

Original articles

Numerical solution of a generalized boundary value problem for the modified Helmholtz equation in two dimensions

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Abstract

We propose numerical schemes for solving the boundary value problem for the modified Helmholtz equation and generalized impedance boundary condition. The approaches are based on the reduction of the problem to the boundary integral equation with a hyper-singular kernel. In the first scheme the hyper-singular integral operator is treated by splitting off the singularity technique whereas in the second scheme the idea of numerical differentiation is employed. The solvability of the boundary integral equation and convergence of the first method are established. Exponential convergence for analytic data is exhibited by numerical examples.

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1. Introduction

Boundary value problems for the modified Helmholtz equation occupy an important place in heating and cooling materials, in implicit marching schemes for the heat equation, in Debye–Hückel theory, and in the linearization of the Poisson–Boltzmann equation associated with electrostatic interactions and electric potential governed by modified Helmholtz equation, see [5] and references therein. The generalized impedance boundary condition (GIBC) is used primarily to model a thin coating. Additional motivation for generalized impedance boundary condition can be found in [6].

Mathematically, the problem we study is formulated as following. Let D be a simply connected and bounded domain in \mathbb{R}^2 with boundary ∂D of class C^3 . Given $f \in H^{-\frac{1}{2}}(\partial D)$, $\lambda > 0$ and $\mu > 0$, $\lambda \in C(\partial D)$, $\mu \in C^1(\partial D)$ with $k > 0$ we seek a solution $u \in H^2(D)$ to the modified Helmholtz equation

$$\Delta u - k^2 u = 0 \quad \text{in } D \tag{1.1}$$

that satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} + k \left(\lambda u - \frac{d}{ds} \mu \frac{du}{ds} \right) = f \quad \text{on } \partial D \tag{1.2}$$

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in the weak sense, i.e.

$$\int_{\partial D} \left(\zeta \frac{\partial u}{\partial \nu} + k\lambda \zeta u + k\mu \frac{d\zeta}{ds} \frac{du}{ds} \right) ds = \int_{\partial D} \zeta f ds, \quad \forall \zeta \in H^{\frac{3}{2}}(\partial D). \tag{1.3}$$

Here ν denotes the unit normal vector directed into the exterior of D and $\frac{d}{ds}$ is a tangential derivative.

Theorem 1.1. *The boundary value problem (1.1)–(1.2) has at most one solution.*

Proof. Assume that u_1 and u_2 are solutions to problem (1.1)–(1.2) with their difference $u = u_1 - u_2$. Multiplying (1.1) by u and integrating over D , we have

$$\int_D u \Delta u dx - k^2 \int_D u^2 dx = 0. \tag{1.4}$$

From the Green’s first theorem and the boundary condition (1.3) with $\zeta = u|_{\partial D}$ we obtain

$$- \int_D (\nabla u)^2 dx - k^2 \int_D u^2 dx - k \int_{\partial D} \lambda u^2 ds - k \int_{\partial D} \mu \left(\frac{du}{ds} \right)^2 = 0.$$

Since k, λ, μ are positive, the last equality implies $u = 0$ in D . \square

There is a variety of numerical methods in the literature for boundary value problems for the modified Helmholtz equation, for example in [16] the author introduced a method based on by plane wave functions, the method of fundamental solution and a singular boundary method were used in [2,4], respectively, [15] suggested a fast multipole-based iterative solution, in [5] a fast multipole-accelerated integral equation is presented, the Trefftz method is considered in [8]. In all these papers the classical boundary conditions are considered. The boundary value problems with GIBC recently gained much attention in the area of direct and inverse problem. The most relevant publications to the current work are the paper by Cakoni and Kress [3] where a solution method was suggested for the Laplace equation, and the paper by Kress [14] where the direct and inverse problems for 2D Helmholtz equation were investigated .

We propose two numerical methods for solving the boundary value problem for the modified Helmholtz equation with GIBC. Applying the indirect integral equation approach, a single-layer potential representation, we reformulate the boundary value problem as a Fredholm integral equation of the first kind which is solved by a projection method. As an alternative approach, one may represent the solution via a combination of the single- and double-layer potential as was suggested and analyzed in [9] for solving the impedance boundary value problem for the Helmholtz equation in three dimensions. For the numerical evaluation of the integral operators with continuous or weakly singular kernels we employ quadratures based on trigonometric interpolation. To evaluate an integral with a hyper-singular kernel we consider two schemes. In the first scheme we split off a hyper-singular part of the kernel and evaluate the corresponding integral operator analytically, [11]. In the second scheme we take advantage of the numerical differentiation, [13].

In Section 2 the boundary integral equation equivalent to the boundary value problem is derived. Existence of the solution is proved with aid of the Riesz theory. In Section 3 the parametrized version of the integral equation is presented, all singularities in the kernels of the integral operators are split off and the existence of the corresponding solution is analyzed. Section 4 is devoted to the convergence of the first numerical scheme. In the last section the feasibility of the two proposed methods is illustrated by numerical examples.

