Quantum Calculus of Fibonacci Divisors and Infinite Hierarchy of Bosonic-Fermionic Golden Quantum Oscillators

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Abstract

Starting from divisibility problem for Fibonacci numbers we introduce Fibonacci divisors, related hierarchy of Golden derivatives in powers of the Golden Ratio and develop corresponding quantum calculus. By this calculus, the infinite hierarchy of Golden quantum oscillators with integer spectrum determined by Fibonacci divisors, the hierarchy of Golden coherent states and related Fock-Bargman representations in space of complex analytic functions are derived. It is shown that Fibonacci divisors with even and odd k describe Golden deformed bosonic and fermionic quantum oscillators, correspondingly. By the set of translation operators we find the hierarchy of Golden binomials and related Golden analytic functions, conjugate to Fibonacci number F_k. In the limit $k \to 0$, Golden analytic functions reduce to classical holomorphic functions and quantum calculus of Fibonacci divisors to the usual one. Several applications of the calculus to quantum deformation of bosonic and fermionic oscillator algebras, R-matrices, hydrodynamic images and quantum computations are discussed.

Keywords: Fibonacci numbers, Fibonacci divisors, Golden ratio, Golden quantum oscillator, Fibonomials, coherent states, Golden quantum calculus, Golden analytic functions

1 Introduction

The Golden calculus is the quantum or q-calculus with Golden Ratio bases φ and φ', so that Binet formula for Fibonacci numbers becomes the q-number in this calculus [1]. The corresponding Golden derivative appears as a finite q-difference derivative with Golden ratio base, allowing one to construct Golden binomials and Taylor expansion in terms of these binomials. According to this expansion, the Golden exponential functions and corresponding trigonometric and hyperbolic functions, as solutions of Golden ODE of oscillator type and PDE of wave type have been derived [1].

It was shown that quantized Fibonacci number operator determines the deformed Golden quantum oscillator with discrete energy spectrum in the form of Fibonacci numbers [1]. The relative difference of the energy levels for this oscillator asymptotically, for big n becomes the Golden ratio. It turns out that Golden calculus as q-calculus with fixed bases φ, φ' does not allow limit $q \to 1$ to the usual calculus. In the present paper we address this problem by working with Fibonacci divisors, instead of Fibonacci numbers. In general, the ratio of two Fibonacci numbers is not an integer number. However, surprising fact is that Fibonacci numbers of the form F_{kn}, where k and n are arbitrary integers, is dividable by Fibonacci number F_k. The infinite sequence of integer numbers $F_{kn}/F_k \equiv F_n^{(k)}$ we call the Fibonacci divisors conjugate to F_k or shortly, the $k-th$ Fibonacci divisors. According to number k we have an infinite hierarchy of Fibonacci divisor sequences. An intriguing fact is that the Binet type formula for $F_n^{(k)}$ divisor is determined by integer k-th power of the Golden ratio φ and $\varphi' = -1/\varphi$,

$$F_n^{(k)} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k}. \tag{1}$$

For $k = 1$ this formula gives the sequence of usual Fibonacci numbers, while for $k = 2, 3, 4, ...$ we have generalized Fibonacci sequences, with three term recurrence relations, depending on value of k. The goal of the present paper is to develop quantum calculus for these hierarchy of number sequences and construct an infinite hierarchy of Golden oscillators, with discrete spectrum determined by these numbers.

The paper is organized as follows. In Section 2 we discuss briefly some physical examples from quantum mechanics, hydrodynamics, quantum information theory and quantum integrable systems, where Fibonacci divisors

$F_n^{(k)}$ and corresponding calculus can be applied. Division of Fibonacci numbers and algebraic properties of corresponding integers, denoted as Fibonacci divisors are subject of Section 3. In Section 4 we develop corresponding calculus and introduce the hierarchy of Golden derivatives, corresponding $k-th$ Golden periodic functions and generating function for Fibonacci divisors. Section 5 deals with entire generating function for Fibonacci divisors and several interesting identities for them. The $k-th$ hierarchy of Fibonomials, constructed from $F_n^{(k)}$ is subject of Section 6. The corresponding hierarchy of Golden binomials, the $k-th$ Golden binomial expansion and Golden Taylor formula are developed in Section 7. By this formula the set of exponential functions, the translation operator and $k-th$ hierarchy of Golden analytic functions are introduced. In Section 8 we apply calculus of Fibonacci divisors to hierarchy of bosonic and fermionic Golden quantum oscillators. Finally, the hierarchy of Golden coherent states and Fock-Bargman representations are obtained as realization of quantum calculus of Fibonacci divisors in complex plane.

2 Physical Motivations of $F_n^{(k)}$

Below we describe several physical systems in which $F_n^{(k)}$ appears naturally.

2.1 $F_N^{(k)}$ and the Golden-deformed bosons and fermions

For even k equation (1) in the limit $k \to 0$ gives just natural numbers, $\lim_{k \to 0} F_n^{(k)} = n$. In contrast, for odd k in this limit we have the binary numbers 0 and 1, for even and odd n correspondingly. This shows that depending on k, Fibonacci divisor numbers (1) can describe deformed bosonic and fermionic quantum oscillators. These oscillators are determined by the hierarchy of quantum deformed algebras

$$b_k b_k^+ - \varphi'^k b_k^+ b_k = \varphi^{kN}, \tag{2}$$

parameterized by integer k, with number operator

$$b_k^+ b_k = F_N^{(k)} = \frac{(\varphi^k)^N - (\varphi'^k)^N}{\varphi^k - \varphi'^k}. \tag{3}$$

Depending on k we have two type of models.

1) For odd $k = 2l + 1$ the algebra

$$b_k b_k^+ + \frac{1}{\varphi^k} b_k^+ b_k = \varphi^{kN}, \tag{4}$$

becomes a non-trivial q-deformed fermionic algebra [2],[3] for the q-deformed quantum oscillators. It was applied to several problems, as the dynamic mass generation of quarks and nuclear pairing [4], [5], and as descriptive of higher order effects in many-body interactions in nuclei [6], [7]. When $0 < q < 1$, an arbitrary number of q-fermions in this algebra can occupy a given state. If we denote $q = \frac{1}{\varphi^{2k}}$, where k is positive odd number, the inequality is satisfied and algebra (4) coincides with the one in [2]. The Fock space construction for the Golden-deformed fermionic algebra requires to introduce the "fermionic q-numbers" [2], in the form (1), for odd k,

$$[n]_{\frac{1}{\varphi^{2k}}}^{F_1} = \frac{\varphi^{kn} - (-1)^n \varphi^{-kn}}{\varphi^k + \varphi^{-k}} = F_n^{(k)}. \tag{5}$$

It shows that for odd k the Fibonacci divisors number operator (3) is a specific realization of fermionic q-number operator of Parthasarathy and Viswanathan [2]. Statistical properties of corresponding q-deformed fermions, as descriptive of fractional statistics were investigated in [8]. As was shown [9], thermodynamics of these generalized fermions should involve the q-calculus with q-derivative in the form

$$D_x f(x) = \frac{1}{x} \frac{f(q^{-1}x) - f(-qx)}{q + q^{-1}}, \tag{6}$$

which for $q = \frac{1}{\varphi}$, becomes the Golden Derivative [1]. In a more general case $q = \frac{1}{\varphi^k}$, for odd k it becomes the k-th Golden derivative (9), studied in the present paper.

