_{\Sigma_{2,0}}, \Gamma\right)}\right|^{1/2}.$$

Here, $\mathbf{h}^1_{\Sigma_{2,0}} = \{\omega_i\}_1^4$ is the Poincaré dual basis of $H^1(\Sigma_{2,0})$ corresponding to the basis $\mathbf{h}_1^{\Sigma_{2,0}}$ of $H_1(\Sigma_{2,0})$, where $\wp(\mathbf{h}^1, \Gamma) = \left[\int_{\Gamma_i} \omega_j\right]$ is the period matrix of $\mathbf{h}^1_{\Sigma_{2,0}}$ with respect to Γ.

Proof Let us first explain the method we use to show that there exists a basis $\mathbf{h}_i^{\Sigma_{2,0}}$ of $H_i(\Sigma_{2,0})$, $i = 0, 1, 2$ such that the Reidemeister torsion of the long exact sequence $\mathcal{H}_*$ in the corresponding bases, namely corrective term, becomes 1. For $p \in \{0, 1, \ldots, 6\}$, let us denote by $C_p(\mathcal{H}_*)$ the vector spaces in the long exact sequence $\mathcal{H}_*$. Consider the following short exact sequences

$$0 \to Z_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \overset{\partial_p}{\to} B_{p-1}(\mathcal{H}_*) \to 0, \tag{5.2}$$

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow Z_p(\mathcal{H}_*) \overset{\varphi_p}{\twoheadrightarrow} H_p(\mathcal{H}_*) \to 0. \tag{5.3}$$

Here, "$\hookrightarrow$" and "$\twoheadrightarrow$" are the inclusion and the natural projection, respectively. Assume that $s_p : B_{p-1}(\mathcal{H}_*) \to C_p(\mathcal{H}_*)$ and $\ell_p : H_p(\mathcal{H}_*) \to Z_p(\mathcal{H}_*)$ are sections of $\partial_p : C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$ and $\varphi_p : Z_p(\mathcal{H}_*) \to H_p(\mathcal{H}_*)$, respectively. The exactness of $\mathcal{H}_*$ implies that $Z_p(\mathcal{H}_*) = B_p(\mathcal{H}_*)$ for all p. Hence, the sequence (5.2) becomes

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*) \to 0. \tag{5.4}$$

Let $\mathbf{h_p}$, $\mathbf{b_p}$, and $\mathbf{h}_p^*$ be respectively bases of $C_p(\mathcal{H}_*)$, $B_p(\mathcal{H}_*)$, and $H_p(\mathcal{H}_*)$ for all p. Since $H_p(\mathcal{H}_*)$ is a trivial space, its homology basis $\mathbf{h}_p^*$ is $\{0\}$ and ℓ_p is the zero map for each p. Then the Reidemeister torsion of $\mathcal{H}_*$ with respect to bases $\{\mathbf{h}_p\}_0^6$, $\{\mathbf{h}_p^*\}_0^6$ is given as follows

$$\begin{aligned}
\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^6, \{\mathbf{h}_p^*\}_{p=0}^6\right) &= \prod_{p=0}^{6} \left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p^*) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{h}_p\right]^{(-1)^{(p+1)}} \\
&= \prod_{p=0}^{6} \left[\mathbf{b}_p \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{h}_p\right]^{(-1)^{(p+1)}} \\
&= \prod_{p=0}^{6} \left[\mathbf{h}'_p, \mathbf{h}_p\right]^{(-1)^{(p+1)}}.
\end{aligned}$$

Here, $\mathbf{h}'_p$ denotes the obtained basis $\mathbf{b}_p \sqcup s_p(\mathbf{b}_{p-1})$ for each p. Note that Reidemeister torsion is independent of the bases $\mathbf{b}_p$ and sections s_p, ℓ_p. Therefore, in the following method we will choose suitable bases $\mathbf{b}_p$ and sections s_p such that the Reidemeister torsion of $\mathcal{H}_*$ in the corresponding bases becomes 1.

Consider the space $C_0(\mathcal{H}_*) = H_0(\Sigma_{2,0})$ in the sequence (5.4). Then we get

$$0 \to \mathrm{Im}(k) \hookrightarrow C_0(\mathcal{H}_*) \xrightarrow{\ell} \mathrm{Im}(\ell) \to 0. \tag{5.5}$$

The zero map $s_0 : \mathrm{Im}(\ell) \to C_0(\mathcal{H}_*)$ can be considered as a section of ℓ because the space $\mathrm{Im}(\ell)$ is trivial. From the Splitting Lemma it follows

$$C_0(\mathcal{H}_*) = \mathrm{Im}(k) \oplus s_0(\mathrm{Im}(\ell)) = \mathrm{Im}(k). \tag{5.6}$$

Let us take the basis of $\mathrm{Im}(k)$ as

$$\mathbf{h}^{\mathrm{Im}(k)} = \left\{ a_{21} k(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + a_{22} k(0, \mathbf{h}_0^{\Sigma_{1,1}}) \right\},$$

where (a_{21}, a_{22}) is a non-zero vector. By equation (5.6), $\mathbf{h}^{\mathrm{Im}(k)}$ becomes the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. Taking the initial basis $\mathbf{h}_0$ (namely, $\mathbf{h}_0^{\Sigma_{2,0}}$) of $C_0(\mathcal{H}_*)$ as $\mathbf{h}'_0$, we obtain

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{5.7}$$

For the space $C_1(\mathcal{H}_*) = H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1})$, the sequence (5.4) becomes as follows

$$0 \to \mathrm{Im}(j) \hookrightarrow C_1(\mathcal{H}_*) \xrightarrow{k} \mathrm{Im}(k) \to 0 \tag{5.8}$$

By the First Isomorphism Theorem, $\mathrm{Im}(k)$ and $(H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1}))/\mathrm{Ker}(k)$ are isomorphic. Therefore, we can consider the inverse of this isomorphism

$$s_1 : \mathrm{Im}(k) \to (H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1}))/\mathrm{Ker}(k)$$

as a section of k. By the Splitting Lemma, the space $C_1(\mathcal{H}_*)$ can be expressed as the following direct sum

$$C_1(\mathcal{H}_*) = \mathrm{Im}(j) \oplus s_1(\mathrm{Im}(k)). \tag{5.9}$$

Note that the initial basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$ is

$$\left\{(\mathbf{h}_0^{\Sigma_{1,1}}, 0), (0, \mathbf{h}_0^{\Sigma_{1,1}})\right\}.$$

Using the fact that $\mathrm{Im}(j)$ is isomorphic to $H_0(\gamma)$, $j(\mathbf{h}_0^{\gamma})$ becomes a basis of $\mathrm{Im}(j)$. Since $\mathrm{Im}(j)$ and $s_1(\mathrm{Im}(k))$ are one-dimensional subspaces of the 2-dimensional space $C_1(\mathcal{H}_*)$, there exists a non-zero vector (a_{11}, a_{12}) such that

$$\begin{aligned} j(\mathbf{h}_0^{\gamma}) &= a_{11}(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + a_{12}(0, \mathbf{h}_0^{\Sigma_{1,1}}), \\ s_1(\mathbf{h}^{\mathrm{Im}(k)}) &= a_{21}(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + a_{22}(0, \mathbf{h}_0^{\Sigma_{1,1}}). \end{aligned} \tag{5.10}$$

Let us choose the basis $\mathbf{h}^{\mathrm{Im}(j)}$ of $\mathrm{Im}(j)$ as $j((\det A)^{-1}\mathbf{h}_0^{\gamma})$, where $A = (a_{ij})$ is the (2×2)-matrix over $\mathbb{R}$. By equation (5.9) and equation (5.10),

$$\left\{(\det A)^{-1}[a_{11}(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + a_{12}(0, \mathbf{h}_0^{\Sigma_{1,1}})], a_{21}(\mathbf{h}_0^{\Sigma_{1,1}}, 0) + a_{22}(0, \mathbf{h}_0^{\Sigma_{1,1}})\right\}$$

becomes the obtained basis $\mathbf{h}'_1$ for $C_1(\mathcal{H}_*)$. Hence, we conclude that the determinant of the transition matrix is 1. That is,

$$[\mathbf{h}'_1, \mathbf{h}_1] = 1. \tag{5.11}$$

We now consider the short exact sequence (5.4) for $C_2(\mathcal{H}_*) = H_0(\gamma)$. By the equalities $B_2(\mathcal{H}_*) = \mathrm{Im}(i)$ and $B_1(\mathcal{H}_*) = \mathrm{Im}(j)$, the sequence (5.4) becomes

$$0 \to \mathrm{Im}(i) \hookrightarrow C_2(\mathcal{H}_*) \xrightarrow{j} \mathrm{Im}(j) \to 0. \tag{5.12}$$

Since $j : H_0(\gamma) \to \mathrm{Im}(j)$ is an isomorphism, we can take the inverse of j as a section

$s_2 : \mathrm{Im}(j) \to H_0(\gamma)$ of j. From the Splitting Lemma it follows

$$C_2(\mathcal{H}_*) = \mathrm{Im}(i) \oplus s_2(\mathrm{Im}(j)). \tag{5.13}$$

Recall that in the previous step, we chose $j((\det A)^{-1}\mathbf{h}_0^{\gamma})$ as a basis of $\mathrm{Im}(j)$. By equation (5.13) and the fact that $\mathrm{Im}(i)$ is trivial, we get that the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$ as follows

$$s_2(j((\det A)^{-1}\mathbf{h}_0^{\gamma})) = (\det A)^{-1}\mathbf{h}_0^{\gamma}.$$

Since the initial basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$ is also $\mathbf{h}_0^{\gamma}$, the determinant of the transition matrix satisfies the following equality

$$[\mathbf{h}_2', \mathbf{h}_2] = (\det A)^{-1}. \tag{5.14}$$

Considering the space $C_3(\mathcal{H}_*) = H_1(\Sigma_{2,0})$ in the sequence (5.4) and using the fact that $B_3(\mathcal{H}_*) = \mathrm{Im}(h)$, $B_2(\mathcal{H}_*) = \mathrm{Im}(i)$, we obtain

$$0 \to \mathrm{Im}(h) \hookrightarrow C_3(\mathcal{H}_*) \xrightarrow{i} \mathrm{Im}(i) \to 0. \tag{5.15}$$

Since $\mathrm{Im}(i) = \{0\}$, we can take the zero map $s_3 : \mathrm{Im}(i) \to H_1(\Sigma_{2,0})$ as a section of i. By the Splitting Lemma, we have

$$C_3(\mathcal{H}_*) = \mathrm{Im}(h) \oplus s_3(\mathrm{Im}(i)) = \mathrm{Im}(h). \tag{5.16}$$

The given basis $\mathbf{h}^{\Sigma_{1,1}\oplus\Sigma_{1,1}}$ of $H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$ is

$$\left\{(\mathbf{h}_{1,1}^{\Sigma_{1,1}}, 0), (0, \mathbf{h}_{1,1}^{\Sigma_{1,1}}), (\mathbf{h}_{1,2}^{\Sigma_{1,1}}, 0), (0, \mathbf{h}_{1,2}^{\Sigma_{1,1}})\right\}.$$

Because of the isomorphism between $\mathrm{Im}(h)$ and $H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$, we can choose the basis $\mathbf{h}^{\mathrm{Im}(h)}$ of $\mathrm{Im}(h)$ as

$$\left\{h(\mathbf{h}_{1,1}^{\Sigma_{1,1}},0),h(0,\mathbf{h}_{1,1}^{\Sigma_{1,1}}),h(\mathbf{h}_{1,2}^{\Sigma_{1,1}},0),h(0,\mathbf{h}_{1,2}^{\Sigma_{1,1}})\right\}.$$

By equation (5.16), $\mathbf{h}^{\mathrm{Im}(h)}$ becomes the obtained basis $\mathbf{h}'_3$ of $C_3(\mathcal{H}_*)$. If we let the initial basis $\mathbf{h}_3$ (namely, $\mathbf{h}_1^{\Sigma_{2,0}}$) of $C_3(\mathcal{H}_*)$ as $(\det A)^{-1}\mathbf{h}'_3$, then we get

$$[\mathbf{h}'_3,\mathbf{h}_3]=((\det A)^{-1})^{-1}=\det A. \tag{5.17}$$

If we consider the sequence (5.4) for the space $C_4(\mathcal{H}_*)=H_1(\Sigma_{1,1})\oplus H_1(\Sigma_{1,1})$, then, by the equalities $B_4(\mathcal{H}_*)=\mathrm{Im}(g)$ and $B_3(\mathcal{H}_*)=\mathrm{Im}(h)$, we get

$$0\to\mathrm{Im}(g)\hookrightarrow C_4(\mathcal{H}_*)\xrightarrow{h}\mathrm{Im}(h)\to 0. \tag{5.18}$$

Since h is an isomorphism, we can consider the inverse of h as a section

$$s_4:\mathrm{Im}(h)\to H_1(\Sigma_{1,1})\oplus H_1(\Sigma_{1,1})$$

of h. As $\mathrm{Im}(g)$ is trivial, the Splitting Lemma gives

$$C_4(\mathcal{H}_*)=\mathrm{Im}(g)\oplus s_4(\mathrm{Im}(h))=s_4(\mathrm{Im}(h)). \tag{5.19}$$

Recall that $\mathbf{h}^{\Sigma_{1,1}\oplus\Sigma_{1,1}}$ is the initial basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$. Moreover, in the previous step, we chose the basis $\mathbf{h}^{\mathrm{Im}(h)}$ of $\mathrm{Im}(h)$ as

$$\left\{h(\mathbf{h}_{1,1}^{\Sigma_{1,1}},0),h(0,\mathbf{h}_{1,1}^{\Sigma_{1,1}}),h(\mathbf{h}_{1,2}^{\Sigma_{1,1}},0),h(0,\mathbf{h}_{1,2}^{\Sigma_{1,1}})\right\}.$$

It follows from equation (5.19) that $s_4(\mathbf{h}^{\mathrm{Im}(h)})=\mathbf{h}^{\Sigma_{1,1}\oplus\Sigma_{1,1}}$ is the obtained basis $\mathbf{h}'_4$ of $C_4(\mathcal{H}_*)$.

Hence, the determinant of the transition matrix satisfies the following equation

$$[\mathbf{h}'_4, \mathbf{h}_4] = 1. \tag{5.20}$$

Now we consider the space $C_5(\mathcal{H}_*) = H_1(\gamma)$ in the short exact sequence (5.4). Using the fact that $B_5(\mathcal{H}_*) = \mathrm{Im}(f)$ and $B_4(\mathcal{H}_*) = \mathrm{Im}(g)$, we get

$$0 \to \mathrm{Im}(f) \hookrightarrow C_5(\mathcal{H}_*) \xrightarrow{g} \mathrm{Im}(g) \to 0. \tag{5.21}$$

Since $\mathrm{Im}(g)$ is trivial, the zero map $s_5 : \mathrm{Im}(g) \to C_5(\mathcal{H}_*)$ can be considered as a section of g. From the Splitting Lemma it follows that

$$C_5(\mathcal{H}_*) = \mathrm{Im}(f) \oplus s_5(\mathrm{Im}(g)) = \mathrm{Im}(f). \tag{5.22}$$

The initial basis $\mathbf{h}_5$ of $C_5(\mathcal{H}_*)$ is $\mathbf{h}_1^{\gamma}$. By equation (5.22), we choose the basis $\mathbf{h}^{\mathrm{Im}(f)}$ of $\mathrm{Im}(f)$ as $\mathbf{h}_1^{\gamma}$, which is also the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. Thus, we obtain

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{5.23}$$

Finally, considering the space $C_6(\mathcal{H}_*) = H_2(\Sigma_{2,0})$ in the sequence (5.4) and using the fact that $B_6(\mathcal{H}_*) = \{0\}$ and $B_5(\mathcal{H}_*) = \mathrm{Im}(f)$, we get

$$0 \to B_6(\mathcal{H}_*) \hookrightarrow C_6(\mathcal{H}_*) \xrightarrow{f} \mathrm{Im}(f) \to 0. \tag{5.24}$$

Since $\mathrm{Im}(f)$ is isomorphic to $H_2(\Sigma_{2,0})$, the inverse of f

$$s_6 : \mathrm{Im}(f) \to H_2(\Sigma_{2,0})$$

can be considered as a section of f. By the Splitting Lemma, the space $C_6(\mathcal{H}_*)$ satisfies the following equation

$$C_6(\mathcal{H}_*) = B_6(\mathcal{H}_*) \oplus s_6(\mathrm{Im}(f)) = s_6(\mathrm{Im}(f)). \tag{5.25}$$

From equation (5.25) it follows that $s_6(\mathbf{h}^{\mathrm{Im}(f)})$ is the obtained basis $\mathbf{h}'_6$ of $C_6(\mathcal{H}_*)$. If we take the basis $\mathbf{h}_6$ (namely, $\mathbf{h}_2^{\Sigma_{2,0}}$) of $C_6(\mathcal{H}_*)$ as $s_6(\mathbf{h}^{\mathrm{Im}(f)})$, then the determinant of the transition matrix satisfies the following equality

$$[\mathbf{h}'_6, \mathbf{h}_6] = 1. \tag{5.26}$$

Combining equations (5.7), (5.11), (5.14), (5.17), (5.20), (5.23), and (5.26) gives that the corrective term is 1. More precisely,

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^6, \{0\}_{p=0}^6\right) = \prod_{p=0}^{6} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{5.27}$$

Clearly, the natural bases are compatible in the sequence (5.1). Then Theorem 4.1 and equation (5.27) yield

$$\mathbb{T}\left(\Sigma_{1,1} \oplus \Sigma_{1,1}, \{\mathbf{h}_p^{\Sigma_{1,1}\oplus\Sigma_{1,1}}\}_{p=0}^1\right) = \mathbb{T}\left(\gamma_1, \{\mathbf{h}_p^{\gamma}\}_{p=0}^1\right)\ \mathbb{T}\left(\Sigma_{2,0}, \{\mathbf{h}_i^{\Sigma_{2,0}}\}_{i=0}^2\right). \tag{5.28}$$

By Lemma 4.1 and equation (5.28), the following formula holds

$$\mathbb{T}\left(\Sigma_{1,1}, \{\mathbf{h}_p^{\Sigma_{1,1}}\}_{p=0}^1\right)^2 = \mathbb{T}\left(\gamma_1, \{\mathbf{h}_p^{\gamma}\}_{p=0}^1\right)\ \mathbb{T}\left(\Sigma_{2,0}, \{\mathbf{h}_i^{\Sigma_{2,0}}\}_{i=0}^2\right). \tag{5.29}$$

From Remark 4.1 and equation(5.29), it follows that

$$\left|\mathbb{T}\left(\Sigma_{1,1}, \{\mathbf{h}_p^{\Sigma_{1,1}}\}_{p=0}^1\right)\right| = \left|\mathbb{T}\left(\Sigma_{2,0}, \{\mathbf{h}_i^{\Sigma_{2,0}}\}_{i=0}^2\right)\right|^{1/2}. \tag{5.30}$$

Theorem 4.4 (i) and equation (5.30) give the following formula

$$\left|\mathbb{T}\left(\Sigma_{1,1},\{\mathbf{h}_p^{\Sigma_{1,1}}\}_{p=0}^{1}\right)\right| = \left|\triangle_{0,2}^{\Sigma_{2,0}}\left(\mathbf{h}_0^{\Sigma_{2,0}},\mathbf{h}_2^{\Sigma_{2,0}}\right)\right|^{1/2} \left|\triangle_{1,1}^{\Sigma_{2,0}}\left(\mathbf{h}_1^{\Sigma_{2,0}},\mathbf{h}_1^{\Sigma_{2,0}}\right)\right|^{-1/4}.$$

Then, by Theorem 4.3, we have

$$\left|\mathbb{T}\left(\Sigma_{1,1},\{\mathbf{h}_p^{\Sigma_{1,1}}\}_{p=0}^{1}\right)\right| = \left|\frac{\triangle_{0,2}^{\Sigma_{2,0}}\left(\mathbf{h}_0^{\Sigma_{2,0}},\mathbf{h}_2^{\Sigma_{2,0}}\right)}{\det\wp\left(\mathbf{h}^1_{\Sigma_{2,0}},\Gamma\right)}\right|^{1/2}.$$

□

The following proposition gives a formula that computes the Reidemeister torsion of closed ball $\overline{\mathbb{D}^n}$ for arbitrary $n \in \{2,3,\ldots\}$ by considering the double of $\overline{\mathbb{D}^n}$.

Proposition 5.1 *Let $d(\overline{\mathbb{D}^n})$ be the double of $\overline{\mathbb{D}^n}$. Then there is the natural short exact sequence of the chain complexes*

$$0 \to C_*(\mathbb{S}^{n-1}) \to C_*(\overline{\mathbb{D}^n}) \oplus C_*(\overline{\mathbb{D}^n}) \to C_*(d(\overline{\mathbb{D}^n})) \to 0. \tag{5.31}$$

Associated to the sequence (5.31), there exists the following Mayer-Vietoris sequence

$$\mathcal{H}_*:\quad 0 \longrightarrow H_n(d(\overline{\mathbb{D}^n})) \xrightarrow{f} H_{n-1}(\mathbb{S}^{n-1}) \xrightarrow{g} 0 \xrightarrow{\partial}$$

$$H_0(\mathbb{S}^{n-1}) \xrightarrow{h} H_0(\overline{\mathbb{D}^n}) \oplus H_0(\overline{\mathbb{D}^n}) \xrightarrow{\alpha} H_0(d(\overline{\mathbb{D}^n})) \longrightarrow 0.$$

Let $\mathbf{h}_0^{\overline{\mathbb{D}^n}}$ be a basis of $H_0(\overline{\mathbb{D}^n})$ and $f: H_n(d(\overline{\mathbb{D}^n})) \to H_{n-1}(\mathbb{S}^{n-1})$ the isomorphism obtained by the sequence $\mathcal{H}_*$.

(i) For odd n, let $\mathbf{h}_p^{d(\overline{\mathbb{D}^n})}$ be a basis of $H_p(d(\overline{\mathbb{D}^n}))$, $p = 0,\ldots,n$ and let $\mathbf{h}_{n-1}^{\mathbb{S}^{n-1}} = f(\mathbf{h}_{n-1}^{d(\overline{\mathbb{D}^n})})$ be a basis of $H_{n-1}(\mathbb{S}^{n-1})$. Then there exists a basis $\mathbf{h}_0^{\mathbb{S}^{n-1}}$ of $H_0(\mathbb{S}^{n-1})$ such that the

following formula holds

$$\left|\mathbb{T}\left(\overline{\mathbb{D}^n}, \{\mathbf{h}_0^{\overline{\mathbb{D}^n}}\}\right)\right| = \sqrt{\left|\triangle_{0,n-1}^{\mathbb{S}^{n-1}}\left(\mathbf{h}_0^{\mathbb{S}^{n-1}}, \mathbf{h}_{n-1}^{\mathbb{S}^{n-1}}\right)\right|},$$

(ii) For even n, let $\mathbf{h}_n^{\mathbb{S}^{n-1}}$ be a basis of $H_n(\mathbb{S}^{n-1})$ and $\mathbf{h}_n^{d(\overline{\mathbb{D}^n})} = f^{-1}(\mathbf{h}_n^{\mathbb{S}^{n-1}})$ a basis of $H_n(d(\overline{\mathbb{D}^n}))$. Then there are bases $\mathbf{h}_0^{\mathbb{S}^{n-1}}$ and $\mathbf{h}_0^{d(\overline{\mathbb{D}^n})}$ of $H_0(\mathbb{S}^{n-1})$ and $H_0(d(\overline{\mathbb{D}^n}))$, respectively so that the following formula is valid

$$\left|\mathbb{T}\left(\overline{\mathbb{D}^n}, \{\mathbf{h}_0^{\overline{\mathbb{D}^n}}\}\right)\right| = \sqrt{\left|\triangle_{0,n}^{\mathbb{S}^{n}}\left(\mathbf{h}_0^{\mathbb{S}^{n}}, \mathbf{h}_{n}^{\mathbb{S}^{n}}\right)\right|}.$$

Here, $[\varphi_p] : H_p(d(\overline{\mathbb{D}^n})) \to H_p(\mathbb{S}^n)$ is an isomorphism defined by $[\varphi_p](\mathbf{h}_p^{d(\overline{\mathbb{D}^n})}) = \mathbf{h}_p^{\mathbb{S}^n}$ for $p \in \{0, \dots, n\}$ which is induced by the homeomorphism $\varphi : d(\overline{\mathbb{D}^n}) \to \mathbb{S}^n$.