2. The boundary integral equations

In this section we introduce a boundary integral equation method for solving the problem (1.1)–(1.2) and establish the existence of the solution. We seek the solution of (1.1)–(1.2) in the form of a single layer potential

$$u(x) = \int_{\partial D} \Phi(x, y) \varphi(y) ds(y), \quad x \in D, \tag{2.1}$$

where $\varphi \in H^{\frac{1}{2}}(\partial D)$ and

$$\Phi(x, y) = \frac{1}{2\pi} K_0(k|x - y|) \tag{2.2}$$

is the fundamental solution of the modified Helmholtz equation in \mathbb{R}^2 with

$$K_0(x) = -\left(\ln \frac{x}{2} + \alpha\right) I_0(x) + 2 \sum_{k=1}^{\infty} \frac{I_{2k}(x)}{k}. \tag{2.3}$$

Here $\alpha = 0.5772156\dots$ is the Euler constant and K_0, I_0 are modified Bessel functions of the second kind and the first kind of order zero, respectively, [1]. The modified Bessel function of the first kind with order zero is given by

$$I_0(x) = 1 + \frac{\frac{1}{4}x^2}{(1!)^2} + \frac{\left(\frac{1}{4}x^2\right)^2}{(2!)^2} + \frac{\left(\frac{1}{4}x^2\right)^3}{(3!)^2} + \dots \tag{2.4}$$

Approaching the boundary ∂D from the interior of D , with the aid of jump relations for the single layer potential, [12], we conclude that the boundary condition (1.2) is satisfied provided φ solves the boundary integral equation

$$K'\varphi + \frac{1}{2}\varphi + k\left(\lambda - \frac{d}{ds}\mu\frac{d}{ds}\right)S\varphi = f, \tag{2.5}$$

where $S : H^{\frac{1}{2}}(\partial D) \rightarrow H^{\frac{3}{2}}(\partial D)$ and $K' : H^{\frac{1}{2}}(\partial D) \rightarrow H^{\frac{1}{2}}(\partial D)$ are bounded integral operators defined by

$$(S\varphi)(x) = \int_{\partial D} \Phi(x, y)\varphi(y)ds(y), \quad x \in \partial D \tag{2.6}$$

and

$$(K'\varphi)(x) = \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(x)}\varphi(y)ds(y), \quad x \in \partial D. \tag{2.7}$$

For more details we refer to [12,14,17].

Theorem 2.1. For each $f \in H^{-\frac{1}{2}}(\partial D)$, the boundary integral equation (2.5) has a unique solution $\varphi \in H^{\frac{1}{2}}(\partial D)$ provided $\lambda > 0, \mu > 0, k > 0$.

Proof. Since $\mu > 0$ Eq. (2.5) can be rewritten in the equivalent form

$$(A_1 + A_2)\varphi = -\frac{1}{\mu}f,$$

where

$$A_1\varphi = k\left(\frac{d^2}{ds^2}S\varphi + \int_{\partial D} S\varphi ds\right),$$

$$A_2\varphi = \frac{k}{\mu}\frac{d\mu}{ds}\frac{d}{ds}S\varphi - \frac{k\lambda}{\mu}S\varphi - \frac{1}{\mu}(K'\varphi + \frac{1}{2}\varphi) - k \int_{\partial D} S\varphi ds.$$

The operator $A_1 : H^{\frac{1}{2}}(\partial D) \rightarrow H^{-\frac{1}{2}}(\partial D)$ is invertible with a bounded inverse, [3,14], and the operator $A_2 : H^{\frac{1}{2}}(\partial D) \rightarrow H^{-\frac{1}{2}}(\partial D)$ is compact, that follows from the boundedness defined in (2.6)–(2.7) and a compact embedding $H^{\frac{1}{2}}(\partial D) \hookrightarrow H^{-\frac{1}{2}}(\partial D)$, [12]. Assume that $\varphi \in H^{\frac{1}{2}}(\partial D)$ such that

$$(A_1 + A_2)\varphi = 0$$

and construct a function

$$u(x) = \int_{\partial D} \Phi(x, y)\varphi(y)ds(y), \quad x \in \mathbb{R}^2 \setminus \partial D.$$

Since u satisfies the modified Helmholtz equation and the homogenous GIBC by Theorem 1.1, we have $u = 0$ in D . Furthermore, u solves the modified Helmholtz equation in the exterior of D , $u(x) = O\left(\frac{1}{|x|}\right)$ as $|x| \rightarrow \infty$. Uniqueness of the exterior Dirichlet problem, e.g. [18], implies $u = 0$ in $\mathbb{R}^2 \setminus \bar{D}$. By the jump relations for the normal derivative of the single layer potential we have $\varphi = 0$. Since $A_1^{-1}A_2$ is compact, by the Riesz theory the equation $(I + A_1^{-1}A_2)\varphi = A_1^{-1}f$ has a unique solution and hence the boundary integral equation (2.5) is uniquely solvable. \square