2) For even $k = 2l$ the algebra

$$b_k b_k^+ - \frac{1}{\varphi^k} b_k^+ b_k = \varphi^{kN}, \tag{7}$$

becomes the q-deformed bosonic algebra, with number operator

$$F_N^{(k)} = \frac{(\varphi^k)^N - (\varphi^{-k})^N}{\varphi^k - \varphi^{-k}} = \frac{\sinh N(k \ln \varphi)}{\sinh(k \ln \varphi)}. \tag{8}$$

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This algebra and the number operator correspond to quantum algebra with symmetric q-numbers, where $q = \varphi^k$ or in notation of paper [2], $q = \frac{1}{\varphi^{2k}}$.

The above results indicate on intriguing relation between divisibility of Fibonacci numbers for odd and even index k, and hierarchy of deformed fermionic and bosonic quantum algebras correspondingly. As would be shown below all these numbers are integer, and the spectrum of corresponding oscillators is also integer.

The Fibonacci divisors number operator, rewritten in the Fock-Bargman representation $F_N^{(k)} = F_{z\frac{d}{dz}}^{(k)}$, becomes the k-th Golden derivative dilatation operator

$$z_{(k)}D_F^z[f(z)] = \frac{(\varphi^k)^{z\frac{d}{dz}} - (\varphi'^k)^{z\frac{z}{dz}}}{\varphi^k - \varphi'^k} f(z) = \frac{f(\varphi^k z) - f(\varphi'^k z)}{(\varphi^k - \varphi'^k)}, \qquad (9)$$

acting on holomorphic wave function $f(z)$. In this form it can be related with the method of images in hydrodynamics, described in next section.

2.2   Hydrodynamic images and k-th Golden derivatives

As was proposed in [10], an arbitrary complex analytic function describing quantum state in Fock-Bargman representation, can be interpreted as the complex potential of incompressible and irrotational hydrodynamic flow in two dimensions. In several bounded domains, according to the method of images, this flow is described by q-periodic functions and corresponding theorems, such as the two circle theorem [11], the wedge theorem [12] and the strip theorem [13]. Depending on geometry of the domain, parameter q has different geometrical meanings and values. For the two circle theorem it is given by squared ratio of two circle radiuses $q = r_2^2/r_1^2$. By choosing annular domain between concentric circles with radiuses in the Golden ratio, we able to give then hydrodynamic interpretation of our k-th Golden derivatives and corresponding periodic functions. For even and odd k results are different and would be considered separately.

1) For even $k = 2l$, derivative (9) is determined by finite difference

$$z_{(k)}D_F^z[f(z)] = \frac{f(\varphi^k z) - f(\frac{1}{\varphi^k}z)}{\left(\varphi^k - \frac{1}{\varphi^k}\right)} \qquad (10)$$

between values of function f at points $\varphi^k z$ and $\frac{1}{\varphi^k}z$. Geometrically, numbers φ^k and $\frac{1}{\varphi^k}$ are symmetric points for unit circle at origin. The analytic

function $f(z)$ describes irrotational and incompressible hydrodynamic flow in two dimensions. If this flow is in annular domain bounded by two concentric circles $1 < |z| < \sqrt{\varphi}$, then the method of images in the form of two circle theorem can be applied [11]. Replacing the boundary by an infinite set of images it gives complex potential of the flow in following form

$$G_\varphi(z) = f_\varphi(z) + \bar{f}_\varphi\left(\frac{1}{z}\right), \tag{11}$$

where

$$f_\varphi(z) = \sum_{n=-\infty}^{\infty} f(\varphi^n z), \qquad \bar{f}_\varphi\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \bar{f}\left(\varphi^n \frac{1}{z}\right). \tag{12}$$

Here, the first sum describes the image flow in even annular domains and the second sum, in odd ones. Then, the whole complex plane is divided to infinite set of circles with Golden ratio of successive radiuses. The complex function $f_\varphi(z)$ is the Golden periodic function, $f_\varphi(\varphi z) = f_\varphi(z)$ and as follows, it valid also for function $G_\varphi(\varphi z) = G_\varphi(z)$. The last relation implies that

$$_{(k)}D_F^z G_\varphi(z) = 0, \tag{13}$$

for any integer k. The complex velocity corresponding to flow (11),

$$\bar{V}(z) = \frac{dG(z)}{dz} = \sum_{n=-\infty}^{\infty} \varphi^n \bar{v}(\varphi^n z) - \frac{1}{z^2}\sum_{n=-\infty}^{\infty} \varphi^n v\left(\varphi^n \frac{1}{z}\right), \tag{14}$$

is then the Golden self-similar function $\bar{V}(\varphi z) = \frac{1}{\varphi}\bar{V}(z)$.

In more general case, the annular domain is bounded by circles $1 < |z| < \varphi^{\frac{k}{2}}$ with fixed $k > 1$, so that the flow is k-th Golden periodic $G_{\varphi^k}(\varphi^k z) = G_{\varphi^k}(z)$, and as follows

$$_{(k)}D_F^z G_{\varphi^k}(z) = 0. \tag{15}$$

Corresponding complex velocity is the k-Golden self-similar function $\bar{V}(\varphi^k z) = \frac{1}{\varphi^k}\bar{V}(z)$, but the flow is not the Golden periodic one $G_{\varphi^k}(\varphi z) \neq G_{\varphi^k}(z)$.

As an example we consider the point vortex of strength Γ at position z_0 in the annular domain, $1 < |z_0| < \sqrt{\varphi}$, with complex potential [11], [14],

$$G(z) = \frac{\Gamma}{2\pi i}\sum_{n=-\infty}^{\infty} \ln \frac{z - \varphi^n z_0}{z - \varphi^n \frac{1}{\bar{z}_0}}. \tag{16}$$

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It describes an infinite set of vortex images with Golden ratio φ of neighbouring image distances from origin: for even reflections we have

$$..., \frac{z_0}{\varphi^n}, ..., \frac{z_0}{\varphi^2}, \frac{z_0}{\varphi}, z_0, \varphi z_0, \varphi^2 z_0, ..., \varphi^n z_0, ...$$

and for odd reflections

$$..., \frac{1}{\varphi^n \bar{z}_0}, ..., \frac{1}{\varphi^2 \bar{z}_0}, \frac{1}{\varphi \bar{z}_0}, \frac{1}{\bar{z}_0}, \frac{\varphi}{\bar{z}_0}, \frac{\varphi^2}{\bar{z}_0}, ..., \frac{\varphi^n}{\bar{z}_0}, ...$$

Due to characteristic equation for Golden ratio, $\varphi - 1 = \frac{1}{\varphi}$, the relative distance between two neighbouring images is given by power of Golden ratio, $|\varphi^{n+1} - \varphi^n| = \varphi^{n-1}$ - for even reflections, and $\left|\frac{1}{\varphi^n} - \frac{1}{\varphi^{n+1}}\right| = \frac{1}{\varphi^{n+2}}$ - for odd reflections.

