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ES-*w*-stability

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ABSTRACT

We introduce and study the notion of ES-w-stability for an integral domain R. A nonzero ideal I of R is called *ES-w-stable* if $(I^2)_w = (JI)_w$ for some t-invertible ideal J of R contained in I, and I is called *weakly ES-w-stable* if $I_w = (JE)_w$ for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R. We define R to be an *ES-w-stable domain* (resp., a *weakly ES-w-stable domain*) if every nonzero ideal of R is ES-w-stable (resp., weakly ES-w-stable). These notions allow us to generalize some well-known properties of ES-stable and weakly ES-stable domains.

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1. Introduction

Let R be an integral domain with quotient field K, $\bar{F}(R)$ the set of nonzero R-submodules of K, $F(R)$ the set of nonzero fractional ideals of R, and $f(R)$ the set of finitely generated fractional ideals of R. For $I \in F(R)$, we call I simply an ideal if $I \subseteq R$. For $I, J \in F(R)$, let $(I:_K J) = \{x \in K \mid xJ \subseteq I\}$, then $(I:_K J) \in F(R)$. Hence, if $I^{-1} = (R:_K I)$, then $I^{-1}, I_v = (I^{-1})^{-1}, I_t = \cup\{J_v \mid J \subseteq I$ and $J \in f(R)\}$, and $I_w = \{x \in K \mid xJ \subseteq I$ for some $J \in f(R)$ with $J_v = R\}$ are well-defined nonzero fractional ideals of R. Let $\star = d, w, t$ or v, where $I_d = I$ for all $I \in F(R)$. Then the following properties hold for all nonzero $x \in K$ and $I, J \in F(R)$:

(1) $R_\star = R$ and $(xI)_\star = xI_\star$.
(2) $I \subseteq I_\star$; $I \subseteq J$ implies $I_\star \subseteq J_\star$.
(3) $(I_\star)_\star = I_\star$.
(4) $(IJ)_\star = (I_\star J_\star)_\star = (I_\star J)_\star$ and $(I + J)_\star = (I_\star + J_\star)_\star$.
(5) $(I_\star :_K J_\star) = (I_\star :_K J) = (I_\star :_K J)_\star$.
(6) $I_d \subseteq I_w \subseteq I_t \subseteq I_v$.

A fractional ideal I of R is called a *$\star$-ideal* if $I = I_\star$, and a $\star$-ideal I of R is of *finite type* if $I = J_\star$ for some $J \in f(R)$. A $\star$-ideal is a *maximal $\star$-ideal* if it is maximal among all proper integral $\star$-ideals of R. Let $\star$-Max(R) denote the set of all maximal $\star$-ideals of R; so d-Max$(R) :=$ Max(R) is the set of maximal ideals of R. Each maximal $\star$-ideal is a prime ideal. Two ideals I, J of R are said to be *$\star$-comaximal* if $(I + J)_\star = R$. For all $I \in F(R), I_w = \cap_{P \in t-\mathrm{Max}(R)} IR_P$, hence $I_w R_P = IR_P$

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for each $P \in t\text{-Max}(R)$. Moreover, $(I \cap J)_w = I_w \cap J_w$ for all $I, J \in F(R)$, and a w-ideal I of R is of finite type if and only if $I = J_w$ for some finitely generated ideal J of R contained in I. A fractional ideal I of R is said to be $\star$-*invertible* if there is a $J \in F(R)$ such that $(IJ)_\star = R$. Clearly if $(IJ)_\star = R$ for some $J \in F(R)$, then $J_\star = I^{-1}$. Also, $t\text{-Max}(R) = w\text{-Max}(R)$ [3, Corollary 2.17], and $(II^{-1})_t = R \iff (II^{-1})_w = R$; so the t-invertibility is identical to the w-invertibility. An integral domain R is said to be a *Prüfer domain* (resp., *Prüfer v-multiplication domain* (for short PvMD)) if every nonzero finitely generated ideal of R is invertible (resp., t-invertible). It is known that R is a PvMD if and only if R is integrally closed and $t = w$ on R [42, Theorem 3.5]. Also, R is a Prüfer domain if and only if R is a PvMD whose nonzero maximal ideals are t-ideal. We say that R is of *finite t-character* if every nonzero nonunit of R is contained in only finitely many maximal t-ideals of R. Noetherian domains and Krull domains (i.e., integral domains in which each nonzero ideal is t-invertible) are domains of finite t-character. An ideal I of R is said to be *t-locally principal* if IR_P is principal for all maximal t-ideals P of R. A *t-LPI domain* is an integral domain in which every nonzero t-locally principal t-ideal is t-invertible. An integral domain of finite t-character is t-LPI [5, Lemma 2.2], and a PvMD R is of finite t-character if and only if R is a t-LPI domain [53, Proposition 5].

Sally and Vasconcelos defined a Noetherian ring R to be *SV-stable* if each nonzero ideal of R is projective over its endomorphism ring $\text{End}_R(I)$ [51]. The notion of stability is studied in [4] for arbitrary integral domains; an integral domain R with quotient field K is *SV-stable* if each nonzero ideal I of R is invertible in the overring $\text{End}_R(I) = (I{:}_K I)$, an overring of R means a subring of K containing R. For references about stable domains, the reader may consult [49, 50]. In [30], the notion of $\star$-stability with respect to a semistar operation $\star$ is introduced. We recall that a *semistar operation* on an integral domain R is a map $\star : \bar{F}(R) \to \bar{F}(R)$ such that for each $E, F \in \bar{F}(R)$ and for each nonzero $x \in K$, $(xE)_\star = xE_\star$; $E \subseteq F$ implies $E_\star \subseteq F_\star$; $E \subseteq E_\star$, and $(E_\star)_\star = E_\star$. When $R_\star = R$, the restriction of $\star$ to $F(R)$ is called a *star operation* on R. The reader is referred to [34, Section 32] for more properties of star operations. Consider the overring $T := (I_\star : I_\star)$ of R. Since $T_\star = T$, the restriction of $\star$ to the set of the T-submodules of K is a star operation on T, denoted by $\dot{\star}$. As in [30], we say that a nonzero fractional ideal I of R is $\star$-*stable* if $I_\star$ is $\dot{\star}$-invertible in T, and R is called $\star$-*stable* if every nonzero (fractional) ideal of R is $\star$-stable. It is clear that $\star$-invertible ideals are $\star$-stable. In [18], another type of stability, *ES-stability*, is introduced for local rings. In an integral domain R, an ideal I is called *ES-stable* if $I^2 = IJ$ for some invertible ideal J of R such that $J \subseteq I$, and R is called an *ES-stable domain* if each nonzero ideal of R is an ES-stable ideal. It is known that if I is a nonzero ES-stable ideal of R, then I is stable [25, Lemma 7.4.1]. In [47], a weak form of ES-stability for integral domains is defined. An ideal I of an integral domain R is said to be a *weakly ES-stable ideal* if there is an invertible fractional ideal J and an idempotent fractional ideal E of R such that $I = JE$. Recentley, the concepts of SV-stability, ES-stability and weakly ES-stability are extended to commutative rings with zero-divisors in [7, 8].

The purpose of this paper is to study w-operation analogue of some facts that have been proven for ES-stable and weakly ES-stable domains in [8, 47]. A nonzero ideal I of an integral domain R is called *weakly ES-w-stable* if $I_w = (JE)_w$ for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R. We define R to be a *weakly ES-w-stable domain* if every nonzero ideal of R is weakly ES-w-stable. An ideal I of R is called *ES-w-stable* if $(I^2)_w = (JI)_w$ for some t-invertible ideal J of R such that $J \subseteq I$; and R is called an *ES-w-stable domain* (resp., a *finitely ES-w-stable domain*) if every nonzero (resp., finitely generated) ideal of R is ES-w-stable. We say that an integral domain R has the *w-local stability property* if each nonzero fractional ideal I of R that is t-locally stable (i.e., IR_P is stable, for each $P \in t\text{-Max}(R)$) is indeed w-stable. More precisely, in Section 2, we prove preliminary results for weakly ES-w-stable and ES-w-stable domains and investigate when these two concepts coincide. In Section 3, we show that if (a) R is a completely integrally closed PvMD of finite t-character or (b) R is a weakly Matlis PvMD, then

R is a weakly ES-w-stable domain if and only if R is t-locally weakly ES-stable, that is, R_P is weakly ES-stable for all $P \in t\text{-Max}(R)$. In Section 4, we investigate the transfer of the weakly ES-w-stability to polynomial rings and pullback constructions. In Section 5, we focus on integral domains in which each finitely generated ideal is weakly ES-w-stable, and we show that any finitely weakly ES-w-stable domain with w-local stability property is of finite t-character.

2. ES-w-stability

Let R be an integral domain. A nonzero ideal I of R is called an *ES-w-stable ideal* if $(I^2)_w = (IJ)_w$ for some t-invertible ideal J of R contained in I, and R is called an *ES-w-stable domain* if each nonzero ideal of R is ES-w-stable. The class of ES-w-stable domains includes ES-stable domains and Krull domains. However, an ES-w-stable domain need not be ES-stable. Take, for instance, $D = K[X, Y]$ where K is any field and X, Y are two indeterminates over K. Then D is a non-Prüfer Krull domain and hence D is an ES-w-stable domain that is not ES-stable by Mimouni [47, Theorem 4.1].

Proposition 2.1. *Let R be an integral domain and I a nonzero ideal of R.*

(1) *If I is an ES-w-stable, then I is a w-stable ideal.*
(2) *Let I be a w-stable ideal. Then I is ES-w-stable if one of the following conditions is satisfied:*
  (a) *R is a PvMD.*
  (b) *$R = (I_w : I_w)$ (in particular, if R is completely integrally closed).*

Proof. (1) Let $(I^2)_w = (IJ)_w$ for some t-invertible ideal J of R contained in I. Then $(I^2J^{-1})_w = I_w$. Hence, $(IJ^{-1})_w \subseteq (I_w : I) = (I_w : I_w)$. On the other hand, if $xI_w \subseteq I_w$, then $xJ_w \subseteq I_w$, and so $x \in (IJ^{-1})_w$. Therefore, $(IJ^{-1})_w = (I_w : I_w)$ and hence $(I_w(J^{-1}(I_w : I_w)))_w = (I_w : I_w)$.
(2) (a) Since I_w is $\dot{w}$-invertible in $(I_w : I_w)$, I_w is $\dot{w}$-finite in $(I_w : I_w)$ by Kang [42, Proposition 2.6]. Hence, there exists a finitely generated ideal J of R contained in I such that $I_w = (J(I_w : I_w))_w$. Thus, $(I^2)_w = (IJ(I_w : I_w))_w = (IJ)_w$, where J is t-invertible.
(b) Trivial since I is t-invertible. □

Corollary 2.2. *An integral domain R is ES-w-stable if and only if R_P is ES-stable for each $P \in t\text{-Max}(R)$ and R is of finite t-character if one of the following conditions is satisfied:*

(a) *R is a PvMD.*
(b) *R is a completely integrally closed domain. In particular, R is a Krull domain if and only if R is a completely integrally closed ES-w-stable domain.*

Proof. (a) follows from Proposition 2.1, [25, Lemma 7.4.1] and [30, Corollary 1.10], and (b) follows from Proposition 2.1 and [30, Corollaries 1.10 and 2.5]. □

Let R be an integral domain with quotient field K. A nonzero ideal I of R is called a *weakly ES-w-stable ideal* if $I_w = (JE)_w$ for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R, i.e., $(E^2)_w = E_w$, and R is called a *weakly ES-w-stable domain* if each nonzero ideal of R is weakly ES-w-stable.