3. Parametrization of the integral equations

We assume that the boundary ∂D is analytic and has a 2π -periodic parametric representation of the form

$$\partial D = \{z(t) = (z_1(t), z_2(t)) : 0 \leq t \leq 2\pi\} \tag{3.1}$$

where $z : \mathbb{R} \rightarrow \mathbb{R}^2$ is analytic and 2π -periodic with $|z'(t)| > 0$ for all t . We introduce the parametrized single layer operator by

$$(\tilde{S}\psi)(t) = \frac{1}{2\pi} \int_0^{2\pi} K_0(k|z(t) - z(\tau)|)\psi(\tau)d\tau \tag{3.2}$$

where $\psi(t) = \varphi(z(t))|z'(t)|$. The kernel of the operator \tilde{S} can be written in the form

$$F_1(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) + F_2(t, \tau)$$

with

$$F_1(t, \tau) = -\frac{1}{4\pi} I_0(k|z(t) - z(\tau)|),$$

$$F_2(t, \tau) = \frac{1}{2\pi} K_0(k|z(t) - z(\tau)|) - F_1(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2} \right)$$

and both F_1 and F_2 are smooth with the diagonal terms

$$F_1(t, t) = -\frac{1}{4\pi}, \quad F_2(t, t) = -\frac{1}{2\pi} \left(\alpha + \ln \frac{k}{2} |z'(t)| \right).$$

The parametrized operator K' reads

$$(\tilde{K}'\psi)(t) = \int_0^{2\pi} M(t, \tau)\psi(\tau)d\tau \tag{3.3}$$

with the continuous kernel

$$M(t, \tau) = \frac{1}{|z'(t)|} \begin{cases} [z'(t)]^\perp \cdot \frac{z''(t)}{4\pi}, & t = \tau, \\ \frac{1}{2\pi} k [z'(t)]^\perp \cdot (z(t) - z(\tau)) \frac{K_1(k|z(t) - z(\tau)|)}{|z(t) - z(\tau)|}, & t \neq \tau, \end{cases}$$

and $[z'(t)]^\perp = (z_2'(t), -z_1'(t))$. Denoting $\mu \circ z$ as $\tilde{\mu}$, $\lambda \circ z$ as $\tilde{\lambda}$ and using the parametrization

$$\frac{d}{ds} \mu \frac{d}{ds} S = \frac{1}{|z'|} \frac{d}{dt} \tilde{\mu} \frac{d}{dt} \tilde{S}$$

we rewrite the boundary integral equation (2.5) as follows

$$\frac{1}{b} \tilde{K}'\psi + \frac{\psi}{2b} + \frac{k}{b} \tilde{\lambda} |z'| \tilde{S}\psi + \frac{a}{b} \frac{d\tilde{S}\psi}{dt} + \frac{d^2\tilde{S}\psi}{dt^2} = g, \tag{3.4}$$

where

$$a(t) = \left(\frac{k\tilde{\mu}(t)z'(t) \cdot z''(t)}{|z'(t)|^3} - \frac{k}{|z'(t)|} \frac{d\tilde{\mu}(t)}{dt} \right), \quad b(t) = -\frac{k\tilde{\mu}(t)}{|z'(t)|}, \quad g(t) = \frac{|z'(t)|f(z(t))}{b(t)}.$$

The integral equation (3.4) contains operators with continuous, weakly singular and hyper singular kernels. To this end, with the aid of the expansions (2.3) and (2.4) we find

$$\frac{d\tilde{S}\psi}{dt}(t) = \frac{1}{4\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} \psi(\tau)d\tau + \int_0^{2\pi} L(t, \tau)\psi(\tau)d\tau. \tag{3.5}$$

The kernel L can be represented in the form of

$$L(t, \tau) = L_1(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) + L_2(t, \tau)$$

where the continuous terms

$$L_1(t, \tau) = \frac{k}{4\pi} \frac{z'(t) \cdot (z(\tau) - z(t)) I'_0(k|z(t) - z(\tau)|)}{|z(\tau) - z(t)|},$$

$$L_2(t, \tau) = \frac{1}{2\pi} \frac{dK_0(k|z(t) - z(\tau)|)}{dt} - L_1(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) - \frac{1}{4\pi} \cot \frac{\tau - t}{2}$$

have the limit values

$$L_1(t, t) = 0, \quad L_2(t, t) = -\frac{1}{4\pi} \frac{z'(t) \cdot z''(t)}{|z'(t)|^2}.$$

The operator $\frac{d^2 \tilde{S}\psi}{dt^2}$ defined by

$$\frac{d^2 \tilde{S}\psi}{dt^2}(t) = \frac{1}{2\pi} \frac{d^2}{dt^2} \int_0^{2\pi} K_0(k|z(t) - z(\tau)|) \psi(\tau) d\tau$$

can be rewritten, with the aid of partial integration and the series expansion (2.3) for K_0 , as follows

$$\frac{d^2 \tilde{S}\psi}{dt^2}(t) = \frac{1}{4\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} \psi'(\tau) d\tau + \int_0^{2\pi} N(t, \tau) \psi(\tau) d\tau. \tag{3.6}$$