Proof The exactness of $\mathcal{H}_*$ gives the following isomorphisms:

$$H_n(d(\overline{\mathbb{D}^n})) \overset{f}{\cong} H_{n-1}(\mathbb{S}^{n-1}) \tag{5.32}$$

$$H_0(\overline{\mathbb{D}^n}) \oplus H_0(\overline{\mathbb{D}^n}) \cong H_0(\mathbb{S}^{n-1}) \oplus H_0(d(\overline{\mathbb{D}^n})). \tag{5.33}$$

If n is odd, then $|\mathbb{T}(d(\overline{\mathbb{D}^n}), \{\mathbf{h}_p^{d(\overline{\mathbb{D}^n})}\}_{p=0}^n)| = 1$ by Theorem 4.4 (ii). Since f is an isomorphism given in equation (5.32), $\mathbf{h}_{n-1}^{\mathbb{S}^{n-1}} = f(\mathbf{h}_{n-1}^{d(\overline{\mathbb{D}^n})})$ becomes a basis of $H_{n-1}(\mathbb{S}^{n-1})$ for the given basis $\mathbf{h}_{n-1}^{d(\overline{\mathbb{D}^n})}$ of $H_{n-1}(d(\overline{\mathbb{D}^n}))$. Then there exists a basis $\mathbf{h}_0^{\mathbb{S}^{n-1}}$ of $H_0(\mathbb{S}^{n-1})$ by the isomorphism in equation (5.33) such that the corrective term is 1 and the following formula is valid

$$\mathbb{T}\left(\overline{\mathbb{D}^n}, \{\mathbf{h}_0^{\overline{\mathbb{D}^n}}\}\right)^2 = \mathbb{T}\left(d(\overline{\mathbb{D}^n}), \{\mathbf{h}_p^{d(\overline{\mathbb{D}^n})}\}_{p=0}^n\right)\ \mathbb{T}\left(\mathbb{S}^{n-1}, \{\mathbf{h}_p^{\mathbb{S}^{n-1}}\}_{p=0}^{n-1}\right). \tag{5.34}$$

By taking the absolute value of both sides of equation (5.34), we get

$$\left|\mathbb{T}\left(\overline{\mathbb{D}^n}, \{\mathbf{h}_0^{\overline{\mathbb{D}^n}}\}\right)\right| = \sqrt{\left|\mathbb{T}\left(\mathbb{S}^{n-1}, \{\mathbf{h}_p^{\mathbb{S}^{n-1}}\}_{p=0}^{n-1}\right)\right|}. \tag{5.35}$$

By Theorem 4.4 (i) and equation (5.35), we get

$$\left|\mathbb{T}\left(\overline{\mathbb{D}^n}, \{\mathbf{h}_0^{\overline{\mathbb{D}^n}}\}\right)\right| = \sqrt{\left|\triangle_{0,n-1}^{\mathbb{S}^{n-1}}\left(\mathbf{h}^{\mathbb{S}^{n-1}}, \mathbf{h}_{n-1}^{\mathbb{S}^{n-1}}\right)\right|}.$$

If n is even, then $|\mathbb{T}(\mathbb{S}^{n-1}, \{\mathbf{h}_p^{\mathbb{S}^{n-1}}\}_{p=0}^{n-1})| = 1$ by Theorem 4.4 (ii). It follows from the isomorphism given in equation (5.32) that $\mathbf{h}_n^{d(\overline{\mathbb{D}^n})} = f^{-1}(\mathbf{h}_n^{\mathbb{S}^{n-1}})$ is a basis of $H_n(d(\overline{\mathbb{D}^n}))$ for the given basis $\mathbf{h}_n^{\mathbb{S}^{n-1}}$ of $H_n(\mathbb{S}^{n-1})$. By the isomorphism in equation (5.33), there are bases $\mathbf{h}_0^{\mathbb{S}^{n-1}}$ and $\mathbf{h}_0^{d(\overline{\mathbb{D}^n})}$ of $H_0(\mathbb{S}^{n-1})$ and $H_0(d(\overline{\mathbb{D}^n}))$, respectively so that the corrective term disappears and the following formula holds

$$\mathbb{T}\left(\overline{\mathbb{D}^n}, \{\mathbf{h}_0^{\overline{\mathbb{D}^n}}\}\right)^2 = \mathbb{T}\left(d(\overline{\mathbb{D}^n}), \{\mathbf{h}_p^{d(\overline{\mathbb{D}^n})}\}_{p=0}^{n}\right)\ \mathbb{T}\left(\mathbb{S}^{n-1}, \{\mathbf{h}_p^{\mathbb{S}^{n-1}}\}_{p=0}^{n-1}\right). \tag{5.36}$$

If we take the absolute value of both sides of equation (5.36), we obtain

$$\left|\mathbb{T}\left(\overline{\mathbb{D}^n}, \{\mathbf{h}_0^{\overline{\mathbb{D}^n}}\}\right)\right| = \sqrt{\left|\mathbb{T}\left(d(\overline{\mathbb{D}^n}), \{\mathbf{h}_p^{d(\overline{\mathbb{D}^n})}\}_{p=0}^{n}\right)\right|}. \tag{5.37}$$

Since $d(\overline{\mathbb{D}^n})$ is homeomorphic to $\mathbb{S}^n$, there exists a homeomorphism $\varphi : d(\overline{\mathbb{D}^n}) \to \mathbb{S}^n$. Then, for $p \in \{0, \dots, n\}$ there is an isomorphism $[\varphi_p] : H_p(d(\overline{\mathbb{D}^n})) \to H_p(\mathbb{S}^n)$ defined by $[\varphi_p](\mathbf{h}_p^{d(\overline{\mathbb{D}^n})}) = \mathbf{h}_p^{\mathbb{S}^n}$ which is induced by φ. From this result and equation (5.37) it follows

$$\left|\mathbb{T}\left(\overline{\mathbb{D}^n}, \{\mathbf{h}_0^{\overline{\mathbb{D}^n}}\}\right)\right| = \sqrt{\left|\mathbb{T}\left(\mathbb{S}^n, \{\mathbf{h}_p^{\mathbb{S}^n}\}_{p=0}^{n}\right)\right|}. \tag{5.38}$$

Combining Theorem 4.4 (i) and equation (5.38), we obtain the following formula

$$\left|\mathbb{T}\left(\overline{\mathbb{D}^n}, \{\mathbf{h}_0^{\overline{\mathbb{D}^n}}\}\right)\right| = \sqrt{\left|\triangle_{0,n}^{\mathbb{S}^n}\left(\mathbf{h}_0^{\mathbb{S}^n}, \mathbf{h}_n^{\mathbb{S}^n}\right)\right|}.$$

□

For the rest of this thesis, $\mathbb{D}^{2n}$ denotes the open unit ball in $\mathbb{R}^{2n}$ and $\overline{\mathbb{D}^{2n}}$ also denotes the closed unit ball in $\mathbb{R}^{2n}$. For proofs of the results given in Chapter 4, we use the arguments presented in the proof of Theorem 5.1.

Proposition 5.2 *Let $\Sigma_{1,2}$ be a 2-holed genus one surface with boundary circles $\mathbb{S}^1_1, \mathbb{S}^1_2$. For $i = 1,2$, let $\overline{\mathbb{D}^2_i}$ denote the closed disk with boundary $\mathbb{S}^1_i$. Consider the surface $\Sigma_{1,1}$ obtained by gluing the surfaces $\Sigma_{1,2}$ and $\overline{\mathbb{D}^2_1}$ along the common boundary circle $\mathbb{S}^1_1$ (see, Figure 5.2).*

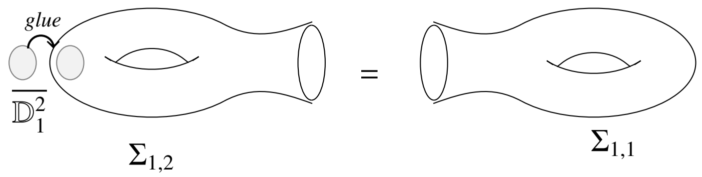


Figure 5.2. One-holed genus one surface $\Sigma_{1,1}$ is obtained by gluing $\Sigma_{1,2}$ and $\overline{\mathbb{D}^2_1}$ along the common boundary circle.

Consider also the associated short exact sequence of chain complexes

$$0 \to C_*(\mathbb{S}^1_1) \longrightarrow C_*(\Sigma_{1,2}) \oplus C_*(\overline{\mathbb{D}^2_1}) \longrightarrow C_*(\Sigma_{1,1}) \to 0 \tag{5.39}$$

and corresponding Mayer-Vietoris sequence

$$\begin{aligned} \mathcal{H}_*: \quad 0 \longrightarrow H_1(\mathbb{S}^1_1) \xrightarrow{f} H_1(\Sigma_{1,2}) \xrightarrow{g} H_1(\Sigma_{1,1}) \\ \xrightarrow{h} H_0(\mathbb{S}^1_1) \xrightarrow{i} H_0(\Sigma_{1,2}) \oplus H_0(\overline{\mathbb{D}^2_1}) \xrightarrow{j} H_0(\Sigma_{1,1}) \xrightarrow{k} 0. \end{aligned}$$

Let $\mathbf{h}^{\Sigma_{1,2}}_p$ and $\mathbf{h}^{\overline{\mathbb{D}^2_1}}_0$ be bases of $H_p(\Sigma_{1,2})$ and $H_0(\overline{\mathbb{D}^2_1})$ for $p = 0,1$. Then there exist bases $\mathbf{h}^{\Sigma_{1,1}}_p$ and $\mathbf{h}^{\mathbb{S}^1_1}_p$ of $H_p(\Sigma_{1,1})$ and $H_p(\mathbb{S}^1_1)$, respectively such that the corrective term disappears and the following multiplicative gluing formula holds

$$\mathbb{T}\left(\Sigma_{1,2}, \{\mathbf{h}^{\Sigma_{1,2}}_p\}^1_{p=0}\right) = \mathbb{T}\left(\Sigma_{1,1}, \{\mathbf{h}^{\Sigma_{1,1}}_p\}^1_{p=0}\right) \mathbb{T}\left(\mathbb{S}^1_1, \{\mathbf{h}^{\mathbb{S}^1_1}_p\}^1_{p=0}\right) \mathbb{T}\left(\overline{\mathbb{D}^2_1}, \{\mathbf{h}^{\mathbb{D}_1}_0\}\right)^{-1}.$$

Proof By the exactness of the sequence $\mathcal{H}_*$ and the First Isomorphism Theorem, we have the followings

$$
\begin{aligned}
\mathrm{Im}(j) &= H_0(\Sigma_{1,1}) \\
\mathrm{Im}(f) &\cong H_1(\mathbb{S}^1_1) \\
\mathrm{Im}(i) &\cong H_0(\mathbb{S}^1_1).
\end{aligned}
$$

For $p \in \{0, \ldots, 5\}$, we denote the vector spaces in the long exact sequence $\mathcal{H}_*$ by $C_p(\mathcal{H}_*)$ and consider the short exact sequence

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*) \to 0. \tag{5.40}$$

For each p, let us consider the isomorphism $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*))$ obtained by the First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$. Then we obtain

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{5.41}$$

We first consider the vector space $C_0(\mathcal{H}_*) = H_0(\Sigma_{1,1})$ in equation (5.41). From the fact that $\mathrm{Im}(k)$ is a trivial space it follows

$$C_0(\mathcal{H}_*) = \mathrm{Im}(j) \oplus s_0(\mathrm{Im}(k)) = \mathrm{Im}(j). \tag{5.42}$$

As $\mathrm{Im}(j)$ is a one-dimensional space, there is a non-zero vector (a_{11}, a_{12}) such that

$$\mathbf{h}^{\mathrm{Im}(j)} = \left\{ a_{11}\, j(\mathbf{h}_0^{\Sigma_{1,2}}) + a_{12}\, j(\mathbf{h}_0^{\overline{\mathbb{D}^2_1}}) \right\}$$

is the basis of $\mathrm{Im}(j)$. From equation (5.42) it follows that $\mathbf{h}^{\mathrm{Im}(j)}$ is the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. If we choose the initial basis $\mathbf{h}_0$ (namely, $\mathbf{h}_0^{\Sigma_{1,1}}$) of $C_0(\mathcal{H}_*)$ as $\mathbf{h}^{\mathrm{Im}(j)}$, then we get

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{5.43}$$

Considering equation (5.41) for $C_1(\mathcal{H}_*) = H_0(\Sigma_{1,2}) \oplus H_0(\overline{\mathbb{D}_1^2})$, the space $C_1(\mathcal{H}_*)$ can be expressed as follows

$$C_1(\mathcal{H}_*) = \mathrm{Im}(i) \oplus s_1(\mathrm{Im}(j)). \tag{5.44}$$

Recall that in the previous step we chose the basis of $\mathrm{Im}(j)$ as $\mathbf{h}^{\mathrm{Im}(j)}$. Since s_1 is a section of j, the following equality holds

$$s_1(\mathbf{h}^{\mathrm{Im}(j)}) = a_{11}\mathbf{h}_0^{\Sigma_{1,2}} + a_{12}\mathbf{h}_0^{\overline{\mathbb{D}_1^2}}.$$

As $\mathrm{Im}(i)$ is a one-dimensional subspace of $C_1(\mathcal{H}_*)$, there is a non-zero vector (a_{21}, a_{22}) such that

$$\left\{a_{21}\mathbf{h}_0^{\Sigma_{1,2}} + a_{22}\mathbf{h}_0^{\overline{\mathbb{D}_1^2}}\right\}$$

is a basis of $\mathrm{Im}(i)$ and clearly $A = (a_{ij})$ is (2×2)-real matrix with non-zero determinant. If we take the basis of $\mathrm{Im}(i)$ as follows

$$\mathbf{h}^{\mathrm{Im}(i)} = \left\{-(\det A)^{-1}\left[a_{21}\mathbf{h}_0^{\Sigma_{1,2}} + a_{22}\mathbf{h}_0^{\overline{\mathbb{D}_1^2}}\right]\right\},$$

then by equation (5.44),

$$\mathbf{h}_1' = \left\{\mathbf{h}^{\mathrm{Im}(i)}, s_1(\mathbf{h}^{\mathrm{Im}(j)})\right\}$$

becomes the obtained basis of $C_1(\mathcal{H}_*)$. Since the initial basis of $C_1(\mathcal{H}_*)$ is

$$\mathbf{h}_1 = \left\{\mathbf{h}_0^{\Sigma_{1,2}}, \mathbf{h}_0^{\overline{\mathbb{D}_1^2}}\right\},$$

the determinant of the transition matrix becomes 1; that is,

$$[\mathbf{h}_1', \mathbf{h}_1] = 1. \tag{5.45}$$

Next, let us consider the space $C_2(\mathcal{H}_*) = H_0(\mathbb{S}_1^1)$ in equation (5.41). Using the fact that Im(h) is a trivial, we get

$$C_2(\mathcal{H}_*) = \mathrm{Im}(h) \oplus s_2(\mathrm{Im}(i)) = s_2(\mathrm{Im}(i)). \tag{5.46}$$

Recall that the basis $\mathbf{h}^{\mathrm{Im}(i)}$ of Im(i) was chosen as

$$\left\{-(\det A)^{-1}\left[a_{21}\mathbf{h}_0^{\Sigma_{1,2}} + a_{22}\mathbf{h}_0^{\overline{\mathbb{D}_1^2}}\right]\right\}$$

in the previous step. It follows from equation (5.46) that $s_2(\mathbf{h}^{\mathrm{Im}(i)})$ is the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_2$ (namely, $\mathbf{h}_0^{S_1^1}$) of $C_2(\mathcal{H}_*)$ as $s_2(\mathbf{h}^{\mathrm{Im}(i)})$, then we obtain

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{5.47}$$

We now consider the case of $C_3(\mathcal{H}_*) = H_1(\Sigma_{1,1})$ in equation (5.41). Because Im(h) is trivial, we have the following equality

$$C_3(\mathcal{H}_*) = \mathrm{Im}(g) \oplus s_3(\mathrm{Im}(h)) = \mathrm{Im}(g). \tag{5.48}$$

By the fact that Im(g) is a 2-dimensional space and $\mathbf{h}_1^{\Sigma_{1,2}} = \{\mathbf{h}_{1,j}^{\Sigma_{1,2}}\}_{j=1}^3$ is the given basis of $H_1(\Sigma_{1,2})$, for $i = 1, 2, 3$, there exist non-zero vectors (b_{i1}, b_{i2}, b_{i3}) such that

$$\mathbf{h}^{\mathrm{Im}(g)} = \left\{\sum_{j=1}^{3} b_{ij}\, g(\mathbf{h}_{1,j}^{\Sigma_{1,2}})\right\}_{i=1}^{2}$$

is a basis of Im(g). By equation (5.48), $\mathbf{h}^{\mathrm{Im}(g)}$ becomes the obtained basis $\mathbf{h}_3'$ of $C_3(\mathcal{H}_*)$. Since $C_3(\mathcal{H}_*)$ is equal to Im(g), we can take initial basis $\mathbf{h}_3$ (namely, $\mathbf{h}_1^{\Sigma_{1,1}}$) of $C_3(\mathcal{H}_*)$ as $\mathbf{h}^{\mathrm{Im}(g)}$. Therefore, the determinant of the transition matrix is satisfies the following equality

$$[\mathbf{h}_3', \mathbf{h}_3] = 1. \tag{5.49}$$

Now we consider equation (5.41) for $C_4(\mathcal{H}_*) = H_1(\Sigma_{1,2})$. Then we get

$$C_4(\mathcal{H}_*) = \mathrm{Im}(f) \oplus s_4(\mathrm{Im}(g)). \tag{5.50}$$

By the previous step, we obtain the basis $\mathbf{h}^{\mathrm{Im}(g)}$ of $\mathrm{Im}(g)$ as $\{\sum_{j=1}^{3} b_{ij} g(\mathbf{h}_{1,j}^{\Sigma_{1,2}})\}_{i=1}^{2}$. By the fact that s_4 is a section of g, we get the basis of $s_4\,(\mathrm{Im}(g))$ as

$$s_4(\mathbf{h}^{\mathrm{Im}(g)}) = \left\{ \sum_{j=1}^{3} b_{ij} \mathbf{h}_{1,j}^{\Sigma_{1,2}} \right\}_{i=1}^{2}.$$

Note that $\mathrm{Im}(f)$ is a one-dimensional subspace of $C_4(\mathcal{H}_*)$, so there is a non-zero vector $(b_{3\,1}, b_{3\,2}, b_{3\,3})$ such that $\{b_{3\,1}\mathbf{h}_{1,1}^{\Sigma_{1,2}} + b_{3\,2}\mathbf{h}_{1,1}^{\Sigma_{1,2}} + b_{3\,3}\mathbf{h}_{1,3}^{\Sigma_{1,2}}\}$ is a basis of $\mathrm{Im}(f)$. Clearly, the determinant of the matrix $B = (b_{ij})$ is non-zero. Take the basis of $\mathrm{Im}(f)$ as follows

$$\mathbf{h}^{\mathrm{Im}(f)} = \left\{ (\det B)^{-1} \left[b_{3\,1}\mathbf{h}_{1,1}^{\Sigma_{1,2}} + b_{3\,2}\mathbf{h}_{1,1}^{\Sigma_{1,2}} + b_{3\,3}\mathbf{h}_{1,3}^{\Sigma_{1,2}} \right] \right\}.$$

By equation (5.50),

$$\left\{ \mathbf{h}^{\mathrm{Im}(f)}, s_4(\mathbf{h}^{\mathrm{Im}(g)}) \right\}$$

becomes the obtained basis $\mathbf{h}'_4$ of $C_4(\mathcal{H}_*)$. Since $\mathbf{h}_1^{\Sigma_{1,2}}$ is the initial basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$, we get the following equality

$$[\mathbf{h}'_4, \mathbf{h}_4] = 1. \tag{5.51}$$

Finally, let us consider the case of $C_5(\mathcal{H}_*) = H_1(\mathbb{S}_1^1)$ in equation (5.41). Since $B_5(\mathcal{H}_*)$ is trivial, the following equality holds

$$C_5(\mathcal{H}_*) = B_5(\mathcal{H}_*) \oplus s_5(\mathrm{Im}(f)) = s_5(\mathrm{Im}(f)). \tag{5.52}$$

Recall that the basis $\mathbf{h}^{\mathrm{Im}(f)}$ of $\mathrm{Im}(f)$ was chosen in the previous step. By equation (5.52), $s_5(\mathbf{h}^{\mathrm{Im}(f)})$ becomes the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_5$ (namely,

$\mathbf{h}_0^{\mathbb{S}^1_1}$) of $C_5(\mathcal{H}_*)$ as $s_5(\mathbf{h}^{\mathrm{Im}(f)})$, then the transition matrix satisfies the following equation

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{5.53}$$

By equations (5.43), (5.45), (5.47), (5.49), (5.51), and (5.53), it is concluded that the corrective term equals to 1; that is,

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^5, \{0\}_{p=0}^5\right) = \prod_{p=0}^{5} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{5.54}$$

By the compatibility of the natural bases in the short exact sequence (5.39), Theorem 4.1, and equation (5.54), it is concluded that

$$\mathbb{T}\left(\Sigma_{1,2} \oplus \overline{\mathbb{D}^2_1}, \{\mathbf{h}_p^{\Sigma_{1,2}}\}_{p=0}^1 \sqcup \{\mathbf{h}_0^{\overline{\mathbb{D}^2_1}}\}\right) = \mathbb{T}\left(\Sigma_{1,1}, \{\mathbf{h}_p^{\Sigma_{1,1}}\}_{p=0}^1\right)\ \mathbb{T}\left(\mathbb{S}^1_1, \{\mathbf{h}_p^{\mathbb{S}^1_1}\}_{p=0}^1\right). \tag{5.55}$$

Then Lemma 4.1 and equation (5.55) finish the proof of Proposition 5.2. □

Combining Remark 4.1 and Proposition 5.2, we obtain

Proposition 5.3 *Let $\Sigma_{1,2}, \Sigma_{1,1}, \mathbb{S}^1_1, \overline{\mathbb{D}^2_1}$, $\mathbf{h}_p^{\Sigma_{1,2}}$, $\mathbf{h}_p^{\Sigma_{1,1}}$, $\mathbf{h}_0^{\overline{\mathbb{D}^2_1}}$, $\mathbf{h}_p^{\mathbb{S}^1_i}$ be as in Proposition 5.2. Then the following formula holds*

$$\left|\mathbb{T}\left(\Sigma_{1,2}, \{\mathbf{h}_p^{\Sigma_{1,1}}\}_{p=0}^1\right)\right| = \left|\mathbb{T}\left(\Sigma_{1,1}, \{\mathbf{h}_p^{\Sigma_{1,1}}\}_{p=0}^1\right)\right| \left|\mathbb{T}\left(\overline{\mathbb{D}^2_1}, \{\mathbf{h}_0^{\overline{\mathbb{D}^2_1}}\}\right)\right|^{-1}.$$

The following result provides a formula that computes the Reidemeister torsion of $\Sigma_{g,1}$ in terms of the Reidemeister torsion of the surfaces $\Sigma_{g-1,1}$ and $\Sigma_{1,2}$ and boundary circle γ_1. More precisely,

Proposition 5.4 *Consider the surface $\Sigma_{g,1}$ ($g \geq 2$) obtained by gluing the surfaces $\Sigma_{g-1,1}$ and $\Sigma_{1,2}$ along the common boundary circle γ_1 (see, Figure 5.3).*

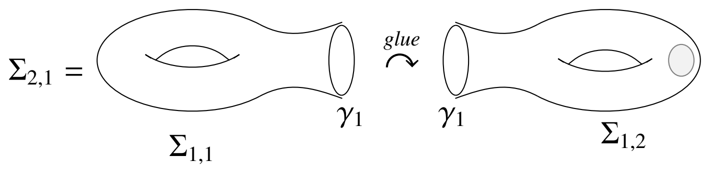


Figure 5.3. Orientable surface $\Sigma_{2,1}$ is obtained by gluing $\Sigma_{1,1}$ and $\Sigma_{1,2}$ along common boundary circle γ_1.