Proposition 2.3 is the w-analogue of Mimouni [47, Proposition 2.2 (ii), Lemma 2.4 (i) and Proposition 2.2 (iii)], Corollaries 2.4 and 2.5 are the w-analogues for Mimouni [47, Corollaries 2.5 and 2.6].

Proposition 2.3. *Let R be an integral domain and I a nonzero ideal of R.*

(1) *I is a weakly ES-w-stable ideal if and only if* $(I^2)_w = (JI)_w$ *for some t-invertible ideal J of R.*
(2) *If* $I_w = (JE)_w$ *for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R, then* $(I_w : I) = (E_w : E)$ *and* $E_w = (I(I_w : I^2))_w$.
(3) *I is ES-w-stable if and only if* $I_w = (JE)_w$ *for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R with* $J \subseteq I \subseteq E$.

Proof. (1) Let I be a weakly ES-w-stable ideal. Then $I_w = (JE)_w$ for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R. Hence, $(I^2)_w = ((I_w)^2)_w = (((JE)_w)^2)_w = (J^2E)_w = (I_wJ)_w = (IJ)_w$. For the converse, if $(I^2)_w = (JI)_w$ for some t-invertible ideal J of R, then $I_w = (I_w(JJ^{-1})_w)_w = (JIJ^{-1})_w$ where IJ^{-1} is w-idempotent.

(2) If $xI_w \subseteq I$, then $x(JE)_w \subseteq (JE)_w$. Hence, $(xJ^{-1}(JE)_w)_w \subseteq (J^{-1}(JE)_w)_w$ and so $xE_w \subseteq E_w$. Conversely, if $xE_w \subseteq E$, then $xI_w = x(JE)_w = (xJE)_w = (xJE_w)_w \subseteq (JE)_w = I_w$. Thus, $x \in (I_w : I_w) = (I_w : I)$. To show that $E_w = (I(I_w : I^2))_w$, let $x \in (I_w : I^2)$. Then $x(I^2)_w \subseteq I_w$, hence $x((JE)_w(JE)_w)_w \subseteq (JE)_w$. Thus, $x(J^2E)_w \subseteq (JE)_w$. Since J is t-invertible, $x(JE)_w \subseteq E_w$. Thus, $xI_w \subseteq E_w$, and so $(I_w(I_w : I^2))_w \subseteq E_w$. On the other hand, since I is weakly ES-w-stable, $(I^2)_w = (JI)_w$ for some t-invertible ideal J of R by (1). Hence, $(J^{-1}I^2)_w = I_w$, and so $J^{-1} \subseteq (I_w : I^2)$. Thus, $E_w = (J^{-1}I)_w \subseteq ((I_w : I)I)_w$.

(3) Let I be ES-w-stable. Then $(I^2)_w = (IJ)_w$ for some t-invertible ideal J of R contained in I. Set $E := J^{-1}I$. Then $(E^2)_w = E_w$ and $(JE)_w = I_w$. Since $J \subseteq I, I \subseteq II^{-1} \subseteq IJ^{-1} = E$. The converse follows from (1). □

Corollary 2.4. *Let R be an integral domain and I a nonzero ideal of R. Then I is ES-w-stable if and only if I is w-stable and weakly ES-w-stable. In particular, if R is a Krull domain, then weakly ES-w-stability and ES-w-stability coincide.*

Proof. Assume that I is w-stable and weakly ES-w-stable. Then $I_w = (JE)_w$ for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R. Hence, $E_w = (I(I_w : I^2))_w = (I(I_w : (I_w)^2))_w = (I_w : I_w)$, where the first equality follows from Proposition 2.3, and the last equality follows because I is w-stable. Thus, $I_w = (J(I_w : I_w))_w$. We note that if J is a fractional ideal of R, then $xJ \subseteq R$ for some nonzero $x \in K$. Since $\frac{1}{x}(R : J) = (R : xJ)$, J is t-invertible if and only if xJ is t-invertible. So we may assume $J \subseteq R$. Hence, $J \subseteq (I_w : (I_w : I_w)) = I_w = (JE)_w \subseteq E_w$. By Proposition 2.3, I_w and hence I is ES-w-stable. The converse follows from Propositions 2.1 and 2.3. □

Let $F_w(R) = \{I \in F(R) \mid I_w = I\}$ and $P(R) = \{I \in F(R) \mid I \text{ is principal}\}$. Note that $F_w(R)$ is a commutative semigroup with identity R under the usual ideal multiplication and $P(R)$ is a subsemigroup of $F_w(R)$. We say that the factor semigroup $\mathcal{I}_w(R) = F_w(R)/P(R)$ is the *w-class semigroup* of R. A commutative semigroup S is said to be *Clifford* if every element $s \in S$ is regular (in the sense of Von Neumann), i.e., $s^2a = s$ for some $a \in S$. An integral domain R is called a *Clifford w-regular domain* if $\mathcal{I}_w(R)$ is a Clifford semigroup. In [31, Proposition 1.5], it has been proven that a w-stable domain is Clifford w-regular.

Corollary 2.5. *Let R be a weakly ES-w-stable domain. Then R is a Clifford w-regular domain. In particular, R is of finite t-character.*

Proof. Let I be a nonzero ideal of R such that $I_w = (JE)_w$ for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R. By Proposition 2.3, $E_w = (I(I_w : I^2))_w$ and hence $(IE)_w = (I^2(I_w : I^2))_w$. Also, $(IE)_w = (I_wE_w)_w = ((JE)_wE_w)_w = (JE^2)_w = (JE)_w = I_w$. Hence, I is Clifford w-regular by [31, Lemma 1.2]. Therefore, R is of finite t-character by Gabelli and Picozza [31, Theorem 5.2]. □

We recall that an integral domain R is called *Mori* if the ascending chain condition on v-ideals of R holds; equivalently, each nonzero fractional ideal of R is v-finite. A Mori domain R such that R_P is Noetherian for each maximal t-ideal P of R is called a *strong Mori domain*. They are precisely the domains satisfying the ascending chain condition on w-ideals. Trivially, a Noetherian domain is strong Mori and a strong Mori domain is Mori. The *t-dimension* of R (denoted by t-dim$R$) is defined by $\sup\{\text{ht } P \mid P \in t\text{-Spec}(R)\}$.

Corollary 2.6. *Let R be an integral domain.*

(1) *If R is a Mori weakly ES-w-stable domain, then R is ES-w-stable of t-dimension one.*
(2) *If R is a strong Mori w-stable domain, then R_P is ES-stable for each $P \in t\text{-Max}(R)$.*

Proof. (1) Since a Mori Clifford w-regular domain is w-stable of t-dimension one by Gabelli and Picozza [32, Theorem 4.3], the result follows from Corollaries 2.4 and 2.5.

(2) For each $P \in t\text{-Max}(R)$, R_P is a Noetherian stable domain by Fangui and Casland [56, Theorem 1.9] and [30, Corollary 1.10]. Hence, R_P is ES-stable by Fontana et al. [25, Corollary 7.4.2]. □

We recall that an overring T of R is called *t-linked* if for each nonzero finitely generated ideal I of R, $I^{-1} = R$ implies $(IT)^{-1} = T$. For a nonzero ideal I of R, the overring $T := (I_w : I_w)$ of R is t-linked because $T_w = T$ [16, Proposition 2.13].

Lemma 2.7. *Let R be an integral domain and T a t-linked overring of R. If I is a fractional ideal of R, then $(I_wT)_{w'} = (IT)_{w'}$ where w' denotes the w-operation on T.*

Proof. Let $x \in (I_wT)_{w'}$. Then $xJ \subseteq I_wT$ for some finitely generated ideal J of T with $(T : J) = T$. Pick $j \in J$. Then there exist $a_i \in I_w$ and $t_i \in T$ such that $xj = \sum_{i=1}^{n} a_it_i$. For each $a_i \in I_w$, there exists a finitely generated ideal B_i of R with $B_i^{-1} = R$ such that $a_iB_i \subseteq I$. Set $B = B_1 \cdots B_n$. Then $B^{-1} = R$, and $xJBT \subseteq IT$. Since T is a t-linked overring of R, $(T : BT) = T$, and so $(T : JBT) = T$. Hence, $x \in (IT)_{w'}$. The reverse containment is clear. □

Theorem 2.8. *Let R be a weakly ES-w-stable domain and T a t-linked overring of R. Then T is a weakly ES-w'-stable where w' denotes the w-operation on T.*

Proof. Assume that I is a nonzero ideal of T. Then I is a fractional ideal of R. Let $A := xI$ for some nonzero $x \in R$. Then A is weakly ES-w-stable, so is I. Hence, $I_w = (JE)_w$ for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R. By Lemma 2.7, $I_{w'} = (JTET)_{w'}$ where JT is t'-invertible ideal of T by Baghdadi and Fontana [20, Proposition 3.2] and $(ET)_{w'} = (E^2T)_{w'}$. □

Corollary 2.9. *Let R be an ES-w-stable domain and T a t-linked overring of R. Then T is ES-$\dot{w}$-stable.*

Proof. If R is ES-w-stable, then $\dot{w} = w'$ and T is $\dot{w}$-stable by Gabelli and Picozza [30, Corollary 2.2]. Hence, the result follows from Theorem 2.8 and Corollary 2.4. □

Recall from [14] that the *w-integral closure* of R is the integrally closed overring of R defined by $R^w = \cup\{(I_w : I_w) \mid I \in f(R)\}$. We say that R is w-integrally closed if $R^w = R$. Clearly $R \subseteq \bar{R} \subseteq R^w \subseteq \tilde{R}$, where $\bar{R}$ (resp., $\tilde{R}$) is the integral closure (resp., the complete integral closure) of R. Let X be an indeterminate over an integral domain R. A nonzero prime ideal Q of $R[X]$ is called an *upper to zero* if $Q \cap R = 0$. A *UMt domain* is an integral domain R in which every upper to zero in $R[X]$ is a maximal t-ideal (hence t-invertible).

Corollary 2.10. *Let R be a weakly ES-w-stable domain. Then the complete integral closure $\tilde{R}$ (resp., the w-integral closure R^w of R) is a Pv'MD where v' denotes the v-operation on $\tilde{R}$ (resp., R^w).*

Proof. By [14, Lemma 1.2] and Dobbs et al. [16, Corollary 2.3], R^w and $\tilde{R}$ are t-linked overrings of R. Hence, the results follow from Theorem 2.8 and the facts that a weakly ES-w-stable domain is a UMt domain because Clifford w-regular domains are UMt [32, Proposition 3.9], and an integrally closed UMt domain is a PvMD [38, Proposition 3.2]. □

The next example shows that the concept of w-stability and ES-w-stability do not necessarily coincide.

Example 2.11. Let T be a Krull domain which is not Noetherian and which has a maximal ideal M such that T_M is Noetherian. Let $K = T/M$ and k be a proper subfield of K such that $[K : k]$ is finite. (To see a concrete example of T, let p be a prime number. Then there is a non-finitely generated abelian group G of rank two such that each rank one subgroup of G is cyclic and such that G/H is a p-group for some finitely generated subgroup H of G (see [28, Chapter XIII, Section 88]). Let K be a field of characteristic distinct from p, and let $T = K[X; G]$ be the group ring of G over K. Then T is a UFD by Gilmer [35, Theorem 1] which is not Noetherian since G is a non-finitely generated abelian group. Consider a maximal ideal M of T which is generated by $\{1 - X^g \mid g \in G\}$. Then MT_M is finitely generated by Gilmer [35, Theorem 3], and it implies that T_M is Noetherian [39, Proposition 4].)