Employing the modified Bessel differential equation for K_0 , we can deduce the following expression

$$N(t, \tau) = N_1(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) + N_2(t, \tau)$$

where

$$N_1(t, \tau) = \frac{1}{4\pi} k I_1(k|z(t) - z(\tau)|) \frac{(z'(t) \cdot (z(t) - z(\tau)))^2}{|z(t) - z(\tau)|^3}$$

$$- k^2 I_0(k|z(t) - z(\tau)|) \frac{(z'(t) \cdot (z(t) - z(\tau)))^2}{4\pi |z(t) - z(\tau)|^2}$$

$$+ k I_1(k|z(t) - z(\tau)|) \left(\frac{z''(t) \cdot (z(t) - z(\tau)) - |z'(t)|^2}{4\pi |z(t) - z(\tau)|} + \frac{(z'(t) \cdot (z(t) - z(\tau)))^2}{4\pi |z(t) - z(\tau)|^3} \right)$$

and

$$N_2(t, \tau) = \frac{1}{2\pi} \frac{d^2 K_0(k|t - \tau|)}{dt^2} + \frac{1}{8\pi} \frac{1}{\sin^2 \frac{t - \tau}{2}} - N_1(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2} \right)$$

with the diagonal terms

$$N_1(t, t) = -k^2 \frac{|z'(t)|^2}{8\pi},$$

$$N_2(t, t) = \frac{1}{2\pi} \left(-k^2 \frac{|z'(t)|^2}{4} - \alpha \frac{k^2 |z'(t)|^2}{2} - k^2 \frac{|z'(t)|^2}{2} \ln \left(\frac{k}{2} |z'(t)| \right) \right)$$

$$+ \frac{6(z'(t) \cdot z''(t))^2 - |z'(t)|^4 - 4|z'(t)|^2 z'(t) \cdot z'''(t) - 3|z'(t)|^2 |z''(t)|^2}{12|z'(t)|^4}.$$

We note that for the static case the continuous representation of regular part of a mixed second order tangential derivative is found in [7, p. 151].

The parametrized equation (3.4) can be written in the form, where all singularities appear explicitly

$$\frac{1}{4\pi} \int_0^{2\pi} \cot \left(\frac{\tau - t}{2} \right) \psi'(\tau) d\tau + \frac{1}{4\pi} \frac{a(t)}{b(t)} \int_0^{2\pi} \cot \left(\frac{\tau - t}{2} \right) \psi(\tau) d\tau \tag{3.7}$$

$$+ \frac{1}{b(t)} \int_0^{2\pi} \left(H_1(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) + H_2(t, \tau) + |z'(t)| M(t, \tau) \right) \psi(\tau) d\tau$$

$$+ \frac{\psi(t)}{2b(t)} = g(t), \quad 0 \leq t \leq 2\pi.$$

Here

$$H_i(t, \tau) = k\tilde{\lambda}(t)|z'(t)|F_i(t, \tau) + a(t)L_i(t, \tau) + b(t)N_i(t, \tau)$$

are analytic functions for $i = 1, 2$.

Theorem 3.1. For any $g \in H^{-\frac{1}{2}}[0, 2\pi]$ and $\tilde{\lambda} \in C[0, 2\pi], \tilde{\mu} \in C^1[0, 2\pi], \tilde{\lambda} > 0, \tilde{\mu} > 0$, the integral equation (3.4) has a unique solution $\psi \in H^{\frac{3}{2}}[0, 2\pi]$ which depends continuously on the data.

Proof. In order to investigate solvability of the parametrized integral equation (3.4) we define the operators

$$\begin{aligned} (T\psi)(t) &= \frac{1}{4\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} \psi'(\tau) d\tau + \frac{1}{2\pi} \int_0^{2\pi} \psi(\tau) d\tau, \\ (B_1\psi)(t) &= \frac{1}{b(t)} \int_0^{2\pi} \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) H_1(t, \tau) \psi(\tau) d\tau, \\ (B_2\psi)(t) &= \frac{1}{b(t)} \int_0^{2\pi} H_2(t, \tau) \psi(\tau) d\tau + \frac{|z'(t)|}{b(t)} \int_0^{2\pi} M(t, \tau) \psi(\tau) d\tau + \frac{\psi(t)}{2b(t)} - \frac{1}{2\pi} \int_0^{2\pi} \psi(\tau) d\tau, \\ (B_3\psi)(t) &= \frac{1}{4\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} \psi(\tau) d\tau \end{aligned} \tag{3.8}$$

and set $B = B_1 + B_2 + \frac{a(t)}{b(t)}B_3$.

The operator $T : H^p[0, 2\pi] \rightarrow H^{p-1}[0, 2\pi]$ is bounded and has a bounded inverse for all $p \geq 0$. From [11], considering the trigonometric monomials $u_m(t) = e^{imt}$ we have

$$Tu_m = \beta_m u_m \tag{3.9}$$

for $m \in \mathbb{Z}$ with $\beta_m = -\frac{|m|}{2}, m \neq 0$ and $\beta_0 = 1$. This indicates the boundedness of $T : H^p[0, 2\pi] \rightarrow H^{p-1}[0, 2\pi]$ and the existence of the inverse operator $T^{-1} : H^{p-1}[0, 2\pi] \rightarrow H^p[0, 2\pi]$ given by

$$T^{-1}u_m = \frac{1}{\beta_m} u_m, m \in \mathbb{Z}$$

The operator $B : H^p[0, 2\pi] \rightarrow H^{p-1}[0, 2\pi]$ is compact, since $B_3 : H^p[0, 2\pi] \rightarrow H^p[0, 2\pi]$ is bounded. From Theorem 2.1. we can conclude that $T + B$ is injective. By the Riesz theory [12, Corollary 3.6], $T + B$ has a bounded inverse. \square