Complex potential (16) is the Golden periodic one, which means that any vortex image can be chosen as the original vortex, so that the full flow in the plane will not change by this choice. In addition, following [14] the infinite sum for this potential can be expressed by ratio of first Jacobi elliptic theta functions, with Golden ratio as a parameter.

2) For odd $k = 2l + 1$, derivative (9) is given by the finite difference

$$z_{\,(k)}D_F^z[f(z)] = \frac{f(\varphi^k z) - f(-\frac{1}{\varphi^k}z)}{\left(\varphi^k + \frac{1}{\varphi^k}\right)} \tag{17}$$

between values of function f at points $\varphi^k z$ and $-\frac{1}{\varphi^k}z$. To interpret these image points geometrically, we consider the flow in double-circular wedge domain, with boundary restricted by two coordinate lines $\Gamma_1 : z = x$ and $\Gamma_2 : z = xe^{i\frac{\pi}{2}} = ix$ in first quadrant of complex plane z; and by two concentric circles at origin with radiuses 1 and φ; $C_1 : z = e^{it}$, and $C_2 : z = \sqrt{\varphi}\,e^{it}$, where $0 < t < \frac{\pi}{2}$. Then, application of the double-circular wedge theorem [12] gives complex potential as

$$G(z) = f_\varphi(z) + \bar{f}_\varphi\left(\frac{1}{z}\right), \tag{18}$$

where

$$f_\varphi(z) = \sum_{n=-\infty}^{\infty}[f(\varphi^n z) + f(-\varphi^n z)], \quad \bar{f}_\varphi\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty}[\bar{f}\left(\varphi^n\frac{1}{z}\right) + \bar{f}\left(-\varphi^n\frac{1}{z}\right)].$$

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This function is Golden periodic $G(\varphi z) = G(z)$ and in addition, it is even $G(-z) = G(z)$, so that,

$$_{(k)}D_F^z G(z) = 0. \tag{19}$$

For point vortex in the double-circular wedge domain it gives the sum

$$G(z) = \frac{\Gamma}{2\pi i} \sum_{n=-\infty}^{\infty} \ln \frac{(z^2 - \varphi^{2n} z_0^2)(z^2 - \varphi^{2(n+1)} \frac{1}{z_0^2})}{(z^2 - \varphi^{2n} \bar{z}_0^2)(z^2 - \varphi^{2(n+1)} \frac{1}{\bar{z}_0^2})}, \tag{20}$$

which is even and arbitrary k-th Golden periodic function. The set of images in this sum includes an infinite number of pairs of images at antipodal points, $\varphi^n z_0$ and $-\frac{1}{\varphi^n \bar{z}_0}$, As we show in next section, these antipodal points play an important role in quantum information theory and can be related to the pair of orthogonal qubit coherent states.

2.3 $F_n^{(k)}$ and the $n-$qubit coherent states

The pair of orthogonal qubit states in coherent state representation

$$|\psi\rangle = \frac{|0\rangle + \psi|1\rangle}{\sqrt{1 + |\psi|^2}}, \qquad |-\frac{1}{\bar{\psi}}\rangle = \frac{-\bar{\psi}|0\rangle + |1\rangle}{\sqrt{1 + |\psi|^2}}, \tag{21}$$

is parameterized by complex number $\psi \in C$, given by the stereographic projection $\psi = \tan \frac{\theta}{2} e^{i\phi}$, of the Bloch sphere state $|\theta, \phi\rangle$. The states are determined by point ψ, and its antipodal one $-\frac{1}{\psi}$ for unit circle in C [15]. Geometrically, they correspond to antipodal points on the Bloch sphere, $M(x, y, z)$ and $M^*(-x, -y, -z)$. For the Golden ratio $\varphi = \tan \frac{\theta}{2}$ the states $|\varphi\rangle$ and $|\varphi'\rangle = |-\frac{1}{\varphi}\rangle$ are orthogonal and correspond to antipodal qubit states $|\theta, 0\rangle$ and $|\pi - \theta, \pi\rangle$.

Now we introduce the pair of normalized orthogonal antipodal states

$$|\varphi^k\rangle = \frac{|0\rangle + \varphi^k|1\rangle}{\sqrt{1 + \varphi^{2k}}}, \qquad |-\frac{1}{\varphi^k}\rangle = \frac{-\varphi^k|0\rangle + |1\rangle}{\sqrt{1 + \varphi^{2k}}}, \tag{22}$$

characterized by integer number $k \in Z$. In the limiting case, the states become computational basis states: $|1\rangle$ and $|0\rangle$ for $k \to \infty$, and $|0\rangle$ and $|1\rangle$ for $k \to -\infty$.

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From these states (without normalization) the set of n-qubit states $|\Psi\rangle_k$, characterized by Fibonacci divisors $F_n^{(k)}$ can be constructed. For odd $k = 2l + 1$, we define the state

$$|\Psi\rangle_k = c_0 \frac{(|0\rangle + \varphi^k|1\rangle)^n - (|0\rangle + \varphi'^k|1\rangle)^n}{\varphi^k - \varphi'^k}, \tag{23}$$

which is expanded in computational basis as

$$|\Psi\rangle_k = c_0[F_1^{(k)}(|10...0\rangle + ...|00...1\rangle) + F_2^{(k)}(|110...0\rangle + ...|00...11\rangle) ... + F_n^{(k)}|111...1\rangle].$$

Normalization constant for this state is determined by sum

$$c_0^{-1} = \sqrt{\sum_{s=1}^{n} \binom{n}{s} (F_s^{(k)})^2} \tag{24}$$

and probabilities of measurement are

$$P_l = \frac{(F_l^{(k)})^2}{\sum_{s=1}^{n} \binom{n}{s} (F_s^{(k)})^2}. \tag{25}$$

The states are entangled. For two qubit state with $n = 2$ and arbitrary odd k, we have the state

$$|\Psi\rangle_k = \frac{|01\rangle + |10\rangle + L_k|11\rangle}{\sqrt{2 + L_k^2}}, \tag{26}$$

where $L_k = \varphi^k + \varphi'^k = F_2^{(k)}$ - are the Lucas numbers. The level of entanglement in this state is determined by concurrence, expressed in terms of Lucas numbers as decreasing function of k,

$$C_k = \frac{2}{2 + L_k^2}. \tag{27}$$

The maximal value $C_1 = 2/3$ this concurrence reaches for $k = \pm 1$.