Let $R = \phi^{-1}(k)$ be the pullback issued from the following diagram:

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ \phi\ } & K. \end{array}$$

Then R is a strong Mori domain [45, Theorem 3.11] which is neither Noetherian nor Krull. Since M is the largest common ideal of R and T, R and T have the same quotient field and hence the same complete integral closure by [33, Lemma 5]. Since $\tilde{R}$, the complete integral closure of R, is a Krull domain by Fangui and Casland [56, Theorem 3.5], we may assume that $T = \tilde{R}$. Furthermore, M is a maximal t-ideal of R such that $(R : M) = (M : M)$ by [36, Corollaries 3 and 5]. Since $(R : \tilde{R}) = M, (R : M) = \tilde{R}$. Hence, M is a non t-invertible ideal of R and $MM^{-1} = M = M\tilde{R}$. Also, any maximal t-ideal of R distinct from M is t-invertible. To see this, let $N \neq M$ be a maximal t-ideal of R which is not t-invertible. Then $(R : N) = (N : N)$ and $(N : N) \subset \tilde{R} = (R : M)$. It follows that $M = M_v \subset N_v = N$; a contradiction. Now, we claim that M is a w-stable ideal which is not ES-w-stable. Since $\tilde{R}$ is a Krull domain, M is t-invertible in $\tilde{R} = (M : M)$. Thus, M is a w-stable ideal of R. Suppose on the contrary that M is a weakly ES-w-stable ideal. Then $M = (JE)_w$ for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R. By Proposition 2.3, $E_w = (E_w : E_w) = (M : M)$. Since $\tilde{R}$ is a t-linked overring of R, $(M\tilde{R}J^{-1})_w = \tilde{R}$ and hence $(MJ^{-1})_w = (MM^{-1}J^{-1})_w = (M\tilde{R}M^{-1}J^{-1})_w = \tilde{R}$. Therefore, $(J^{-1})_w = (M^{-1}J^{-1})_w$, which implies that $R = \tilde{R}$; a contradiction because a completely integrally closed Mori domain is Krull.

Remark 2.12. *By [14, Corollary 1.4], $R \subseteq R^w$ satisfies (w, w')-INC property (i.e., if whenever Q_1 and Q_2 are nonzero prime ideals of R^w such that $Q_1 \cap R = Q_2 \cap R$ and $(Q_1 \cap R)_w \subsetneq R$, then Q_1 and Q_2 are incomparable) and (w, w')-LO property (i.e., for each prime w-ideal P of R, then there exists a prime w'-ideal Q of R^w such that $P = Q \cap R$). Therefore, if P is a maximal t-ideal of R, then there exists a prime w'-ideal Q of R^w such that $P = Q \cap R$. Assume that $Q' \in t'$-Max(R^w)*

such that $Q \subsetneq Q'$. Then $P = Q \cap R \subsetneq Q' \cap R$. Since R^w is a t-linked overring of R, $(Q' \cap R)_t \neq R$. Hence, $P \subsetneq (Q' \cap R)_t \subsetneq R$; a contradiction. Thus, $Q = Q'$.

As in [17], an integral domain R with quotient filed K is said to be *conducive* if $(R : T) \neq 0$ for each overring T of R with $T \neq K$. Valuation domains are conducive, and the complete integral closure of a conducive domain is a valuation domain [9, Theorem 3.3]. Conducive domains may have infinitely many maximal ideals.

Proposition 2.13. *Let R be a conducive domain which is weakly ES-w-stable. Then R is a weakly ES-stable domain.*

Proof. Let R^w be the w-integral closure of R and v' the v-operation on R^w. Then R^w is a Pv' MD which is a w-integrally closed conducive domain by Corollary 2.10, [14, Corollary 1.4], and [17, Lemma 2.0]. By Remark 2.12, it is enough to show that the set of maximal t'-ideals of R^w is finite. Without loss of generality, we assume that $R = R^w$ is a PvMD. Let M be a maximal t-ideal of R, then R_M is a valuation domain by Kang [42, Theorem 3.2]. Hence, $(R : R_M) \neq 0$ by [17, Lemma 2.0]. Thus, there exists a nonzero prime ideal P of R such that $P = PR_P = PR_M$ by [17, Lemma 2.10]. Since $(R_M)_{PR_M}$ is a valuation domain, PR_M is a prime t-ideal of R_M. Hence, $P_t = (PR_M \cap R)_t \neq R$ since R_M is a t-linked overring of R [16, Proposition 2.2]. Thus, there is a maximal t-ideal Q of R such that $P \subseteq P_t \subseteq Q$. Let N be an arbitrary maximal t-ideal of R such that $N \neq Q$. Then there is a $a \in N \setminus Q$. Hence, for each $x \in P$, $\frac{x}{a} \in PR_Q = P$ which implies $x \in aP \subseteq NP \subseteq N$. Therefore, P is contained in all maximal t-ideals of R. Since R is of finite t-character by Corollary 2.5, the set of maximal t-ideals of R is finite. Therefore, R is a semi-local domain with each maximal ideal a t-ideal by Zafrullah [54, Proposition 3.5]. Hence, the d- and w-operations coincide in R by [13, Corollary 1.3], and R is a weakly ES-stable domain. □

The next theorem is the w-operation analogue of [52, Theorem 2.6] that an integral domain R is a stable domain if and only if R is Clifford regular and every nonzero idempotent fractional ideal of R is a ring.

Theorem 2.14. *An integral domain R is a w-stable domain if and only if R is Clifford w-regular and w-closure of each nonzero w-idempotent fractional ideal of R is a ring.*

Proof. Assume that R is a w-stable domain. Then clearly R is Clifford w-regular. Let I be a nonzero w-idempotent fractional ideal of R. Consider the overring $T := (I_w : I_w)$ of R. Then $T = (I(T : I_w))_w = (I(I_w : I_w))_w = I_w$. Hence, I_w is a ring. For the converse, let I be a nonzero ideal of R. Then $I_w = (I^2(I_w : I^2))_w$ since R is a Clifford w-regular domain. Set $L := I(I_w : I^2)$. Then L is a w-idempotent fractional ideal of R. By assumption, L_w is a ring and hence $L_w = (L_w : L_w)$. Since $I_w = (IL)_w$, clearly $(L_w : L_w) = (I_w : I_w)$. Hence, $(I((I_w : I_w) : I_w))_w = (I(I_w : I^2))_w = L_w = (I_w : I_w)$. Therefore, I is w-stable. □

Corollary 2.15. *Assume that R is an integral domain such that w-closure of each nonzero w-idempotent fractional ideal of R is a ring. Then R is a weakly ES-w-stable domain if and only if R is ES-w-stable.*

Proof. Let R be a weakly ES-w-stable domain. Then R is Clifford w-regular by Corollary 2.5. Hence, R is a w-stable domain by Theorem 2.14. Therefore, R is ES-w-stable by Corollary 2.4. The converse follows from Proposition 2.3(1). □

Proposition 2.16. *Let R be a PvMD of finite t-character and I a nonzero ideal of R. Then I is a weakly ES-w-stable ideal if and only if there is a t-invertible fractional ideal J of R such that either $I_w = (J(I_w : I_w))_w$ or $I_w = (JP_1 \cdots P_n(I_w : I_w))_w$, where P_i is a nonzero w-idempotent prime t-ideal of R.*

Proof. Assume that I is a weakly ES-w-stable ideal. Then $I_w = (JE)_w$ for some t-invertible fractional ideal J of R and w-idempotent fractional ideal E of R. Set $T := (I_w : I_w)$ and let $E \subsetneq T$. Since R is a PvMD of finite t-character, R is a Clifford w-regular domain by Gabelli and Picozza [31, Corollary 4.5]. Thus, T is a PvMD of finite t-character and $w = t = t' = w'$ on T, where w' and t' denote respectively the w-operation and the t-operation on T by [22, Proposition 1.5] and [31, Corollary 2.6 and Theorem 5.2]. Hence, $E_w = (Q_1 \cdots Q_n)_w$, where Q_i is a nonzero w-idempotent prime t-ideal of T by [27, Corollary 3.7]. Set $P_i := Q_i \cap R$. Then $Q_i = (P_i T)_t$ by [43, Proposition 2.5 and Corollary 2.11]. Therefore, $E_w = (P_1 \cdots P_n T)_w$, where P_i is a nonzero w-idempotent prime t-ideal of R by [41, Lemma 2.3]. □

Following [19], an integral domain R is said to be *strongly t-discrete* if it has no t-idempotent prime t-ideals, i.e., for every prime t-ideal P of R, $(P^2)_t \subsetneq P$.

Corollary 2.17. *Let R be a strongly t-discrete PvMD. Then R is a weakly ES-w-stable domain if and only if R is ES-w-stable.*

Proof. Let R be a weakly ES-w-stable domain. Then R is of finite t-character by Corollary 2.5. Since R has no w-idempotent prime t-ideals, $I_w = (J(I_w : I_w))_w$ for some t-invertible fractional J of R by Proposition 2.16. Thus, I is ES-w-stable by Proposition 2.3(3). The converse follows from Proposition 2.3(1). □

3. Some results on *t*-locally weakly ES-stability

Let R be an integral domain. We say that R is *t-locally weakly ES-stable* if R_P is weakly ES-stable for each $P \in t\text{-Max}(R)$. It is clear that if R is a weakly ES-w-stable domain, then R is t-locally weakly ES-stable. However, Example 2.11 shows that a t-locally weakly ES-stable ideal in a domain of finite t-character need not be weakly ES-w-stable in general. We introduce a tool. Let $J \neq 0$ be an ideal of a valuation domain R. We associate a prime ideal $J^\sharp$ as follows. First, set $U(J) = \{r \in R | rJ = J\}$ is a submonoid of the group of units of R. We define $J^\sharp = R - U(J) = \{r \in R | rJ \subset J\}$.

Lemma 3.1. *Let $I \neq 0$ be an ideal and P a maximal t-ideal of a PvMD R. For a prime ideal L of R, the following are equivalent.*

(1) $LR_P = (IR_P)^\sharp$.
(2) $R_L = \text{End}_R(IR_P) = \text{End}_{R_P}(IR_P)$.
(3) *L is the smallest prime t-ideal of R contained in P such that $IR_L = IR_P$.*
(4) $(IR_P : I) = R_L$.