4. Numerical methods

The parametrized integral equation of the first kind is solved by the collocation method with trigonometric polynomials. We introduce the trigonometric interpolation operator $P_n : H^p[0, 2\pi] \rightarrow \mathbb{T}_n$ with $2n$ equidistant interpolation points $t_i^{(n)} = \frac{i\pi}{n}, i = 0, 2n - 1$ and recall the following error estimate

$$\|P_n\varphi - \varphi\|_q \leq \frac{C}{n^{p-q}} \|\varphi\|_p, \quad 0 \leq q \leq p, p > \frac{1}{2} \tag{4.1}$$

where the constant C depends on p and q , [12]. In the case of 2π -periodic and analytic function the interpolation error decays exponentially. The integrals in (3.7) are approximated by the following quadrature rules based on the trigonometric interpolation, [12],

$$\begin{aligned} \int_0^{2\pi} h(\tau) d\tau &\approx \frac{\pi}{n} \sum_{i=0}^{2n-1} h(t_i^{(n)}), \\ \frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} \varphi'(\tau) d\tau &\approx \sum_{i=0}^{2n-1} T_{1,i}(t) \varphi(t_i^{(n)}), \\ \frac{1}{2\pi} \int_0^{2\pi} \ln \left(4 \sin^2 \frac{\tau - t}{2} \right) \varphi(\tau) d\tau &\approx \sum_{i=0}^{2n-1} R_i(t) \varphi(t_i^{(n)}), \end{aligned} \tag{4.2}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} \varphi(\tau) d\tau \approx \sum_{i=0}^{2n-1} T_{2,i}(t) \varphi(t_i^{(n)})$$

where the quadrature weights are given by

$$T_{1,i}^{(n)}(t) = -\frac{1}{n} \sum_{m=1}^{n-1} m \cos m(t - t_i^{(n)}) - \frac{1}{2} \cos n(t - t_i^{(n)}),$$

$$R_i^{(n)}(t) = -\frac{1}{n} \sum_{m=1}^{n-1} \frac{1}{m} \cos m(t - t_i^{(n)}) - \frac{1}{2n^2} \cos n(t - t_i^{(n)}),$$

$$T_{2,i}^{(n)}(t) = \frac{1}{2n} \left\{ 1 - \cos n(t - t_i^{(n)}) \right\} \cot \frac{t - t_i^{(n)}}{2}.$$

Recalling the operator T defined in (3.8) we have that $P_n T \varphi = T P_n \varphi = T \varphi$, $\varphi \in \mathbb{T}_n$ and the fully discrete system for (3.7) reads

$$(T + P_n B_n) \psi_n(t_i) = (P_n g_n)(t_i), \quad i = \overline{1, 2n}, \tag{4.3}$$

where $B_n = B_{1,n} + B_{2,n} + B_{3,n}$,

$$(B_{1,n} \psi)(t) = \frac{1}{b(t)} \int_0^{2\pi} \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) (P_n (H_1(t, \cdot) \psi))(\tau) d\tau,$$

$$(B_{2,n} \psi)(t) = \frac{1}{b(t)} \int_0^{2\pi} (P_n (H_2(t, \cdot) \psi))(\tau) d\tau + \frac{1}{b(t)} \int_0^{2\pi} (P_n (M(t, \cdot) \psi))(\tau) d\tau + \frac{1}{2b(t)} \psi(t),$$

$$(B_{3,n} \psi)(t) = \frac{1}{4\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} (P_n \psi)(\tau) d\tau.$$

Note, since we want to compare two methods, as the collocation points we choose t_i for $i = \overline{1, 2n}$. This choice guarantees that we can approximate the derivative of 2π periodic function by the derivative the unique interpolatory trigonometric polynomial of degree n without the term $\sin(nt)$.

Theorem 4.1. *Under the assumption that λ, μ and Γ are analytic, the fully discrete collocation method (4.3) converges in $H^p[0, 2\pi]$ for each $p > 3/2$.*

Proof. From uniform convergence of weakly singular kernels of the operators $B_{1,n}$ to the kernel of B_1 we have

$$\|B_{1,n} \psi - B_1 \psi\|_{q+1} \leq \frac{c}{n^{p-q}} \|\psi\|_p, \quad 0 \leq q \leq p, \quad \frac{1}{2} < p \tag{4.4}$$

and by the same way this estimate can be seen to be valid for analytic kernel of $B_{2,n}$. By construction $B_{3,n}$ integrates trigonometric polynomials of degree less than or equal to n exactly [10,12], we have $B_n \psi \rightarrow B \psi$ for all $\psi \in \mathbb{T}_n \subset H^p[0, 2\pi]$. By Banach–Steinhaus theorem it follows $B_n \psi \rightarrow B \psi$ as $n \rightarrow \infty$ for all $\psi \in H^p[0, 2\pi]$.