From another side, from normalized states (22) following [15], the set of maximally entangled two qubit states can be derived,

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$$|P_\pm\rangle = \frac{|\varphi^k\rangle|\varphi^k\rangle \pm |-\frac{1}{\varphi^k}\rangle|-\frac{1}{\varphi^k}\rangle}{\sqrt{2}}, \qquad |G_\pm\rangle = \frac{|\varphi^k\rangle|-\frac{1}{\varphi^k}\rangle \pm |-\frac{1}{\varphi^k}\rangle|\varphi^k\rangle}{\sqrt{2}}. \tag{28}$$

Two of these states are independent of k and in explicit form are just the Bell states

$$|P_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad |G_-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}, \tag{29}$$

while another pair of states is superposition of the Bell states, depending on Fibonacci and Lucas numbers:

1) for even $k = 2l$,

$$|P_-\rangle = -\frac{\sqrt{5}F_k}{L_k}\frac{|00\rangle - |11\rangle}{\sqrt{2}} + \frac{2}{L_k}\frac{|01\rangle + |10\rangle}{\sqrt{2}}, \tag{30}$$

$$|G_+\rangle = -\frac{2}{L_k}\frac{|00\rangle - |11\rangle}{\sqrt{2}} - \frac{\sqrt{5}F_k}{L_k}\frac{|01\rangle + |10\rangle}{\sqrt{2}}, \tag{31}$$

2) for odd $k = 2l + 1$,

$$|P_-\rangle = -\frac{L_k}{\sqrt{5}F_k}\frac{|00\rangle - |11\rangle}{\sqrt{2}} + \frac{2}{\sqrt{5}F_k}\frac{|01\rangle + |10\rangle}{\sqrt{2}}, \tag{32}$$

$$|G_+\rangle = -\frac{2}{\sqrt{5}F_k}\frac{|00\rangle - |11\rangle}{\sqrt{2}} - \frac{L_k}{\sqrt{5}F_k}\frac{|01\rangle + |10\rangle}{\sqrt{2}}. \tag{33}$$

By using identities: $5F_k^2 + 4 = L_k^2$, for even $k = 2l$, and $L_k^2 + 4 = 5F_k^2$, for odd $k = 2l + 1$; we find that the concurrence for these states is maximal $C = 1$ and in the limit $k \to \pm\infty$ the states reduce to the second pair of Bell states.

2.4 $F_N^{(k)}$ and Hecke characteristic equation for R matrix

The Fibonacci divisors $F_n^{(k)}$ are connected also with quantum integrable systems approach to the theory of quantum groups, via solution of the Yang-Baxter equation for the R-matrix [17]. The invertible $\hat{R}$ matrix obeys a characteristic equation, known as the Hecke condition. For two roots φ^k and $-\frac{1}{\varphi^k}$,

$$(\hat{R} - \varphi^k)(\hat{R} + \frac{1}{\varphi^k}) = 0 \tag{34}$$

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it is
$$\hat{R}^2 = B_k \hat{R} + I, \tag{35}$$

where $B_k = \varphi^k - \frac{1}{\varphi^k}$. For odd $k = 2l + 1$ these numbers are the Lucas numbers $B_k = \varphi^k + \varphi'^k = L_k$ and equation (35) is characteristic equation for Fibonacci divisors $F_n^{(k)}$, satisfying recursion relation (51),

$$F_{n+1}^{(k)} = L_k F_n^{(k)} + (-1)^{k-1} F_{n-1}^{(k)}. \tag{36}$$

For even $k = 2l$ these numbers can be written by Fibonacci numbers $B_k = \varphi^k - \varphi'^k = F_k \sqrt{5}$.

By studying representations of the braid group, satisfying this quadratic relation a polynomial invariant in two variables for oriented links was obtained by Jones [18]. Calculating higher powers of matrix $\hat{R}$ by repeated application of the Hecke condition (35), we find

$$\hat{R}^n = F_n^{(k)} \hat{R} + F_{n-1}^{(k)} I, \tag{37}$$

in terms of Fibonacci divisors.

3   Fibonacci Divisors

3.1   Addition and Division of Fibonacci Numbers

The addition formula for Fibonacci numbers is given by following proposition.

Proposition 3.1.1 (Addition formula)

$$F_{n+m} = F_m F_{n+1} + F_n F_{m-1}, \quad where \quad m, n \in \mathcal{Z} \tag{38}$$

The proof is straightforward by using Binet formula for Fibonacci numbers. If in (38) we put $m = n \equiv k$, then Fibonacci numbers with even index $n = 2k$ can be factorized as

$$F_{2k} = F_k L_k, \tag{39}$$

where $L_k = F_{k-1} + F_{k+1}$ are the Lucas numbers. By repeating application of this formula

$$
\begin{aligned}
F_{3k} &= F_{k+2k} = F_k F_{2k-1} + F_{k+1}(F_{2k}) = \\
&= F_k F_{2k-1} + F_{k+1}(F_k L_k) = F_k(F_{2k-1} + F_{k+1} L_k),
\end{aligned}
$$

we get factorization

$$F_{3k} = F_k \ (F_{2k-1} + F_{k+1} \ L_k). \tag{40}$$

This can be continued

$$\begin{aligned} F_{4k} &= F_{2k+2k} = F_{2k} \ F_{2k-1} + F_{2k+1} \ F_{2k} = F_{2k}(F_{2k-1} + F_{2k+1}) = \\ &= F_{2k} \ L_{2k} = F_k \ L_k \ L_{2k}, \end{aligned} \tag{41}$$

with following factorization,

$$F_{4k} = F_k \ L_k \ L_{2k}. \tag{42}$$

The above results suggest the following divisibility property of F_{nk}.

Proposition 3.1.2 F_{nk} *is divisible by* F_k.

Proof 3.1.3 *The proof can be done by induction on n. For given n, suppose $F_{nk} = F_k \ X(k, n)$, where $X(k, n) \in \mathcal{Z}$. Then, for $n + 1$, by using (38) we have:*

$$\begin{aligned} F_{(n+1)k} &= F_{k+nk} = F_k \ F_{nk-1} + [F_{nk}] \ F_{k+1} \\ &= F_k \ F_{nk-1} + \ [F_k \ X(k, n)] \ F_{k+1} = F_k(\ F_{nk-1} + \ X(k, n) \ F_{k+1}). \end{aligned}$$

3.2 Fibonacci divisors conjugate to F_k

Since F_{nk} is divisible by F_k, the ratio $\frac{F_{nk}}{F_k}$ is an integer number. We call this number as Fibonacci divisor, conjugate to F_k.

Definition 3.2.1 *The Fibonacci divisor, conjugate to F_k is an integer number*

$$F_n^{(k)} = \frac{F_{nk}}{F_k}. \tag{43}$$

Proposition 3.2.2 *The Binet type formula for Fibonacci divisor is,*

$$F_n^{(k)} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k}. \tag{44}$$

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Proof 3.2.3 *It is derived simply by using the Binet formula for Fibonacci numbers,*

$$F_{nk} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi - \varphi'} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k} \frac{\varphi^k - \varphi'^k}{\varphi - \varphi'} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k} F_k,$$

so that

$$\frac{F_{nk}}{F_k} = F_n^{(k)} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k}. \tag{45}$$

The Fibonacci divisor numbers $F_n^{(k)}$ give factorization of Fibonacci numbers with factorized index nk:

$$F_{nk} = F_k \; F_n^{(k)}.$$

The first few sequences of Fibonacci divisors $F_n^{(k)}$ for $k = 1, 2, 3, 4, 5$ and $n = 1, 2, 3, 4, 5, ...$ are given below

$$k \;=\; 1; F_n^{(1)} = F_n = 1, 1, 2, 3, 5, ... \tag{46}$$

$$k \;=\; 2; F_n^{(2)} = F_{2n} = 1, 3, 8, 21, 55, ... \tag{47}$$

$$k \;=\; 3; F_n^{(3)} = \frac{1}{2} F_{3n} = 1, 4, 17, 72, 305, ... \tag{48}$$

$$k \;=\; 4; F_n^{(4)} = \frac{1}{3} F_{4n} = 1, 7, 48, 329, 2255, ... \tag{49}$$

$$k \;=\; 5; F_n^{(5)} = \frac{1}{5} F_{5n} = 1, 11, 122, 1353, 15005, ... \tag{50}$$

The next theorem gives the three terms recursion formula for Fibonacci divisors.