Proof. Since R_P is a valuation domain, (1) $\iff$ (2) follows easily (see [29, Chapter II, Section 4] for details). Clearly, (2) $\iff$ (3) and (2) $\iff$ (4) hold. □

Remark 3.2. *Let I be an ideal and P a maximal t-ideal of a PvMD R. From now on, we use the notation $Z_P(I)$ for the uniquely determined prime t-ideal L of R in the preceding lemma. We observe the following.*

(1) *Clearly, $Z_P(I) \subseteq P$. By Lemma 3.1(3), $Z_P(I) = P$ if I is not contained in P.*
(2) *Let Q be a prime ideal of R such that $Q \subset Z_P(I)$. Then there exists $q \in R_{Z_P(I)}$ such that $PR_P \subseteq q^{-1}IR_P \subseteq R_P$. Hence, $(q^{-1}IR_P)_{QR_P} = R_P$. So, $IQR_M \subset IR_P$ and $IR_Q = qR_Q$ for $q \in R_P$.*
(3) *Let $Z_P(I) \subseteq Q \subseteq P$. By Lemma 3.1(2), $IR_P = IR_Q = IR_{Z_P(I)}$.*

We will use the symbol $P \wedge P'$ to denote the largest prime t-ideal contained in the prime t-ideals P and P'; this makes sense since t-Spec(R), the set of all prime t-ideals of R, is a tree under inclusion by [48, Proposition 4.4]. We observe that $R_P R_{P'} = R_{P \wedge P'}$.

Lemma 3.3. *Let I be an ideal of a PvMD R and P, Q are distinct maximal t-ideals of R. Then $P \wedge Z_Q(I) = Q \wedge Z_P(I)$.*

Proof. We claim $P \wedge Z_Q(I) \subseteq Z_P(I)$. Suppose that $P \wedge Z_Q(I) = Z_Q(I)$. So, $Z_Q(I) \subseteq P$. Since both $Z_P(I)$ and $Z_Q(I)$ are contained in the same maximal t-ideal and t-Spec(R) is linearly ordered by inclusion, without loss of generality, assume that $Z_P(I) \subsetneq Z_Q(I)$. By Remark 3.2(2), $IR_{Z_P(I)} = qR_{Z_P(I)}$, and by Remark 3.2(3), $IR_{Z_P(I)} = IR_{Z_Q(I)}$. Since $IR_{Z_P(I)}$ and $IR_{Z_Q(I)}$ are fractional ideals of R_Q, $Z_P(I)R_Q = Z_Q(I)R_Q$ by Lemma 3.1(1) so that $Z_P(I) = Z_Q(I)$ by Lemma 3.1(3), which is a contradiction. Now let $A = Z_Q(I) \wedge P \subset Z_Q(I)$ and $Z_P(I) \subsetneq A$. Since $A \subset P, IR_A = qR_A = IR_{Z_P(I)}$ by Remark 3.2(2,3). Thus, $A = Z_P(I)$, a contradiction. Hence, we are done. □

Lemma 3.4. *If I is a fractional ideal in a PvMD R, then*

$$\mathrm{End}(I_w) = (I_w : I_w) = \bigcap_{P \in t-\mathrm{Max}(R)} R_{Z_P(I)}.$$

Proof. Clearly, we have

$$(I_w : I_w) = \left(\bigcap_{P \in t-\mathrm{Max}(R)} IR_P : I_w \right) = \bigcap_{P \in t-\mathrm{Max}(R)} (IR_P : I_w) = \bigcap_{P \in t-\mathrm{Max}(R)} R_{Z_P(I)};$$

the last equality follows from Lemma 3.1. □

Lemma 3.5. *Let R be a PvMD of finite t-character and I an ideal of R. Set $T = (I_w : I_w) = \mathrm{End}(I_w)$. Then the following hold.*

(1) $TR_P = R_{Z_P(I)}$ for all $P \in t$-Max(R).
(2) *The maximal t-ideals of T are precisely the t-ideals XT where X ranges over the maximal members of the set $Z = \{Z_P(I) | P \in t\text{-Max}(R)\}$.*

Proof. (1) If $\Omega(I) = \{P_1, ..., P_n\}$ is the set of all maximal t-ideals containing I, then by Lemma 3.4 we have $T = \cap_{Q \notin \Omega(I)} R_Q \cap R_{Z_{P_1}(I)} \cap ... \cap R_{Z_{P_n}(I)}$. So, $TR_P = R_P$ for all maximal t-ideals such that $P \notin \Omega(I)$ and $TR_P \subseteq R_{Z_P(I)}$ for $P \in \Omega(I)$. Multiply T by IR_P and note that $(\cap_{Q \notin \Omega(I)} R_Q)R_P$ is an overring of the valuation domain R_P so that $(\cap_{Q \notin \Omega(I)} R_Q)R_P = R_L$ for some prime ideal $L \subseteq P$. So,

$$\begin{aligned} TIR_P &= IR_P R_L \cap R_{Z_{P_1}(I)} IR_P \cap ... \cap R_{Z_P(I)} IR_P \cap ... \cap R_{Z_{P_n}(I)} \\ &= IR_P R_L \cap IR_P R_{Z_{P_1}(I)} \cap ... \cap IR_{Z_P(I) \wedge P} \cap ... \cap IR_P R_{Z_{P_n}(I)} \\ &= IR_P R_L \cap IR_P R_{Z_{P_1}(I)} \cap ... \cap IR_{Z_P(I)} \cap ... \cap IR_P R_{Z_{P_n}(I)} \\ &= IR_P R_L \cap IR_P R_{Z_{P_1}(I)} \cap ... \cap IR_P \cap ... \cap IR_P R_{Z_{P_n}(I)} \\ &= IR_P \end{aligned}$$

Thus, $(IR_P : IR_P) = TR_P = R_{Z_P(I)}$ by Lemma 3.1(2).

(2) A maximal t-ideal of T is of the form PT, where P is a prime t-ideal of R and $T \subseteq R_P$. Let $N \in t$-Max(R) satisfying $P \subseteq N$. Then $Z_N(I)$ is comparable with P, and $Z_N(I)T$ is a proper prime ideal of T. Hence, $Z_N(I)T \subseteq PT \in t$-Max$(T)$. In R, we have $Z_N(I) \subseteq P \subseteq N$. If $N \notin \Omega(I)$, then $N = Z_N(I) = P$. Otherwise, $TR_N = R_{Z_N(I)}$. Since $T \subseteq R_P$ and $R_N \subseteq R_P, R_{Z_N(I)} \subseteq R_P$ implying that $P \subseteq Z_N(I)$. Thus, $P = Z_N(I)$. Conversely, let $P \in t$-Max(R) and $Z_P(I)$ maximal in Z. By virtue of

Lemma 3.4 and the fact that the prime t-ideals of T are exactly the ideals PT where P is a prime t-ideal of R such that $T \subseteq R_P, Z_P(I)$ survives as a prime t-ideal in T. If R contains a prime t-ideal $Z_P(I) \subsetneq P'$, then by the definition of $Z_P(I)$ there is an $r \in P'$ such that $rI = I$. So, $rI_w = I_w$, and hence $r^{-1} \in T$ but $r^{-1} \notin R_{P'}$. Thus, P' does not survive in T. □

Lemma 3.6. *Assume that R is a PvMD of finite t-character. If I and J are t-locally isomorphic ideals of R, then there exists a t-invertible ideal B of $T = (I_w : I_w)$ such that $I_w = (BJ)_w$.*

Proof. We observe that $T = (I_w : I_w) = (J_w : J_w)$. Without loss of generality, suppose that $I \subseteq J$. Let $\Omega(I) = \{P_1, ..., P_n\}$ be the set of maximal t-ideals of R which contain I. Hence, $\Omega(J) \subseteq \Omega(I)$. By hypothesis, for every $i = 1, ..., n$, we can write $IR_{P_i} = a_iJR_{P_i}$ for some $a_i \in R_{P_i}$, in deed, we may assume that $a_i \in R$. By Lemma 3.1, $IR_{P_i} = IR_{Z_{P_i}(I)} = a_iJR_{Z_{P_i}(I)}$. Let

$$B = T \cap a_1R_{Z_{P_1}(I)} \cap ... \cap a_nR_{Z_{P_n}(I)}.$$

We observe that $a_iR_{Z_{P_i}(I)}J_{P_i} = R_{Z_{P_i}(I)}I_{P_i} = I_{P_i}$ by Lemma 3.1 and, for $i \neq j, a_iR_{Z_{P_i(I)}}J_{P_j} = a_iTR_{P_i}J_{P_j} = a_iTJ_{P_j}R_{P_i}$ by Lemma 3.4. Also, for all maximal t-ideals P, $TJR_P = TJ_wR_P = J_wR_P = JR_P$. Thus, $BJ_{P_i} = J_{P_i} \cap I_{P_i} \cap \cap_{j \neq i} I_{P_i}R_{P_j} = J_{P_i} \cap I_{P_i}$ by [29, Lemma VI.9.9]. Furthermore, for all maximal t-ideals such that $P \neq P_i$, $TJR_P = R_P$, implying that $BJR_P = (T \cap a_1R_{Z_{P_1}(I)} \cap ... \cap a_nR_{Z_{P_n}(I)})JR_P = TJR_P \cap a_1R_{Z_{P_1}(I)}JR_P \cap ... \cap a_nR_{Z_{P_n}(I)}JR_P$ since JR_P is flat. Thus, $BJR_P = R_P \cap a_1TR_{P_1}JR_P \cap ... \cap a_nTR_{P_n}JR_P$ by Lemma 3.5 implying that $BJR_P = R_P$. Hence, we have

$$\begin{aligned}(BJ)_w &= \bigcap_{P \in t-\mathrm{Max}(R)} BJR_P \\ &= BJ_{P_1} \cap BJ_{P_2} \cap ... \cap BJ_{P_n} \cap \bigcap_{P \neq P_i} BJ_P \\ &= \bigcap_{i=1}^{n} (J_{P_i} \cap I_{P_i}) \cap \bigcap_{P \neq P_i} R_P \\ &= \bigcap_{i=1}^{n} I_{P_i} \cap \bigcap_{P \neq P_i} R_P \\ &= \bigcap_{i=1}^{n} I_{P_i} \cap \bigcap_{P \neq P_i} I_P \\ &= I_w\end{aligned}$$

From Lemma 3.5, we observe that B is an ideal of the overring T. Next we claim that the localizations of B at maximal t-ideals of T (see Lemma 3.5) are principal. If the maximal t-ideal does not contain I, then it is obvious. Let us consider

$$BR_{Z_{P_i}(I)} = a_iR_{Z_{P_i}(I)} \cap \bigcap_{j \neq i} a_jR_{Z_{P_i}(I) \wedge Z_{P_j}(I)}.$$

We observe that $Z_{P_i(I)} \wedge Z_{P_j(I)} \subsetneq Z_{P_i(I)}$, so $JR_{Z_{P_i(I)} \wedge Z_{P_j(I)}} \cong R_{Z_{P_i(I)} \wedge Z_{P_j(I)}}$ by Remark 3.2(2). For $j \neq i, IR_{Z_{P_i}(I) \wedge Z_{P_j}(I)} = a_iJR_{Z_{P_i}(I) \wedge Z_{P_j}(I)} = a_jJR_{Z_{P_i}(I) \wedge Z_{P_j}(I)}$. Hence, $a_ia_j^{-1}$ is a unit in the valuation domain $R_{Z_{P_i}(I) \wedge Z_{P_j}(I)}$ so $a_iR_{Z_{P_i}(I) \wedge Z_{P_j}(I)} = a_jR_{Z_{P_i}(I) \wedge Z_{P_j}(I)}$. Also, $a_iR_{Z_{P_i}(I)} \subseteq a_jR_{Z_{P_i}(I)}$. Therefore, $BR_{Z_{P_i}(I)} = a_iR_{Z_{P_i}(I)}$ for each i. Since T is of finite t-character, B is a t-invertible ideal of T. □