Interpolation operators $P_n : H^p[0, 2\pi] \rightarrow H^p[0, 2\pi]$ are bounded for $p > \frac{1}{2}$, (see [12, Theorem 11.8]). From (4.4), it can be readily seen that

$$\|P_n(B_{1,n} - B_1) \psi\|_{p-1} \leq \frac{c}{n} \|\psi\|_p, \quad p > \frac{3}{2}. \tag{4.5}$$

By the same approach, this estimate can be done for the operator B_2 with analytic kernels. The boundedness of P_n , for $p > \frac{3}{2}$ and [12, Corollary 8.8] implies

$$\left\| P_n \left(\frac{a}{b} (B_{3,n} - B_3) \psi \right) \right\|_{p-1} \leq c \|(B_{3,n} - B_3) \psi\|_{p-1}$$

where a/b is analytic. Since $B_{3,n}$ is obtained by integrating trigonometric polynomial exactly, we have convergence for all trigonometric polynomials. By the Banach–Steinhaus theorem $P_n B_n \psi \rightarrow P_n B \psi$, $n \rightarrow \infty \quad \forall \psi \in H^p[0, 2\pi]$. The convergence follows from [12, Corollary 13.13]. \square

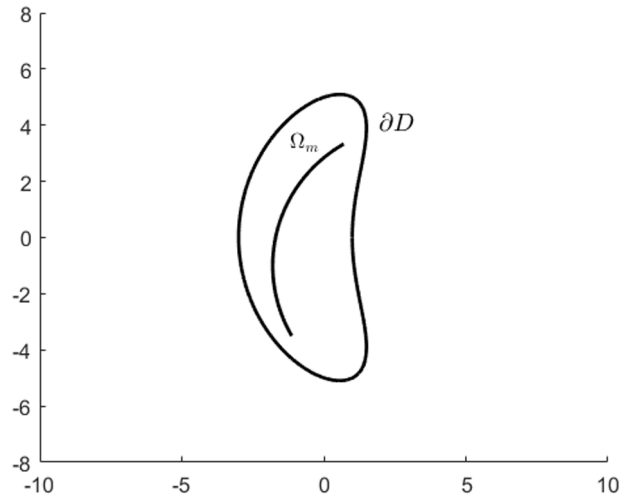


Fig. 1. Domain D with the measurement region Ω_m .

For the second numerical scheme instead of splitting off the singularities in the kernels of integral operators arising from the tangential derivatives of the single-layer operator we employ the idea of numerical differentiation suggested in [13]. In particular, defining the derivative $D_n = P'_n$ of trigonometric interpolation operator, the integro-differential operator is approximated as follows

$$\left(\frac{1}{|z'|} \frac{d}{dt} \tilde{\mu} \frac{d}{|z'|} \tilde{S} \psi_n \right) (t_i) \Big|_{i=1,2n} \approx \frac{1}{|z'(t_i)|} D_n \text{diag} \left(\frac{\tilde{\mu}(t_i)}{|z'(t_i)|} \right) D_n (\tilde{S}_n \psi_n)(t_i) \Big|_{i=1,2n},$$

where $\psi_n \in \mathbb{T}_n$,

$$(D_n g)(t_i) = \sum_{k=0}^{2n-1} d_{k-i}^{(n)} g(t_k), \quad i = 0, \dots, 2n - 1, \tag{4.6}$$

and

$$d_i^{(n)} = \begin{cases} \frac{(-1)^i}{2} \cot \frac{i\pi}{2n}, & i = \pm 1, \dots, \pm(2n - 1). \\ 0, & i = 0. \end{cases}$$

In the next section we present numerical examples for both methods.

5. Numerical examples

Assume that the boundary ∂D is parametrized by the function

$$z(t) = (2 \cos(t) - 2 \cos^2(t) + 1, 5 \sin(t) - \cos(t) \sin(t)), \quad 0 \leq t \leq 2\pi,$$

$k = \frac{1}{2}$, the impedance functions are chosen as follows

$$\lambda(z(t)) = -\sin(|z(t)|) + 4.5 \quad \text{and} \quad \mu(z(t)) = -2 \cos(|z(t)|) + 4.5.$$

To analyze numerical convergence of the proposed methods, we introduce the measurement curve $\Omega_m \subset D$ parametrized by

$$\Omega_m = \left\{ z(t) = (5 \cos(t), \sin(t)), t \in \left[\frac{2\pi}{3}, \frac{7\pi}{6} \right] \right\}.$$

The boundary ∂D and the impedance function are shown in Figs. 1 and 2, correspondingly. Below we present numerical examples that illustrate the effectiveness and accuracy of the proposed methods.