Theorem 3.2.4 *(Recurrence relation for Fibonacci divisors)*

$$F_{n+1}^{(k)} = L_k F_n^{(k)} + (-1)^{k-1} F_{n-1}^{(k)}. \tag{51}$$

Here, L_k are Lucas numbers. The formula is particular case of more general relation, given by following theorem.

Theorem 3.2.5

$$F_{k(n+1)+\alpha} = L_k F_{kn+\alpha} + (-1)^{k-1} F_{k(n-1)+\alpha}, \quad where \quad \alpha = 0, 1, \ldots, k-1. \tag{52}$$

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The proof of both theorems is given in Appendix.

The total set of Fibonacci numbers $\{F_n\}$ is the sum of subsets $\{F_{kn+\alpha}\}$ for each k and $\alpha = 0, 1, \ldots, k-1$;

$k = 2; \ \{F_{2n}, \ F_{2n+1}\}$

$k = 3; \ \{F_{3n}, \ F_{3n+1}, \ F_{3n+2}\}$

$\vdots$

$k = k; \ \{F_{kn}, \ F_{kn+1}, \ldots, \ F_{kn+(k-1)}\}$

Then equation (52) state that for given k, the subsequences $F_{kn}, \ F_{kn+1}, \ F_{kn+2}, \ldots,$ $F_{kn+(k-1)}$, satisfy the same recursion formula. As an example we consider following sequences with $k = 3$;

$\alpha = 0 \quad \Rightarrow \quad F_{3n} = 0, 2, 8, 34, \ldots$

$\alpha = 1 \quad \Rightarrow \quad F_{3n+1} = 1, 3, 13, 55, \ldots$

$\alpha = 2 \quad \Rightarrow \quad F_{3n+2} = 1, 5, 21, 89, \ldots$

These sequences satisfy the same recursion formula,

$$F_{3(n+1)+\alpha} = L_3 F_{3n+\alpha} + (-1)^{3-1} F_{3(n-1)+\alpha},$$

where $\alpha = 0, 1, 2$, but with different initial values, so that their union set covers the whole Fibonacci sequence. The corresponding Fibonacci divisors $F_n^{(3)}$ conjugate to F_3 are determined by recursion with Lucas number $L_3 = 4$,

$$F_{n+1}^{(3)} = 4F_n^{(3)} + F_{n-1}^{(3)},$$

and the initial values $F_0^{(3)} = 0$ and $F_1^{(3)} = 1$. This gives the sequence of numbers $0, 1, 4, 17, 72, 305, \ldots$

Proposition 3.2.6 *The Fibonacci divisors $F_n^{(k)}$ can be extended to negative integers according to formulas*

$$
\begin{aligned}
F_{-n}^{(k)} &= (-1)^{kn+1} F_n^{(k)}, & (53)\\
F_n^{(-k)} &= (-1)^{(n+1)k} F_n^{(k)}, & (54)\\
F_{-n}^{(-k)} &= (-1)^{k+1} F_n^{(k)}. & (55)
\end{aligned}
$$

These formulas can be proved by direct application of the Binet representation. The formulas determine $F_n^{(k)}$ for each $k \in \mathcal{Z}$, and each $n \in \mathcal{Z}$.

The following proposition relates powers of the Golden ratio with Fibonacci divisors.

Proposition 3.2.7 *For $k \in Z$ and $n \in Z$,*

$$(\varphi^k)^n = \varphi^k F_n^{(k)} + (-1)^{k+1} F_{n-1}^{(k)}, \tag{56}$$

$$(\varphi'^k)^n = \varphi'^k F_n^{(k)} + (-1)^{k+1} F_{n-1}^{(k)}. \tag{57}$$

The proof is straightforward.

Corollary 3.2.8 *Numbers φ^k and φ'^k are roots of quadratic characteristic equation*

$$\lambda_k^2 = L_k \lambda_k + (-1)^{k+1}, \tag{58}$$

so that

$$\left(\varphi^k\right)^2 = L_k \varphi^k + (-1)^{k+1}, \quad \left(\varphi'^k\right)^2 = L_k \varphi'^k + (-1)^{k+1}. \tag{59}$$

The last relations follow from the proposition for $n = 2$ and $F_2^{(k)} = \varphi^k + \varphi'^k = L_k$.

4   Hierarchy of Golden Derivatives

In this section we introduce the Golden derivative operators $_{(k)}D_F^x$ corresponding to Fibonacci divisors, conjugate to F_k.

Definition 4.0.1 *For arbitrary function $f(x)$ and $k \in Z$,*

$$_{(k)}D_F^x[f(x)] = \frac{f(\varphi^k x) - f(\varphi'^k x)}{(\varphi^k - \varphi'^k) \, x}. \tag{60}$$

The linear operator $_{(k)}D_F^x$ we call the kth Golden derivative operator. The complex analytic version of this derivative, acting on an analytic function $f(z)$, can be represented by the Fibonacci divisor number operator $F_N^{(k)}$ in the Fock-Bargman representation, where $N = z\frac{d}{dz}$,

$$F_{z\frac{d}{dz}}^{(k)} f(z) = \frac{\varphi^{kz\frac{d}{dz}} - \varphi'^{kz\frac{d}{dz}}}{\varphi^k - \varphi'^k} f(z) = z \, _{(k)}D_F^z[f(z)]. \tag{61}$$

For even k this formula admits the limit $k \to 0$, giving the usual derivative

$$\lim_{k \to 0} D_F^z[f(z)] = f'(z). \tag{62}$$

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For $k = 1$, the derivative reduces to the Golden derivative [1],

$$_{(1)}D_F^x[f(x)] = D_F^x[f(x)] = \frac{f(\varphi x) - f(\varphi' x)}{(\varphi - \varphi') x}, \tag{63}$$

which by acting on monomial x^n produces Fibonacci numbers,

$$_{(1)}D_F^x(x^n) = \frac{(\varphi x)^n - (\varphi' x)^n}{(\varphi - \varphi') x} = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'} x^{n-1} = F_n \ x^{n-1}.$$

In a similar way, by applying the kth Golden derivative to x^n, we get the Fibonacci divisor numbers $F_n^{(k)}$, conjugate to F_k,

$$_{(k)}D_F^x(x^n) = \frac{(\varphi^k x)^n - (\varphi'^k x)^n}{(\varphi^k - \varphi'^k) x} = \frac{(\varphi^k)^n - (\varphi'^k)^n}{\varphi^k - \varphi'^k} x^{n-1} = F_n^{(k)} x^{n-1},$$

or

$$_{(k)}D_F^x[x^n] = F_n^{(k)} x^{n-1}. \tag{64}$$

This formula for negative k produces numbers, in accordance with (54),

$$F_n^{(-k)} = (-1)^{(n+1)k} F_n^{(k)}.$$

The Leibnitz and quotient rules for k^{th} Golden derivative are subject of next propositions.