Theorem 3.7. *Let R be a PvMD of finite t-character and I a nonzero ideal of R such that IR_P is weakly ES-stable for each $P \in t$-Max(R). Then there is a t-invertible fractional ideal A of $(I_w : I_w)$ such that $(I^2)_w = (AI)_w$.*

Proof. Let $\{P_1, ..., P_n\}$ be the set of maximal t-ideals of R which contain I. Then $I^2R_{P_i} = J_iR_{P_i}IR_{P_i}$ for some invertible ideal $J_iR_{P_i}$ of R_{P_i} for each $i = 1, ..., n$ by the definition of weakly ES-stability. We observe that these are the only maximal t-ideals which contain I^2, also. For all other maximal

t-ideals $N \neq P_i$, for each i, $I_N^2 = R_N = I_N$. So, for each i, $(I^2)_{P_i} = j_i I_{P_i}$ for some $j_i \in J$. Thus, by Lemma 3.6, there exists a t-invertible ideal A of $(I_w : I_w)$ such that $(I^2)_w = (AI)_w$. □

Corollary 3.8. *Assume that R is a completely integrally closed PvMD of finite t-character. If R is a t-locally weakly ES-stable domain, then R is weakly ES-w-stable.*

Proof. Let R_P be weakly ES-stable for each $P \in t\text{-Max}(R)$ and I a nonzero ideal of R. Then $(I^2)_w = (AI)_w$ for some t-invertible fractional ideal A of $(I_w : I_w)$ by Theorem 3.7. Since R is completely integrally closed, $\tilde{R} = \cup_{I \in F(R)}(I_v : I_v) = R$. Hence, $(I_w : I_w) = R$ and so A is a t-invertible fractional ideal of R. Therefore, I is a weakly ES-w-stable by Proposition 2.3. □

Recall from [6] that an integral domain R is a *weakly Matlis domain* if R is of finite t-character and each prime t-ideal of R is contained in a unique maximal t-ideal. Clearly, Krull domains are weakly Matlis, and an integral domain of t-dimension one is a weakly Matlis domain if and only if it is of finite t-character.

Theorem 3.9. *Assume that R is a weakly Matlis PvMD. If R is a t-locally weakly ES-stable domain, then R is weakly ES-w-stable.*

Proof. Let I be a nonzero ideal of R. Since R is a PvMD, I_w and so I is a w-flat ideal (i.e., IR_P is flat for each $P \in t\text{-Max}(R)$) by [44, Proposition 2]. Since IR_P is weakly ES-stable for each $P \in t\text{-Max}(R)$, $I^2R_P = JR_PIR_P$ for some invertible ideal JR_P of R_P by [8, Proposition 2.1]. Let $\{P_1, \dots, P_n\}$ be the set of maximal t-ideals of R which contain I. Then $I^2R_{P_i} = a_iIR_{P_i}$ for some $a_i \in R$ and $I^2R_P = JR_PIR_P = R_P$ for all maximal t-ideal $P \neq P_i$ for $i = 1, \dots, n$. Set $A_i := a_iR_{P_i} \cap R$ for $i = 1, \dots, n$, and $A := A_1 \cap \dots \cap A_n$. Since R is a weakly Matlis domain, P_i is the unique maximal t-ideal of R which contains A_i and A_i is w-ideal by [6, Corollary 4.4 and Lemma 2.3]. Hence, $AR_P = A_1R_P \cap \dots \cap A_nR_P$ for each $P \in t\text{-Max}(R)$ by [6, Proposition 4.7]. Since IR_P is flat for each $P \in t\text{-Max}(R), AIR_P = A_1IR_P \cap \dots \cap A_nIR_P$ by [29, Chapter VI, Lemma 9.9]. We note that if P is a maximal t-ideal of R such that $P \notin \{P_1, \dots, P_n\}$, then $A_iR_P = R_P$, so $AR_P = R_P$. If $i, j \in \{1, \dots, n\}$ with $j \neq i$, then $A_jR_{P_i} = R_{P_i}$, so $AR_{P_i} = A_iR_{P_i} = a_iR_{P_i}$. Therefore, AR_P is principal for each $P \in t\text{-Max}(R)$ and

$$\begin{aligned}
(AI)_w &= \bigcap_{P \in t-\text{Max}(R)} AIR_P \\
&= AIR_{P_1} \cap \dots \cap AIR_{P_n} \cap \bigcap_{P \neq P_i} R_P \\
&= I^2R_{P_1} \cap \dots \cap I^2R_{P_n} \cap \bigcap_{P \neq P_i} R_P \\
&= (I^2)_w.
\end{aligned}$$

Since R is of finite t-character, R is t-LPI. Therefore, A is t-invertible and hence I is a weakly ES-w-stable ideal by Proposition 2.3. □

Corollary 3.10. *Let R be an integrally closed w-divisorial domain, i.e., the w- and v- operations are the same on R. Then R is weakly ES-w-stable if and only if R is t-locally weakly ES-stable.*

Proof. Since an integrally closed w-divisorial domain is a weakly Matlis PvMD [21, Theorem 3.3], the result follows from Theorem 3.9. □

Let $\star$ be a star operation on an integral domain R. As in [46], we say that R is a *$\star$-RTP domain* if for each nonzero fractional ideal I of R, either $(II^{-1})_\star = R$ or a radical ideal of R.

Example 3.11. Let Y, Z be indeterminates over a field K and let $D := K[Y, Z]$. Consider a multiplicatively closed subset $S = \{1, Y, Y^2, Y^3, \dots\}$ of D, and let $R := D + XD_S[X]$, i.e., $R = \{f \in$

$D_S[X] \mid f(0) \in D\}$. Then R is a non-Prüfer non-Krull weakly Matlis PvMD by [1, Theorem 3.6], [44, Corollary 3], and [2, Corollary 2.8]. Let P be a maximal t-ideal of R. If $P \cap S = \emptyset$, then $R_P = D_S[X]_{PD_S[X]}$ is a DVR. Thus we may assume $P \cap S \neq \emptyset$. Lemma 2.1 of [2] implies $P = P \cap D + XD_S[X]$ such that $P \cap D$ is a maximal t-ideal of D. Hence, $P = YK[Y,Z] + XK[Y, Y^{-1}, Z, X]$. Since D is a Krull domain, the maximal t-ideal (Y) of D is not w-idempotent by Gabelli and Picozza [30, Theorem 2.9]. Hence, $(P^2)_w \neq P$ and so $P^2R_P \neq PR_P$. Since R is a weakly Matlis PvMD, R is t-RTP by [22, Theorem 1.12]. Hence, R_P is RTP by [46, Theorem 17] and so PR_P is divisorial by [40, Theorem 6]. Thus, R_P is a Noetherian valuation domain, and hence it is weakly ES-stable by Mimouni [47, Proposition 4.6]. Therefore, R is a weakly ES-w-stable domain by Theorem 3.9.

4. ES-w-stability of polynomial rings

Let R be an integral domain, $\star$ a star operation on R, X an indeterminate over R, and $R[X]$ the polynomial ring over R. Assume that $c(f)$ is the ideal of R generated by the coefficients of $f \in R[X]$. As in [42], let $N(\star) = \{f \in R[X] \mid f(X) \neq 0 \text{ and } (c_R(f))_\star = R\}$. Then $N(\star)$ is a saturated multiplicative subset of $R[X]$, and the domain $\mathrm{Na}(R,\star) := R[X]_{N(\star)}$ is called the *Nagata ring of R with respect to $\star$*. For $\star = d, \mathrm{Na}(R,d) =: R(X)$ is the usual Nagata ring of R [34, Section 33], and $\mathrm{Na}(R,v) = \mathrm{Na}(R,t) = \mathrm{Na}(R,w)$.

Theorem 4.1. *Let R be an integrally closed domain. Then R is a weakly ES-w-stable domain if and only if $\mathrm{Na}(R,v)$ is a weakly ES-stable domain.*

Proof. Assume that R is a weakly ES-w-stable domain and J is a nonzero ideal of $\mathrm{Na}(R,v)$. Since R is an integrally closed weakly ES-w-stable domain, R is a PvMD. Hence, $J = I\mathrm{Na}(R,v)$ for some ideal I of R by Kang [42, Theorem 3.1]. By assumption, there is a t-invertible ideal A of R such that $(I^2)_w = (IA)_w$. Note that $A\mathrm{Na}(R,v)$ is invertible and the d- and w- operations are the same on $\mathrm{Na}(R,v)$ because each maximal ideal of $\mathrm{Na}(R,v)$ is a t-ideal (cf. [42, Proposition 2.1, Corollaries 2.3 and 2.5]). Thus, $I^2\mathrm{Na}(R,v) = IA\mathrm{Na}(R,v)$ since $\mathrm{Na}(R,v)$ is a PvMD [42, Theorem 3.7]. It follows that J is a weakly ES-stable ideal of $\mathrm{Na}(R,v)$.

Conversely, suppose $\mathrm{Na}(R,v)$ is a weakly ES-stable domain. Then $\mathrm{Na}(R,v)$ is a quasi-Prüfer domain by Mimouni [47, Corollary 2.4]. It follows that R is a PvMD. Let I be a nonzero ideal of R. Then $I^2\mathrm{Na}(R,v) = I\mathrm{Na}(R,v)L$ for some invertible ideal L of $\mathrm{Na}(R,v)$. Note that $L = J\mathrm{Na}(R,v)$ for some ideal J of R which is t-invertible by Kang [42, Corollary 2.7]. Therefore, $(I^2)_w = (I^2)_w\mathrm{Na}(R,v) \cap R = (IJ)_w\mathrm{Na}(R,v) \cap R = (IJ)_w$ by Kang [42, Proposition 2.8]. □

Example 4.2. Let V be a rank one discrete valuation domain with quotient field $K \neq V$, M maximal ideal of V, and X an indeterminate over K. Then $D := V + XK[X]$ is an h-local Prüfer domain by [2, Corollary 2.8]. We first show that each nonzero prime ideal of D is not idempotent. *Let* Q be a prime ideal of D and $S := V \setminus \{0\}$. The case $Q \cap S = \emptyset$ is trivial, so assume $Q \cap S \neq \emptyset$. Then $Q = Q \cap V + XK[X]$ by [15, Theorem 2.1]. If $Q \cap V = 0$, then $Q = XK[X]$ which is not idempotent. If $Q \cap V = M$, then $Q = M + XK[X]$ is a maximal ideal of D [2, Lemma 2.1] which is not idempotent since $M^2 \neq M$. Therefore, D is an h-local strongly discrete Prüfer domain and hence D is an ES-stable domain by Gabelli and Picozza [30, Corollary 3.8]. Now, let Y be an indeterminate over D and $R := D[Y]$. Then R is a non-Krull non-Prüfer weakly Matlis PvMD by Gabelli and Picozza [32, Proposition 3.8] which is not a weakly ES-stable domain by Mimouni [47, Corollary 2.7]. We note that Theorem 2.3(e) of [12] implies

$$t\text{-}\mathrm{Max}(R) = \{Q \in \mathrm{Spec}R \mid Q \cap D = (0)\} \cup \{P[Y] \mid P \in \mathrm{Max}(D)\},$$

since D is a Prüfer domain and hence a UMt domain. Let Q be a maximal t-ideal of R and $P := Q \cap D$. *If* $P \neq 0$, then $R_Q = D[Y]_{P[Y]} = D_P(Y)$ is weakly ES-stable by Theorem 4.1. *If* $P = 0$, then R_Q is a DVR and hence a weakly ES-stable domain by Mimouni [47, Proposition 4.6]. Therefore, R is a weakly ES-w-stable by Theorem 3.9.