Consider also the associated short exact sequence of chain complexes

$$0 \to C_*(\gamma_1) \longrightarrow C_*(\Sigma_{g-1,1}) \oplus C_*(\Sigma_{1,2}) \longrightarrow C_*(\Sigma_{g,1}) \to 0, \tag{5.56}$$

and corresponding Mayer-Vietoris sequence

$$\begin{aligned}\mathcal{H}_* : \quad 0 \longrightarrow H_1(\gamma_1) \xrightarrow{f} H_1(\Sigma_{g-1,1}) \oplus H_1(\Sigma_{1,2}) \xrightarrow{g} H_1(\Sigma_{g,1}) \\ \xrightarrow{h} H_0(\gamma_1) \xrightarrow{i} H_0(\Sigma_{g-1,1}) \oplus H_0(\Sigma_{1,2}) \xrightarrow{j} H_0(\Sigma_{g,1}) \xrightarrow{k} 0.\end{aligned}$$

For the given bases $\mathbf{h}_p^{\Sigma_{g,1}}$ *and* $\mathbf{h}_p^{\gamma_1}$ *of* $H_p(\Sigma_{g,1})$ *and* $H_p(\gamma_1)$, $p = 0, 1$, *there exist bases* $\mathbf{h}_p^{\Sigma_{g-1,1}}$ *and* $\mathbf{h}_p^{\Sigma_{1,2}}$ *of* $H_p(\Sigma_{g-1,1})$ *and* $H_p(\Sigma_{1,2})$, *respectively such that the corrective term disappears and the following multiplicative gluing formula holds*

$$\mathbb{T}\left(\Sigma_{g,1}, \{\mathbf{h}_p^{\Sigma_{g,1}}\}_{p=0}^1\right) = \mathbb{T}\left(\Sigma_{g-1,1}, \{\mathbf{h}_p^{\Sigma_{g-1,1}}\}_{p=0}^1\right) \mathbb{T}\left(\Sigma_{1,2}, \{\mathbf{h}_p^{\Sigma_{1,2}}\}_{p=0}^1\right) \mathbb{T}\left(\gamma_1, \{\mathbf{h}_p^{\gamma_1}\}_{p=0}^1\right)^{-1}.$$

Proof Let us denote the vector spaces in the sequence $\mathcal{H}_*$ by $C_p(\mathcal{H}_*)$, $p \in \{0, 1, \dots, 5\}$. For each p, the exactness of $\mathcal{H}_*$ yields the following short exact sequence

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*) \to 0.$$

For all p, considering the isomorphism $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*))$ obtained by the

First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$, we obtain

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{5.57}$$

Let us consider the space $C_0(\mathcal{H}_*) = H_0(\Sigma_{g,1})$ in equation (5.57). From the fact that $\mathrm{Im}(k)$ is trivial it follows

$$C_0(\mathcal{H}_*) = \mathrm{Im}(j) \oplus s_0(\mathrm{Im}(k)) = \mathrm{Im}(j). \tag{5.58}$$

Let us choose the basis of $\mathrm{Im}(j)$ as $\mathbf{h}_0^{\Sigma_{g,1}}$. By equation (5.58), it is concluded that $\mathbf{h}_0^{\Sigma_{g,1}}$ becomes the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. Since the initial basis $\mathbf{h}_0$ of $C_0(\mathcal{H}_*)$ is also $\mathbf{h}_0^{\Sigma_{g,1}}$, we have

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{5.59}$$

Next consider $C_1(\mathcal{H}_*) = H_0(\Sigma_{g-1,1}) \oplus H_0(\Sigma_{1,2})$ in equation (5.57), we get

$$C_1(\mathcal{H}_*) = \mathrm{Im}(i) \oplus s_1(\mathrm{Im}(j)). \tag{5.60}$$

As i is injective, $i(\mathbf{h}_0^{\gamma_1})$ becomes the basis of $\mathrm{Im}(i)$. In the previous step, we chose $\mathbf{h}_0^{\Sigma_{g,1}}$ as the basis of $\mathrm{Im}(j)$. Thus, by equation (5.60),

$$\left\{ i(\mathbf{h}_0^{\gamma_1}), s_1(\mathbf{h}_0^{\Sigma_{g,1}}) \right\}$$

becomes the obtained basis $\mathbf{h}'_1$ of $C_1(\mathcal{H}_*)$. Since $H_0(\Sigma_{g-1,1})$ and $H_0(\Sigma_{1,2})$ are both one-dimensional subspaces of the 2-dimensional space $C_1(\mathcal{H}_*)$, there exist non-zero vectors (a_{i1}, a_{i2}), $i = 1, 2$ such that

$$\left\{ a_{11} i(\mathbf{h}_0^{\gamma_1}) + a_{12} s_1(\mathbf{h}_0^{\Sigma_{g,1}}) \right\}$$

is a basis of $H_0(\Sigma_{g-1,1})$ and

$$\left\{ a_{21} i(\mathbf{h}_0^{\gamma_1}) + a_{22} s_1(\mathbf{h}_0^{\Sigma_{g,1}}) \right\}$$

is a basis of $H_0(\Sigma_{1,2})$. Clearly, the (2×2)-matrix $A=(a_{ij})$ is invertible. Let

$$\begin{aligned}\mathbf{h}_0^{\Sigma_{g-1,1}} &= \left\{(\det A)^{-1}[a_{11}\, i(\mathbf{h}_0^{\gamma_1}) + a_{12}\, s_1(\mathbf{h}_0^{\Sigma_{g,1}})]\right\}\\ \mathbf{h}_0^{\Sigma_{1,2}} &= \left\{a_{21} i(\mathbf{h}_0^{\gamma_1}) + a_{22} s_1(\mathbf{h}_0^{\Sigma_{g,1}})\right\}\end{aligned}$$

be respectively basis of $H_0(\Sigma_{g-1,1})$ and $H_0(\Sigma_{1,2})$. Considering

$$\left\{\mathbf{h}_0^{\Sigma_{g-1,1}}, \mathbf{h}_0^{\Sigma_{1,2}}\right\}$$

as the initial basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$, we have

$$[\mathbf{h}_1', \mathbf{h}_1] = 1. \tag{5.61}$$

Now, consider equation (5.57) for the space $C_2(\mathcal{H}_*) = H_0(\gamma_1)$. Since h is a zero map, we get

$$C_2(\mathcal{H}_*) = \operatorname{Im}(h) \oplus s_2(\operatorname{Im}(i)) = s_2(\operatorname{Im}(i)). \tag{5.62}$$

Recall that the basis of $\operatorname{Im}(i)$ was chosen previously as $i(\mathbf{h}_0^{\gamma_1})$. From this and equation (5.62) it follows that $\mathbf{h}_0^{\gamma_1}$ is the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$. In addition, $\mathbf{h}_0^{\gamma_1}$ is also the initial basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$. Thus, we get

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{5.63}$$

Let us consider $C_3(\mathcal{H}_*) = H_1(\Sigma_{g,1})$ in equation (5.57). Obviously, we have

$$C_3(\mathcal{H}_*) = \operatorname{Im}(g) \oplus s_3(\operatorname{Im}(h)) = \operatorname{Im}(g). \tag{5.64}$$

Let us choose the basis of $\operatorname{Im}(g)$ as

$$\mathbf{h}_1^{\Sigma_{g,1}} = \left\{\mathbf{h}_{1,j}^{\Sigma_{g,1}}\right\}_{j=1}^{2g}.$$

By equation (5.64), we get that $\mathbf{h}_1^{\Sigma_{g,1}}$ is the obtained basis $\mathbf{h}_3'$ of $C_3(\mathcal{H}_*)$. Note that $\mathbf{h}_1^{\Sigma_{g,1}}$ is also the initial basis $\mathbf{h}_3$ of $C_3(\mathcal{H}_*)$. So the determinant of the transition matrix is 1; that is,

$$[\mathbf{h}'_3, \mathbf{h}_3] = 1. \tag{5.65}$$

Considering the space $C_4(\mathcal{H}_*) = H_1(\Sigma_{g-1,1}) \oplus H_1(\Sigma_{1,2})$ in equation (5.57), we have

$$C_4(\mathcal{H}_*) = \mathrm{Im}(f) \oplus s_4(\mathrm{Im}(g)). \tag{5.66}$$

As f is injective, we can take the basis of $\mathrm{Im}(f)$ as $f(\mathbf{h}_1^{\gamma_1})$. In the previous step, we chose the basis of $\mathrm{Im}(g)$ as $\mathbf{h}_1^{\Sigma_{g,1}}$. From equation (5.66) it follows that

$$\left\{ f(\mathbf{h}_1^{\gamma_1}), s_4(\mathbf{h}_1^{\Sigma_{g,1}}) \right\}$$

becomes the obtained basis $\mathbf{h}'_4$ of $C_4(\mathcal{H}_*)$. Since $H_1(\Sigma_{g-1,1})$ and $H_1(\Sigma_{1,2})$ are respectively $(2g-2)$ and (3)-dimensional subspaces of the $(2g+1)$-dimensional space $C_4(\mathcal{H}_*)$, for $i \in \{1, \dots, 2g+1\}$ there exist the non-zero vectors $(b_{i1}, \dots, b_{i(2g+1)})$ such that

$$\left\{ \sum_{j=1}^{2g} b_{ij} s_4(\mathbf{h}_{1,j}^{\Sigma_{g,1}}) + b_{i(2g+1)} f(\mathbf{h}_1^{\gamma_1}) \right\}_{i=1}^{3}$$

is a basis of $H_1(\Sigma_{1,2})$ and

$$\left\{ \sum_{j=1}^{2g} b_{ij} s_4(\mathbf{h}_{1,j}^{\Sigma_{g,1}}) + b_{i(2g+1)} f(\mathbf{h}_1^{\gamma_1}) \right\}_{i=4}^{2g+1}$$

is a basis of $H_1(\Sigma_{g-1,1})$. Moreover, $B = (b_{ij})$ is a $(2g+1) \times (2g+1)$-matrix with non-zero determinant. Let us choose the basis of $H_1(\Sigma_{1,2}^1)$ as

$$\mathbf{h}_1^{\Sigma_{1,2}} = \left\{ (\det B)^{-1} \sum_{j=1}^{2g} \left[b_{1j} s_4(\mathbf{h}_{1,j}^{\Sigma_{g,1}}) + b_{1(2g+1)} f(\mathbf{h}_1^{\gamma_1}) \right], \left\{ \sum_{j=1}^{2g} b_{ij} s_4(\mathbf{h}_{1,j}^{\Sigma_{g,1}}) + b_{i(2g+1)} f(\mathbf{h}_1^{\gamma_1}) \right\}_{i=2}^{3} \right\},$$

and take the basis of $H_1(\Sigma_{g-1,1})$ as

$$\mathbf{h}_1^{\Sigma_{g-1,1}} = \left\{ \sum_{j=1}^{2g} b_{ij} s_4(\mathbf{h}_{i,j}^{\Sigma_{g,1}}) + b_{i(2g+1)} f(\mathbf{h}_1^{\gamma_1}) \right\}_{i=4}^{2g+1}.$$

If we consider $\{\mathbf{h}_1^{\Sigma_{g-1,1}}, \mathbf{h}_1^{\Sigma_{1,2}}\}$ as the initial basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$, then we obtain

$$[\mathbf{h}'_4, \mathbf{h}_4] = 1. \tag{5.67}$$

Finally, we consider equation (5.57) for the space $C_5(\mathcal{H}_*) = H_1(\gamma_1)$. By the fact that $B_5(\mathcal{H}_*)$ is trivial, the following equality holds

$$C_5(\mathcal{H}_*) = B_5(\mathcal{H}_*) \oplus s_5(\mathrm{Im}(f)) = s_5(\mathrm{Im}(f)). \tag{5.68}$$

In the previous step, $f(\mathbf{h}_1^{\gamma_1})$ was chosen as the basis of $\mathrm{Im}(f)$. By equation (5.68), $\mathbf{h}_1^{\gamma_1}$ becomes the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. Note that $\mathbf{h}_1^{\gamma_1}$ is also the initial basis $\mathbf{h}_5$ of $C_5(\mathcal{H}_*)$. Hence, we get

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{5.69}$$

By equations (5.59), (5.61), (5.63), (5.65), (5.67), (5.69) it is concluded that the corrective term satisfies the following equality

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^5, \{0\}_{p=0}^5\right) = \prod_{p=0}^{5} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{5.70}$$

Compatibility of the natural bases in the short exact sequence (5.56), Theorem 4.1, and equation (5.70) end the proof of Proposition 5.4. □

Let $\Sigma_{g,1}$ be a one-holed genus g (≥ 2) surface with boundary circles $\mathbb{S}^1_1$. Consider $\Sigma_{g,1}$ as the connected sum $\#_{j=1}^{g-1}(\Sigma_{1,0})\#\Sigma_{1,1}$ (see, Figure 5.4).

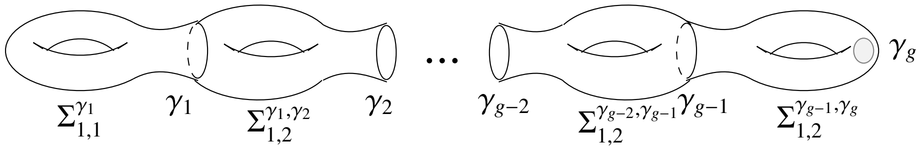


Figure 5.4. Connected sum decomposition of one-holed genus g surface $\Sigma_{g,1}$.

From left to right let $\gamma_1, \ldots, \gamma_{g-1}$ be the boundary circles of the surfaces in the connected sum decomposition of $\Sigma_{g,1}$. This connected sum consists of

- $\overline{\mathbb{D}^2_{\gamma_i}}$, the closed disk with boundary circle γ_i, $i = 1, \ldots, g$,
- $\Sigma_{1,1}^{\gamma_1}$, the one-holed genus one surface with boundary circle γ_1,
- $\Sigma_{1,2}^{\gamma_i,\gamma_{i+1}}$, the 2-holed genus one surface with boundary circles γ_i, γ_{i+1}, $i = 1, \ldots, g-1$.

If $\Sigma_{1,1}^{\gamma_{i+1}}$ denotes the one-holed torus with boundary circle γ_{i+1} which is obtained by gluing $\Sigma_{1,2}^{\gamma_i,\gamma_{i+1}}$ and the closed disk $\overline{\mathbb{D}^2_{\gamma_i}}$ along the common boundary circle γ_i for $i = 1, \ldots, g-1$, then, by Proposition 5.2, there exists the homology basis $(\mathbf{h}_p^{\gamma_i})'$ of $H_p(\gamma_i)$. By using Proposition 5.4 inductively, we obtain the following theorem.

Theorem 5.2 *Let* $\mathbf{h}_p^{\Sigma_{g,1}}$, $\mathbf{h}_p^{\gamma_i}$, *and* $\mathbf{h}_0^{\overline{\mathbb{D}^2_{\gamma_i}}}$ *be respectively bases of* $H_p(\Sigma_{g,1})$, $H_p(\gamma_i)$, *and* $H_0(\overline{\mathbb{D}^2_{\gamma_i}})$ *for* $p = 0, 1$, $i = 1, \ldots, g-1$. *Then there exist bases* $\mathbf{h}_p^{\Sigma_{1,1}^{\gamma_i}}$ *and* $(\mathbf{h}_p^{\gamma_i})'$ *so that the following formula is valid*

$$\begin{aligned}\mathbb{T}\left(\Sigma_{g,1}, \{\mathbf{h}_p^{\Sigma_{g,1}}\}_{p=0}^1\right) = &\prod_{i=1}^{g} \mathbb{T}\left(\Sigma_{1,1}^{\gamma_i}, \{\mathbf{h}_p^{\Sigma_{1,1}^{\gamma_i}}\}_{p=0}^1\right)\\ &\times \prod_{i=1}^{g-1} \mathbb{T}\left(\gamma_i, \{\mathbf{h}_p^{\gamma_i}\}_{p=0}^1\right)^{-1}\\ &\times \prod_{i=1}^{g-1} \left[\mathbb{T}\left(\gamma_i, \{(\mathbf{h}_p^{\gamma_i})'\}_{p=0}^1\right) \mathbb{T}\left(\overline{\mathbb{D}^2_{\gamma_i}}, \{\mathbf{h}_0^{\overline{\mathbb{D}^2_{\gamma_i}}}\}\right)^{-1}\right].\end{aligned}$$

Remark 5.1 *Reidemeister torsion of n-holed genus one surface* $\Sigma_{1,n}$ *is also obtained by following the arguments stated in the Proposition 5.2. Using this result and Theorem 5.2, we obtain Reidmeister torsion of n-holed genus g surface* $\Sigma_{g,n}$ *for* $n > 1$. *These results can be found in (Dirican and Sözen, 2016).*

Theorem 5.3 *Let $\Sigma_{g,0}$ be a genus g (≥ 2) closed orientable surface. From left to right let $\gamma_1, \ldots, \gamma_{g-1}$ be the circles obtained by the connected sum decomposition of $\Sigma_{g,0}$. Let $\Sigma_{1,1}^{\gamma_g}$ be one-holed genus one surface with boundary circle γ_{g-1}. Then the surface $\Sigma_{g,0}$ is obtained by gluing the surfaces $\Sigma_{g-1,1}$ and $\Sigma_{1,1}$ along the common boundary circle γ_{g-1}. Consider the natural short exact sequence of chain complexes*

$$0 \to C_*(\gamma_{g-1}) \to C_*(\Sigma_{g-1,1}) \oplus C_*(\Sigma_{1,1}^{\gamma_g}) \to C_*(\Sigma_{g,0}) \to 0 \tag{5.71}$$

and its corresponding Mayer-Vietoris sequence

$$\begin{array}{l} \mathcal{H}_* : \quad 0 \longrightarrow H_2(\Sigma_{g,0}) \xrightarrow{\delta} H_1(\gamma_{g-1}) \xrightarrow{f} H_1(\Sigma_{g-1,1}) \oplus H_1(\Sigma_{1,1}^{\gamma_g}) \xrightarrow{g} H_1(\Sigma_{g,0}) \\ \xrightarrow{h} H_0(\gamma_{g-1}) \xrightarrow{i} H_0(\Sigma_{g-1,1}) \oplus H_0(\Sigma_{1,1}^{\gamma_g}) \xrightarrow{j} H_0(\Sigma_{g,0}) \xrightarrow{k} 0, \end{array}$$

where the connecting map δ is an isomorphism. Let $\mathbf{h}_p^{\Sigma_{g,0}}$ be a basis of $H_p(\Sigma_{g,0})$, $p = 0, 1, 2$. Let $\mathbf{h}_1^{\gamma_{g-1}} = \delta(\mathbf{h}_2^{\Sigma_{g,0}})$ be the basis of $H_1(\gamma_{g-1})$ and $\mathbf{h}_0^{\gamma_{g-1}}$ be an arbitrary basis of $H_0(\gamma_{g-1})$. Then there are respectively bases $\mathbf{h}_\eta^{\Sigma_{g-1,1}}$ and $\mathbf{h}_\eta^{\Sigma_{1,1}^{\gamma_g}}$ of $H_\eta(\Sigma_{g-1,1})$ and $H_\eta(\Sigma_{1,1}^{\gamma_g})$, $\eta = 0, 1$ such that the corrective term becomes 1 and the following formula holds

$$\mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_p^{\Sigma_{g,0}}\}_{p=0}^2\right) = \mathbb{T}\left(\Sigma_{g-1,1}, \{\mathbf{h}_\eta^{\Sigma_{g-1,1}}\}_{\eta=0}^1\right)\ \mathbb{T}\left(\Sigma_{1,1}^{\gamma_g}, \{\mathbf{h}_\eta^{\Sigma_{1,1}^{\gamma_g}}\}_{\eta=0}^1\right)\ \mathbb{T}\left(\gamma_{g-1}, \{\mathbf{h}_\eta^{\gamma_{g-1}}\}_{\eta=0}^1\right)^{-1}.$$

Proof The exactness of $\mathcal{H}_*$ implies that f and h are zero-maps and thus the following isomorphisms hold

$$\begin{aligned} H_1(\Sigma_{g,0}) &\cong H_1(\gamma_{g-1}) \\ H_1(\Sigma_{g,0}) &\cong H_1(\Sigma_{g-1,1}) \oplus H_1(\Sigma_{1,1}^{\gamma_g}) \\ H_0(\Sigma_{g-1,1}) \oplus H_0(\Sigma_{1,1}^{\gamma_g}) &\cong H_0(\gamma_{g-1}) \oplus H_0(\Sigma_{g,0}). \end{aligned}$$

Using the above isomorphisms together with the arguments presented in the proof of Theorem 5.1, we obtain that the corrective term is 1. The compatibility of the natural bases in the short exact sequence (5.71) and Theorem 4.1 give the following formula

$$\mathbb{T}\left(\Sigma_{g-1,1} \oplus \Sigma_{1,1}^{\gamma_g}, \{\mathbf{h}_\eta^{\Sigma_{g-1,1}}\}_{\eta=0}^1 \sqcup \{\mathbf{h}_\eta^{\Sigma_{1,1}^{\gamma_g}}\}_{\eta=0}^1\right) = \mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_p^{\Sigma_{g,0}}\}_{p=0}^2\right)\ \mathbb{T}\left(\gamma_{g-1}, \{\mathbf{h}_\eta^{\gamma_{g-1}}\}_{\eta=0}^1\right). \quad (5.72)$$

Then Lemma 4.1 and equation (5.72) finish the proof of Proposition 5.3. □

Now we consider the connected sum decomposition of closed genus g surface $\Sigma_{g,0}$ given in Figure 5.5. More precisely, this decomposition consists of

- $\overline{\mathbb{D}^2_{\gamma_i}}$, the closed disk with boundary circle γ_i, $i = 1, \dots, g-1$,
- $\Sigma_{1,1}^{\gamma_1}$, the one-holed genus one surface with boundary circle γ_i,
- $\Sigma_{1,1}^{\gamma_g}$, the one-holed genus one surface with boundary circle γ_{g-1},
- $\Sigma_{1,2}^{\gamma_i,\gamma_{i+1}}$, the 2-holed genus one surface with boundary circles γ_i, γ_{i+1}, $i = 1, \dots, g-2$.

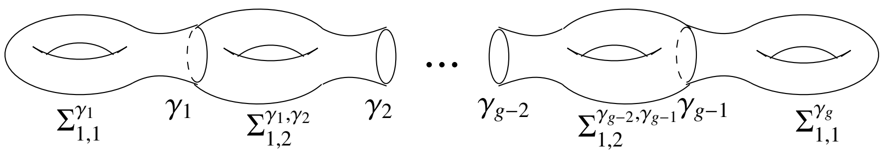


Figure 5.5. Connected sum decomposition of closed genus g surface $\Sigma_{g,0}$.

Combining Theorem 5.2 and Theorem 5.3, we have the main result of this section.

Theorem 5.4 *Consider the connected sum decomposition* $\Sigma_{g,0} = \mathop{\#}_{j=1}^{g}(\Sigma_{1,0})$ *given in Figure 5.5. Let* $\Sigma_{1,1}^{\gamma_{j+1}}$ *be the one-holed torus with boundary circle* γ_{i+1} *which is obtained by gluing* $\Sigma_{1,1}^{\gamma_j,\gamma_{j+1}}$ *and* $\overline{\mathbb{D}^2_{\gamma_j}}$ *along the common boundary circle* γ_i *for* $j = 1, \dots, g-2$. *Assume that* $\mathbf{h}_p^{\Sigma_{g,0}}$ *and* $\mathbf{h}_0^{\overline{\mathbb{D}^2_{\gamma_j}}}$ *are respectively bases of* $H_p(\Sigma_{g,0})$ *and* $H_0(\overline{\mathbb{D}^2_{\gamma_j}})$ *for* $p = 0, 1, 2$. *Assume also that* $\mathbf{h}_\eta^{\gamma_i}$ *is an arbitrary basis of* $H_\eta(\gamma_i)$ *such that* $\mathbf{h}_1^{\gamma_{g-1}} = \delta(\mathbf{h}_2^{\Sigma_{g,0}})$ *for* $\eta = 0, 1$. *Then there are bases* $\mathbf{h}_\eta^{\Sigma_{1,1}^{\gamma_i}}$, $(\mathbf{h}_\eta^{\gamma_j})'$ *so that the following multiplicative gluing formula is valid*

$$\mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_p^{\Sigma_{g,0}}\}_{p=0}^2\right) = \prod_{i=1}^{g} \mathbb{T}\left(\Sigma_{1,1}^{\gamma_i}, \{\mathbf{h}_\eta^{\Sigma_{1,1}^{\gamma_i}}\}_{\eta=0}^1\right)$$
$$\times \prod_{i=1}^{g-1} \mathbb{T}\left(\gamma_i, \{\mathbf{h}_\eta^{\gamma_i}\}_{\eta=0}^1\right)^{-1}$$
$$\times \prod_{j=1}^{g-2} \left[\mathbb{T}\left(\gamma_j, \{(\mathbf{h}_\eta^{\gamma_j})'\}_{\eta=0}^1\right) \mathbb{T}\left(\overline{\mathbb{D}_{\gamma_j}^2}, \{\mathbf{h}_0^{\overline{\mathbb{D}_{\gamma_j}^2}}\}\right)^{-1}\right].$$

Here, δ is obtained in Theorem 5.3.

By Remark 4.1 and Theorem 5.4, we obtain the following result

Theorem 5.5 *Let $\Sigma_{g,0}$, $\Sigma_{1,1}^{\gamma_i}$, $\overline{\mathbb{D}_{\gamma_i}^2}$, $\mathbf{h}_p^{\Sigma_{g,0}}$, $\mathbf{h}_\eta^{\Sigma_{1,1}^{\gamma_i}}$, $\mathbf{h}_0^{\overline{\mathbb{D}_{\gamma_i}^2}}$ be as in Theorem 5.4. Then the following formula is valid*

$$\left|\mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_p^{\Sigma_{g,0}}\}_{p=0}^2\right)\right| = \prod_{i=1}^{g} \left|\mathbb{T}\left(\Sigma_{1,1}^{\gamma_i}, \{\mathbf{h}_\eta^{\Sigma_{1,1}^{\gamma_i}}\}_{\eta=0}^1\right)\right| \prod_{j=1}^{g-2} \left|\mathbb{T}\left(\overline{\mathbb{D}_{\gamma_j}^2}, \{\mathbf{h}_0^{\overline{\mathbb{D}_{\gamma_j}^2}}\}\right)^{-1}\right|$$
$$= \prod_{i=1}^{g} \left|\frac{\triangle_{0,2}^{\Sigma_{2,0}^{\gamma_i}}(\mathbf{h}_0^{\Sigma_{2,0}^{\gamma_i}}, \mathbf{h}_2^{\Sigma_{2,0}^{\gamma_i}})}{\det \wp(\mathbf{h}^1_{\Sigma_{2,0}^{\gamma_i}}, \Gamma)}\right|^{1/2} \prod_{j=1}^{g-2} \left|\mathbb{T}\left(\overline{\mathbb{D}_{\gamma_j}^2}, \{\mathbf{h}_0^{\overline{\mathbb{D}_{\gamma_j}^2}}\}\right)^{-1}\right|.$$

5.2. Reidemeister Torsion of $(n-2)$-Connected $2n$-Dimensional Closed π-Manifold

The purpose of this section is to establish multiplicative gluing formulas for the Reidemeister torsion of $(n-2)$-connected $2n$-dimensional closed π-manifolds by using their connected sum decompositions.

Let $\mathbb{F}_2$ denote the field with two elements and V be a $2n$-dimensional vector space over $\mathbb{F}_2$ for some $n \in \mathbb{Z}^+$. A quadratic form on V is a function $q : V \to \mathbb{F}_2$ such that

- $q(0) = 0$ and
- $q(x + y) - q(x) - q(y) = (x, y)$ is symmetric and $\mathbb{F}_2$-bilinear.

Assume q is non-singular; that is, there exists a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ of V such that $(x_i, y_j) = \delta_{i,j}$, and $(x_i, x_j) = (y_i, y_j) = 0$. This basis is also called *symplectic basis*. Here, $\delta_{i,j}$ denotes the Kronecker delta. The *Arf invariant* of q is given by

$$c(q) = \sum_{i=1}^{n} q(x_i)q(y_i) \in \mathbb{F}_2.$$

Let M be a simply-connected, almost parallelizable, closed, smooth $(4k+2)$-dimensional manifold such that $H_i(M;\mathbb{Z}) = 0$ unless $i \in \{0, 2k+1, 4k+2\}$. Then, by Universal Coefficient Theorem, $H_{2k+1}(M;\mathbb{Z})$ is free abelian. Consider the skew-symmetric intersection form

$$\Phi : H_{2k+1}(M;\mathbb{Z}) \times H_{2k+1}(M;\mathbb{Z}) \to \mathbb{Z}.$$

Define a function

$$\Phi_0 : H_{2k+1}(M;\mathbb{Z}) \to \mathbb{Z}_2$$

as follows: For $k \geqslant 1$ and $x \in H_{2k+1}(M;\mathbb{Z})$, there is a smooth imbedding $\iota_x : \mathbb{S}^{2k+1} \hookrightarrow M$ realizing x. There exists a tubular neighbourhood of $\iota_x(\mathbb{S}^{2k+1})$ in M that is parallelizable which is either trivial or isomorphic to a tubular neighbourhood of the diagonal in $\mathbb{S}^{2k+1} \times \mathbb{S}^{2k+1}$. Then $\Phi_0(0) = 0$ and for any $x, y \in H_{2k+1}(M;\mathbb{Z})$

$$\Phi_0(x+y) \equiv \Phi_0(x) + \Phi_0(y) + \Phi(x,y) \bmod 2.$$

Hence, Φ_0 is a quadratic form. It is also well-known that Φ_0 is non-singular for $2k+1 \neq 1, 3, 7$.