Proposition 4.0.2 *(The Leibnitz Rule)*

$$_{(k)}D_F^x(f(x)g(x)) =_{(k)} D_F^x(f(x)) \ g(\varphi^k x) + f\left(\varphi'^k x\right) \ _{(k)}D_F^x(g(x)). \tag{65}$$

Proposition 4.0.3 *(The Quotient Rule)*

$$_{(k)}D_F^x\left(\frac{f(x)}{g(x)}\right) = \frac{_{(k)}D_F^x(f(x)) \ g(\varphi^k x) - f(\varphi^k x) \ _{(k)}D_F^x(g(x))}{g(\varphi^k x) \ g\left(\varphi'^k x\right)}. \tag{66}$$

By applying Leibnitz rule to (64),

$$_{(k)}D_F^x(x^n) =_{(k)} D_F^x(x^m x^{n-m}) = F_n^{(k)} x^{n-1}, \tag{67}$$

we get the following corollary.

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Corollary 4.0.4

$$F_n^{(k)} = F_m^{(k)} \, \varphi^{k(n-m)} + F_{n-m}^{(k)} \, \varphi'^{km}. \tag{68}$$

This corollary allows us to formulate following proposition.

Proposition 4.0.5

$$F_n^{(k)} = F_{n-m}^{(k)} \, F_{m+1}^{(k)} + (-1)^{k+1} \, F_m^{(k)} \, F_{n-m-1}^{(k)}. \tag{69}$$

In particular case $k = 1$ it gives addition formula for Fibonacci numbers,

$$F_n = F_{n-m} \, F_{m+1} + F_m \, F_{n-m-1}.$$

Proof 4.0.6 *By substituting (56) and (57), to (68) we get*

$$
\begin{aligned}
F_n^{(k)} &= & F_m^{(k)} \left(\varphi^k F_{n-m}^{(k)} + (-1)^{k+1} F_{n-m-1}^{(k)} \right) + F_{n-m}^{(k)} \left(\varphi'^k F_m^{(k)} + (-1)^{k+1} F_{m-1}^{(k)} \right) \\
&= & F_m^{(k)} \, F_{n-m}^{(k)} \left(\varphi^k + \varphi'^k \right) + (-1)^{k+1} F_m^{(k)} \, F_{n-m-1}^{(k)} + (-1)^{k+1} F_{n-m}^{(k)} \, F_{m-1}^{(k)} \\
&= & F_m^{(k)} \, F_{n-m}^{(k)} \, L_k + (-1)^{k+1} F_m^{(k)} \, F_{n-m-1}^{(k)} + (-1)^{k+1} F_{n-m}^{(k)} \, F_{m-1}^{(k)} \\
&= & F_{n-m}^{(k)} \left(L_k \, F_m^{(k)} + (-1)^{k+1} \, F_{m-1}^{(k)} \right) + (-1)^{k+1} F_m^{(k)} \, F_{n-m-1}^{(k)} \\
(51) &= & F_{n-m}^{(k)} \, F_{m+1}^{(k)} + (-1)^{k+1} \, F_m^{(k)} \, F_{n-m-1}^{(k)}.
\end{aligned}
$$

By shifting index n in (69) we get addition formula for $F_n^{(k)}$ in convenient form.

Proposition 4.0.7 *An addition formula for Fibonacci divisor numbers is*

$$F_{n+m}^{(k)} = F_m^{(k)} \, F_{n+1}^{(k)} + (-1)^{k+1} \, F_n^{(k)} \, F_{m-1}^{(k)}.$$

For $k = 1$, it reduces to addition formula for Fibonacci numbers (38).

4.1   Hierarchy of Golden Periodic Functions

The Golden periodic function for Golden derivative plays the same role as a constant for the usual derivative. Similarly, the set of kth Golden derivatives (60) determines the hierarchy of Golden periodic functions for every natural k.

Proposition 4.1.1 *Every Golden periodic function $D_F^x(f(x)) = 0 \iff (f(\varphi x) = f(\varphi' x))$ is also periodic for arbitrary $k - th$ order Golden derivatives,*

$$D_F^x(f(x)) = 0 \quad \Rightarrow \quad {}_{(k)}D_F^x(f(x)) = 0,$$
$$f(\varphi x) = f(\varphi' x) \quad \Rightarrow \quad f(\varphi^k x) = f(\varphi'^k x), \tag{70}$$

for $k = 2, 3, \ldots$

The proof is evident by induction. As an example, $f(x) = \sin\left(\frac{\pi}{\ln \varphi} \ln |x|\right)$ as the Golden periodic function,

$$D_F^x \sin\left(\frac{\pi}{\ln \varphi} \ln |x|\right) = 0,$$

is also the $k - th$ Golden periodic,

$$_{(k)}D_F^x \sin\left(\frac{\pi}{\ln \varphi} \ln |x|\right) = 0.$$

But the opposite to Proposition 4.1.1 is not in general true. If function $f(x)$ is k- periodic, $k = 2, 3...,$

$$_{(k)}D_F^x(f(x)) = 0,$$

then it is not necessarily the Golden periodic one (with $k = 1$). For example, $f(x) = \sin\left(\frac{\pi}{\ln \varphi^2} \ln |x|\right)$ is the Golden periodic function with $k = 2$, i.e $_{(2)}D_F^x(f(x)) = 0$. However, the Golden derivative of this function doesn't vanish,

$$D_F^x(f(x)) = 2 \frac{\cos\left(\frac{\pi}{\ln(\varphi^2)} \ln |x|\right)}{(\varphi - \varphi')x} \neq 0$$

and it is not the Golden periodic function.

4.2   Generating Function for Fibonacci divisors

Definition 4.2.1 *Function,*

$$_{(k)}F(x) = \sum_{n=0}^{\infty} F_n^{(k)} x^n = F_0^{(k)} + F_1^{(k)} x + F_2^{(k)} x^2 + F_3^{(k)} x^3 + \ldots$$

is called the generating function for Fibonacci divisors $F_n^{(k)}$, where

$$F_n^{(k)} = \frac{1}{n!} \frac{d^n}{dx^n} {}_{(k)}F(x) \Big|_{x=0}. \tag{71}$$

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Proposition 4.2.2 *The generating function $_{(k)}F(x)$ is convergent in the disk $|x| < \frac{1}{\varphi^k}$ and has following explicit representation*

$$_{(k)}F(x) = \sum_{n=0}^{\infty} F_n^{(k)} x^n = \frac{1}{1 - L_k x + (-1)^k x^2}. \tag{72}$$