Theorem 4.3. *Let R be an integrally closed domain and X an indeterminate over R. Then R is a weakly ES-w-stable domain if and only if $R[X]$ is a weakly ES-w-stable domain.*

Proof. Assume that R is a weakly ES-w-stable domain. Then R is a PvMD since R is integrally closed. Let J be a nonzero ideal of $R[X]$. We may assume that J is a w-ideal. Set $I := J \cap R$. If $I \neq 0$, then $J = I[X]$ by [37, Lemma 4.5]. By assumption, $(I^2)_w = (IA)_w$ for some t-invertible ideal A of R. Thus, by Kang [42, Corollary 2.3], $(J^2)_w = (I^2)_w[X] = (IA)_w[X] = (JA[X])_w$ where $A[X]$ is t-invertible ideal of $R[X]$. Now suppose that $I = 0$. Then $J = fA[X]$ for some $f \in R[X]$ and a fractional t-ideal A of R by [37, Lemma 4.5]. Then $(A^2)_w = (AB)_w$ for some t-invertible ideal B of R. Thus, $(J^2)_w = (f^2A^2[X])_w = f^2(A^2)_w[X] = f^2(AB)_w[X] = (JfB[X])_w$ where $fB[X]$ is a t-invertible ideal of $R[X]$. Therefore, $R[X]$ is a weakly ES-w-stable domain.

Conversely, suppose $R[X]$ is a weakly ES-w-stable domain. It suffices to show that $\mathrm{Na}(R, v)$ is weakly ES-stable domian. Let J be a nonzero ideal of $\mathrm{Na}(R, v)$. Then $J = A\mathrm{Na}(R, v)$ for some ideal A of $R[X]$. By assumption, $(A^2)_w = (AB)_w$ for some t-invertible ideal B of $R[X]$. Note that R is a PvMD by [26, Theorem 2.4]. Thus, by Kang [42, Lemma 3.4], $A^2\mathrm{Na}(R, v) = AB\mathrm{Na}(R, v)$ where $B\mathrm{Na}(R, v)$ is an invertible ideal of $\mathrm{Na}(R, v)$. Therefore, R is a weakly ES-w-stable domain by Theorem 4.1. □

Now we characterize weakly ES-w-stability in pullback constructions. Let T be an integral domain, M a maximal ideal of T, $K = T/M$, D a proper subring of K, $\phi : T \to K$ the canonical homomorphism, and $R = \phi^{-1}(D)$ the pullback of the following diagram:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K. \end{array}$$

We assume that $R \subset T$, and we refer to the diagram as a pullback diagram of type $(\square^*)$ if K is the quotient field of D.

We first give some examples of which D and T are weakly ES-w-stable, but R is not necessarily a weakly ES-w-stable in a pullback diagram.

Example 4.4. (1) Let D be a rank one discrete valuation domain with quotient field K (e.g., a local Dedekind domain that is not a field), X, Y the indeterminates over K. Set $T = K[[X, Y]] = K + M$ where $M = (X, Y)T$. It is well known that the set of maximal t-ideals of a Krull domain is the set of height one primes. Hence, M is not a t-ideal of T. Therefore, $R = D + M$ the pullback of D in the Krull domain T cannot be a weakly ES-w-stable domain because R is not a UMt domain by [26, Proposition 3.5].

(2) Assume that F is a field and F' is a proper subfield of F. For any integer $n > 1$, let $X_1, \ldots, X_n$ be indeterminates over F and set $T = F[[X_1, \ldots, X_n]] = F + M$ where $M = (X_1, \ldots, X_n)T$. Note that T is a Krull domain and M is not a t-ideal of T. Thus, $R = F' + M$ is not a UMt domain by [26, Proposition 3.6]. It follows that R is not a weakly ES-w-stable.

(3) Let $\mathbb{Q}$ be the field of rational numbers, X, Y indeterminates over $\mathbb{Q}$, $T = \mathbb{Q}[X, Y]$ with maximal ideal $M = (X, Y)T$, and $D = \mathbb{Z}$, the ring of integers. Then M is not a t-ideal of T. Hence, $R = \mathbb{Z} + M$ cannot be a weakly ES-w-stable domain because R is not a UMt domain by [26, Proposition 3.5].

Theorem 4.5. *In a pullback diagram of type* ($\square^{\star}$)*, if R is a weakly ES-w-stable domain, then M is a maximal t-ideal of T, T is a weakly ES-w-stable domain, and D is a semi-local weakly ES-stable domain.*

Proof. By [26, Propositions 3.1 and 3.5], T is a t-linked overring of R and M is a maximal t-ideal of T. Hence, M is a prime t-ideal of R and T is weakly ES-w-stable by Theorem 2.8. Furthermore, the set of maximal t-ideals of D is finite because R is of finite t-character by Corollary 2.5, and q is a maximal t-ideal of D if and only if $\phi^{-1}(q)$ is a maximal t-ideal of R containing M by [24, Propositions 1.6 and 1.8]. Therefore, D is a semi-local domain with each maximal ideal a t-ideal by Zafrullah [54, Proposition 3.5]. First, we show that D is a weakly ES-w-stable domain and hence a UMt domain. Let A be a nonzero ideal of D. Then $I=\phi^{-1}(A)$ is an ideal of R containing M. By assumption, $I_w=(JE)_w$ for some t-invertible fractional ideal of R and w-idempotent fractional ideal E of R. Thus, $E_w\subseteq(E_w:E_w)=(I_w:I_w)=((\phi^{-1}(A))_w:(\phi^{-1}(A))_w)=(\phi^{-1}(A_w):\phi^{-1}(A_w))=\phi^{-1}(A_w:A_w)\subseteq\phi^{-1}(K)=T$, where the third equality follows from [45, Lemma 3.1]. Hence, $I_w\subseteq E_w\subseteq T$. Since $M\subsetneq I$, $IT=T$ and hence $E_wT=T$ and $M\subsetneq E_w$. Therefore, $(JT)_w=(JE_wT)_w=((JE_w)_wT)_w=(IT)_w=T$, and hence $M\subsetneq J_w$. Hence, $J_w=\phi^{-1}(B)$ and $E_w=\phi^{-1}(F)$ for some nonzero fractional ideals B and F of D. Clearly, $A_w=(BF)_w$ such that B is a t-invertible fractional ideal of D and F is a w-idempotent fractional ideal of D. Therefore, D is a weakly ES-stable domain by [13, Corollary 1.3]. □

Corollary 4.6. *Let D be an integral domain with quotient field K, X an indeterminate over K and* $R=D+XK[X]$ *the subring of the polynomial ring* $K[X]$ *consisting of those polynomials with constant term in D. If R is a weakly ES-w-stable domain, then D is a semilocal domain which is a weakly ES-stable domain.*

Proof. The result follows from Theorem 4.5. □

In a pullback diagram of type ($\square^{\star}$), since we do not know any example of a weakly ES-w-stable domain T with a maximal t-ideal M, and a semi-local weakly ES-stable domain D such that R is not of finite t-character, we end this section by considering the following question:

Question 4.7. *In a pullback diagram of type* ($\square^{\star}$)*, assume that T is a weakly ES-w-stable domain, M is a maximal t-ideal of T, and D is a semi-local weakly ES-stable domain. Is R a weakly ES-w-stable domain?*

5. Finitely ES-*w*-stable domains

An integral domain R is said to be a *finitely ES-w-stable domain* (resp., *finitely weakly ES-w-stable*) if every finitely generated ideal of R is ES-w-stable (resp., weakly ES-w-stable).

Proposition 5.1. *An integral domain R is finitely weakly ES-w-stable if and only if R is finitely ES-w-stable. In particular, every finitely generated ideal of a weakly ES-w-stable domain is ES-w-stable.*

Proof. Let R be a finitely weakly ES-w-stable domain and I a finitely generated ideal of R. Then IR_P is weakly ES-stable and hence IR_P is stable for each $P\in t\text{-Max}(R)$ by [8, Lemmas 2.4 and 2.6]. Hence, for each $P\in t\text{-Max}(R)$,

$$I((I_w:I):I)R_P=IR_P((I_wR_P:IR_P):IR_P)=(I_wR_P:IR_P)=(I_w:I)R_P.$$

Therefore, I is w-stable and hence I is ES-w-stable by Corollary 2.4. □

Corollary 5.2. *Let R be a Noetherian domain. Then R is weakly ES-w-stable if and only if R is ES-w-stable.*

We recall that an integral domain R is called *finitely w-stable* if each finitely generated ideal of R is w-stable. We say that an ideal I of R is *w-prestable* (resp., *prestable*) if I^n is w-stable (resp., stable) for some integer $n \geq 1$.

Theorem 5.3. *Let R be an integral domain. Then the following statements are equivalent.*

(1) *R is a UMt domain.*
(2) *Each nonzero finitely generated ideal I of R is w-prestable.*

Proof. (1) ⇒ (2) Let R be a UMt and I a nonzero finitely generated ideal of R. Then R_P is a quasi-Prüfer domain for each $P \in t\text{-Max}(R)$ by [13, Theorem 2.16]. Hence, each nonzero finitely generated ideal of R_P is prestable for each $P \in t\text{-Max}(R)$ by Fontana et al. [25, Theorem 7.4.6]. Thus, I^nR_P is stable for some $n \geq 1$. Set $J := I^n$. Hence,

$$\begin{aligned}(J((J_w : J_w) : J_w))_w &= \bigcap_{P\in t-\text{Max}(R)} J((J_w : J_w) : J_w)R_P \\ &= \bigcap_{P\in t-\text{Max}(R)} JR_P((J_w : J_w)R_P : J_wR_P) \\ &= \bigcap_{P\in t-\text{Max}(R)} JR_P((JR_P : JR_P) : JR_P) \\ &= \bigcap_{P\in t-\text{Max}(R)} (JR_P : JR_P) \\ &= \bigcap_{P\in t-\text{Max}(R)} (J_w : J_w)R_P \\ &= (J_w : J_w).\end{aligned}$$

Therefore, I is w-prestable.

(2) ⇒ (1) Let $P \in t\text{-Max}(R)$ and J a nonzero finitely generated ideal of R_P. Then $J = IR_P$ for some finitely generated ideal I of R. By assumption, I^n is w-stable for some integer $n \geq 1$. Hence, J is prestable. Thus, R_P is a quasi-Prüfer domain for each $P \in t\text{-Max}(R)$ by Fontana et al. [25, Theorem 7.4.6], and hence R is UMt by [13, Theorem 2.16]. □

Corollary 5.4. *Let R be an integrally closed domain. Then R is a PvMD if and only if R is a finitely weakly ES-w-stable domain.*

Proof. Assume that R is a finitely weakly ES-w-stable domain. Then R is a finitely ES-w-stable domain by Proposition 5.1. Hence, R is a UMt domain by Theorem 5.3. Thus, R is a PvMD by Houston and Zafrullah [38, Proposition 3.2]. The converse is trivial. □

In [11, Example 2.14], the authors provide an example of a PvMD which is not of finite t-character. Hence, a finitely weakly ES-w-stable domain need not be of finite t-character.