Definition 5.1 *The Arf-Kervaire invariant of a compact smooth $(4k+2)$-dimensional π-manifold M, denoted by $\kappa(M)$, is defined as the Arf invariant of $\Phi_0 : H_{2k+1}(M;\mathbb{Z}) \to \mathbb{Z}_2$ for a symplectic basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ of $H_{2k+1}(M)$; that is,*

$$\kappa(M) = \sum_{i=1}^{n} \Phi_0(x_i)\Phi_0(y_i) \pmod 2.$$

Note that $\kappa(M)$ is independent of the choice of the symplectic basis.

As an example, for the Kervaire manifold M_K^{2n}, we have $\kappa(M_K^{2n}) = 1$. Note that for the closed manifold $\mathbb{S}^{2n+1} \times \mathbb{S}^{2n+1}$, the Arf-Kervaire invariant is equal to zero.

For given $(4k+2)$-dimensional π-manifolds M_1 and M_2, the middle cohomology of $M_1 \# M_2$ is $H_{2k+1}(M_1) \oplus H_{2k+1}(M_2)$ with the two summands orthogonal for the cup product pairing. Hence, the symplectic bases for $H_{2k+1}(M_1; \mathbb{Z}_2)$ and $H_{2k+1}(M_2; \mathbb{Z}_2)$ yield together a symplectic basis for $H_{2k+1}(M_1 \# M_2; \mathbb{Z}_2)$. By applying the Kervaire form to the following maps

$$M_1 \# M_2 \to M_1 \vee M_2 \to M_1,$$
$$M_1 \# M_2 \to M_1 \vee M_2 \to M_2,$$

it is concluded that

$$\kappa(M_1 \# M_2) = \kappa(M_1) + \kappa(M_2).$$

Let M be an $(n-1)$-connected $2n$-dimensional closed π-manifold ($n \geq 3$). If the Arf-Kervaire invariant of M is zero, then Ishimoto (1969) showed that there exists such a symplectic basis $\{x_1, \ldots, x_p, y_1, \ldots, y_p\}$ for $H_n(M; \mathbb{Z})$ with $\Phi_0(x_i) = \Phi_0(y_i) = 0$ that the imbedded n-spheres $\mathbb{S}^n$, $\mathbb{S}'^n$ representing x_i, y_i respectively have trivial normal bundles. By using this result with the surgery on π-manifolds, Ishimoto (1969) proved that there is a decomposition for an $(n-2)$-connected $2n$-dimensional closed π-manifold M^{2n} as follows:

Theorem 5.6 *Let M^{2n} be an $(n-2)$-connected $2n$-dimensional closed π-manifold ($n \geq 3$) such that $H_{n-1}(M^{2n}; \mathbb{Z})$ has no torsion. Under the assumption $\kappa(M^{2n}) = 0$ when $n = 4k+3$, there exists the decomposition*

$$M^{2n} = M \# M_1^{2n},$$

where $M = \mathop{\#}\limits_{j=1}^{p} (\mathbb{S}^n \times \mathbb{S}^n)$ is the connected sum of p (≥ 2) copies of the product of the original n-spheres and M_1^{2n} is an $(n-2)$-connected $2n$-dimensional closed π-manifold such that

$$H_i(M_1^{2n}; \mathbb{Z}) \simeq \begin{cases} H_i(M^{2n}; \mathbb{Z}), & i = n-1, n+1 \\ 0, & i = n. \end{cases} \tag{5.73}$$

Here, $2p$ is the rank of $H_n(M^{2n}; \mathbb{Z})$.

From the Van Kampen Theorem it follows that M_1^{2n} is simply connected. By using the Mayer-Vietoris sequence, it has such homology groups as

$$H_i(M_1^{2n}) \cong \begin{cases} \mathbb{R}^r , & i = n-1, n+1 \\ \mathbb{R} , & i = 0, 2n \\ 0 , & \text{otherwise.} \end{cases} \tag{5.74}$$

From equation (5.73) and equation (5.74) it follows

$$H_i(M^{2n}) \cong \begin{cases} \mathbb{R}^r , & i = n-1, n+1 \\ \mathbb{R}^{2p} , & i = n \\ \mathbb{R} , & i = 0, 2n \\ 0 , & \text{otherwise.} \end{cases}$$

Note that M_1^{2n} is a π-manifold since the index (Hirzebruch signature) of M_1^{2n} is zero. It is also decomposed as follows

$$M_1^{2n} = \widetilde{\mathbb{S}^{2n}} \# \partial(\mathcal{H}^{2n+1}),$$

where $\widetilde{\mathbb{S}^{2n}}$ is the homotopy $2n$-sphere and $\mathcal{H}^{2n+1}$ is a handlebody

$$\mathbb{D}^{2n+1} \underset{\{\varphi_i\}}{\cup} \{\bigcup_{i=1}^{r} \mathbb{D}_i^{n+1} \times \mathbb{D}_i^n\},$$

$r = \text{rank}(H_{n-1}(M_1^{2n}))$, and $\{\varphi_i : \mathbb{S}^n \times \mathbb{D}^n \to \mathbb{S}^{2n}\}_{i=1}^r$ is the disjoint set of imbeddings.

Throughout this section, we assume that the Arf-Kervaire invariant is zero when $n = 4k + 3$ for manifold M^{2n} and use the notation $M = \overset{p}{\underset{j=1}{\#}} M_j$ instead of $M = \overset{p}{\underset{j=1}{\#}} (\mathbb{S}^n \times \mathbb{S}^n)$, where $M_j = \mathbb{S}^n \times \mathbb{S}^n$ for each $j \in \{1, \ldots, p\}$.

In this section, our aim is to prove Theorem 5.7 which gives a formula that computes the Reidemeister torsion of M^{2n} in terms of the Reidemeister torsion of its building blocks in the decomposition given in Theorem 5.6.

Theorem 5.7 *Suppose that $M^{2n} = M \# M_1^{2n}$ is an $(n-2)$-connected $2n$-dimensional closed π-manifold ($n \geq 3$) such that $H_{n-1}(M^{2n}; \mathbb{Z})$ has no torsion, where $M = \overset{p}{\underset{j=1}{\#}} M_j$ is a connected sum of p (≥ 2) copies of $\mathbb{S}^n \times \mathbb{S}^n$ and M_1^{2n} is an $(n-2)$-connected $2n$-dimensional* *closed π-manifold. Let $\mathbf{h}_\nu^{M^{2n}}$ and $\mathbf{h}_\eta^{\mathbb{S}_j^{2n-1}}$ be bases of $H_\nu(M^{2n})$ and $H_\eta(\mathbb{S}_j^{2n-1})$, $\nu = 0, \cdots, 2n$, $\eta = 0, \cdots, 2n-1$ and let $\mathbf{h}_0^{\overline{\mathbb{D}_j^{2n}}}$ be an arbitrary basis of $H_0(\overline{\mathbb{D}_j^{2n}})$. Then there exist respectively bases $\mathbf{h}_\nu^{M_j}$ and $\mathbf{h}_\nu^{M_1^{2n}}$ of $H_\nu(M_j)$ and $H_\nu(M_1^{2n})$ such that the corrective term becomes 1 and the Reidemeister torsion of M^{2n} satisfies the following formula*

$$\mathbb{T}(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}) = \mathbb{T}\left(M_1^{2n}, \{\mathbf{h}_\nu^{M_1^{2n}}\}_{\nu=0}^{2n}\right) \prod_{j=1}^{p} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^{2n}\right)$$
$$\times \prod_{j=1}^{p} \left[\mathbb{T}\left(\mathbb{S}_j^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}_j^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}_j^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_j^{2n}}}\}\right)^{-2} \right].$$

Now we establish auxiliary results to prove Theorem 5.7.

Proposition 5.5 *Let $M = M_L \# M_R$ be a connected sum of p (≥ 2) copies of $\mathbb{S}^n \times \mathbb{S}^n$, where $M_L = \overset{p-1}{\underset{j=1}{\#}} (\mathbb{S}^n \times \mathbb{S}^n)$ and $M_R = \mathbb{S}^n \times \mathbb{S}^n$. Then there exists the natural short exact sequence of the chain complexes*

$$0 \to C_*(\mathbb{S}^{2n-1}) \longrightarrow C_*(M_L - \mathbb{D}^{2n}) \oplus C_*(M_R - \mathbb{D}^{2n}) \longrightarrow C_*(M) \to 0 \tag{5.75}$$

and its corresponding Mayer-Vietoris sequence

$$\begin{aligned}
\mathcal{H}_*: \quad & 0 \xrightarrow{\alpha} H_{2n}(M) \xrightarrow{\gamma} H_{2n-1}(\mathbb{S}^{2n-1}) \xrightarrow{\sigma} 0 \\
& \xrightarrow{f} H_n(M_L - \mathbb{D}^{2n}) \oplus H_n(M_R - \mathbb{D}^{2n}) \xrightarrow{g} H_n(M) \xrightarrow{h} 0 \\
& \xrightarrow{\beta} H_0(\mathbb{S}^{2n-1}) \xrightarrow{\ell} H_0(M_L - \mathbb{D}^{2n}) \oplus H_0(M_R - \mathbb{D}^{2n}) \xrightarrow{m} H_0(M) \xrightarrow{\rho} 0.
\end{aligned} \tag{5.76}$$

If $\mathbf{h}_\nu^M$, $\mathbf{h}_0^{\mathbb{S}^{2n-1}}$, $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}} = \gamma(\mathbf{h}_{2n}^M)$ are respectively bases of $H_\nu(M)$, $H_0(\mathbb{S}^{2n-1})$, $H_{2n-1}(\mathbb{S}^{2n-1})$ for $\nu = 0, \cdots, 2n$, then there exist respectively bases $\mathbf{h}_\nu^{M_L - \mathbb{D}^{2n}}$, $\mathbf{h}_\nu^{M_R - \mathbb{D}^{2n}}$ of $H_\nu(M_L - \mathbb{D}^{2n})$,

$H_\nu(M_R - \mathbb{D}^{2n})$ *such that the corrective term disappears and the following formula holds*

$$\mathbb{T}\left(M, \{\mathbf{h}_\nu^M\}_{\nu=0}^{2n}\right) = \mathbb{T}\left(M_L - \mathbb{D}^{2n}, \{\mathbf{h}_\nu^{M_L - \mathbb{D}^{2n}}\}_{\nu=0}^{2n}\right)\ \mathbb{T}\left(M_R - \mathbb{D}^{2n}, \{\mathbf{h}_\nu^{M_R - \mathbb{D}^{2n}}\}_{\nu=0}^{2n}\right)$$
$$\times\, \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right)^{-1}.$$

Proof For $p \in \{0, \ldots, 8\}$, let $C_p(\mathcal{H}_*)$ denote the vector spaces in the long exact sequence $\mathcal{H}_*$. Then we consider the short exact sequences

$$0 \to Z_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \overset{\partial_p}{\to} B_{p-1}(\mathcal{H}_*) \to 0, \tag{5.77}$$

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow Z_p(\mathcal{H}_*) \overset{\varphi_p}{\twoheadrightarrow} H_p(\mathcal{H}_*) \to 0. \tag{5.78}$$

For each p, let us consider the isomorphism $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*))$ obtained by the First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$. From the exactness of $\mathcal{H}_*$ it follows

$$B_p(\mathcal{H}_*) = Z_p(\mathcal{H}_*).$$

Hence, the sequence (5.77) becomes

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*) \to 0. \tag{5.79}$$

Applying the Splitting Lemma for the sequence (5.79), we have

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{5.80}$$

Now we consider the vector space $C_0(\mathcal{H}_*) = H_0(M)$ in equation (5.80). By the fact that $\mathrm{Im}(\rho) = \{0\}$, we obtain

$$C_0(\mathcal{H}_*) = \mathrm{Im}(m) \oplus s_0(\mathrm{Im}(\rho)) = \mathrm{Im}(m). \tag{5.81}$$

Let us choose the basis $\mathbf{h}^{\mathrm{Im}(m)}$ of $\mathrm{Im}(m)$ as $\mathbf{h}_0^M$. It follows from equation (5.81) that $\mathbf{h}_0^M$ is the obtained basis $\mathbf{h}_0'$ of $C_0(\mathcal{H}_*)$. Since $\mathbf{h}_0^M$ is also the initial basis $\mathbf{h}_0$ of $C_0(\mathcal{H}_*)$, the following equality holds

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{5.82}$$

By equation (5.80), the space $C_1(\mathcal{H}_*) = H_0(M_L - \mathbb{D}^{2n}) \oplus H_0(M_R - \mathbb{D}^{2n})$ can be expressed as the following direct sum

$$C_1(\mathcal{H}_*) = \mathrm{Im}(\ell) \oplus s_1(\mathrm{Im}(m)). \tag{5.83}$$

In the previous step, the basis $\mathbf{h}^{\mathrm{Im}(m)}$ of $\mathrm{Im}(m)$ was chosen as $\mathbf{h}_0^M$. Note that $\mathrm{Im}(\ell)$ is isomorphic to $H_0(\mathbb{S}^{2n-1})$, so we can choose the basis $\mathbf{h}^{\mathrm{Im}(\ell)}$ of $\mathrm{Im}(\ell)$ as $\ell(\mathbf{h}_0^{\mathbb{S}^{2n-1}})$. If we use equation (5.83), then we get the obtained basis $\mathbf{h}'_1$ of $C_1(\mathcal{H}_*)$ as

$$\left\{\ell(\mathbf{h}_0^{\mathbb{S}^{2n-1}}), s_1(\mathbf{h}_0^M)\right\}.$$

Since $H_0(M_L - \mathbb{D}^{2n})$ and $H_0(M_R - \mathbb{D}^{2n})$ are one-dimensional subspaces of the 2-dimensional space $C_1(\mathcal{H}_*)$, there are non-zero vectors (a_{11}, a_{12}) and (a_{21}, a_{22}) such that

$$\left\{a_{11}\ell(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{12}s_1(\mathbf{h}_0^M)\right\} \text{ and } \left\{a_{21}\ell(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{22}s_1(\mathbf{h}_0^M)\right\}$$

are bases of $H_0(M_L - \mathbb{D}^{2n})$ and $H_0(M_R - \mathbb{D}^{2n})$, respectively. Then we obtain a non-singular (2×2)-matrix $A = (a_{ij})$ with entries in $\mathbb{R}$. Let us take the basis of $H_0(M_L - \mathbb{D}^{2n})$ and $H_0(M_R - \mathbb{D}^{2n})$ as

$$\begin{aligned}
\mathbf{h}_0^{M_L - \mathbb{D}^{2n}} &= \left\{(\det A)^{-1}\left[a_{11}\ell(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{12}s_1(\mathbf{h}_0^M)\right]\right\}, \\
\mathbf{h}_0^{M_R - \mathbb{D}^{2n}} &= \left\{a_{21}\ell(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{22}s_1(\mathbf{h}_0^M)\right\}.
\end{aligned}$$

Taking $\{\mathbf{h}_0^{M_L - \mathbb{D}^{2n}}, \mathbf{h}_0^{M_R - \mathbb{D}^{2n}}\}$ as the initial basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$, we get that the determinant of the transition matrix is 1 as follows

$$[\mathbf{h}'_1, \mathbf{h}_1] = 1. \tag{5.84}$$

If we use equation (5.80) for $C_2(\mathcal{H}_*) = H_0(\mathbb{S}^{2n-1})$ and consider the fact that $\mathrm{Im}(\beta) = \{0\}$, then we have

$$C_2(\mathcal{H}_*) = \mathrm{Im}(\beta) \oplus s_2(\mathrm{Im}(\ell)) = s_2(\mathrm{Im}(\ell)). \tag{5.85}$$

Note that $\mathbf{h}_0^{\mathbb{S}^{2n-1}}$ is the initial basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$. By equation (5.85), we get that

$$s_2(\ell(\mathbf{h}_0^{\mathbb{S}^{2n-1}})) = \mathbf{h}_0^{\mathbb{S}^{2n-1}}$$

is the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$. Hence, the following equality holds

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{5.86}$$

Considering the trivial space $C_3(\mathcal{H}_*)$ in the sequence $\mathcal{H}_*$ and using the convention $1 \cdot 0 = 1$, we have

$$[\mathbf{h}_3', \mathbf{h}_3] = 1. \tag{5.87}$$

We next consider equation (5.80) for $C_4(\mathcal{H}_*) = H_n(M)$. Since $\mathrm{Im}(h)$ is trivial, we get the following equality

$$C_4(\mathcal{H}_*) = \mathrm{Im}(g) \oplus s_4(\mathrm{Im}(h)) = \mathrm{Im}(g). \tag{5.88}$$

Let us choose the basis $\mathbf{h}^{\mathrm{Im}(g)}$ of $\mathrm{Im}(g)$ as $\mathbf{h}_n^M = \{\mathbf{h}_{n,1}^M, \cdots, \mathbf{h}_{n,2p}^M\}$. Then, by equation (5.88), $\mathbf{h}_n^M$ becomes the obtained basis $\mathbf{h}_4'$ of $C_4(\mathcal{H}_*)$. Moreover, the initial basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$ is also $\mathbf{h}_n^M$, we get

$$[\mathbf{h}_4', \mathbf{h}_4] = 1. \tag{5.89}$$

If we consider $C_5(\mathcal{H}_*) = H_n(M_L - \mathbb{D}^{2n}) \oplus H_n(M_R - \mathbb{D}^{2n})$ in equation (5.80) and use the fact that $\mathrm{Im}(f) = \{0\}$, then the following equality holds

$$C_5(\mathcal{H}_*) = \mathrm{Im}(f) \oplus s_5(\mathrm{Im}(g)) = s_5(\mathrm{Im}(g)). \tag{5.90}$$

In the previous step, the basis $\mathbf{h}^{\text{Im}(g)}$ of $\text{Im}(g)$ was chosen as $\mathbf{h}_n^M$. From equation (5.90) it follows that $s_5(\mathbf{h}_n^M)$ becomes the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. Recall that $\{\mathbf{h}_{n,1}^M, \cdots, \mathbf{h}_{n,2p}^M\}$ is the given basis $\mathbf{h}_n^M$ of $H_n(M)$. Since $H_n(M_L - \mathbb{D}^{2n})$ and $H_n(M_R - \mathbb{D}^{2n})$ are respectively $(2p-2)$ and 2-dimensional subspaces of $2p$-dimensional space $C_5(\mathcal{H}_*)$, there are non-zero vectors $(b_{j1}, \cdots, b_{j2p})$ for $j \in \{1, \ldots, 2p\}$ such that

$$\left\{\sum_{i=1}^{2p} b_{ji} s_5(\mathbf{h}_{n,i}^M)\right\}_{j=1}^{2p-2} \text{ and } \left\{\sum_{i=1}^{2p} b_{ji} s_5(\mathbf{h}_{n,i}^M)\right\}_{j=2p-1}^{2p}$$

are bases of $H_n(M_L - \mathbb{D}^{2n})$ and $H_n(M_R - \mathbb{D}^{2n})$, respectively. Clearly, the transition matrix $B = (b_{ji})$ is an invertible-$(2p \times 2p)$ real matrix. If we let

$$\begin{aligned}
\mathbf{h}_n^{M_L - \mathbb{D}^{2n}} &= \left\{\det(B)^{-1} \sum_{i=1}^{2p} b_{1i} s_5(\mathbf{h}_{n,i}^M), \left\{\sum_{i=1}^{2p} b_{ji} s_5(\mathbf{h}_{n,i}^M)\right\}_{j=2}^{2p-2}\right\}, \\
\mathbf{h}_n^{M_R - \mathbb{D}^{2n}} &= \left\{\sum_{i=1}^{2p} b_{ji} s_5(\mathbf{h}_{n,i}^M)\right\}_{j=2p-1}^{2p}
\end{aligned}$$

be the basis of $H_n(M_L - \mathbb{D}^{2n})$ and $H_n(M_R - \mathbb{D}^{2n})$, respectively and if we take the initial basis $\mathbf{h}_5$ of $C_5(\mathcal{H}_*)$ as $\{\mathbf{h}_n^{M_L - \mathbb{D}^{2n}}, \mathbf{h}_n^{M_R - \mathbb{D}^{2n}}\}$, then we get

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{5.91}$$

Considering the trivial space $C_6(\mathcal{H}_*)$ in the sequence $\mathcal{H}_*$ and using the convention $1 \cdot 0 = 0$, the transition matrix satisfies the following equation

$$[\mathbf{h}'_6, \mathbf{h}_6] = 1. \tag{5.92}$$

Let us consider the space $C_7(\mathcal{H}_*) = H_{2n-1}(\mathbb{S}^{2n-1})$ in equation (5.80). Since $\text{Im}(\sigma)$ is trivial, we get

$$C_7(\mathcal{H}_*) = \text{Im}(\gamma) \oplus s_7(\text{Im}(\sigma)) = \text{Im}(\gamma). \tag{5.93}$$

Taking the basis $\mathbf{h}^{\mathrm{Im}(\gamma)}$ of $\mathrm{Im}(\gamma)$ as $\mathbf{h}^{\mathbb{S}^{2n-1}}_{2n-1} = \gamma(\mathbf{h}^{M}_{2n})$ and considering equation (5.93), we get that $\mathbf{h}^{\mathbb{S}^{2n-1}}_{2n-1}$ is the obtained basis $\mathbf{h}'_7$ of $C_7(\mathcal{H}_*)$. As the initial basis $\mathbf{h}_7$ of $C_7(\mathcal{H}_*)$ is also $\mathbf{h}^{\mathbb{S}^{2n-1}}_{2n-1}$, the following equality holds

$$[\mathbf{h}'_7, \mathbf{h}_7] = 1. \tag{5.94}$$

Finally, let us consider equation (5.80) for $C_8(\mathcal{H}_*) = H_{2n}(M)$. Since $\mathrm{Im}(\alpha)$ is trivial, the space $C_8(\mathcal{H}_*)$ can be expressed as follows

$$C_8(\mathcal{H}_*) = \mathrm{Im}(\alpha) \oplus s_8(\mathrm{Im}(\gamma)) = s_8(\mathrm{Im}(\gamma)). \tag{5.95}$$

Recall that $\mathbf{h}^{M}_{2n}$ is the initial basis $\mathbf{h}_8$ of $C_8(\mathcal{H}_*)$ and $\mathbf{h}^{\mathbb{S}^{2n-1}}_{2n-1} = \gamma(\mathbf{h}^{M}_{2n})$ was chosen as the basis $\mathbf{h}^{\mathrm{Im}(\gamma)}$ of $\mathrm{Im}(\gamma)$ in the previous step. By equation (5.95), $s_8(\gamma(\mathbf{h}^{M}_{2n})) = \mathbf{h}^{M}_{2n}$ becomes the obtained basis $\mathbf{h}'_8$ of $C_8(\mathcal{H}_*)$ and satisfies the following equation

$$[\mathbf{h}'_8, \mathbf{h}_8] = 1. \tag{5.96}$$

If we combine equations (5.82), (5.84), (5.86), (5.87), (5.89), (5.91), (5.92), (5.94), and (5.96), then the corrective term satisfies the following equation

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{8}, \{0\}_{p=0}^{8}\right) = \prod_{p=0}^{8} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{5.97}$$

Since the natural bases in the short exact sequence (5.75) are compatible, the following formula holds by Theorem 4.1

$$\begin{aligned}
&\mathbb{T}\left(M_L - \mathbb{D}^{2n} \oplus M_R - \mathbb{D}^{2n}, \{\mathbf{h}^{M_L - \mathbb{D}^{2n}}_{\nu}\}_{\nu=0}^{2n} \sqcup \{\mathbf{h}^{M_R - \mathbb{D}^{2n}}_{\nu}\}_{\nu=0}^{2n}\right) \\
&= \mathbb{T}\left(M, \{\mathbf{h}^{M}_{\nu}\}_{\nu=0}^{2n}\right)\ \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}^{\mathbb{S}^{2n-1}}_{\eta}\}_{\eta=0}^{2n-1}\right)\ \mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{8}, \{0\}_{p=0}^{8}\right).
\end{aligned} \tag{5.98}$$

By equations (5.97) and (5.98), the following formula is valid

$$
\begin{aligned}
&\mathbb{T}\Big(M_L - \mathbb{D}^{2n} \oplus M_R - \mathbb{D}^{2n}, \{\mathbf{h}_\nu^{M_L - \mathbb{D}^{2n}}\}_{\nu=0}^{2n} \sqcup \{\mathbf{h}_\nu^{M_R - \mathbb{D}^{2n}}\}_{\nu=0}^{2n}\Big) \\
&= \mathbb{T}\Big(M, \{\mathbf{h}_\nu^M\}_{\nu=0}^{2n}\Big)\ \mathbb{T}\Big(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\Big).
\end{aligned}
\tag{5.99}
$$

Considering Lemma 4.1 and equation (5.99), we finish the proof of Proposition 5.5. □

Proposition 5.6 *Let* $M = \mathop{\#}\limits_{j=1}^{p} (\mathbb{S}^n \times \mathbb{S}^n)$ *be a connected sum of* p (≥ 1) *copies of product of the original n-spheres* $\mathbb{S}^n \times \mathbb{S}^n$ *and*

$$
0 \to C_*(\mathbb{S}^{2n-1}) \longrightarrow C_*(M - \mathbb{D}^{2n}) \oplus C_*(\overline{\mathbb{D}^{2n}}) \longrightarrow C_*(M) \to 0 \tag{5.100}
$$

be the natural short exact sequence of the chain complexes with the corresponding Mayer-Vietoris sequence

$$
\begin{aligned}
\mathcal{H}_* :\quad & 0 \xrightarrow{\phi} H_{2n}(M) \xrightarrow{\alpha} H_{2n-1}(\mathbb{S}^{2n-1}) \xrightarrow{\varphi} 0 \\
& \xrightarrow{\psi} H_n(M - \mathbb{D}^{2n}) \xrightarrow{\beta} H_n(M) \xrightarrow{\eta} 0 \\
& \xrightarrow{\varsigma} H_0(\mathbb{S}^{2n-1}) \xrightarrow{\theta} H_0(M - \mathbb{D}^{2n}) \oplus H_0(\overline{\mathbb{D}^{2n}}) \xrightarrow{\delta} H_0(M) \xrightarrow{\rho} 0.
\end{aligned}
\tag{5.101}
$$

Assume that $\mathbf{h}_\nu^{M-\mathbb{D}^{2n}}$ *and* $\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}$ *are respectively bases of* $H_\nu(M - \mathbb{D}^{2n})$ *and* $H_\eta(\mathbb{S}^{2n-1})$ *for* $\nu = 0, \cdots, 2n$, $\eta = 0, \cdots, 2n-1$. *Assume also that* $\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}$ *is an arbitrary basis of* $H_0(\overline{\mathbb{D}^{2n}})$. *Then there exists a basis* $\mathbf{h}_\nu^M$ *of* $H_\nu(M)$ *such that the corrective term becomes* 1 *and the following multiplicative gluing formula is valid*

$$
\mathbb{T}\Big(M - \mathbb{D}^{2n}, \{\mathbf{h}_\nu^{M-\mathbb{D}^{2n}}\}_{\nu=0}^{n}\Big) = \mathbb{T}\Big(M, \{\mathbf{h}_\nu^M\}_{\nu=0}^{2n}\Big)\ \mathbb{T}\Big(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\Big)\ \mathbb{T}\Big(\overline{\mathbb{D}^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}\}\Big)^{-1}.
$$

Proof Let us denote the vector spaces in the sequence $\mathcal{H}_*$ by $C_p(\mathcal{H}_*)$ for $p \in \{0, \ldots, 8\}$.