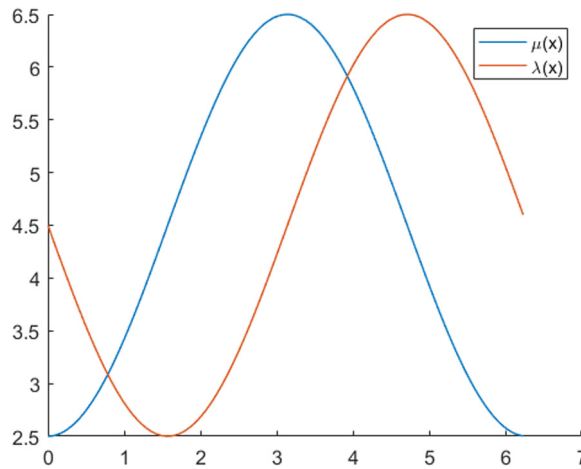


Fig. 2. Impedance functions.

Example 5.1. We test the methods by solving the boundary value problem (1.1)–(1.2) with known exact solution. In particular, we consider the exact solution u^\dagger to be given as point source with the location $x_1 = (2, 0.4)$, i.e.

$$u^\dagger(x) = \Phi(x, x_1), x \in D, x_1 \in \mathbb{R}^2 \setminus \bar{D}.$$

The problem reads

$$\begin{aligned} \Delta u - k^2 u &= 0 \text{ in } D, \\ \frac{\partial u}{\partial \nu} + k \left(\lambda u - \frac{d}{ds} \mu \frac{du}{ds} \right) &= f \text{ on } \partial D, \\ f(x) &= \frac{\partial \Phi(x, x_1)}{\partial \nu(x)} + k \left(\lambda(x) \Phi(x, x_1) - \frac{d}{ds} \mu(x) \frac{d\Phi(x, x_1)}{ds} \right), x \in \partial D. \end{aligned}$$

The maximum absolute errors in the numerical solutions at the points $y \in \Omega_m$ are presented in Table 1; $\|u_1 - u^\dagger\|_{\Omega_m, \infty}$ denotes the maximum error for the first method and $\|u_2 - u^\dagger\|_{\Omega_m, \infty}$ represents the error for the second method based on numerical differentiation. The error of the first numerical scheme decays exponentially as it is predicted by the theoretical investigation for the case of analytic boundary and data, as can be seen in Table 1. For the second method as illustrated in Table 1 the convergence is slower since the approximation of the hyper-singular part is not accurate. Indeed, from (3.9) we have

$$\frac{1}{4\pi} \frac{d^2}{dt^2} \int_0^{2\pi} \ln \left(\sin^2 \frac{t - \tau}{2} \right) \cos n\tau \, d\tau = -\frac{n}{2} \cos nt, \quad n \in \mathbb{N},$$

whereas for its approximation from (4.6) we obtain

$$D_n D_n \frac{1}{2\pi} \int_0^{2\pi} \ln \left(\sin^2 \frac{t - \tau}{2} \right) \cos n\tau \, d\tau = 0, \quad n \in \mathbb{N}.$$

The problem for the trigonometric differentiation can be resolved by introducing extra weights in the evaluation of $D_n D_n \tilde{S}$ or by choosing odd number of interpolation and collocation points, [13].

As it can be observed in Table 1 doubling the number of grid points almost doubles the number of correct digits in the approximate solution. In the next example we present numerical solution of the boundary value problem for the case when the exact solution is unknown.

Example 5.2. We choose the boundary data f to be given by

$$f(x) = \Phi(x, x_1), \quad x \in \partial D, \quad x_1 = (3, 2) \in \mathbb{R}^2 \setminus \bar{D}.$$

Table 1
Error analysis for Example 5.1.

n	$\ u_1 - u^\dagger\ _{\Omega_m, \infty}$	$\ u_2 - u^\dagger\ _{\Omega_m, \infty}$
8	1.24e-03	3.54e-03
16	1.37e-05	4.72e-05
32	9.90e-09	4.23e-07
64	1.00e-15	1.52e-12

Table 2
Numerical solution for Example 5.2.

n	$u_1(y)$
8	0.012063279277905
16	0.012284634729342
32	0.012285858740215
64	0.012285858683054
128	0.012285858683054

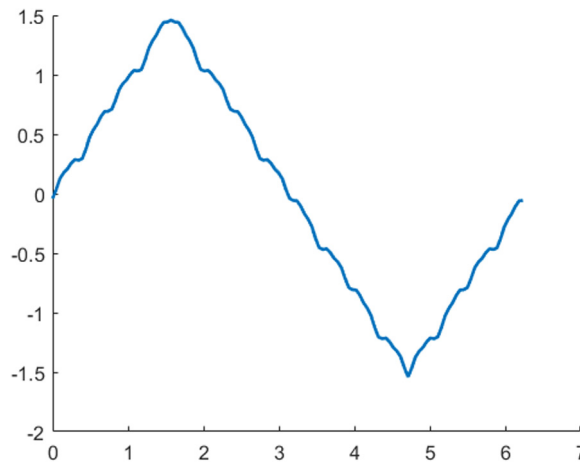


Fig. 3. Function f for Example 5.3.

The rest of input parameters remained unchanged. Table 2 presents value of the solution of the boundary value problem at the point $y = (0, 0.5) \in D$ via the first method. Similar to the error analysis in Table 1, we observe that the number of correct digits of the exact solution doubles when the number of grid points is increased twofold.