Corollary 5.5. *Let R be an integrally closed domain of finite t-character. If R is a finitely ES-w-stable domain and R_P is ES-stable for each $P \in t\text{-Max}(R)$, then R is ES-w-stable.*

Proof. By Gabelli and Picozza [30, Corollary 1.10], R is a w-stable domain. Hence, R is ES-w-stable by Corollary 5.4 and Proposition 2.1. □

Proposition 5.6. *Let R be an integral domain and T a t-linked overring of R. If R is a finitely ES-w-stable domain, then T is finitely ES-w′-stable where w′ denotes the w-operation on T.*

Proof. Let I be a nonzero finitely generated ideal of T. Then there is a nonzero $c \in R$ and a finitely generated ideal J of R such that $cI = JT$. Since R is a finitely ES-w-stable domain, $(J^2)_w = (JA)_w$ for some t-invertible ideal A of R contained in J. Hence, $(J^2T)_{w'} = (JAT)_{w'}$ by Lemma 2.7. Thus, $(I^2)_{w'} = (c^{-1}IAT)_{w'}$. Since T is a t-linked overring of R, AT is t'-invertible ideal of T by Baghdadi and Fontana [20, Proposition 3.2] where t' denotes the t-operation on T. Hence, $c^{-1}AT \subseteq I$ is t-invertible ideal of T. Therefore, I is ES-w'-stable. □

Corollary 5.7. *Assume that R is a finitely ES-w-stable domain. Then the complete integral closure $\tilde{R}$ of R is a Pv'MD where v' denotes the v-operation on $\tilde{R}$.*

Proof. By Dobbs et al. [16, Corollary 2.3], $\tilde{R}$ is a t-linked overring of R. Hence, $\tilde{R}$ is finitely ES-w'-stable by Proposition 5.6. Since $\tilde{R}$ is integrally closed, $\tilde{R}$ is a Pv'MD by Proposition 5.4. □

We say that an integral domain R has the *w-local stability property* if each nonzero fractional ideal I of R that is t-locally stable (i.e., IR_P is stable, for each $P \in t\text{-Max}(R)$) is indeed w-stable.

Proposition 5.8. *Any integral domain R of finite t-character has the w-local stability property.*

Proof. Let I be a nonzero ideal of R such that IR_P is stable for each $P \in t\text{-Max}(R)$. First, we show that $(I_w : I_w)R_P = (IR_P : IR_P)$ for each $P \in t\text{-Max}(R)$. Let x be a nonzero element of $(IR_P : IR_P)$. Since R is of finite t-character, there exist only finitely many maximal t-ideals $P_1, \ldots, P_n$ of R such that $xR_{P_i} \neq R_{P_i}$. Since IR_{P_i} is stable by assumption, $IR_{P_i} = A_i(IR_{P_i} : IR_{P_i})$ for some finitely generated ideal $A_i \subseteq I$. Hence, $d_i x A_i \subseteq I$ for some $d_i \in R \setminus P_i$. Setting $d = d_1 \cdots d_n$, we have $dxA_i \subseteq I$ for each $i = 1, \ldots, n$. Hence, $dxIR_{P_i} \subseteq IR_{P_i}$. If M is a maximal t-ideal of R such that $M \notin \{P_1, \ldots, P_n\}$, then $dxIR_M = dIR_M \subseteq IR_M$. Thus, $dxI_w \subseteq I_w$ and hence $x = xd.d^{-1} \in (I_w : I_w)R_P$ for each $P \in t\text{-Max}(R)$. With a similar method, we observe that $((I_w : I_w) : I_w)R_P = ((I_w : I_w)R_P : IR_P)$ for each $P \in t\text{-Max}(R)$. Hence, for each $P \in t\text{-Max}(R)$, $I((I_w : I_w) : I_w)R_P = IR_P((I_w : I_w)R_P : IR_P) = IR_P((IR_P : IR_P) : IR_P) = (IR_P : IR_P) = (I_w : I_w)R_P$. Therefore, I is w-stable. □

We recall that a *t-LPI domain* is an integral domain in which every nonzero t-locally principal t-ideal is t-invertible. Recently, several properties of t-LPI domains have been surveyed in [23].

Proposition 5.9. *Any integral domain R with the w-local stability property is a t-LPI domain.*

Proof. Let I be a t-locally principal t-ideal of R. Then $(IR_P : IR_P) = R_P$ for each $P \in t\text{-Max}(R)$. Since I is $\dot{w}$-invertible in $(I : I)$ by assumption, I is $\dot{w}$-finite in $(I : I)$. Hence, $(I : I)R_P = (IR_P : IR_P)$ for each $P \in t\text{-Max}(R)$ by Gabelli and Picozza [30, Lemma 1.8]. Thus,

$$R = \bigcap_{P \in t-\text{Max}(R)} R_P = \bigcap_{P \in t-\text{Max}(R)} (IR_P : IR_P) = \bigcap_{P \in t-\text{Max}(R)} (I : I)R_P = (I : I).$$

Hence, I is t-invertible. □

Lemma 5.10. *Let R be an integral domain and I a nonzero ideal of R. Assume that I is w-stable and T is a t-linked overring of R containing $E = (I_w : I_w)$. Then*

(1) *IT is $\dot{w}$-invertible in T and $(T : IT) = ((E : I)T)_w$.*
(2) *$(TR_Q : (T : IT)R_Q) = (T : (T : IT))R_Q$ for each $Q \in t\text{-Max}(R)$.*

Proof. (1) Since I is $\dot{w}$-invertible in E, we have

$$T = ET = (I(E : I))_w T \subseteq ((I(E : I))_w T)_w = (I(E : I)T)_w \subseteq (IT(T : IT))_w \subseteq T.$$

Therefore, $((E : I)T)_w = (T : IT)$.

(2) Since $(T : IT)$ is $\dot{w}$-finite in T by (1), $(T : IT) = (x_1T + ... + x_kT)_w$ for some $x_1, ..., x_k \in (T : IT)$. So, there exists a nonzero element $d \in R$ such that $dx_i \in R$ for $i = 1, ..., k$. Thus, $H := dx_1R + ... + dx_nR$ is a finitely generated ideal of R such that $(HT)_w = d(T : IT)$. Hence, for each $Q \in t\text{-Max}(R)$,

$$(TR_Q : (T : IT)R_Q) = \left(TR_Q : \frac{1}{d}HTR_Q\right) = (dT : HT)R_Q = (T : (T : IT))R_Q.$$

□

The next theorem is the w-operation analogue of [10, Theorem 4.5] that any finitely stable domain with the local stability property is of finite character.

Theorem 5.11. *Let R be a finitely w-stable domain. Then R has the w-local stability property if and only if R is of finite t-character.*

Proof. If R is of finite t-character, then R has the w-local stability property by Proposition 5.8. For the converse, assume to the contrary that R is not of finite t-character. By [55, Theorem 2.6], there exists a w-ideal I of finite type that is contained in infinitely many pairwise w-comaximal w-ideals of finite type, say, $\{A_m \mid m \in \mathbb{N}\}$. Any w-ideal of finite type is w-stable by assumption. Hence, for each $m \in \mathbb{N}$, $(A_m(T_m : A_m))_w = T_m$ where $T_m := (A_m : A_m)$. Let

$$A = \sum_{m\in\mathbb{N}}(T_m : A_m).$$

Then A is a fractional ideal of R because $I^2A \subseteq (I^2A)_w = (\sum_{m\in\mathbb{N}} I^2(T_m : A_m))_w \subseteq (\sum_{m\in\mathbb{N}} A_m^2(T_m : A_m))_w = (\sum_{m\in\mathbb{N}}(A_m^2(T_m : A_m))_w)_w = (\sum_{m\in\mathbb{N}} A_m)_w \subseteq R$. Also, for each $Q \in t\text{-Max}(R)$,

$$AR_Q = \sum_{m\in\mathbb{N}}(T_mR_Q : A_mR_Q)$$

by Gabelli and Picozza [30, Lemma 1.8]. We claim that A is t-locally stable. Let Q be a maximal t-ideal of R. If $A \not\subseteq Q$, then $AR_Q = R_Q$. Suppose that $A \subseteq Q$. If $A_m \not\subseteq Q$ for all $m \in \mathbb{N}$, then $AR_Q = R_Q$. Hence, we assume that $A_k \subseteq Q$ for some $k \in \mathbb{N}$. Then for each $m \neq k, A_m \not\subseteq Q$ because A_m and A_k are w-comaximal. Thus, $AR_Q = (T_kR_Q : A_kR_Q)$. Since A_k is w-stable, AR_Q is invertible in T_kR_Q and hence $(AR_Q : AR_Q) = T_kR_Q$. Therefore, A is w-stable by assumption. Setting $T := (\sum_{m\in\mathbb{N}} T_m)_w$, we observe that $T = (A_w : A_w)$. Since A is $\dot{w}$-finite in T, there exists a finitely generated fractional ideal $B \subseteq A$ of R such that $A_w = (BT)_w$. Let $B = \sum_{m=1}^{q}(T_m : A_m)$ for some $q \in \mathbb{N}$. We note that for each $m \in \mathbb{N}$, A_mT is $\dot{w}$-invertible in T by Lemma 5.10(1). Hence, $((T_m : A_m)T)_w = (T : A_mT)$, and

$$A_w = \left(\sum_{m=1}^{q}(T_m : A_m)T\right)_w = \left(\sum_{m=1}^{q}(T : A_mT)\right)_w.$$

Thus, for every $n \in \mathbb{N}$,

$$(T : A_nT) \subseteq A_w = (\sum_{m=1}^{q}(T : A_mT))_w;$$

so

$$(T : A) = (T : \sum_{m=1}^{q}(T : A_mT)) = \bigcap_{m=1}^{q}(T : (T : A_mT)) \subseteq (T : (T : A_nT)).$$

We also note that since $(T : (T : A_m))$ are pairwise w-comaximal for each $m \in \mathbb{N}$,

$$\left(\bigcap_{m=1}^{q}(T : (T : A_mT))\right)_w = \left(\prod_{m=1}^{q}(T : (T : A_mT))\right)_w.$$

Furthermore, by Lemma 5.10(2), $(T : (T : A_mT))R_Q = (TR_Q : (T : A_mT)R_Q)$ for each $m \in \mathbb{N}$ and for each $Q \in t\text{-Max}(R)$.