Proof 4.2.3 *The ratio test*

$$\rho = \lim_{n\to\infty} \left| \frac{F_{n+1}^{(k)} x^{n+1}}{F_n^{(k)} x^n} \right| = \lim_{n\to\infty} \left| \frac{F_{n+1}^{(k)}}{F_n^{(k)}} \right| |x| = \varphi^k \, |x| < 1$$

implies,

$$|x| < \left(\frac{1}{\varphi}\right)^k < 1,$$

for any positive k. By using $k - th$ Golden derivative

$$
\begin{aligned}
_{(k)}F(x) &= \sum_{n=0}^{\infty} F_n^{(k)} x^n = F_0^{(k)} + \sum_{n=1}^{\infty} x F_n^{(k)} x^{n-1} = 0 + \sum_{n=1}^{\infty} x \, F_n^{(k)} \, x^{n-1} = \\
&= \sum_{n=1}^{\infty} x \,_{(k)} D_F^x(x^n) = x \,_{(k)} D_F^x \sum_{n=1}^{\infty} x^n = \\
&= x \,_{(k)} D_F^x \, x \left(1 + x + x^2 + \dots\right) \\
&= x \,_{(k)} D_F^x \left(\frac{x}{1-x}\right) = x \left[_{(k)} D_F^x \left(\frac{1}{1-x}\right)\right] = x \, \frac{\left(\frac{1}{1-\varphi^k x} - \frac{1}{1-\varphi'^k x}\right)}{(\varphi^k - \varphi'^k)x} = \\
&= \frac{x}{(1 - \varphi^k x)(1 - \varphi'^k x)} = \frac{x}{1 - L_k \, x + (-1)^k \, x^2}.
\end{aligned}
$$

Corollary 4.2.4 $_{(k)}F(x)$ *is rational function with simple zero at $x = 0$ and two simple poles at*

$$x = \frac{1}{\varphi^k}, \quad x = \frac{1}{\varphi'^k}.$$

For $k = 1$ it reduces to generating function for Fibonacci numbers,

$$_{(1)}F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2} = \frac{x}{(1 - \varphi x)(1 - \varphi' x)}. \tag{73}$$

For $k = 2$ and $k = 3$ it gives generating function for "mod 2" and "mod 3" Fibonacci numbers correspondingly,

$$_{(2)}F(x) = \sum_{n=0}^{\infty} F_{2n}x^n = \frac{x}{1 - 3x + x^2} = \frac{x}{(1 - \varphi^2 x)(1 - \varphi'^2 x)}, \quad (74)$$

$$_{(3)}F(x) = \frac{1}{2}\sum_{n=0}^{\infty} F_{3n}x^n = \frac{x}{1 - 4x - x^2} = \frac{x}{(1 - \varphi^3 x)(1 - \varphi'^3 x)}. \quad (75)$$

For arbitrary k it represents generating function for "mod k"-Fibonacci numbers,

$$_{(k)}F(x) = \frac{1}{F_k}\sum_{n=0}^{\infty} F_{kn}x^n = \frac{x}{1 - L_k x + (-1)^k x^2} = \frac{x}{(1 - \varphi^k x)(1 - \varphi'^k x)}. \quad (76)$$

5 Entire generating function for Fibonacci divisors

Applying $_{(k)}D_F^x$ to e^x in power series form

$$
\begin{aligned}
_{(k)}D_F^x(e^x) &= {}_{(k)}D_F^x\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = {}_{(k)}D_F^x\left(\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\\
&= \sum_{n=1}^{\infty}\frac{_{(k)}D_F^x(x^n)}{n!} = \sum_{n=1}^{\infty} F_n^{(k)}\frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} F_{n+1}^{(k)}\frac{x^n}{(n+1)!}
\end{aligned}
$$

we get the series

$$_{(k)}D_F^x(e^x) = \sum_{n=0}^{\infty} F_{n+1}^{(k)}\frac{x^n}{(n+1)!}, \quad (77)$$

convergent for arbitrary x. From another side, by definition (60),

$$_{(k)}D_F^x(e^x) = \frac{e^{\varphi^k x} - e^{\varphi'^k x}}{(\varphi^k - \varphi'^k)x} = e^{\frac{\varphi^k + \varphi'^k}{2}x}\frac{e^{\left(\frac{\varphi^k - \varphi'^k}{2}\right)x} - e^{-\left(\frac{\varphi^k - \varphi'^k}{2}\right)x}}{(\varphi^k - \varphi'^k)x},$$

and relations $\varphi^k + \varphi'^k = L_k$, $\varphi^k - \varphi'^k = F_k(\varphi - \varphi')$, we have

$$_{(k)}D_F^x(e^x) = 2e^{\frac{L_k}{2}x}\frac{\sinh\left(\frac{F_k}{2}(\varphi - \varphi')x\right)}{F_k(\varphi - \varphi')x}.$$

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Replacing $\varphi - \varphi' = \sqrt{5}$, we get explicit formula for derivative of exponential function

$$_{(k)}D_F^x(e^x) = e^{\frac{L_k}{2}x} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2}x\right)}{\left(F_k \frac{\sqrt{5}}{2}x\right)}. \tag{78}$$

By comparing (77) and (78), finally we obtain identity:

$$\sum_{n=1}^{\infty} \frac{F_n^{(k)}}{n!}x^n = e^{\frac{L_k}{2}x} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2}x\right)}{F_k \frac{\sqrt{5}}{2}}. \tag{79}$$

In particular case $k = 1$, it reduces to the one for Fibonacci numbers [1],

$$\sum_{n=1}^{\infty} \frac{F_n}{n!}x^n = e^{\frac{1}{2}x} \frac{\sinh\left(\frac{\sqrt{5}}{2}x\right)}{\frac{\sqrt{5}}{2}}. \tag{80}$$

Relation (79) represents the entire generating function for Fibonacci divisors, and allows us to obtain several interesting identities.

5.1 Some identities for $F_n^{(k)}$

From (79) for $x = 1$ follow summation formulas

$$\sum_{n=0}^{\infty} \frac{F_n^{(k)}}{n!} = e^{\frac{L_k}{2}} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2}\right)}{\left(F_k \frac{\sqrt{5}}{2}\right)}, \tag{81}$$

and

$$\sum_{n=0}^{\infty} \frac{F_{nk}}{n!} = e^{\frac{L_k}{2}} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2}\right)}{\left(\frac{\sqrt{5}}{2}\right)}. \tag{82}$$

By replacing $x \to ix$ in (79) we get,

$$\sum_{n=1}^{\infty} \frac{F_n^{(k)}}{n!}(i)^n x^n = e^{i\frac{L_k}{2}x} \frac{\sinh\left(F_k \frac{\sqrt{5}}{2}ix\right)}{F_k \frac{\sqrt{5}}{2}}. \tag{83}$$

Substituting $\sinh(ix) = i\sin(x)$ and splitting sum in the l.h.s. to even and odd parts gives,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l x^{2l+1} + i\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l x^{2l+2} = e^{i\frac{L_k}{2}x} \frac{\sin(F_k \frac{\sqrt{5}}{2}x)}{F_k \frac{\sqrt{5}}{2}} \tag{84}$$

By using Euler formula $e^{i\frac{L_k}{2}x} = \cos(\frac{L_k}{2}x) + i\sin(\frac{L_k}{2}x)$, and splitting equality to real and imaginary parts, we get generating functions for even and odd Fibonacci divisors:

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l x^{2l+1} = \cos\left(\frac{L_k}{2}x\right) \frac{\sin(F_k\frac{\sqrt{5}}{2}x)}{F_k\frac{\sqrt{5}}{2}}, \tag{85}$$

and

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l x^{2l+2} = \sin\left(\frac{L_k}{2}x\right) \frac{\sin(F_k\frac{\sqrt{5}}{2}x)}{F_k\frac{\sqrt{5}}{2}}. \tag{86}$$