Example 5.3. In the last example we consider the disk of radius 2 centered at the origin as a domain D , the constant impedance functions $\mu = 1, \lambda = 1$ and the boundary data $f(z(t)) = \frac{3}{\pi} \arcsin(\sin t) + 0.04 \cos(16t) + 0.02 \cos(8t) - 0.02 \cos(32t)$ which can be represented accurately only by high degree trigonometric polynomials (see Fig. 3).

We compare the error between the Dirichlet traces of the solutions obtained by the two method, which is presented in Table 3 by the column $\|u_1 - u_2\|_{\partial D, \infty}$. Furthermore, since the first method converges what is guaranteed by Theorem 4.1. and verified by the previous two examples we can consider the solution obtained by the first method with $n = 256$ as the exact solution and denote it by u^\dagger . Having introduced the value of the exact solution, we present the corresponding errors in Table 3.

6. Conclusion

In this paper, existence and uniqueness of the solution to GIBC problem associated with two-dimensional modified Helmholtz equation is proved by employing boundary integral equations. Exploiting boundary integral

Table 3
Error analysis for Example 5.3.

n	$\ u_1 - u_2\ _{\partial D, \infty}$	$\ u_1 - u^\dagger\ _{\Omega_m, \infty}$	$\ u_2 - u^\dagger\ _{\Omega_m, \infty}$
8	2.69e-03	1.03e-01	1.00e-01
16	3.60e-03	3.35e-02	3.71e-02
32	1.06e-03	1.60e-03	2.65e-03
64	7.00e-10	3.93e-04	3.93e-04
128	2.31e-11	7.99e-05	7.99e-05

equations, the numerical solution method is proposed. The method is based on splitting off the singularities in the kernels of integral operators. The convergence in Sobolev spaces is proved and also verified by numerical examples. In particular, the numerical solution converges super-algebraically if all input data are analytic. The numerical method is compared with the approach based on trigonometric differentiation. The possible issues with the second approach are mentioned and illustrated by the examples.

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References

- [1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables, Vol. 55, Courier Corporation, 1965.
- [2] B. Bin-Mohsin, D. Lesnic, Determination of inner boundaries in modified Helmholtz inverse geometric problems using the method of fundamental solutions, *Math. Comput. Simulation* 82 (8) (2012) 1445–1458.
- [3] F. Cakoni, R. Kress, Integral equation methods for the inverse obstacle problem with generalized impedance boundary condition, *Inverse Problems* 29 (1) (2012) 015005.
- [4] W. Chen, J.-Y. Zhang, Z.-J. Fu, Singular boundary method for modified Helmholtz equations, *Eng. Anal. Bound. Elem.* 44 (2014) 112–119.
- [5] H. Cheng, J. Huang, T.J. Leiterman, An adaptive fast solver for the modified Helmholtz equation in two dimensions, *J. Comput. Phys.* 211 (2) (2006) 616–637.
- [6] M. Duruflé, H. Haddar, P. Joly, Higher order generalized impedance boundary conditions in electromagnetic scattering problems, *C. R. Phys.* 7 (5) (2006) 533–542.
- [7] K. Erhard, Point Source Approximation Methods in Inverse Obstacle Reconstruction Problems (Dissertation Thesis), Göttingen, 2005.
- [8] C. Fan, Y. Liu, H. Chan, S. Hsiao, Solutions of boundary detection problem for modified Helmholtz equation by trefftz method, *Inverse Probl. Sci. Eng.* 22 (2) (2014) 267–281.
- [9] E. Khalilov, Constructive method for solving a boundary value problem with impedance boundary condition for the Helmholtz equation, *Differential Equations* 54 (4) (2018) 539–550.
- [10] A. Kirsch, S. Ritter, The Nystrom method for solving a class of singular integral equations and applications in 3D-plate elasticity, *Math. Methods Appl. Sci.* 22 (2) (1999) 177–197.
- [11] R. Kress, On the numerical solution of a hypersingular integral equation in scattering theory, *J. Comput. Appl. Math.* 61 (3) (1995) 345–360.
- [12] R. Kress, *Linear Integral Equations*, Springer, New York, 2014.
- [13] R. Kress, A collocation method for a hypersingular boundary integral equation via trigonometric differentiation, *J. Integral Equations Appl.* 26 (2) (2014) 197–213.
- [14] R. Kress, Integral equation methods in inverse obstacle scattering with a generalized impedance boundary condition, in: *Contemporary Computational Mathematics—a Celebration of the 80th Birthday of Ian Sloan*, Springer, 2018, pp. 721–740.
- [15] M.C.A. Kropinski, B.D. Quaipe, Fast integral equation methods for the modified Helmholtz equation, *J. Comput. Phys.* 230 (2) (2011) 425–434.
- [16] X. Li, On solving boundary value problems of modified Helmholtz equations by plane wave functions, *J. Comput. Appl. Math.* 195 (1–2) (2006) 66–82.
- [17] W.D. McLean, W.C.H. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge university press, 2000.
- [18] B.D. Quaipe, *Fast Integral Equation Methods for the Modified Helmholtz Equation* (Ph.D. thesis), Science: Department of Mathematics, 2011.