Now, let $n > q$ and let Q be a maximal t-ideal of R containing A_n. Then for every $1 \leq m \leq q$, A_m is not contained in Q and hence

$$\begin{aligned}(T : (T : A_mT))R_Q &= (TR_Q : (T : A_mT)R_Q)\\ &= (TR_Q : (T_m : A_m)TR_Q)\\ &= (T_nR_Q : (T_mR_Q : A_mR_Q)T_nR_Q)\\ &= T_nR_Q.\end{aligned}$$

Thus,

$$(\bigcap_{m=1}^{q}(T : (T : A_mT)))R_Q = \prod_{m=1}^{q}(T : (T : A_mT))R_Q = T_nR_Q.$$

As a consequence,

$$\begin{aligned}T_nR_Q &\subseteq (T : (T : A_nT))R_Q\\ &= (TR_Q : (T : A_nT)R_Q)\\ &= (TR_Q : (T_n : A_n)TR_Q)\\ &= (T_nR_Q : (T_nR_Q : A_nR_Q)T_nR_Q)\\ &= A_nR_Q,\end{aligned}$$

where the last equality follows because A_n is a w-stable ideal. Hence, $R_Q \subseteq T_nR_Q \subseteq A_nR_Q \subsetneq R_Q$; a contradiction. Therefore, R is of finite t-character. □

Corollary 5.12. *Let R be an integrally closed conducive domain. If R is a finitely weakly ES-w-stable domain with w-local stability property, then R is finitely weakly ES-stable.*

Proof. By Propositions 5.1 and 2.1, R is finitely w-stable. Hence, R is a PvMD of finite t-character by Corollary 5.4 and Theorem 5.11. Since any t-linked overring of R is finitely weakly ES-w'-stable by Proposition 5.6, by using the same method as Proposition 2.13, we observe that R is a semi-local domain whose maximal ideals are t-ideal. Therefore, R is a Prüfer domain and hence R is finitely weakly ES-stable. □

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References

[1] Anderson, D. D., Anderson, D.F., Zafrullah, M. (2001). The ring $D+XD_S[X]$ and t-splitting sets. *Commut. Algebra Arab. J. Sci. Eng. Sect. C Theme Issues*. 26(1):3–16.
[2] Anderson, D. D., Chang, G.W., Zafrullah, M. (2013). Integral domains of finite t-character. *J. Algebra*. 396: 169–183. DOI: 10.1016/j.jalgebra.2013.08.014.
[3] Anderson, D. D., Cook, S. J. (2000). Two star-operations and their induced lattices. *Commun. Algebra* 28(5):2461–2475. DOI: 10.1080/00927870008826970.
[4] Anderson, D. D., Huckaba, J. A., Papick, I. J. (1987). A note on stable domains. *Houston J. Math.* 13(1): 13–17.
[5] Anderson, D. D., Mott, J., Zafrullah, M. (1992). Finite character representations for integral domains. *Bull. Math. Ital.* 6-B(7):613–630.
[6] Anderson, D. D., Zafrullah, M. (1999). Independent locally-finite intersections of localizations. *Houston J. Math.* 25:433–452.
[7] Ay Saylam, B. (2019). Integrally closed rings which are Prüfer. *Commun. Algebra* 47(3):1271–1277. DOI: 10.1080/00927872.2018.1503282.
[8] Ay Saylam, B. (2020). Stability in commutative rings. *Turk. J. Math.* 44(3):801–812. DOI: 10.3906/mat-1911-101.
[9] Badawi, A. (2003). On the complete integral closure of rings that admit a φ-strongly prime ideal. In: *Lecture Notes in Pure and Appl. Math.*, Vol. 231, New York: Marcel Dekker, pp. 15–22.
[10] Bazzoni, S. (2011). Finite character of finitely stable domains. *J. Pure Appl. Algebra* 215(6):1127–1132. DOI: 10.1016/j.jpaa.2010.07.011.
[11] Chang, G. W., Kim, H., Lim, J. W. (2013). Integral domains in which every nonzero t-locally principal ideal is t-invertible. *Commun. Algebra* 41(10):3805–3819. DOI: 10.1080/00927872.2012.678022.
[12] Chang, G. W., Fontana, M. (2007). Uppers to zero and semistar operations in polynomial rings. *J. Algebra* 318(1):484–493. DOI: 10.1016/j.jalgebra.2007.06.010.
[13] Chang, G. W., Fontana, M. (2009). Uppers to zero in polynomial rings and Prüfer-like domains. *Commun. Algebra* 37(1):164–192. DOI: 10.1080/00927870802243564.
[14] Chang, G. W., Zafrullah, M. (2006). The w-integral closure of integral domains. *J. Algebra* 295(1):195–210. DOI: 10.1016/j.jalgebra.2005.04.025.
[15] Costa, D., Mott, J. L., Zafrullah, M. (1978). The construction $D+XD_S[X]$. *J. Algebra*. 53:423–439.
[16] Dobbs, D. E., Houston, E. G., Lucas, T. G., Zafrullah, M. (1989). t-Linked overrings and Prüfer v-multiplication domains. *Commun. Algebra* 17(11):2835–2852. DOI: 10.1080/00927878908823879.
[17] Dobbs, D. E., Fedder, R. (1984). Conducive integral domains. *J. Algebra* 86(2):494–510. DOI: 10.1016/0021-8693(84)90044-9.
[18] Eakin, P., Sathaye, A. (1976). Prestable ideals. *J. Algebra* 41(2):439–454. DOI: 10.1016/0021-8693(76)90192-7.
[19] Baghdadi, S. E. (2002). On a class of Prüfer v-multiplication domains. *Commun. Algebra* 30(8):3723–3742. DOI: 10.1081/AGB-120005815.
[20] Baghdadi, S. E., Fontana, M. (2004). Semistar linkedness and flatness, Prüfer semistar multiplication domains. *Commun. Algebra* 32(3):1101–1126. DOI: 10.1081/AGB-120027969.
[21] Baghdadi, S. E., Gabelli, S. (2005). w-divisorial domains. *J. Algebra* 285(1):335–355. DOI: 10.1016/j.jalgebra.2004.11.016.
[22] Baghdadi, S. E., Gabelli, S. (2007). Ring-theoretic properties of PvMDs. *Commun. Algebra* 35(5):1607–1625. DOI: 10.1080/00927870601169283.
[23] Finocchiaro, C. A., Tartarone, F. (2018). A survey on the local invertibility of ideals in commutative rings. in: *Advances in Rings, Modules and Factorizations. Rings and Factorizations* Springer Proceedings in Mathematics and Statistics, Vol 321. Springer, Cham, 2020, p. 153–168.
[24] Fontana, M., Gabelli, S. (1996). On the class group and local class group of a pullback. *J. Algebra* 181(3): 803–835. DOI: 10.1006/jabr.1996.0147.
[25] Fontana, M., Huckaba, J., and Papick, I. (1997), *Prüfer Domains*, New York: Marcel Dekker.
[26] Fontana, M., Gabelli, S., Houston, E. (1998). UMT-domains and domains with Prüfer integral closure. *Commun. Algebra* 26(4):1017–1039. DOI: 10.1080/00927879808826181.
[27] Fontana, M., Houston, E., Park, M. H. Idempotence and divisoriality in Prüfer-like domains, (2018). In: *Advances in rings, modules and factorizations. Rings and factorizations. Springer Proceedings in Mathematics and Statistics*, Vol 321. Cham: Springer, pp. 169–182.

[28] Fuchs, L. (1973). *Infinite Abelian Groups*, Vol. II. New York: Academic Press.

[29] Fuchs, L., Salce, L. (2001). Modules over non-Noetherian domains. *Mathematical Surveys and Monographs*, No. 84. USA: American Mathematical Society.

[30] Gabelli, S., Picozza, G. (2007). Star stable domains. *J. Pure Appl. Algebra* 208(3):853–866. DOI: 10.1016/j.jpaa.2006.03.020.

[31] Gabelli, S., Picozza, G. (2012). Stability and Clifford regularity with respect to star operations. *Commun. Algebra* 40(9):3558–3582. DOI: 10.1080/00927872.2011.591863.

[32] Gabelli, S., Picozza, G. (2015). w-Stability and Clifford w-regularity of polynomial rings. *Commun. Algebra* 43(1):59–75. DOI: 10.1080/00927872.2014.897194.

[33] Gilmer, R., Heinzer, W.J. (1966). On the complete integral closure of an integral domain. *J. Aust. Math. Soc*. 6(3):351–361. DOI: 10.1017/S1446788700004304.

[34] Gilmer, R. (1972). *Multiplicative ideal theory*. New York: Marcel Dekker.

[35] Gilmer, R. (1974). A two-dimensional non-Noetherian factorial ring. *Proc. Amer. Math. Soc*. 44(1):25–30. DOI: 10.1090/S0002-9939-1974-0335500-0.

[36] Houston, E., Kabbaj, S., Lucas, T. G., Mimouni, A. (1995). Duals of ideals in pullback constructions. *Dekker Lect. Notes Pure Appl. Math*. 171:263–276.

[37] Houston, E., Zafrullah, M. (1988). Integral domains in which each t-ideal is divisorial. *Michigan Math. J*. 35(2):291–300. DOI: 10.1307/mmj/1029003756.

[38] Houston, E., Zafrullah, M. (1989). On t-invertibility II. *Commun. Algebra* 17(8):1955–1969. DOI: 10.1080/00927878908823829.

[39] Hungerford, T. W. (1968). On the structure of principal ideal rings. *Pacific J. Math*. 25(3):543–547. DOI: 10.2140/pjm.1968.25.543.

[40] Kabbaj, S., Lucas, T. G., Mimouni, A. (2003). Trace properties and pullbacks. *Commun. Algebra* 31(3):1085–1111. DOI: 10.1081/AGB-120017753.

[41] Kabbaj, S., Mimouni, A. (2009). Constituent groups of Clifford semigroups arising from t-closure. *J. Algebra* 321(5):1443–1452. DOI: 10.1016/j.jalgebra.2008.11.029.

[42] Kang, B.G. (1989). Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. *J. Algebra* 123:151–170.

[43] Kwak, D. J., Park, Y. S. (1995). On t-flat overrings. *Chinese J. Math*. 23:17–24.

[44] Kim, H., Lim, J. W. (2020). Integral domains in which every nonzero w-flat ideal is w-invertible. *Mathematics* 8(2):247. DOI: 10.3390/math8020247.

[45] Mimouni, A. (2003). TW-domains and Strong Mori domains. *J. Pure and Appl. Algebra* 177(1):79–93. DOI: 10.1016/S0022-4049(02)00171-8.

[46] Mimouni, A. (2004). Star-operations and trace properties. In: Badawi, A., (ed). *Trends in Commutative Rings Research*. New York: Nova Science Publishers, Inc., pp. :77–92.

[47] Mimouni, A. (2017). Factoring ideals and stability in integral domains. *J. Commut. Algebra* 9(2):263–290. DOI: 10.1216/JCA-2017-9-2-263.

[48] Mott, J.L., Zafrullah, M. (1981). On Prüfer v-multiplication domains. *Manuscripta Math*. 35(1-2):1–26. DOI: 10.1007/BF01168446.

[49] Olberding, B. (1999). Stability of ideals and its applications. In: *Ideal Theoretic Methods in Commutative Algebra (Columbia, MO). Lecture Notes in Pure and Appl. Math.* vol. 220. USA: CRC Press, pp. 319–342.

[50] Olberding, B. (2002). On the structure of stable domains. *Commun. Algebra* 30(2):877–895. DOI: 10.1081/AGB-120013188.

[51] Sally, J. D., Vasconcelos, W. V. (1974). Stable rings. *J. Pure Appl. Algebra* 4(3):319–336. DOI: 10.1016/0022-4049(74)90012-7.

[52] Sega, L. (2007). Ideal class semigroups of overrings. *J. Algebra* 311(2):702–713. DOI: 10.1016/j.jalgebra.2006.10.030.

[53] Zafrullah, M. (2010). t-invertibility and Bazzoni-like statements. *J. Pure Appl. Algebra.* 214(5):654–657. DOI: 10.1016/j.jpaa.2009.07.004.

[54] Zafrullah, M. (2006). What v-coprimality can do for you. In: Brewer, J. W., Glaz, S., Heinzer, W. J., Olberding, B. M., (eds). *Multiplicative ideal theory in commutative algebra*. Boston, MA: Springer, pp. 387–404.

[55] Zafrullah, M. On *-homogeneous ideals. https://arxiv.org/pdf/1907.04384.pdf.

[56] Fangui, W., Casland, R.L M. (1999). On strong Mori domains. *J. Pure Appl. Algebra* 135:155–165.