Following the arguments presented in the proof of Theorem 5.1, we obtain the equation

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{5.102}$$

Here, $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*))$ is the isomorphism obtained by the First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$ for each p.

First, we use equation (5.102) for the vector space $C_0(\mathcal{H}_*) = H_0(M)$. Since $\mathrm{Im}(\rho)$ is trivial, we get

$$C_0(\mathcal{H}_*) = \mathrm{Im}(\delta) \oplus s_0(\mathrm{Im}(\rho)) = \mathrm{Im}(\delta). \tag{5.103}$$

Let us take the basis of $\mathrm{Im}(\delta)$ as

$$\mathbf{h}^{\mathrm{Im}(\delta)} = \left\{ a_{21}\delta(\mathbf{h}^{M-\mathbb{D}^{2n}}) + a_{22}\delta(\mathbf{h}^{\overline{\mathbb{D}^{2n}}}) \right\}$$

for non-zero vector (a_{21}, a_{22}). It follows from equation (5.103) that $\mathbf{h}^{\mathrm{Im}(\delta)}$ is the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_0$ of $C_0(\mathcal{H}_*)$ as $\mathbf{h}^{\mathrm{Im}(\delta)}$, then the following equality holds

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{5.104}$$

By using equation (5.102), $C_1(\mathcal{H}_*) = H_0(M - \mathbb{D}^{2n}) \oplus H_0(\overline{\mathbb{D}^{2n}})$ can be expressed as the following direct sum

$$C_1(\mathcal{H}_*) = \mathrm{Im}(\theta) \oplus s_1(\mathrm{Im}(\delta)). \tag{5.105}$$

In the previous step, the basis of $\mathrm{Im}(\delta)$ was chosen as $\mathbf{h}^{\mathrm{Im}(\delta)}$. Note also that $\mathrm{Im}(\theta)$ is isomorphic to $H_0(\mathbb{S}^{2n-1})$, so we can choose the basis $\mathbf{h}^{\mathrm{Im}(\theta)}$ of $\mathrm{Im}(\theta)$ as $\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}})$. As $\mathrm{Im}(\theta)$ is one-dimensional subspace of $C_1(\mathcal{H}_*)$, there is a non-zero vector (a_{11}, a_{12}) such that

$$\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) = a_{11}\mathbf{h}_0^{M-\mathbb{D}^{2n}} + a_{12}\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}.$$

Clearly, $A = (a_{ij})$ is the non-singular (2×2)-real matrix. By equation (5.105),

$$\left\{ \theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}), s_1(\mathbf{h}^{\mathrm{Im}(\delta)}) \right\}$$

becomes the obtained basis $\mathbf{h}'_1$ of $C_1(\mathcal{H}_*)$. Since the initial basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$ is

$$\left\{\mathbf{h}_0^{M-\mathbb{D}^{2n}}, \mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}\right\},$$

we get the following equality

$$[\mathbf{h}'_1, \mathbf{h}_1] = \det A. \tag{5.106}$$

If we use equation (5.102) for the space $C_2(\mathcal{H}_*) = H_0(\mathbb{S}^{2n-1})$ and consider the fact that $\mathrm{Im}(\varsigma) = \{0\}$, then we have

$$C_2(\mathcal{H}_*) = \mathrm{Im}(\varsigma) \oplus s_2(\mathrm{Im}(\theta)) = s_2(\mathrm{Im}(\theta)). \tag{5.107}$$

By equation (5.107), we obtain that $s_2(\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}})) = \mathbf{h}_0^{\mathbb{S}^{2n-1}}$ is the obtained basis $\mathbf{h}'_2$ of $C_2(\mathcal{H}_*)$. Note that $\mathbf{h}_0^{\mathbb{S}^{2n-1}}$ is the initial basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$. Hence, we have

$$[\mathbf{h}'_2, \mathbf{h}_2] = 1. \tag{5.108}$$

By using the convention $1 \cdot 0 = 1$ for the trivial space $C_3(\mathcal{H}_*)$, we obtain that the determinant of the transition matrix is 1; that is,

$$[\mathbf{h}'_3, \mathbf{h}_3] = 1. \tag{5.109}$$

Let us consider equation (5.102) for the space $C_4(\mathcal{H}_*) = H_n(M)$. Since the space $\mathrm{Im}(\eta)$ is trivial, we have

$$C_4(\mathcal{H}_*) = \mathrm{Im}(\beta) \oplus s_4(\mathrm{Im}(\eta)) = \mathrm{Im}(\beta). \tag{5.110}$$

Since $\mathrm{Im}(\beta)$ is isomorphic to $H_n(M - \mathbb{D}^{2n})$, we can take the basis $\mathbf{h}^{\mathrm{Im}(\beta)}$ of $\mathrm{Im}(\beta)$ as $\beta(\mathbf{h}_n^{M-\mathbb{D}^{2n}})$. By equation (5.110), $\mathbf{h}^{\mathrm{Im}(\beta)}$ becomes the obtained basis $\mathbf{h}'_4$ of $C_4(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_4$ (namely, $\mathbf{h}_n^M$) of $C_4(\mathcal{H}_*)$ as $\mathbf{h}'_4$, then the following equality holds

$$[\mathbf{h}'_4, \mathbf{h}_4] = 1. \tag{5.111}$$

From equation (5.102) and the trivial space $\mathrm{Im}(\psi) = \{0\}$ it follows that the space $C_5(\mathcal{H}_*) = H_n(M - \mathbb{D}^{2n})$ can be written as follows

$$C_5(\mathcal{H}_*) = \mathrm{Im}(\psi) \oplus s_5(\mathrm{Im}(\beta)) = s_5(\mathrm{Im}(\beta)). \tag{5.112}$$

The basis $\mathbf{h}^{\mathrm{Im}(\beta)}$ of $\mathrm{Im}(\beta)$ was chosen as $\beta(\mathbf{h}_n^{M-\mathbb{D}^{2n}})$ in the previous step. By equation (5.112),

$$s_5(\beta(\mathbf{h}_n^{M-\mathbb{D}^{2n}})) = \mathbf{h}_n^{M-\mathbb{D}^{2n}}$$

becomes the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. As the initial basis $\mathbf{h}_5$ of $C_5(\mathcal{H}_*)$ is also $\mathbf{h}_n^{M-\mathbb{D}^{2n}}$, we have the following equality

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{5.113}$$

If we use $C_6(\mathcal{H}_*) = \{0\}$ in the sequence $\mathcal{H}_*$ and take the convention $1 \cdot 0 = 0$, then the determinant of the transition matrix satisfies the following equality

$$[\mathbf{h}'_6, \mathbf{h}_6] = 1. \tag{5.114}$$

Let us consider the space $C_7(\mathcal{H}_*) = H_{2n-1}(\mathbb{S}^{2n-1})$ in equation (5.102). The trivial space $\mathrm{Im}(\varphi)$ yields

$$C_7(\mathcal{H}_*) = \mathrm{Im}(\alpha) \oplus s_7(\mathrm{Im}(\varphi)) = \mathrm{Im}(\alpha). \tag{5.115}$$

Recall that $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}}$ is the initial basis $\mathbf{h}_7$ of $C_7(\mathcal{H}_*)$. Taking the basis $\mathbf{h}^{\mathrm{Im}(\alpha)}$ of $\mathrm{Im}(\alpha)$ as $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}}$ and considering equation (5.115), we get that $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}}$ is the obtained basis $\mathbf{h}'_7$ of $C_7(\mathcal{H}_*)$. Therefore, we have

$$[\mathbf{h}'_7, \mathbf{h}_7] = 1. \tag{5.116}$$

Finally, we use equation (5.102) for the space $C_8(\mathcal{H}_*) = H_{2n}(M)$. If we consider the trivial space $\mathrm{Im}(\phi)$, then we get

$$C_8(\mathcal{H}_*) = \mathrm{Im}(\phi) \oplus s_8(\mathrm{Im}(\alpha)) = s_8(\mathrm{Im}(\alpha)). \tag{5.117}$$

Since $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}}$ was chosen as the basis $\mathbf{h}^{\mathrm{Im}(\alpha)}$ of $\mathrm{Im}(\alpha)$ in the previous step, $s_8(\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}})$ becomes the obtained basis $\mathbf{h}'_8$ of $C_8(\mathcal{H}_*)$ by equation (5.117). Let us take the initial basis $\mathbf{h}_8$ (namely, $\mathbf{h}_{2n}^{M}$) of $C_8(\mathcal{H}_*)$ as

$$\left\{(\det A)^{-1} s_8(\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}})\right\}.$$

Then the following equality holds

$$[\mathbf{h}'_8, \mathbf{h}_8] = (\det A)^{-1}. \tag{5.118}$$

From equations (5.104), (5.106), (5.108), (5.109), (5.111), (5.113), (5.114), (5.116), and (5.118) it follows that

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{8}, \{0\}_{p=0}^{8}\right) = \prod_{p=0}^{8} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{5.119}$$

Let us consider the compatibility of the natural bases in the short exact sequence (5.100). If we use equation (5.119) with Theorem 4.1 and apply Lemma 4.1 for the direct sum of the chain comlexes $C_*(M - \mathbb{D}^{2n}) \oplus C_*(\overline{\mathbb{D}^{2n}})$, then we finish the proof of Proposition 5.6. □

Proposition 5.7 *Let $M = \#_{j=1}^{p} M_j$ be a connected sum of p (≥ 2) copies of $\mathbb{S}^n \times \mathbb{S}^n$ and $N_i = \#_{j=1}^{i} M_j$ for $i = 1, \cdots, p$. Consider the following short exact sequences of the chain complexes*

$$0 \to C_*(\mathbb{S}_i^{2n-1}) \longrightarrow C_*(N_{i-1} - \mathbb{D}_i^{2n}) \oplus C_*(M_i - \mathbb{D}_i^{2n}) \longrightarrow C_*(N_i) \to 0,$$
$$0 \to C_*(\mathbb{S}_i^{2n-1}) \longrightarrow C_*(N_{i-1} - \mathbb{D}_i^{2n}) \oplus C_*(\overline{\mathbb{D}_i^{2n}}) \longrightarrow C_*(N_{i-1}) \to 0,$$
$$0 \to C_*(\mathbb{S}_i^{2n-1}) \longrightarrow C_*(M_i - \mathbb{D}_i^{2n}) \oplus C_*(\overline{\mathbb{D}_i^{2n}}) \longrightarrow C_*(M_i) \to 0,$$

and their associated Mayer-Vietoris sequences as in Proposition 5.5 and Proposition 5.6. If $\mathbf{h}_\nu^{N_i}$, $\mathbf{h}_\eta^{\mathbb{S}_i^{2n-1}}$, *and* $\mathbf{h}_0^{\overline{\mathbb{D}_i^{2n}}}$ *are respectively bases of* $H_\nu(N_i)$, $H_\eta(\mathbb{S}_i^{2n-1})$, *and* $H_0(\overline{\mathbb{D}_i^{2n}})$, $\nu = 0, \ldots, 2n$, $\eta = 0, \ldots, 2n-1$, *then there are respectively bases* $\mathbf{h}_\nu^{N_{i-1}}$ *and* $\mathbf{h}_\nu^{M_i}$ *of* $H_\nu(N_{i-1})$ *and* $H_\nu(M_i)$ *such that the corrective term equals to* 1 *and the Reidemeister torsion of* N_i *satisfies the following formula*

$$\mathbb{T}\left(N_i, \{\mathbf{h}_\nu^{N_i}\}_{\nu=0}^{2n}\right) = \mathbb{T}\left(N_{i-1}, \{\mathbf{h}_\nu^{N_{i-1}}\}_{\nu=0}^{2n}\right)\ \mathbb{T}\left(M_i, \{\mathbf{h}_\nu^{M_i}\}_{\nu=0}^{2n}\right)$$
$$\times\, \mathbb{T}\left(\mathbb{S}_i^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}_i^{2n-1}}\}_{\eta=0}^{2n-1}\right)\ \mathbb{T}\left(\overline{\mathbb{D}_i^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_i^{2n}}}\}\right)^{-2}.$$

If we follow the arguments in the Proposition 5.7 inductively, then we have

Theorem 5.8 *Suppose* $M = \mathop{\#}_{j=1}^{p} M_j$ *is a connected sum of p-copies of* $\mathbb{S}^n \times \mathbb{S}^n$ *and* $\mathbf{h}_\nu^M$, $\mathbf{h}_\eta^{\mathbb{S}_i^{2n-1}}$, *and* $\mathbf{h}_0^{\overline{\mathbb{D}_i^{2n}}}$ *are respectively bases of* $H_\nu(M)$, $H_\eta(\mathbb{S}_i^{2n-1})$, *and* $H_0(\overline{\mathbb{D}_i^{2n}})$, $\nu = 0, \ldots, 2n$, $\eta = 0, \cdots, 2n-1$. *Then there is a basis* $\mathbf{h}_\nu^{M_j}$ *of* $H_\nu(M_j)$ *for each* j *such that the following formula holds*

$$\mathbb{T}\left(M, \{\mathbf{h}_\nu^M\}_{\nu=0}^{2n}\right) = \prod_{j=1}^{p} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^{2n}\right) \prod_{i=1}^{p-1} \left[\mathbb{T}\left(\mathbb{S}_i^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}_i^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}_i^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_i^{2n}}}\}\right)^{-2}\right].$$

Proposition 5.8 *Let* $M^{2n} = M \# M_1^{2n}$ *be an (n–2)-connected* $2n$*-dimensional closed* π*-manifold* ($n \geq 3$) *such that* $H_{n-1}(M^{2n}; \mathbb{Z})$ *has no torsion, where* $M = \mathop{\#}_{j=1}^{p} (\mathbb{S}^n \times \mathbb{S}^n)$ *and* M_1^{2n} *is an (n–2)-connected* $2n$*-dimensional closed* π*-manifold. Consider the natural short exact sequence of the chain complexes*

$$0 \to C_*(\mathbb{S}^{2n-1}) \longrightarrow C_*(M - \mathbb{D}^{2n}) \oplus C_*(M_1^{2n} - \mathbb{D}^{2n}) \longrightarrow C_*(M^{2n}) \to 0. \qquad (5.120)$$

Associated to the sequence (5.120), there exists the corresponding Mayer-Vietoris sequence

$$\mathcal{H}_* : 0 \to \mathcal{H}_*^3 \to \mathcal{H}_*^2 \to \mathcal{H}_*^1 \to 0,$$

where, for $j = 1, \cdots, n-1, n+1, \cdots, 2n-2$

$$\mathcal{H}^3_*: \quad 0 \xrightarrow{\beta} H_{2n}(M^{2n}) \xrightarrow{\gamma} H_{2n-1}(\mathbb{S}^{2n-1}) \xrightarrow{\partial'_{2n-1}} H_{2n-1}(M_1^{2n} - \mathbb{D}^{2n}) \xrightarrow{\partial_{2n-1}} H_{2n-1}(M^{2n}) \xrightarrow{\partial''_{2n-1}} 0,$$

$$\begin{aligned} \mathcal{H}^2_*: \quad & 0 \xrightarrow{\partial'_j} H_j(M_1^{2n} - \mathbb{D}^{2n}) \xrightarrow{\partial_j} H_j(M^{2n}) \xrightarrow{\partial''_j} 0, \\ & 0 \xrightarrow{\partial'_n} H_n(M - \mathbb{D}^{2n}) \xrightarrow{\partial_n} H_n(M^{2n}) \xrightarrow{\partial''_n} 0, \end{aligned}$$

$$\mathcal{H}^1_*: \quad 0 \xrightarrow{\alpha} H_0(\mathbb{S}^{2n-1}) \xrightarrow{\theta} H_0(M - \mathbb{D}^{2n}) \oplus H_0(M_1^{2n} - \mathbb{D}^{2n}) \xrightarrow{\psi} H_0(M^{2n}) \xrightarrow{\phi} 0.$$

Let $\mathbf{h}^{M^{2n}}_\nu$, $\mathbf{h}^{\mathbb{S}^{2n-1}}_0$, *and* $\mathbf{h}^{\mathbb{S}^{2n-1}}_{2n-1} = \gamma(\mathbf{h}^{M^{2n}}_{2n})$ *be respectively bases of* $H_\nu(M^{2n})$, $H_0(\mathbb{S}^{2n-1})$, *and* $H_{2n-1}(\mathbb{S}^{2n-1})$ *for* $\nu = 0, \cdots, 2n$. *Then there exist bases* $\mathbf{h}^{M-\mathbb{D}^{2n}}_\nu$ *and* $\mathbf{h}^{M_1^{2n}-\mathbb{D}^{2n}}_\nu$ *of* $H_\nu(M - \mathbb{D}^{2n})$ *and* $H_\nu(M_1^{2n} - \mathbb{D}^{2n})$ *such that the corrective term disappears from the following multiplicative gluing formula*

$$\begin{aligned} \mathbb{T}\left(M^{2n}, \{\mathbf{h}^{M^{2n}}_\nu\}_{\nu=0}^{2n}\right) = \mathbb{T}\left(M - \mathbb{D}^{2n}, \{\mathbf{h}^{M-\mathbb{D}^{2n}}_\nu\}_{\nu=0}^{2n}\right) \; \mathbb{T}\left(M_1^{2n} - \mathbb{D}^{2n}, \{\mathbf{h}^{M_1^{2n}-\mathbb{D}^{2n}}_\nu\}_{\nu=0}^{2n}\right) \\ \times \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}^{\mathbb{S}^{2n-1}}_\eta\}_{\eta=0}^{2n-1}\right)^{-1}. \end{aligned}$$

Proof Let $C_p(\mathcal{H}_*)$ denote the vector spaces in the sequence $\mathcal{H}_*$ for $p \in \{0, 1, \ldots, 6n-2\}$. Then we have the following equation for each p

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{5.121}$$

Now we consider the first part of the long exact sequence $\mathcal{H}_*$ given as follows

$$\mathcal{H}^1_*: 0 \xrightarrow{\alpha} H_0(\mathbb{S}^{2n-1}) \xrightarrow{\theta} H_0(M - \mathbb{D}^{2n}) \oplus H_0(M_1^{2n} - \mathbb{D}^{2n}) \xrightarrow{\psi} H_0(M^{2n}) \xrightarrow{\phi} 0.$$

First, we use equation (5.121) for the vector space $C_0(\mathcal{H}_*) = H_0(M^{2n})$. Since $\mathrm{Im}(\phi)$ is trivial, we get

$$C_0(\mathcal{H}_*) = \mathrm{Im}(\psi) \oplus s_0(\mathrm{Im}(\phi)) = \mathrm{Im}(\psi). \tag{5.122}$$

If we choose the basis $\mathbf{h}^{\mathrm{Im}(\psi)}$ of $\mathrm{Im}(\psi)$ as $\mathbf{h}^{M^{2n}}_0$, then $\mathbf{h}^{M^{2n}}_0$ becomes the obtained basis $\mathbf{h}'_0$ of

$C_0(\mathcal{H}_*)$ by equation (5.122). Since $\mathbf{h}_0^{M^{2n}}$ is also the initial basis $\mathbf{h}_0$ of $C_0(\mathcal{H}_*)$, we get that the transition matrix is the identity matrix, and thus the following equality holds

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{5.123}$$

Let us consider equation (5.121) for $C_1(\mathcal{H}_*) = H_0(M - \mathbb{D}^{2n}) \oplus H_0(M_1^{2n} - \mathbb{D}^{2n})$. Then the space $C_1(\mathcal{H}_*)$ can be written as the following direct sum

$$C_1(\mathcal{H}_*) = \operatorname{Im}(\theta) \oplus s_1(\operatorname{Im}(\psi)). \tag{5.124}$$

Recall that the basis $\mathbf{h}^{\operatorname{Im}(\psi)}$ of $\operatorname{Im}(\psi)$ was chosen as $\mathbf{h}_0^{M^{2n}}$ in the previous step. Note also that $\operatorname{Im}(\theta)$ is isomorphic to $H_0(\mathbb{S}^{2n-1})$, so we can take the basis $\mathbf{h}^{\operatorname{Im}(\theta)}$ of $\operatorname{Im}(\theta)$ as $\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}})$. By equation (5.124), we get the obtained basis of $C_1(\mathcal{H}_*)$ as follows

$$\mathbf{h}'_1 = \left\{\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}), s_1(\mathbf{h}_0^{M^{2n}})\right\}.$$

As $H_0(M - \mathbb{D}^{2n})$ and $H_0(M_1^{2n} - \mathbb{D}^{2n})$ are one-dimensional subspaces of the 2-dimensional space $C_1(\mathcal{H}_*)$, there are non-zero vectors (a_{11}, a_{12}) and (a_{21}, a_{22}) such that

$$\left\{a_{11}\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{12}s_1(\mathbf{h}_0^{M^{2n}})\right\},$$
$$\left\{a_{21}\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{22}s_1(\mathbf{h}_0^{M^{2n}})\right\}$$

are bases of $H_0(M - \mathbb{D}^{2n})$ and $H_0(M_1^{2n} - \mathbb{D}^{2n})$, respectively. Moreover, $A = (a_{ij})$ is the (2×2)-invertible matrix over $\mathbb{R}$. Let us take the bases of $H_0(M - \mathbb{D}^{2n})$ and $H_0(M_1^{2n} - \mathbb{D}^{2n})$ as follows

$$\begin{aligned}
\mathbf{h}_0^{M-\mathbb{D}^{2n}} &= \left\{(\det A)^{-1}[a_{11}\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{12}s_1(\mathbf{h}_0^{M^{2n}})]\right\}, \\
\mathbf{h}_0^{M_1^{2n}-\mathbb{D}^{2n}} &= \left\{a_{21}\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{22}s_1(\mathbf{h}_0^{M^{2n}})\right\}.
\end{aligned}$$

Then $\mathbf{h}_1 = \{\mathbf{h}_0^{M-\mathbb{D}^{2n}}, \mathbf{h}_0^{M_1^{2n}-\mathbb{D}^{2n}}\}$ becomes the initial basis of $C_1(\mathcal{H}_*)$ and we have

$$[\mathbf{h}_1', \mathbf{h}_1] = 1. \tag{5.125}$$

Considering the space $C_2(\mathcal{H}_*) = H_0(\mathbb{S}^{2n-1})$ in equation (5.121) and using the fact that $\mathrm{Im}(\alpha)$ is trivial, we can express the space $C_2(\mathcal{H}_*)$ as follows

$$C_2(\mathcal{H}_*) = \mathrm{Im}(\alpha) \oplus s_2(\mathrm{Im}(\theta)) = s_2(\mathrm{Im}(\theta)). \tag{5.126}$$

By equation (5.126), $s_2(\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}})) = \mathbf{h}_0^{\mathbb{S}^{2n-1}}$ becomes the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$. Since the initial basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$ is $\mathbf{h}_0^{\mathbb{S}^{2n-1}}$, we get the following equality

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{5.127}$$

If we use the convention $1 \cdot 0 = 1$ for the space $C_3(\mathcal{H}_*) = \{0\}$ in the sequence $\mathcal{H}_*$, then the determinant of the transition matrix becomes 1; that is

$$[\mathbf{h}_3', \mathbf{h}_3] = 1. \tag{5.128}$$

Now we consider the following parts $\mathcal{H}_*^2$ of the sequence $\mathcal{H}_*$

- for $j = 1, 2, \ldots, n-1, n+1, \ldots, 2n-2$

$$0 \xrightarrow{\partial_j'} H_j(M_1^{2n} - \mathbb{D}^{2n}) \xrightarrow{\partial_j} H_j(M^{2n}) \xrightarrow{\partial_j''} 0,$$

- for $j = n$

$$0 \xrightarrow{\partial_j'} H_j(M - \mathbb{D}^{2n}) \xrightarrow{\partial_j} H_j(M^{2n}) \xrightarrow{\partial_j''} 0.$$

Let us denote the vector spaces in the above short exact sequences as $C_{3j}(\mathcal{H}_*)$, $C_{3j+1}(\mathcal{H}_*)$ and $C_{3j+2}(\mathcal{H}_*)$ for each $j \in \{1, 2, \ldots, 2n-2\}$. Note that the space $C_{3j}(\mathcal{H}_*)$ is trivial. If we use the convention $1 \cdot 0 = 1$ for each $j \in \{2, \cdots, 2n-2\}$, then we get

$$[\mathbf{h}'_{3j}, \mathbf{h}_{3j}] = 1. \tag{5.129}$$

By the exactness of $\mathcal{H}_*$, we obtain the following isomorphisms

$$\begin{aligned} H_j(M_1^{2n} - \mathbb{D}^{2n}) &\overset{\partial_j}{\cong} H_j(M^{2n}), \\ H_n(M - \mathbb{D}^{2n}) &\overset{\partial_n}{\cong} H_n(M^{2n}). \end{aligned}$$

We then use equation (5.121) for the space $C_{3j+1}(\mathcal{H}_*) = H_j(M^{2n})$. Since $\mathrm{Im}(\partial''_j)$ is a trivial space, the following equality holds

$$C_{3j+1}(\mathcal{H}_*) = \mathrm{Im}(\partial_j) \oplus s_{3j+1}(\mathrm{Im}(\partial''_j)) = \mathrm{Im}(\partial_j). \tag{5.130}$$

Since $\mathrm{Im}(\partial_j)$ equals to $H_j(M^{2n})$, we can take the basis $\mathbf{h}^{\mathrm{Im}(\partial_j)}$ of $\mathrm{Im}(\partial_j)$ as $\mathbf{h}_j^{M^{2n}}$. By equation (5.130), $\mathbf{h}_j^{M^{2n}}$ becomes the obtained basis $\mathbf{h}'_{3j+1}$ of $C_{3j+1}(\mathcal{H}_*)$. Since the initial basis $\mathbf{h}_{3j+1}$ of $C_{3j+1}(\mathcal{H}_*)$ is also $\mathbf{h}_j^{M^{2n}}$, we obtain

$$[\mathbf{h}'_{3j+1}, \mathbf{h}_{3j+1}] = 1. \tag{5.131}$$

Considering equation (5.121) for $C_{3j+2}(\mathcal{H}_*) = H_j(M_1^{2n} - \mathbb{D}^{2n})$ and using the fact that $\mathrm{Im}(\partial'_j) = \{0\}$, we obtain

$$C_{3j+2}(\mathcal{H}_*) = \mathrm{Im}(\partial'_j) \oplus s_{3j+2}(\mathrm{Im}(\partial_j)) = s_{3j+2}(\mathrm{Im}(\partial_j)). \tag{5.132}$$

Since $H_j(M_1^{2n} - \mathbb{D}^{2n})$ and $H_j(M^{2n})$ are isomorphic, the section s_{3j+2} can be considered as the inverse of the isomorphism ∂_j. In the previous step, the basis $\mathbf{h}^{\mathrm{Im}(\partial_j)}$ of $\mathrm{Im}(\partial_j)$ was chosen as $\mathbf{h}_j^{M^{2n}}$. By equation (5.132), $s_{3j+2}(\mathbf{h}_j^{M^{2n}})$ becomes the obtained basis $\mathbf{h}'_{3j+2}$ of $C_{3j+2}(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_{3j+2}$ of $C_{3j+2}(\mathcal{H}_*)$ as $s_{3j+2}(\mathbf{h}_j^{M^{2n}})$, then we get

$$[\mathbf{h}'_{3j+2}, \mathbf{h}_{3j+2}] = 1. \tag{5.133}$$

For $j = n$, if we apply the above two steps to the following part of the Mayer-Vietoris sequence $\mathcal{H}_*$

$$0 \xrightarrow{\partial'_j} H_j(M - \mathbb{D}^{2n}) \xrightarrow{\partial_j} H_j(M^{2n}) \xrightarrow{\partial''_j} 0,$$

then we get the following equalities

$$\begin{aligned} [\mathbf{h}'_{3n+1}, \mathbf{h}_{3n+1}] &= 1, \\ [\mathbf{h}'_{3n+2}, \mathbf{h}_{3n+2}] &= 1. \end{aligned} \tag{5.134}$$

Now we consider the last part of the sequence $\mathcal{H}_*$

$$\mathcal{H}^3_* : 0 \xrightarrow{\beta} H_{2n}(M^{2n}) \xrightarrow{\gamma} H_{2n-1}(\mathbb{S}^{2n-1}) \xrightarrow{\partial'_{2n-1}} H_{2n-1}(M_1^{2n} - \mathbb{D}^{2n}) \xrightarrow{\partial_{2n-1}} H_{2n-1}(M^{2n}) \xrightarrow{\partial''_{2n-1}} 0.$$

By the exactness of $\mathcal{H}^3_*$, ∂'_{2n-1} becomes a zero map. So, the following isomorphism exists

$$H_{2n-1}(M_1^{2n} - \mathbb{D}^{2n}) \overset{\partial_{2n-1}}{\cong} H_{2n-1}(M^{2n}).$$

By using equation (5.121) for the space $C_{6n-5}(\mathcal{H}_*) = H_{2n-1}(M^{2n})$ and the fact that $\mathrm{Im}(\partial''_{2n-1})$ is trivial, the following equality holds

$$C_{6n-5}(\mathcal{H}_*) = \mathrm{Im}(\partial_{2n-1}) \oplus s_{6n-5}(\mathrm{Im}(\partial''_{2n-1})) = \mathrm{Im}(\partial_{2n-1}). \tag{5.135}$$

As $\mathrm{Im}(\partial_{2n-1})$ equals to $H_{2n-1}(M^{2n})$, we can take the basis $\mathbf{h}^{\mathrm{Im}(\partial_{2n-1})}$ of $\mathrm{Im}(\partial_{2n-1})$ as $\mathbf{h}^{M^{2n}}_{2n-1}$.

By equation (5.135), $\mathbf{h}^{M^{2n}}_{2n-1}$ becomes the obtained basis $\mathbf{h}'_{6n-5}$ of $C_{6n-5}(\mathcal{H}_*)$. Since the initial basis $\mathbf{h}_{6n-5}$ of $C_{6n-5}(\mathcal{H}_*)$ is also $\mathbf{h}^{M^{2n}}_{2n-1}$, we conclude that the determinant of the transition matrix is 1; that is,

$$[\mathbf{h}'_{6n-5}, \mathbf{h}_{6n-5}] = 1. \tag{5.136}$$

If we consider equation (5.121) for $C_{6n-4}(\mathcal{H}_*) = H_{2n-1}(M^{2n}_1 - \mathbb{D}^{2n})$ and use the trivial space $\text{Im}(\partial'_{2n-1})$, then we have

$$C_{6n-4}(\mathcal{H}_*) = \text{Im}(\partial'_{2n-1}) \oplus s_{6n-4}(\text{Im}(\partial_{2n-1})) = s_{6n-4}(\text{Im}(\partial_{2n-1})). \tag{5.137}$$

Since $H_{2n-1}(M^{2n}_1 - \mathbb{D}^{2n})$ and $H_{2n-1}(M^{2n})$ are isomorphic, the section s_{6n-4} can be considered as the inverse of the isomorphism ∂_{2n-1}. In the previous step, the basis $\mathbf{h}^{\text{Im}(\partial_{2n-1})}$ of $\text{Im}(\partial_{2n-1})$ was chosen as $\mathbf{h}^{M^{2n}}_{2n-1}$. By equation (5.137), $s_{6n-4}(\mathbf{h}^{M^{2n}}_{2n-1})$ becomes the obtained basis $\mathbf{h}'_{6n-4}$ of $C_{6n-4}(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_{6n-4}$ of $C_{6n-4}(\mathcal{H}_*)$ as $s_{6n-4}(\mathbf{h}^{M^{2n}}_{2n-1})$, then we get

$$[\mathbf{h}'_{6n-4}, \mathbf{h}_{6n-4}] = 1. \tag{5.138}$$

Let us consider the space $C_{6n-3}(\mathcal{H}_*) = H_{2n-1}(\mathbb{S}^{2n-1})$ in equation (5.121). The trivial space $\text{Im}(\partial'_{2n-1})$ yields

$$C_{6n-3}(\mathcal{H}_*) = \text{Im}(\gamma) \oplus s_{6n-3}(\text{Im}(\partial'_{2n-1})) = \text{Im}(\gamma). \tag{5.139}$$

Recall that $\mathbf{h}^{\mathbb{S}^{2n-1}}_{2n-1} = \gamma(\mathbf{h}^{M^{2n}}_{2n})$ is the initial basis $\mathbf{h}_{6n-3}$ of $C_{6n-3}(\mathcal{H}_*)$. If we take the basis $\mathbf{h}^{\text{Im}(\gamma)}$ of $\text{Im}(\gamma)$ as $\mathbf{h}^{\mathbb{S}^{2n-1}}_{2n-1}$ and consider equation (5.139), then $\mathbf{h}^{\mathbb{S}^{2n-1}}_{2n-1}$ becomes the obtained basis $\mathbf{h}'_{6n-3}$ of $C_{6n-3}(\mathcal{H}_*)$. Therefore, we get

$$[\mathbf{h}'_{6n-3}, \mathbf{h}_{6n-3}] = 1. \tag{5.140}$$

Finally, let us consider equation (5.121) for $C_{6n-2}(\mathcal{H}_*) = H_{2n}(M^{2n})$. By the fact that $\text{Im}(\beta)$ is trivial, the following equality holds

$$C_{6n-2}(\mathcal{H}_*) = \text{Im}(\beta) \oplus s_{6n-2}(\text{Im}(\gamma)) = s_{6n-2}(\text{Im}(\gamma)). \tag{5.141}$$

Note that $\mathbf{h}^{\mathbb{S}^{2n-1}}_{2n-1} = \gamma(\mathbf{h}^{M^{2n}}_{2n})$ was chosen as the basis $\mathbf{h}^{\mathrm{Im}(\gamma)}$ of $\mathrm{Im}(\gamma)$ in the previous step. By equation (5.141),

$$s_{6n-2}(\gamma(\mathbf{h}^{M^{2n}}_{2n})) = \mathbf{h}^{M^{2n}}_{2n}$$

becomes the obtained basis $\mathbf{h}'_{6n-2}$ of $C_{6n-2}(\mathcal{H}_*)$. Since the initial basis $\mathbf{h}_{6n-2}$ of $C_{6n-2}(\mathcal{H}_*)$ is $\mathbf{h}^{M^{2n}}_{2n}$, the determinant of the transition matrix satisfies the equality

$$[\mathbf{h}'_{6n-2}, \mathbf{h}_{6n-2}] = 1. \tag{5.142}$$

Equations (5.123), (5.125), (5.127), (5.128), (5.131), (5.133), (5.129) (5.134), (5.136), (5.138), (5.140), and (5.142) yield

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{6n-2}, \{0\}_{p=0}^{6n-2}\right) = \prod_{p=0}^{6n-2} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{5.143}$$

Since the natural bases in the short exact sequence (5.120) are compatible, Theorem 4.1 yields the following formula

$$\begin{aligned}&\mathbb{T}\left(M - \mathbb{D}^{2n} \oplus M_1^{2n} - \mathbb{D}^{2n}, \{\mathbf{h}_\nu^{M-\mathbb{D}^{2n}} \sqcup \mathbf{h}_\nu^{M_1^{2n}-\mathbb{D}^{2n}}\}_{\nu=0}^{2n}\right)\\ &= \mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right)\, \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right)\, \mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{6n-2}, \{0\}_{p=0}^{6n-2}\right).\end{aligned} \tag{5.144}$$

By equation (5.143) and equation (5.144), we have

$$\begin{aligned}&\mathbb{T}\left(M - \mathbb{D}^{2n} \oplus M_1^{2n} - \mathbb{D}^{2n}, \{\mathbf{h}_\nu^{M-\mathbb{D}^{2n}} \sqcup \mathbf{h}_\nu^{M_1^{2n}-\mathbb{D}^{2n}}\}_{\nu=0}^{2n}\right)\\ &= \mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right)\, \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right).\end{aligned} \tag{5.145}$$

Combining Lemma 4.1 and equation (5.145) finishes the proof of Proposition 5.8. □

By using similar arguments in the proof of Proposition 5.8, we obtain the following result.

Proposition 5.9 *Suppose that M_1^{2n} is an $(n-2)$-connected $2n$-dimensional closed π-manifold stated in Proposition 5.6. Then there is the following short exact sequence of* *the chain complexes*

$$0 \to C_*(\mathbb{S}^{2n-1}) \longrightarrow C_*(M_1^{2n} - \mathbb{D}^{2n}) \oplus C_*(\overline{\mathbb{D}^{2n}}) \longrightarrow C_*(M_1^{2n}) \to 0$$

and its corresponding Mayer-Vietoris sequence

$$\mathcal{H}_* : \quad 0 \stackrel{\beta}{\longrightarrow} H_{2n}(M_1^{2n}) \stackrel{\gamma}{\longrightarrow} H_{2n-1}(\mathbb{S}^{2n-1}) \stackrel{\partial'_{2n-1}}{\longrightarrow}$$

$$H_{2n-1}(M_1^{2n} - \mathbb{D}^{2n}) \stackrel{\partial_{2n-1}}{\longrightarrow} H_{2n-1}(M_1^{2n}) \stackrel{\partial''_{2n-1}}{\to} 0 \dashrightarrow$$

$$H_2(M_1^{2n} - \mathbb{D}^{2n}) \stackrel{\partial_2}{\longrightarrow} H_2(M_1^{2n}) \stackrel{\partial''_2}{\to} 0 \stackrel{\partial'_2}{\longrightarrow}$$

$$H_1(M_1^{2n} - \mathbb{D}^{2n}) \stackrel{\partial_1}{\longrightarrow} H_1(M_1^{2n}) \stackrel{\partial''_1}{\longrightarrow}$$

$$H_0(\mathbb{S}^{2n-1}) \stackrel{\theta}{\longrightarrow} H_0(M_1^{2n} - \mathbb{D}^{2n}) \oplus H_0(\overline{\mathbb{D}^{2n}}) \stackrel{\psi}{\longrightarrow} H_0(M_1^{2n}) \stackrel{\phi}{\longrightarrow} 0.$$

Suppose also that $\mathbf{h}_\nu^{M_1^{2n}-\mathbb{D}^{2n}}$ *and* $\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}$ *are respectively bases of* $H_\nu(M_1^{2n} - \mathbb{D}^{2n})$, $H_\eta(\mathbb{S}^{2n-1})$ *for* $\nu = 0, \cdots, n$, $\eta = 0, \cdots, 2n-1$, *and* $\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}$ *is an arbitrary basis of* $H_0(\overline{\mathbb{D}^{2n}})$. *Then there exists a basis* $\mathbf{h}_\nu^{M_1^{2n}}$ *of* $H_\nu(M_1^{2n})$ *such that the corrective term is* 1 *and the formula holds*

$$\mathbb{T}\left(M_1^{2n} - \mathbb{D}^{2n}, \{\mathbf{h}_\nu^{M_1^{2n}-\mathbb{D}^{2n}}\}_{\nu=0}^{n}\right) = \mathbb{T}\left(M_1^{2n}, \{\mathbf{h}_\nu^{M_1^{2n}}\}_{\nu=0}^{2n}\right) \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}\}\right)^{-1}.$$

Now we give the proof of the Main Theorem of this section.

Proof of Theorem 5.7 Let M^{2n} be an $(n-2)$-connected $2n$-dimensional closed π-manifold $(n \geq 3)$ such that $H_{n-1}(M^{2n}; \mathbb{Z})$ has no torsion. By Theorem 5.6, there exists a decomposition

$$M^{2n} = M \# M_1^{2n},$$

where $M = \mathop{\#}\limits_{j=1}^{p} M_j$ is a connected sum of p copies of $\mathbb{S}^n \times \mathbb{S}^n$ and M_1^{2n} is an $(n-2)$-connected $2n$-dimensional closed π-manifold.

Let $\mathbf{h}_\nu^{M^{2n}}$ and $\mathbf{h}_0^{\mathbb{S}^{2n-1}}$ be respectively bases of $H_\nu(M^{2n})$ and $H_0(\mathbb{S}^{2n-1})$, $\nu = 0, \cdots, 2n$. Let also $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}} = \gamma(\mathbf{h}_{2n}^{M^{2n}})$ be a basis of $H_{2n-1}(\mathbb{S}^{2n-1})$ and let $\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}$ be an arbitrary basis of $H_0(\overline{\mathbb{D}^{2n}})$. By Proposition 5.6, Proposition 5.8, and Proposition 5.9, there exist respectively bases $\mathbf{h}_\nu^M$, $\mathbf{h}_\nu^{M_1^{2n}}$ of $H_\nu(M)$, $H_\nu(M_1^{2n})$ such that the following formula is valid

$$\begin{aligned}\mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right) &= \mathbb{T}\left(M, \{\mathbf{h}_\nu^{M}\}_{\nu=0}^{2n}\right)\, \mathbb{T}\left(M_1^{2n}, \{\mathbf{h}_\nu^{M_1^{2n}}\}_{\nu=0}^{2n}\right) \\ &\quad \times \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right)\, \mathbb{T}\left(\overline{\mathbb{D}^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}\}\right)^{-2}. \qquad (5.146)\end{aligned}$$

By applying Theorem 5.8 for $M = \mathop{\#}\limits_{j=1}^{p} M_j$, there exists a basis $\mathbf{h}_\nu^{M_j}$ of $H_\nu(M_j)$ such that

$$\begin{aligned}\mathbb{T}\left(M, \{\mathbf{h}_\nu^{M}\}_{\nu=0}^{2n}\right) &= \prod_{j=1}^{p} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^{2n}\right) \\ &\quad \times \prod_{j=1}^{p-1} \left[\mathbb{T}\left(\mathbb{S}_j^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}_j^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}_j^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_j^{2n}}}\}\right)^{-2}\right]. \qquad (5.147)\end{aligned}$$

Combining equation (5.146) and equation (5.147), we obtain the main formula

$$\begin{aligned}\mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right) &= \mathbb{T}\left(M_1^{2n}, \{\mathbf{h}_\nu^{M_1^{2n}}\}_{\nu=0}^{2n}\right) \prod_{j=1}^{p} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^{2n}\right) \\ &\quad \times \prod_{j=1}^{p} \left[\mathbb{T}\left(\mathbb{S}_j^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}_j^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}_j^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_j^{2n}}}\}\right)^{-2}\right].\end{aligned}$$

This formula finishes the proof of Theorem 5.7.

From Remark 4.1 and Theorem 5.7 it follows that

Corollary 5.1 *Suppose that $M^{2n} = M \# M_1^{2n}$ is an $(n-2)$-connected $2n$-dimensional closed π-manifold ($n \geq 3$) such that $H_{n-1}(M^{2n}; \mathbb{Z})$ has no torsion. Let $\mathbf{h}_\nu^{M^{2n}}$ be a basis of $H_\nu(M^{2n})$ for $\nu = 0, \cdots, 2n$ and $\mathbf{h}_0^{\overline{\mathbb{D}_j^{2n}}}$ be an arbitrary basis of $H_0(\overline{\mathbb{D}_j^{2n}})$. Then there exist bases $\mathbf{h}_\nu^{M_j}$, $\mathbf{h}_\nu^{M_1^{2n}}$ for the homologies $H_\nu(M_j)$, $H_\nu(M_1^{2n})$ such that the following formula holds*

$$\left|\mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right)\right| = \left|\mathbb{T}\left(M_1^{2n}, \{\mathbf{h}_\nu^{M_1^{2n}}\}_{\nu=0}^{2n}\right)\right| \prod_{j=1}^{p} \left|\mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^{2n}\right) \mathbb{T}\left(\overline{\mathbb{D}_j^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_j^{2n}}}\}\right)^{-2}\right|.$$

By using Theorem 3.6, Ishimoto (1969) proved the following result:

Theorem 5.9 *Let M^{2n} be an $(n-2)$-connected $2n$-dimensional closed parallelizable manifold $(n \geq 4)$ such that $H_{n-1}(M^{2n};\mathbb{Z})$ has no torsion. Assume that $\kappa(M^{2n}) = 0$ if $n = 4k+3$. Let $r = \operatorname{rank}(H_{n-1}(M))$ and $2p = \operatorname{rank}(H_n(M))$. Then $p = r + (-1)^{n-1}$ and if n is even, then M^{2n} is a connected sum of $(r-1)$-copies of $\mathbb{S}^n \times \mathbb{S}^n$ or if n is odd, then M^{2n} is a connected sum of $(r+1)$-copies of $\mathbb{S}^n \times \mathbb{S}^n$.*

Combining Theorem 5.9 and Theorem 5.8, we obtain the following corollary.

Corollary 5.2 *Let M^{2n} be an $(n-2)$-connected $2n$-dimensional closed parallelizable manifold $(n \geq 4)$ such that $H_{n-1}(M^{2n};\mathbb{Z})$ has no torsion. Assume that $\kappa(M^{2n}) = 0$ if $n = 4k+3$. Let $\mathbf{h}_\nu^{M^{2n}}$, $\mathbf{h}_\eta^{\mathbb{S}_i^{2n-1}}$, and $\mathbf{h}_0^{\overline{\mathbb{D}_i^{2n}}}$ be respectively bases of $H_\nu(M^{2n})$, $H_\eta(\mathbb{S}_i^{2n-1})$, and $H_0(\overline{\mathbb{D}_i^{2n}})$ for $\nu = 0, \dots, 2n$, $\eta = 0, \cdots, 2n-1$. Then there is a homology basis $\mathbf{h}_\nu^{M_j}$ for each j such that the following formulas hold.*

- *If n is even, then*

$$\mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right) = \prod_{j=1}^{r-1} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^{2n}\right) \prod_{i=1}^{r-2} \left[\mathbb{T}\left(\mathbb{S}_i^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}_i^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}_i^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_i^{2n}}}\}\right)^{-2}\right].$$

- *If n is odd, then*

$$\mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right) = \prod_{j=1}^{r+1} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^{2n}\right) \prod_{i=1}^{r} \left[\mathbb{T}\left(\mathbb{S}_i^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}_i^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}_i^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_i^{2n}}}\}\right)^{-2}\right].$$

Here, M_j denotes the product manifold $\mathbb{S}^1 \times \mathbb{S}^1$ for each $j \in \{1, \dots, p\}$.

5.3. Reidemeister Torsion of $(n-1)$-Connected $2n$-Dimensional Closed π-Manifold

In this section, we establish a formula to compute the Reidemeister torsion of an $(n-1)$-connected $2n$-dimensional closed π-manifold $(n \geq 3)$ by using its decomposition presented by Ishimoto (1969) as follows $M^{2n} = M\#\widetilde{\mathbb{S}^{2n}}$, where $M = \overset{p}{\underset{j=1}{\#}}(\mathbb{S}^n \times \mathbb{S}^n)$ is the connected sum of p-copies of $\mathbb{S}^n \times \mathbb{S}^n$ and $\widetilde{\mathbb{S}^{2n}}$ is a homotopy $2n$-sphere. Note that this decomposition exists under the assumption that the Arf-Kervaire invariant is zero when $n = 4k + 3$. Using this decomposition, we get the following proposition.

Proposition 5.10 *Let $M^{2n} = M\#\widetilde{\mathbb{S}^{2n}}$ be an $(n-1)$-connected $2n$-dimensional π-manifold. Then there is the natural short exact sequence of chain complexes*

$$0 \to C_*(\mathbb{S}^{2n-1}) \longrightarrow C_*(M - \mathbb{D}^{2n}) \oplus C_*(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n}) \longrightarrow C_*(M^{2n}) \to 0 \tag{5.148}$$

with the corresponding Mayer-Vietoris sequence

$$\begin{aligned}
\mathcal{H}_*: \quad 0 &\xrightarrow{\alpha} H_{2n}(M^{2n}) \xrightarrow{\beta} H_{2n-1}(\mathbb{S}^{2n-1}) \xrightarrow{\gamma} 0 \\
&\xrightarrow{\varphi} H_n(M - \mathbb{D}^{2n}) \xrightarrow{\psi} H_n(M^{2n}) \xrightarrow{\theta} 0 \\
&\xrightarrow{\phi} H_0(\mathbb{S}^{2n-1}) \xrightarrow{\eta} H_0(M - \mathbb{D}^{2n}) \oplus H_0(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n}) \\
&\xrightarrow{\rho} H_0(M^{2n}) \xrightarrow{\delta} 0.
\end{aligned} \tag{5.149}$$

Let $\mathbf{h}_\nu^{M^{2n}}$ and $\mathbf{h}_0^{\mathbb{S}^{2n-1}}$ be respectively bases of $H_\nu(M^{2n})$ and $H_0(\mathbb{S}^{2n-1})$ for $\nu = 0, \ldots, 2n$. Let also $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}} = \beta(\mathbf{h}_{2n}^{M})$ be a basis of $H_{2n-1}(\mathbb{S}^{2n-1})$ and $\mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}}-\mathbb{D}^{2n}}$ be an arbitrary basis of $H_0(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n})$. Then there exists a basis $\mathbf{h}_\nu^{M-\mathbb{D}^{2n}}$ of $H_\nu(M - \mathbb{D}^{2n})$ so that the corrective term disappears from the multiplicative gluing formula as follows

$$\begin{aligned}
\mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right) &= \mathbb{T}\left(M - \mathbb{D}^{2n}, \{\mathbf{h}_\nu^{M-\mathbb{D}^{2n}}\}_{\nu=0}^{n}\right) \mathbb{T}\left(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n}, \{\mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}}-\mathbb{D}^{2n}}\}\right) \\
&\quad \times \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right)^{-1}.
\end{aligned}$$

Proof First, we denote the vector spaces in the long exact sequence $\mathcal{H}_*$ by $C_p(\mathcal{H}_*)$ for

$p \in \{0, 1, \ldots, 8\}$. Then we can write the space $C_p(\mathcal{H}_*)$ as a direct sum of the spaces $B_p(\mathcal{H}_*)$ and $s_p(B_{p-1}(\mathcal{H}_*))$ for each p as follows

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{5.150}$$

Let us consider the space $C_0(\mathcal{H}_*) = H_0(M^{2n})$ in equation (5.150). From the fact that $\mathrm{Im}(\delta)$ is a trivial space it follows

$$C_0(\mathcal{H}_*) = \mathrm{Im}(\rho) \oplus s_0(\mathrm{Im}(\delta)) = \mathrm{Im}(\rho). \tag{5.151}$$

Let us choose the basis of $\mathrm{Im}(\rho)$ as $\mathbf{h}_0^{M^{2n}}$. From equation (5.151) it follows that $\mathbf{h}_0^{M^{2n}}$ becomes the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. Since the initial basis $\mathbf{h}_0$ of $C_0(\mathcal{H}_*)$ is also $\mathbf{h}_0^{M^{2n}}$, we have

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{5.152}$$

Next consider $C_1(\mathcal{H}_*) = H_0(M - \mathbb{D}^{2n}) \oplus H_0(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n})$ in equation (5.150). Then the space $C_1(\mathcal{H}_*)$ can be expressed as the following direct sum

$$C_1(\mathcal{H}_*) = \mathrm{Im}(\eta) \oplus s_1(\mathrm{Im}(\rho)). \tag{5.153}$$

As η is injective, $\eta(\mathbf{h}_0^{\mathbb{S}^{2n-1}})$ can be taken as the basis of $\mathrm{Im}(\eta)$. In the previous step, we chose $\mathbf{h}_0^{M^{2n}}$ as the basis of $\mathrm{Im}(\rho)$. By equation (5.153), we get that

$$\left\{\eta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}), s_1(\mathbf{h}_0^{M^{2n}})\right\}$$

is the obtained basis $\mathbf{h}'_1$ of $C_1(\mathcal{H}_*)$. Note that $H_0(M - \mathbb{D}^{2n})$ and $H_0(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n})$ are both one-dimensional subspaces of the 2-dimensional space $C_1(\mathcal{H}_*)$. Thus, there exist non-zero vectors (a_{i1}, a_{i2}), $i = 1, 2$ such that

$$\left\{a_{11}\eta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{12}s_1(\mathbf{h}_0^{M^{2n}})\right\}$$

is a basis of $H_0(M - \mathbb{D}^{2n})$ and

$$\left\{a_{21}\eta(\mathbf{h}_0^{S^{2n-1}}) + a_{22}s_1(\mathbf{h}_0^{M^{2n}})\right\}$$

is a basis of $H_0(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n})$. Clearly, $A = (a_{ij})$ is a non-singular (2×2)-matrix over $\mathbb{R}$. Let

$$\begin{aligned}\mathbf{h}_0^{M-\mathbb{D}^{2n}} &= \left\{(\det A)^{-1}\left[a_{11}\eta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{12}s_1(\mathbf{h}_0^{M^{2n}})\right]\right\}\\ \mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}}-\mathbb{D}^{2n}} &= \left\{a_{21}\eta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) + a_{22}s_1(\mathbf{h}_0^{M^{2n}})\right\}\end{aligned}$$

be basis of $H_0(M - \mathbb{D}^{2n})$ and $H_0(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n})$, respectively. Considering $\{\mathbf{h}_0^{M-\mathbb{D}^{2n}}, \mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}}-\mathbb{D}^{2n}}\}$ as the initial basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$, we conclude that the determinant of the transition matrix is 1; that is,

$$[\mathbf{h}_1', \mathbf{h}_1] = 1. \tag{5.154}$$

If we use equation (5.150) for $C_2(\mathcal{H}_*) = H_0(\mathbb{S}^{2n-1})$ and consider the trivial space $\mathrm{Im}(\phi)$, then we obtain

$$C_2(\mathcal{H}_*) = \mathrm{Im}(\phi) \oplus s_2(\mathrm{Im}(\eta)) = s_2(\mathrm{Im}(\eta)). \tag{5.155}$$

Recall that the basis of $\mathrm{Im}(\eta)$ was chosen previously as $\eta(\mathbf{h}_0^{\mathbb{S}^{2n-1}})$. From equation (5.155) it follows that $\mathbf{h}_0^{\mathbb{S}^{2n-1}}$ becomes the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$. By the fact that $\mathbf{h}_0^{\mathbb{S}^{2n-1}}$ is the initial basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$, the determinant of the transition matrix becomes 1 as follows

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{5.156}$$

Considering the trivial space $C_3(\mathcal{H}_*)$ in the sequence $\mathcal{H}_*$ and using the convention $1 \cdot 0 = 1$, we obtain

$$[\mathbf{h}_3', \mathbf{h}_3] = 1. \tag{5.157}$$

Let us consider $C_4(\mathcal{H}_*) = H_n(M^{2n})$ in equation (5.150). From the fact that $\mathrm{Im}(\theta)$

is trivial it follows

$$C_4(\mathcal{H}_*) = \mathrm{Im}(\psi) \oplus s_4(\mathrm{Im}(\theta)) = \mathrm{Im}(\psi). \tag{5.158}$$

Let us choose the basis of $\mathrm{Im}(\psi)$ as $\mathbf{h}_n^{M^{2n}} = \{\mathbf{h}_{ni}^{M^{2n}}\}_{i=1}^{2p-2}$. By equation (5.158), we get that $\mathbf{h}_n^{M^{2n}}$ is the obtained basis $\mathbf{h}'_4$ of $C_4(\mathcal{H}_*)$. Since the initial basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$ is also $\mathbf{h}_n^{M^{2n}}$, the transition matrix becomes the identity matrix. Hence, we get

$$[\mathbf{h}'_4, \mathbf{h}_4] = 1. \tag{5.159}$$

If we use equation (5.150) for $C_5(\mathcal{H}_*) = H_n(M - \mathbb{D}^{2n})$, then $\mathrm{Im}(\varphi) = \{0\}$ yields

$$C_5(\mathcal{H}_*) = \mathrm{Im}(\varphi) \oplus s_5(\mathrm{Im}(\psi)) = s_5(\mathrm{Im}(\psi)). \tag{5.160}$$

In the previous step, we chose the basis of $\mathrm{Im}(\psi)$ as $\mathbf{h}_n^{M^{2n}}$. Then, by equation (5.160), it is concluded that $s_5(\mathbf{h}_n^{M^{2n}})$ becomes the obtained basis $\mathbf{h}'_5$ of $C_5(\mathcal{H}_*)$. If we take $s_5(\mathbf{h}_n^{M^{2n}})$ as the initial basis $\mathbf{h}_5$ (namely, $\mathbf{h}^{M-\mathbb{D}^{2n}}$) of $C_5(\mathcal{H}_*)$, then we obtain

$$[\mathbf{h}'_5, \mathbf{h}_5] = 1. \tag{5.161}$$

Let us use the convention $1 \cdot 0 = 1$ for the trivial space $C_6(\mathcal{H}_*)$ in the long exact sequence (5.149). Then we get

$$[\mathbf{h}'_6, \mathbf{h}_6] = 1. \tag{5.162}$$

Next we consider equation (5.150) for the space $C_7(\mathcal{H}_*) = H_{2n-1}(\mathbb{S}^{2n-1})$. By the fact that the space $\mathrm{Im}(\gamma)$ is trivial, the following equalities hold

$$C_7(\mathcal{H}_*) = \mathrm{Im}(\beta) \oplus s_7(\mathrm{Im}(\gamma)) = \mathrm{Im}(\beta). \tag{5.163}$$

Since $\mathrm{Im}(\beta)$ is isomorphic to $H_{2n}(M^{2n})$, we can take the basis of $\mathrm{Im}(\beta)$ as $\beta(\mathbf{h}_{2n}^{M^{2n}})$. By equation (5.163), $\beta(\mathbf{h}_{2n}^{M^{2n}})$ becomes the obtained basis $\mathbf{h}'_7$ of $C_7(\mathcal{H}_*)$. As the initial basis $\mathbf{h}_7$

of $C_7(\mathcal{H}_*)$ is $\beta(\mathbf{h}_{2n}^{M^{2n}})$, we have

$$[\mathbf{h}'_7, \mathbf{h}_7] = 1. \tag{5.164}$$

Finally, let us consider equation (5.150) for the space $C_8(\mathcal{H}_*) = H_{2n}(M^{2n})$. Since $\mathrm{Im}(\alpha)$ is trivial, we get

$$C_8(\mathcal{H}_*) = \mathrm{Im}(\alpha) \oplus s_8(\mathrm{Im}(\beta)) = s_8(\mathrm{Im}(\beta)). \tag{5.165}$$

In the previous step, $\beta(\mathbf{h}_{2n}^{M^{2n}})$ was chosen as the basis of $\mathrm{Im}(\beta)$. From equation (5.165) it follows that

$$s_8(\beta(\mathbf{h}_{2n}^{M^{2n}})) = \mathbf{h}_{2n}^{M^{2n}}$$

becomes the obtained basis $\mathbf{h}'_8$ of $C_8(\mathcal{H}_*)$. As the initial basis $\mathbf{h}_8$ of $C_8(\mathcal{H}_*)$ is also $\mathbf{h}_{2n}^{M^{2n}}$, we get that the determinant of the transition matrix is 1; that is,

$$[\mathbf{h}'_8, \mathbf{h}_8] = 1. \tag{5.166}$$

By equations (5.152), (5.154), (5.156), (5.157), (5.159), (5.161), (5.162), (5.164), and (5.166), the corrective term becomes 1 as follows

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^8, \{0\}_{p=0}^8\right) = \prod_{p=0}^{8} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{5.167}$$

Compatibility of the natural bases in the short exact sequence (5.148), Theorem 4.1, and equation (5.167) end the proof of Proposition 5.10. □

Proposition 5.11 *Let $\widetilde{\mathbb{S}^{2n}}$ be a homotopy $2n$-sphere. Then there exists the following short exact sequence of the chain complexes*

$$0 \to C_*(\mathbb{S}^{2n-1}) \longrightarrow C_*(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n}) \oplus C_*(\overline{\mathbb{D}^{2n}}) \longrightarrow C_*(\widetilde{\mathbb{S}^{2n}}) \to 0 \tag{5.168}$$

with the corresponding Mayer-Vietoris sequence as follows

$$\mathcal{H}_*: \quad 0 \xrightarrow{\phi} H_{2n}(\widetilde{\mathbb{S}^{2n}}) \xrightarrow{\alpha} H_{2n-1}(\mathbb{S}^{2n-1}) \xrightarrow{\gamma} 0 \xrightarrow{\varsigma} H_0(\mathbb{S}^{2n-1}) \xrightarrow{\theta} H_0(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n}) \oplus H_0(\overline{\mathbb{D}^{2n}}) \xrightarrow{\delta} H_0(\widetilde{\mathbb{S}^{2n}}) \xrightarrow{\rho} 0.$$

Assume that $\mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}}-\mathbb{D}^{2n}}$ *and* $\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}$ *are respectively bases of* $H_0(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n})$, $H_\eta(\mathbb{S}^{2n-1})$, $\eta = 0, \cdots, 2n-1$. *Assume also that* $\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}$ *is an arbitrary basis of* $H_0(\overline{\mathbb{D}^{2n}})$. *Then there is a basis* $\mathbf{h}_\nu^{\widetilde{\mathbb{S}^{2n}}}$ *of* $H_\nu(\widetilde{\mathbb{S}^{2n}})$ *for* $\nu = 0, \ldots, 2n$ *such that the corrective term disappears and the following multiplicative gluing formula is valid*

$$\mathbb{T}\left(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n}, \{\mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}}-\mathbb{D}^{2n}}\}\right) = \mathbb{T}\left(\widetilde{\mathbb{S}^{2n}}, \{\mathbf{h}_\nu^{\widetilde{\mathbb{S}^{2n}}}\}_{\nu=0}^{2n}\right) \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}\}\right)^{-1}.$$

Proof For $p \in \{0, 1, \ldots, 5\}$, let $C_p(\mathcal{H}_*)$ denote the vector spaces in the long exact sequence $\mathcal{H}_*$. Then the following equality holds for each p

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{5.169}$$

First, we consider the vector space $C_0(\mathcal{H}_*) = H_0(\widetilde{\mathbb{S}^{2n}})$ in equation (5.169). Since $\mathrm{Im}(\rho)$ is trivial, we get

$$C_0(\mathcal{H}_*) = \mathrm{Im}(\delta) \oplus s_0(\mathrm{Im}(\rho)) = \mathrm{Im}(\delta). \tag{5.170}$$

For $(a_{11}, a_{12}) \neq (0, 0)$, let us take the basis of $\mathrm{Im}(\delta)$ as

$$\mathbf{h}^{\mathrm{Im}(\delta)} = \left\{ a_{11}\delta(\mathbf{h}^{\widetilde{\mathbb{S}^{2n}}-\mathbb{D}^{2n}}) + a_{12}\delta(\mathbf{h}^{\overline{\mathbb{D}^{2n}}}) \right\}.$$

It follows from equation (5.170) that $\mathbf{h}^{\mathrm{Im}(\delta)}$ is the obtained basis $\mathbf{h}'_0$ of $C_0(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_0$ of $H_0(\widetilde{\mathbb{S}^{2n}})$ as $\mathbf{h}^{\mathrm{Im}(\delta)}$, the transition matrix becomes the identity matrix and its determinant is given as follows

$$[\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{5.171}$$

Let us consider equation (5.169) for $C_1(\mathcal{H}_*) = H_0(\widetilde{\mathbb{S}^{2n} - \mathbb{D}^{2n}}) \oplus H_0(\overline{\mathbb{D}^{2n}})$. As the spaces $B_1(\mathcal{H}_*)$ and $B_0(\mathcal{H}_*)$ are equal to $\mathrm{Im}(\theta)$ and $\mathrm{Im}(\delta)$, respectively, we get

$$C_1(\mathcal{H}_*) = \mathrm{Im}(\theta) \oplus s_1(\mathrm{Im}(\delta)). \tag{5.172}$$

The initial basis $\mathbf{h}_1$ of $C_1(\mathcal{H}_*)$ is $\{\mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}-\mathbb{D}^{2n}}}, \mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}\}$. Recall that in the previous step, the basis $\mathbf{h}^{\mathrm{Im}(\delta)}$ of $\mathrm{Im}(\delta)$ was chosen as

$$\left\{a_{11}\delta(\mathbf{h}^{\widetilde{\mathbb{S}^{2n}-\mathbb{D}^{2n}}}) + a_{12}\delta(\mathbf{h}^{\overline{\mathbb{D}^{2n}}})\right\}.$$

Note also that $\mathrm{Im}(\theta)$ is isomorphic to $H_0(\mathbb{S}^{2n-1})$, so we can choose the basis $\mathbf{h}^{\mathrm{Im}(\theta)}$ of $\mathrm{Im}(\theta)$ as $\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}})$. Since $\mathrm{Im}(\theta)$ is one-dimensional subspace of $C_1(\mathcal{H}_*)$, there is a non-zero vector (a_{21}, a_{22}) such that

$$\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}) = a_{21}\mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}-\mathbb{D}^{2n}}} + a_{22}\mathbf{h}_0^{\mathbb{D}^{2n}}.$$

Hence, by equation (5.172),

$$\left\{\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}}), s_1(\mathbf{h}^{\mathrm{Im}(\delta)})\right\}$$

is the obtained basis $\mathbf{h}'_1$ of $C_1(\mathcal{H}_*)$ such that $A = (a_{ij})$ is the invertible (2×2)-real matrix. Thus, the determinant of the transition matrix satisfies the following equality

$$[\mathbf{h}'_1, \mathbf{h}_1] = \det A. \tag{5.173}$$

Let us now consider equation (5.169) for the space $C_2(\mathcal{H}_*) = H_0(\mathbb{S}^{2n-1})$. Using the fact that $\mathrm{Im}(\varsigma) = \{0\}$, we have

$$C_2(\mathcal{H}_*) = \mathrm{Im}(\varsigma) \oplus s_2(\mathrm{Im}(\theta)) = s_2(\mathrm{Im}(\theta)). \tag{5.174}$$

Note that $\mathbf{h}_0^{\mathbb{S}^{2n-1}}$ is the initial basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$. By equation (5.174), we obtain that

$$s_2(\theta(\mathbf{h}_0^{\mathbb{S}^{2n-1}})) = \mathbf{h}_0^{\mathbb{S}^{2n-1}}$$

is the obtained basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$. So, the transition matrix becomes the identity matrix, and thus we have

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{5.175}$$

Considering $C_3(\mathcal{H}_*) = \{0\}$ in the sequence $\mathcal{H}_*$, and using the convention $1 \cdot 0 = 1$, we get

$$[\mathbf{h}_3', \mathbf{h}_3] = 1. \tag{5.176}$$

Let us consider the space $C_4(\mathcal{H}_*) = H_{2n-1}(\mathbb{S}^{2n-1})$ in equation (5.169). The equalities $B_4(\mathcal{H}_*) = \text{Im}(\alpha)$ and $B_3(\mathcal{H}_*) = \text{Im}(\gamma) = \{0\}$ yield

$$C_4(\mathcal{H}_*) = \text{Im}(\alpha) \oplus s_4(\text{Im}(\gamma)) = \text{Im}(\alpha). \tag{5.177}$$

Recall that $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}}$ is the initial basis $\mathbf{h}_4$ of $C_4(\mathcal{H}_*)$. Taking the basis $\mathbf{h}^{\text{Im}(\alpha)}$ of $\text{Im}(\alpha)$ as $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}}$ and considering equation (5.177), $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}}$ becomes the obtained basis $\mathbf{h}_4'$ of $C_4(\mathcal{H}_*)$. Therefore, we obtain

$$[\mathbf{h}_4', \mathbf{h}_4] = 1. \tag{5.178}$$

Finally, let us consider equation (5.169) for $C_5(\mathcal{H}_*) = H_{2n}(\widetilde{\mathbb{S}^{2n}})$ and use the equalities $B_5(\mathcal{H}_*) = \text{Im}(\phi) = \{0\}$ and $B_4(\mathcal{H}_*) = \text{Im}(\alpha)$. Then the following equation holds

$$C_5(\mathcal{H}_*) = \text{Im}(\phi) \oplus s_5(\text{Im}(\alpha)) = s_5(\text{Im}(\alpha)). \tag{5.179}$$

Recall that $\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}}$ was chosen as the basis of $\text{Im}(\alpha)$ in the previous step. By equation (5.179), we conclude that $s_5(\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}})$ is the obtained basis $\mathbf{h}_5'$ of $C_5(\mathcal{H}_*)$. If we take the initial basis $\mathbf{h}_5$ (namely, $\mathbf{h}_{2n}^{\widetilde{\mathbb{S}^{2n}}}$) of $C_5(\mathcal{H}_*)$ as $\{(\det A)^{-1} s_5(\mathbf{h}_{2n-1}^{\mathbb{S}^{2n-1}})\}$, then the following equality holds

$$[\mathbf{h}_5', \mathbf{h}_5] = (\det A)^{-1}. \tag{5.180}$$

By equations (5.171), (5.173), (5.175), (5.176), (5.178), and (5.180), we get

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^5, \{0\}_{p=0}^5\right) = \prod_{p=0}^{5} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{5.181}$$

Using compatibility of the natural bases in the short exact sequence (5.168), Theorem 4.1 yields the following formula

$$\begin{aligned}
&\mathbb{T}\left(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n} \oplus \overline{\mathbb{D}^{2n}}, \{\mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}}-\mathbb{D}^{2n}} \sqcup \mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}\}\right)\\
&= \mathbb{T}\left(\widetilde{\mathbb{S}^{2n}}, \{\mathbf{h}_\nu^{\widetilde{\mathbb{S}^{2n}}}\}_{\nu=0}^{2n}\right) \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^5, \{0\}_{p=0}^5\right).
\end{aligned} \tag{5.182}$$

By equation (5.181) and equation (5.182), the following formula holds

$$\mathbb{T}\left(\widetilde{\mathbb{S}^{2n}} - \mathbb{D}^{2n} \oplus \overline{\mathbb{D}^{2n}}, \{\mathbf{h}_0^{\widetilde{\mathbb{S}^{2n}}-\mathbb{D}^{2n}} \sqcup \mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}\}\right) = \mathbb{T}\left(\widetilde{\mathbb{S}^{2n}}, \{\mathbf{h}_\nu^{\widetilde{\mathbb{S}^{2n}}}\}_{\nu=0}^{2n}\right) \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right). \tag{5.183}$$

The proof of Proposition 5.11 is finished by using Lemma 4.1 and equation (5.183). □

By Theorem 5.8, Proposition 5.10 and Proposition 5.11, we get the following theorem.

Theorem 5.10 *Let M^{2n} be an $(n-1)$-connected $2n$-dimensional π-manifold. Let $\mathbf{h}_\nu^{M^{2n}}$ and $\mathbf{h}_\eta^{\mathbb{S}_j^{2n-1}}$ be respectively bases of $H_\nu(M^{2n})$ and $H_\eta(\mathbb{S}_j^{2n-1})$ for $\nu = 0, \cdots, 2n$ and $\eta = 0, \cdots, 2n-1$ and $\mathbf{h}_0^{\overline{\mathbb{D}_j^{2n}}}$ an arbitrary basis of $H_0(\overline{\mathbb{D}_j^{2n}})$. Then there exist the homology bases $\mathbf{h}_\nu^{M_j}$ and $\mathbf{h}_\nu^{\widetilde{\mathbb{S}^{2n}}}$ such that the following formula holds*

$$\begin{aligned}
\mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right) &= \mathbb{T}\left(\widetilde{\mathbb{S}^{2n}}, \{\mathbf{h}_\nu^{\widetilde{\mathbb{S}^{2n}}}\}_{\nu=0}^{2n}\right) \prod_{j=1}^{p} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^{2n}\right)\\
&\times \prod_{j=1}^{p} \left[\mathbb{T}\left(\mathbb{S}_j^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}_j^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}_j^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_j^{2n}}}\}\right)^{-2}\right].
\end{aligned}$$

Let M^{2n} be an $(n-1)$-connected $2n$-dimensional closed parallelizable manifold ($n \geq 3$). By Theorem 3.6, n must be odd and Ishimoto (1969) showed that M^{2n} has the form as

$$M^{2n} = (\mathbb{S}^n \times \mathbb{S}^n)\#\widetilde{\mathbb{S}^{2n}},$$

under the assumption that $\kappa(M^{2n}) = 0$ when $n = 4k+3$. Combining this result and Remark 5.10 gives the following corollary.

Corollary 5.3 *Let M^{2n} be an $(n-1)$-connected $2n$-dimensional closed parallelizable manifold ($n \geq 3$), where n is odd. Assume that $\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}$ is an arbitrary basis of $H_0(\overline{\mathbb{D}^{2n}})$. If $\mathbf{h}_\nu^{M^{2n}}$ and $\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}$ are respectively bases of $H_\nu(M^{2n})$ and $H_\eta(\mathbb{S}^{2n-1})$, $\nu = 0, \ldots, 2n$, $\eta = 0, \cdots, 2n-1$, then there are respectively bases $\mathbf{h}_\nu^{\mathbb{S}^n\times\mathbb{S}^n}$ and $\mathbf{h}_\nu^{\widetilde{\mathbb{S}^{2n}}}$ of $H_\nu(\mathbb{S}^n \times \mathbb{S}^n)$ and $H_\nu(\widetilde{\mathbb{S}^{2n}})$ such that the following formula holds*

$$\begin{aligned}\mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right) &= \mathbb{T}\left(\mathbb{S}^n \times \mathbb{S}^n, \{\mathbf{h}_\nu^{\mathbb{S}^n\times\mathbb{S}^n}\}_{\nu=0}^{2n}\right) \mathbb{T}\left(\widetilde{\mathbb{S}^{2n}}, \{\mathbf{h}_\nu^{\widetilde{\mathbb{S}^{2n}}}\}_{\nu=0}^{2n}\right)\\ &\quad \times \mathbb{T}\left(\mathbb{S}^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}^{2n}}}\}\right)^{-2}.\end{aligned}$$

By De Sapio (1965), an $(n-1)$-connected $2n$-dimensional manifold M^{2n} ($n \geq 3$) which bounds a π-manifold is diffeomorphic to a connected sum $\overset{p}{\underset{j=1}{\#}}(\mathbb{S}^n \times \mathbb{S}^n)$, where p is the rank of $H_n(M^{2n})$. From De Sapios's result and Theorem 5.8 it follows :

Corollary 5.4 *Let M^{2n} be an $(n-1)$-connected $2n$-dimensional manifold ($n \geq 3$) which bounds a π-manifold. Let $\mathbf{h}_\nu^{M^{2n}}$, $\mathbf{h}_\eta^{\mathbb{S}_i^{2n-1}}$, and $\mathbf{h}_0^{\overline{\mathbb{D}_i^{2n}}}$ be respectively bases of $H_\nu(M^{2n})$, $H_\eta(\mathbb{S}_i^{2n-1})$, and $H_0(\overline{\mathbb{D}_i^{2n}})$ for $\nu = 0, \ldots, 2n$, $\eta = 0, \cdots, 2n-1$. Then there is a homology basis $\mathbf{h}_\nu^{M_j}$ for each j such that the Reidemeister torsion of M^{2n} satisfies the following formula*

$$\mathbb{T}\left(M^{2n}, \{\mathbf{h}_\nu^{M^{2n}}\}_{\nu=0}^{2n}\right) = \prod_{j=1}^{p} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^{2n}\right) \prod_{i=1}^{p-1} \left[\mathbb{T}\left(\mathbb{S}_i^{2n-1}, \{\mathbf{h}_\eta^{\mathbb{S}_i^{2n-1}}\}_{\eta=0}^{2n-1}\right) \mathbb{T}\left(\overline{\mathbb{D}_i^{2n}}, \{\mathbf{h}_0^{\overline{\mathbb{D}_i^{2n}}}\}\right)^{-2}\right].$$

Here, we use the notation $\overset{p}{\underset{j=1}{\#}} M_j$ for $\overset{p}{\underset{j=1}{\#}}(\mathbb{S}^n \times \mathbb{S}^n)$.

CHAPTER 6

APPLICATIONS

This chapter applies Theorem 5.5 and Theorem 5.8 to establish Reidemeister torsion formulas for handlebodies (Corollary 6.2, Remark 6.1). It then applies Corollary 5.1 to obtain Reidemeister torsion formula for compact, orientable, smooth $(2n+1)$-dimensional manifolds whose boundary is $(n-2)$-connected $2n$-dimensional closed π-manifold (Corollary 6.3). Moreover, it provides Reidemeister torsion formulas for product manifolds (Corollary 6.4).

6.1. Heegaard Splitting and Handlebodies

In this section, we show that the Reidemeister torsion of a handlebody $\mathcal{H}$ can be expressed in terms of the Reidemeister torsion of its boundary surface. Moreover, we apply Theorem 5.5 to give an explicit formula that computes the Reidemeister torsion of $\mathcal{H}$.

Heegaard splittings are one of the main ingredients in the construction of Heegaard Floer homology (Ozsváth and Szabó, 2006), and multiplicativity with respect to gluings is one of the fundamental axioms of Topological Quantum Field Theories (Atiyah, 1988). Seifert and Threlfall (1934) proved that a closed, connected, orientable 3-dimensional manifold M is decomposed into two homeomorphic handlebodies. More precisely,

Proposition 6.1 *(Seifert and Threlfall, 1934) For every closed, connected, orientable* 3*-dimensional manifold M, there exist handlebodies $\mathcal{H}_1$ and $\mathcal{H}_2$ in M such that*

(i) $\mathcal{H}_1 \cong \mathcal{H}_2$, *that is,* $\text{genus}(\mathcal{H}_1) = \text{genus}(\mathcal{H}_2) = g$,

(ii) $M = \mathcal{H}_1 \cup \mathcal{H}_2$, *and*

(iii) $\mathcal{H}_1 \cap \mathcal{H}_2 = \partial(\mathcal{H}_1) \cap \partial(\mathcal{H}_2) = \partial(\mathcal{H}_1) = \partial(\mathcal{H}_2) = \Sigma_{g,0}$, *the Heegaard surface.*

$(M; \mathcal{H}_1, \mathcal{H}_2; \Sigma_{g,0})$ is called a *Heegaard splitting* for M of genus g, and the minimum genus of such splittings for M is called the *Heegaard genus* of M and denoted by $\mathcal{H}_g(M)$.

Note that the closed surface $\Sigma_{g,0}$ is expressed as a connected sum of g-copies of $\mathbb{S}^1 \times \mathbb{S}^1$. Namely,

$$\Sigma_{g,0} = \mathop{\#}_{j=1}^{g} (\mathbb{S}^1 \times \mathbb{S}^1).$$

By Proposition 3.5, $\Sigma_{g,0}$ is a closed 2-dimensional π-manifold. Moreover, Theorem 5.8 yields the following result.

Corollary 6.1 *Let $\Sigma_{g,0} = \mathop{\#}_{j=1}^{g} M_j$ be a connected sum of g-copies of $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbf{h}_\nu^M$, $\mathbf{h}_\eta^{\mathbb{S}_i^1}$, and $\mathbf{h}_0^{\overline{\mathbb{D}_i^2}}$ be respectively bases of $H_\nu(M)$, $H_\eta(\mathbb{S}_i^1)$, and $H_0(\overline{\mathbb{D}_i^2})$, $\nu = 0, 1, 2$, $\eta = 0, 1$. Then there is a basis $\mathbf{h}_\nu^{M_j}$ of $H_\nu(M_j)$ for each $j \in \{1, \ldots, g\}$ such that the following formula holds*

$$\mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_\nu^{\Sigma_{g,0}}\}_{\nu=0}^2\right) = \prod_{j=1}^{g} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^2\right) \prod_{i=1}^{g-1} \left[\mathbb{T}\left(\mathbb{S}_i^1, \{\mathbf{h}_\eta^{\mathbb{S}_i^1}\}_{\eta=0}^1\right) \mathbb{T}\left(\overline{\mathbb{D}_i^2}, \{\mathbf{h}_0^{\overline{\mathbb{D}_i^2}}\}\right)^{-2}\right].$$

Moreover, Theorem 4.4 yields

$$\left|\mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_\nu^{\Sigma_{g,0}}\}_{\nu=0}^2\right)\right| = \prod_{j=1}^{g} \left|\mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^2\right)\right| \prod_{i=1}^{g-1} \left|\mathbb{T}\left(\overline{\mathbb{D}_i^2}, \{\mathbf{h}_0^{\overline{\mathbb{D}_i^2}}\}\right)\right|^{-2}.$$

Here, $M_j = \mathbb{S}^1 \times \mathbb{S}^1$ for each $j \in \{1, \ldots, g\}$.

Every compact, connected, orientable 3-dimensional manifold is a π-manifold. By the consequence of this result, handlebodies are also π-manifolds with their boundary surfaces $\Sigma_{g,0}$. The following result provides a formula that computes the Reidemeister torsion of a handlebody with regard to the Reidemeister torsion of its boundary surface.

Corollary 6.2 *For a closed, connected, orientable 3-dimensional manifold M with the Heegaard splitting $(M; \mathcal{H}_1, \mathcal{H}_2; \Sigma_g)$ such that $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $\mathcal{H}_g(M) \geqslant 2$, there exists the following short exact sequence of the chain complexes*

$$0 \to C_*(\Sigma_{g,0}) \to C_*(\mathcal{H}) \oplus C_*(\mathcal{H}) \to C_*(M) \to 0. \tag{6.1}$$

For the given bases $\mathbf{h}_p^M$ and $\mathbf{h}_p^{\mathcal{H}}$, $p = 0, 1, 2, 3$, there exists a basis $\mathbf{h}_i^{\Sigma_{g,0}}$ $i = 0, 1, 2$ such that *the corrective term becomes* 1 *and*

(i) If $\mathbb{T}(M,\{\mathbf{h}_p^M\}_{p=0}^3)=1$, *then*

$$\mathbb{T}\left(\mathcal{H},\{\mathbf{h}_p^{\mathcal{H}}\}_{p=0}^3\right)=\prod_{j=1}^{g}\mathbb{T}\left(M_j,\{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^2\right)^{1/2}\prod_{i=1}^{g-1}\left[\mathbb{T}\left(\mathbb{S}_i^1,\{\mathbf{h}_\eta^{\mathbb{S}_i^1}\}_{\eta=0}^1\right)^{1/2}\mathbb{T}\left(\overline{\mathbb{D}_i^2},\{\mathbf{h}_0^{\overline{\mathbb{D}_i^2}}\}\right)^{-1}\right].$$

(ii) If $\mathbb{T}(M,\{\mathbf{h}_p^M\}_{p=0}^3)=-1$, *then*

$$\left|\mathbb{T}\left(\mathcal{H},\{\mathbf{h}_p^{\mathcal{H}}\}_{p=0}^3\right)\right|=\prod_{j=1}^{g}\left|\mathbb{T}\left(M_j,\{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^2\right)\right|^{1/2}\prod_{i=1}^{g-1}\left|\mathbb{T}\left(\overline{\mathbb{D}_i^2},\{\mathbf{h}_0^{\overline{\mathbb{D}_i^2}}\}\right)\right|^{-1}.$$

Proof Let us consider the Mayer-Vietoris sequence

$$\begin{aligned}\mathcal{H}_*:\quad 0 &\xrightarrow{\alpha_3''} H_3(\Sigma_{g,0})\xrightarrow{\alpha_3} H_3(\mathcal{H})\oplus H_3(\mathcal{H})\xrightarrow{\alpha_3'} H_3(M)\\ &\xrightarrow{\partial_3} H_2(\Sigma_{g,0})\xrightarrow{\alpha_2} H_2(\mathcal{H})\oplus H_2(\mathcal{H})\xrightarrow{\alpha_2'} H_2(M)\\ &\xrightarrow{\partial_2} H_1(\Sigma_{g,0})\xrightarrow{\alpha_1} H_1(\mathcal{H})\oplus H_1(\mathcal{H})\xrightarrow{\alpha_1'} H_1(M)\\ &\xrightarrow{\partial_1} H_0(\Sigma_{g,0})\xrightarrow{\alpha_0} H_0(\mathcal{H})\oplus H_0(\mathcal{H})\xrightarrow{\alpha_0'} H_0(M)\xrightarrow{\alpha_0''} 0.\end{aligned}$$

Since M is closed, connected, orientable 3-dimensional manifold, $H_3(M)=\mathbb{R}$. Since $H_2(\Sigma_{g,0})=\mathbb{R}$, the exactness of $\mathcal{H}_*$ implies that α_2 is a zero-map, and hence $H_3(M)\cong H_2(\Sigma_{g,0})$. Moreover, the exactness of $\mathcal{H}_*$ also gives the following isomorphism

$$H_0(\mathcal{H})\oplus H_0(\mathcal{H})\cong H_0(\Sigma_{g,0})\oplus H_0(M).$$

By using the arguments presented in the proof of Theorem 5.1, we conclude that the corrective term is 1. Since the bases $\mathbf{h}_p^M$ and $\mathbf{h}_i^{\Sigma_{g,0}}$ are compatible, Theorem 4.1 and

Lemma 4.1 yield

$$\mathbb{T}\left(\mathcal{H}, \{\mathbf{h}_p^{\mathcal{H}}\}_{p=0}^3\right)^2 = \mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_i^{\Sigma_{g,0}}\}_{i=0}^2\right)\ \mathbb{T}\left(M, \{\mathbf{h}_p^M\}_{p=0}^3\right). \tag{6.2}$$

From Theorem 4.4 it follows that $|\mathbb{T}(M, \{\mathbf{h}_p^M\}_{p=0}^3)| = 1$. Then, by Corollary 6.1 and equation (6.2), the followings hold

- If $\mathbb{T}(M, \{\mathbf{h}_p^M\}_{p=0}^3) = 1$, then

$$\begin{aligned}\mathbb{T}\left(\mathcal{H}, \{\mathbf{h}_p^{\mathcal{H}}\}_{p=0}^3\right) &= \sqrt{\mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_i^{\Sigma_{g,0}}\}_{i=0}^2\right)} \\ &= \prod_{j=1}^{g} \mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^2\right)^{1/2} \prod_{i=1}^{g-1}\left[\mathbb{T}\left(\mathbb{S}_i^1, \{\mathbf{h}_\eta^{\mathbb{S}_i^1}\}_{\eta=0}^1\right)^{1/2} \mathbb{T}\left(\overline{\mathbb{D}_i^2}, \{\mathbf{h}_0^{\overline{\mathbb{D}_i^2}}\}\right)^{-1}\right].\end{aligned}$$

- If $\mathbb{T}(M, \{\mathbf{h}_p^M\}_{p=0}^3) = -1$, then

$$\begin{aligned}\left|\mathbb{T}\left(\mathcal{H}, \left\{\mathbf{h}_p^{\mathcal{H}}\right\}_{p=0}^3\right)\right| &= \sqrt{\left|\mathbb{T}\left(\Sigma_{g,0}, \{\mathbf{h}_i^{\Sigma_{g,0}}\}_{i=0}^2\right)\right|} \\ &= \prod_{j=1}^{g} \left|\mathbb{T}\left(M_j, \{\mathbf{h}_\nu^{M_j}\}_{\nu=0}^2\right)\right|^{1/2} \prod_{i=1}^{g-1}\left|\mathbb{T}\left(\overline{\mathbb{D}_i^2}, \{\mathbf{h}_0^{\overline{\mathbb{D}_i^2}}\}\right)\right|^{-1}.\end{aligned}$$

□

Remark 6.1 *Observe that the formula obtained in Corollary 6.1 is the same as the one in Theorem 5.5, which reads as*

$$\left|\mathbb{T}\left(\mathcal{H}, \{\mathbf{h}_p^{\mathcal{H}}\}_{p=0}^3\right)\right| = \prod_{i=1}^{g}\left|\mathbb{T}\left(\Sigma_{1,1}^{\gamma_i}, \{\mathbf{h}_\eta^{\Sigma_{1,1}^{\gamma_i}}\}_{\eta=0}^1\right)\right|^{1/2} \prod_{j=1}^{g-2}\left|\mathbb{T}\left(\overline{\mathbb{D}_{\gamma_j}^2}, \{\mathbf{h}_0^{\overline{\mathbb{D}_{\gamma_j}^2}}\}\right)\right|^{-1/2}.$$

6.2. Compact Manifolds with Boundary

Corollary 6.3 *Let M^{2n} be an $(n-2)$-connected $2n$-dimensional closed π-manifold ($n \geq 3$) such that $H_{n-1}(M^{2n};\mathbb{Z})$ has no torsion. Suppose that $\kappa(M^{2n}) = 0$ if $n = 4k+3$. Let W be a* *compact, orientable, smooth* $(2n+1)$*-dimensional manifold with boundary* $\partial(W) = M^{2n}$. *There exists the following short exact sequence of the chain complexes*

$$0 \to C_*(M^{2n}) \to C_*(W) \oplus C_*(W) \to C_*(d(W)) \to 0 \tag{6.3}$$

with the corresponding Mayer-Vietoris sequence

$$\begin{array}{l}
\mathcal{H}_*: \quad 0 \xrightarrow{\alpha''_{2n+1}} H_{2n+1}(M^{2n}) \xrightarrow{\alpha_{2n+1}} H_{2n+1}(W) \oplus H_{2n+1}(W) \xrightarrow{\alpha'_{2n+1}} H_{2n+1}(d(W)) \\
\qquad \xrightarrow{\partial_{2n+1}} H_{2n}(M^{2n}) \xrightarrow{\alpha_{2n}} H_{2n}(W) \oplus H_{2n}(W) \xrightarrow{\alpha'_{2n}} H_{2n}(d(W)) \\
\qquad \cdots\cdots \\
\qquad H_1(M^{2n}) \xrightarrow{\alpha_1} H_1(W) \oplus H_1(W) \xrightarrow{\alpha'_1} H_1(d(W)) \\
\qquad \xrightarrow{\partial_1} H_0(M^{2n}) \xrightarrow{\alpha_0} H_0(W) \oplus H_0(W) \xrightarrow{\alpha'_0} H_0(d(W)) \xrightarrow{\alpha''_0} 0.
\end{array}$$

For a given basis $\mathbf{h}^W_\nu$, $\nu = 0, \ldots, 2n+1$ *there exist respectively the bases* $\mathbf{h}^{d(W)}_\nu$ *and* $\mathbf{h}^{M^{2n}}_i$ *with* $\mathbf{h}^{M^{2n}}_{2n} = \partial_{2n+1}(\mathbf{h}^{d(W)}_{2n+1})$ *such that the following formula is valid*

$$\left|\mathbb{T}\left(W, \{\mathbf{h}^W_\nu\}_{\nu=0}^{2n+1}\right)\right| = \prod_{j=1}^{p} \left|\mathbb{T}\left(M_j, \{\mathbf{h}^{M_j}_\nu\}_{\nu=0}^{2n}\right) \mathbb{T}\left(\overline{\mathbb{D}^{2n}_j}, \{\mathbf{h}_0^{\overline{\mathbb{D}^{2n}_j}}\}\right)^{-2}\right|^{1/2} \left|\mathbb{T}\left(M_1^{2n}, \{\mathbf{h}^{M_1^{2n}}_\nu\}_{\nu=0}^{2n}\right)\right|^{1/2}.$$

Here, $M = \mathop{\#}\limits_{j=1}^{p} (\mathbb{S}^n \times \mathbb{S}^n)$ *and* M_1^{2n} *is an* $(n-2)$*-connected* $2n$*-dimensional closed* π*-manifold.*

Proof Let us denote the vector spaces in the sequence $\mathcal{H}_*$ as $C_{3j}(\mathcal{H}_*)$, $C_{3j+1}(\mathcal{H}_*)$ and $C_{3j+2}(\mathcal{H}_*)$ for each $j \in \{1, 2, \ldots, 2n+1\}$.

Note that we use the convention $1 \cdot 0 = 1$ for the trivial space $\{0\}$. Since the space $C_{3j}(\mathcal{H}_*) = H_j(M^{2n})$ is trivial for each $j \in \{1, 2, \ldots, n-2, n+2, n+3, \ldots, 2n-1, 2n+1\}$, we get

$$[\mathbf{h}'_{3j}, \mathbf{h}_{3j}] = 1. \tag{6.4}$$

By the exactness of $\mathcal{H}_*$, we obtain the following results:

- For $j \in \{1, 2, \ldots, n-2, n+3, n+4, \ldots, 2n-1, 2n, 2n+1\}$, we have the following isomorphisms

$$\begin{aligned} H_j(W) \oplus H_j(W) &\overset{\alpha'_j}{\cong} H_j(d(W)), \\ H_{2n}(M^{2n}) &\overset{\partial_{2n+1}}{\cong} H_{2n+1}(d(W)). \end{aligned}$$

From the arguments presented in the proof of Proposition 5.8 and the condition on the basis $\mathbf{h}_{2n}^{M^{2n}} = \partial_{2n+1}(\mathbf{h}_{2n+1}^{d(W)})$ it follows

$$\begin{aligned} [\mathbf{h}'_{3j+1}, \mathbf{h}_{3j+1}] &= 1, \\ [\mathbf{h}'_{3j+2}, \mathbf{h}_{3j+2}] &= 1. \end{aligned} \tag{6.5}$$

- For $j = 0$, there exists the isomorphism $H_0(W) \oplus H_0(W) \cong H_0(M^{2n}) \oplus H_0(d(W))$. Hence, the determinant of the transition matrices are all equal to 1; that is

$$[\mathbf{h}'_2, \mathbf{h}_2] = [\mathbf{h}'_1, \mathbf{h}_1] = [\mathbf{h}'_0, \mathbf{h}_0] = 1. \tag{6.6}$$

- For $j \in \{n-1, n, n+1, n+2\}$, we get the middle part of $\mathcal{H}_*$

$$0 \xrightarrow{\alpha''_{n+2}} H_{n+2}(M^{2n}) \xrightarrow{\alpha_{n+2}} H_{n+2}(W) \oplus H_{n+2}(W) \xrightarrow{\alpha'_{n+2}} H_{n+2}(d(W))$$

$$\xrightarrow{\partial_{n+2}} H_{n+1}(M^{2n}) \xrightarrow{\alpha_{n+1}} H_{n+1}(W) \oplus H_{n+1}(W) \xrightarrow{\alpha'_{n+1}} H_{n+1}(d(W))$$

$$\xrightarrow{\partial_{n+1}} H_n(M^{2n}) \xrightarrow{\alpha_n} H_n(W) \oplus H_n(W) \xrightarrow{\alpha'_n} H_n(d(W))$$

$$\xrightarrow{\partial_n} H_{n-1}(M^{2n}) \xrightarrow{\alpha_{n-1}} H_{n-1}(W) \oplus H_{n-1}(W) \xrightarrow{\alpha'_{n-1}} H_{n-1}(d(W)) \xrightarrow{\alpha''_{n-1}} 0.$$

By using the arguments presented in the proof of Theorem 5.1, we conclude that

$$
\begin{aligned}
&[\mathbf{h}'_{3j}, \mathbf{h}_{3j}] = 1,\\
&[\mathbf{h}'_{3j+1}, \mathbf{h}_{3j+1}] = 1,\\
&[\mathbf{h}'_{3j+2}, \mathbf{h}_{3j+2}] = 1.
\end{aligned} \tag{6.7}
$$

Combining equations (6.4),(6.5), (6.6), and (6.7), we get

$$
\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{6n+5}, \{0\}_{p=0}^{6n+5}\right) = \prod_{p=0}^{6n+5} [\mathbf{h}'_p, \mathbf{h}_p]^{(-1)^{(p+1)}} = 1. \tag{6.8}
$$

Note that the bases in the sequence (6.3) are compatible. By Theorem 4.1 and Lemma 4.1, the following formula holds

$$
\mathbb{T}\left(W, \{\mathbf{h}_\nu^W\}_{\nu=0}^{2n+1}\right)^2 = \mathbb{T}\left(M^{2n}, \{\mathbf{h}_i^{M^{2n}}\}_{i=0}^{2n}\right)\ \mathbb{T}\left(d(W), \{\mathbf{h}_\nu^{d(W)}\}_{\nu=0}^{2n+1}\right)\ \mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{6n+5}\right). \tag{6.9}
$$

From Theorem 4.4, and equation (6.8), and equation (6.9) it follows

$$
\left|\mathbb{T}\left(W, \{\mathbf{h}_\nu^W\}_{\nu=0}^{2n+1}\right)\right| = \left|\mathbb{T}\left(M^{2n}, \{\mathbf{h}_i^{M^{2n}}\}_{i=0}^{2n}\right)\right|^{1/2}. \tag{6.10}
$$

Corollary 5.1 and equation (6.10) finish the proof of Corollary 6.3. □

6.3. Product of Closed π-Manifolds

Corollary 6.4 *Let M_1^{2n} and M_2^{2n} be $(n-2)$-connected $2n$-dimensional closed π-manifolds ($n \geq 3$) such that $H_{n-1}(M_1^{2n};\mathbb{Z})$ and $H_{n-1}(M^{2n};\mathbb{Z})$ have no torsion. Assume that $\kappa(M_1^{2n}) = \kappa(M_2^{2n}) = 0$ if $n = 4k+3$. Let M be a compact, orientable, smooth $(2n+1)$-dimensional manifold with boundary $\partial(M) = M_2^{2n}$. Consider the product manifold $W = M_1^{2n} \times M$ and* *its double $d(W)$. Then there exists the natural short exact sequence of the chain complexes*

$$0 \to C_*(M_1^{2n} \times M_2^{2n}) \to C_*(W) \oplus C_*(W) \to C_*(d(W)) \to 0 \qquad (6.11)$$

and corresponding the Mayer-Vietoris sequence $\mathcal{H}_*$ corresponding to (6.11). Let $\mathbf{h}_i^W$, $\mathbf{h}_i^{d(W)}$, $\mathbf{h}_k^{M_1^{2n}}$, and $\mathbf{h}_k^{M_2^{2n}}$ be given bases for $i = 0, \cdots, 4n+1$, $k = 0, \ldots, 2n$. Let $\mathbf{h}_\nu^{M_1^{2n} \times M_2^{2n}}$ denote the basis $\bigoplus_j (\mathbf{h}_j^{M_1^{2n}} \otimes \mathbf{h}_{\nu-j}^{M_2^{2n}})$ of $H_\nu(M_1^{2n} \times M_2^{2n})$, $\nu = 0, \ldots, 4n$. For $p = 0, \ldots, 12n+5$, let $\mathbf{h}_p$ be the corresponding basis of $\mathcal{H}_*$. Then the following formula holds

$$\begin{aligned}\left|\mathbb{T}\left(W, \{\mathbf{h}_i^W\}_{i=0}^{4n+1}\right)\right| &= \left|\mathbb{T}\left(M_1^{2n}, \{\mathbf{h}_k^{M_1^{2n}}\}_{k=0}^{2n}\right)\right|^{\chi(M_2^{2n})/2} \left|\mathbb{T}\left(M_2^{2n}, \{\mathbf{h}_k^{M_2^{2n}}\}_{k=0}^{2n}\right)\right|^{\chi(M_1^{2n})/2} \\ &\quad \times \left|\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{12n+5}\right)\right|^{1/2}.\end{aligned}$$

Proof Since the bases in the sequence (6.11) are compatible, Theorem 4.1 and Lemma 4.1 yield

$$\begin{aligned}\mathbb{T}\left(W, \{\mathbf{h}_i^W\}_{i=0}^{4n+1}\right)^2 &= \mathbb{T}\left(M_1^{2n} \times M_2^{2n}, \{\mathbf{h}_\nu^{M_1^{2n} \times M_2^{2n}}\}_{\nu=0}^{4n}\right) \mathbb{T}\left(d(W), \{\mathbf{h}_i^{d(W)}\}_{i=0}^{4n+1}\right) \\ &\quad \times \mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{12n+5}\right). \qquad (6.12)\end{aligned}$$

From Theorem 4.4 and (6.12) it follows that

$$\left|\mathbb{T}\left(W, \{\mathbf{h}_i^W\}_{i=0}^{4n+1}\right)\right| = \left|\mathbb{T}\left(M_1^{2n} \times M_2^{2n}, \{\mathbf{h}_\nu^{M_1^{2n} \times M_2^{2n}}\}_{\nu=0}^{4n}\right)\right|^{1/2} \left|\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{12n+5}\right)\right|^{1/2}. \qquad (6.13)$$

By Theorem 4.5, $\left|\mathbb{T}\left(M_1^{2n} \times M_2^{2n}, \{\mathbf{h}_\nu^{M_1^{2n} \times M_2^{2n}}\}_{\nu=0}^{4n}\right)\right|$ is equal to the product

$$\left|\mathbb{T}\left(M_1^{2n}, \{\mathbf{h}_k^{M_1^{2n}}\}_{k=0}^{2n}\right)\right|^{\chi(M^{2n})} \left|\mathbb{T}\left(M_2^{2n}, \{\mathbf{h}_k^{M_2^{2n}}\}_{k=0}^{2n}\right)\right|^{\chi(M_1^{2n})}. \qquad (6.14)$$

Here, χ is the Euler characteristic. Then equation (6.13) and equation (6.14) finish the proof of Corollary 6.4. □

CHAPTER 7

CONCLUSION

In this thesis, we develop multiplicative gluing formulas for the Reidemeister torsion of closed π-manifolds that admit a connected sum decomposition. Milnor (1966) showed that Reidemeister torsion acts multiplicatively with respect to gluings. Namely, given a closed, oriented, smooth manifold M and an embedded submanifold N splitting M into two submanifolds M_1, M_2: the Reidemeister torsion of M is the product of the Reidemeister torsions of M_1, M_2, and N times a corrective term $\mathbb{T}(\mathcal{H}_*)$ coming from the homologies.

Let $\Sigma_{g,0}$ be a closed orientable genus g surface. So, it is a 0-connected closed π-manifold. Moreover, it admits a connected sum decomposition as $\underset{j=1}{\overset{g}{\#}}(\Sigma_{1,0})$. By using the notion of symplectic chain complex and homological algebra techniques, we obtain a multiplicative gluing formula for the Reidemeister torsion of $\Sigma_{g,0}$ so that the corrective term $\mathbb{T}(\mathcal{H}_*)$ becomes 1. Then we focus on the higher dimensional $(n-2)$ and $(n-1)$-connected closed π-manifolds which have connected sum decompositions given in (Ishimoto, 1969). By using these decompositions, we establish multiplicative gluing formulas for the Reidemeister torsion of such manifolds. As an application, we establish Reidemeister torsion formulas for manifolds such as handlebodies, compact orientable smooth $(2n+1)$-dimensional manifolds whose boundary is a $(n-2)$-connected $2n$-dimensional closed π-manifold, and product manifolds by using the main results in Chapter 5.

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