From these entire functions follow several identities. From (85) we have:
1) For $x = \pi$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \pi^{2l} = \cos\left(\frac{\pi}{2}L_k\right) \frac{\sin(F_k\frac{\sqrt{5}}{2}\pi)}{F_k\frac{\sqrt{5}}{2}\pi}. \tag{87}$$

The right hand side of this identity vanishes for odd values of Lucas numbers L_k: $L_1 = 1, L_2 = 3, L_4 = 7, L_5 = 11, etc.$
2) $x = \frac{2\pi}{\sqrt{5}}$,
Since $\sin(F_k\pi) = 0$, then for arbitrary k;

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!} (-1)^l \frac{(2\pi)^{2l}}{5^l} = 0. \tag{88}$$

3) $x = \frac{\pi}{\sqrt{5}}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \frac{\pi^{2l}}{5^l} = \frac{2}{F_k\pi} \cos\left(L_k\frac{\pi}{2\sqrt{5}}\right) \sin\left(F_k\frac{\pi}{2}\right). \tag{89}$$

For even values of F_k; $F_3 = 2, F_6 = 8, F_9 = 34, etc.$ the right hand side vanishes.
4) $x = 2\pi$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l (2\pi)^{2l} = \cos(L_k\pi) \frac{\sin(F_k\sqrt{5}\pi)}{F_k\sqrt{5}\pi}. \tag{90}$$

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5) $x = 1$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l = \cos\left(\frac{L_k}{2}\right) \frac{\sin(F_k\frac{\sqrt{5}}{2})}{F_k\frac{\sqrt{5}}{2}}. \tag{91}$$

6) $x = \frac{\pi}{L_k}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \left(\frac{\pi}{L_k}\right)^{2l} = 0. \tag{92}$$

7) $x = \frac{2\pi}{\sqrt{5}F_k}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \frac{(2\pi)^{2l}}{5^l(F_k)^{2l}} = 0. \tag{93}$$

In a similar way from (86) follow identities:
1) $x = \pi$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l(\pi)^{2l+1} = \sin\left(\frac{\pi}{2}L_k\right) \frac{\sin(F_k\frac{\sqrt{5}}{2}\pi)}{F_k\frac{\sqrt{5}}{2}\pi}. \tag{94}$$

For even Lucas numbers: $L_3 = 4, L_6 = 18, L_9 = 76, etc.$ the right hand side
is zero.
2) $x = \frac{2\pi}{\sqrt{5}}$. Since $\sin(F_k\pi) = 0$, then

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!} (-1)^l \left(\frac{2\pi}{\sqrt{5}}\right)^{2l+1} = 0. \tag{95}$$

3) $x = \frac{\pi}{\sqrt{5}}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l \left(\frac{\pi}{\sqrt{5}}\right)^{2l+1} = \frac{2}{F_k\pi}\sin\left(L_k\frac{\pi}{2\sqrt{5}}\right)\sin\left(F_k\frac{\pi}{2}\right). \tag{96}$$

For even Fibonacci numbers F_k: $F_3 = 2, F_6 = 8, F_9 = 34, etc.$ the right hand
side vanishes.
4) $x = 2\pi$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l(2\pi)^{2l+1} = 0. \tag{97}$$

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5) $x = 1$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l = \sin\left(\frac{L_k}{2}\right) \frac{\sin(F_k \frac{\sqrt{5}}{2})}{F_k \frac{\sqrt{5}}{2}}. \tag{98}$$

6) $x = \frac{\pi}{L_k}$,

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l \left(\frac{\pi}{L_k}\right)^{2l+1} = \frac{\sin\left(\frac{F_k}{L_k}\frac{\sqrt{5}}{2}\pi\right)}{\frac{F_k}{L_k}\frac{\sqrt{5}}{2}\pi}. \tag{99}$$

7) $x = \frac{2\pi}{L_k}$. Since $\sin(\frac{L_k}{2}\frac{2\pi}{L_k}) = \sin(\pi) = 0$, then

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l \left(\frac{2\pi}{L_k}\right)^{2l+1} = 0. \tag{100}$$

For $x = \frac{\pi}{\sqrt{5}F_k}$, two more interesting identities follow from (85) and (86) correspondingly,

$$\sum_{l=0}^{\infty} \frac{F_{2l+1}^{(k)}}{(2l+1)!}(-1)^l \frac{\pi^{2l}}{5^l F_k^{2l}} = \frac{2}{\pi} \cos\left(\frac{L_k}{F_k}\frac{\pi}{2\sqrt{5}}\right), \tag{101}$$

and

$$\sum_{l=0}^{\infty} \frac{F_{2l+2}^{(k)}}{(2l+2)!}(-1)^l \left(\frac{\pi}{\sqrt{5}F_k}\right)^{2l+1} = \frac{2}{\pi} \sin\left(\frac{L_k}{F_k}\frac{\pi}{2\sqrt{5}}\right). \tag{102}$$

6   Fibonacci divisors and Fibonomials

Definition 6.0.1 *The product,*

$$k \cdot 2k \cdot 3k \ldots nk \equiv n!_{mod\, k} \tag{103}$$

is called mod k factorial. It is equal,

$$\prod_{s=1}^{n} sk = n! k^n \tag{104}$$

and for particular case it reduces to usual factorial

$$k = 1 \quad \Rightarrow \quad n!_{mod\, 1} = n!$$

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Definition 6.0.2 *The product of Fibonacci numbers,*

$$F_k F_{2k} \ldots F_{nk} = \prod_{s=1}^{n} F_{sk} \equiv F_n!_{\ mod\ k} \tag{105}$$

is called mod k Fibonacci factorial. For $k = 1$, it gives the Fibonacci factorial,

$$F_n!_{\ mod\ 1} = F_1 F_2 \ldots F_n = F_n! \tag{106}$$

For $k = 2$ and even n, it gives the double Fibonacci factorial,

$$F_n!_{\ mod\ 2} = F_2 F_4 \ldots F_{2n}. \tag{107}$$

Definition 6.0.3 *The product of Fibonacci divisors,*

$$F_1^{(k)} F_2^{(k)} \ldots F_n^{(k)} = \prod_{i=1}^{n} F_i^{(k)} \equiv F_n^{(k)}! \tag{108}$$

is called the Fibonacci divisors factorial. It can be considered as the $k - th$ Fibonorial or generalized Fibonorial. In particular case $k = 1$, it reduces to Fibonacci factorial; $F_n^{(1)}! = F_n!$. For $F_n^{(k)}!$ we have next formula,

$$F_n^{(k)}! = \frac{F_k F_{2k} F_{3k} \ldots F_{nk}}{F_k F_k F_k \ldots F_k} = \frac{F_k F_{2k} F_{3k} \ldots F_{nk}}{(F_k)^n}, \tag{109}$$

or in terms of mod k Fibonacci factorial,

$$F_n^{(k)}! = \frac{F_n!_{\ mod\ k}}{(F_k)^n}. \tag{110}$$

Definition 6.0.4 *The Fibonomial coefficients for Fibonacci divisors or shortly $k - th$ Fibonomials are defined as*

$$_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_1^{(k)} F_2^{(k)} \ldots F_{n-m+1}^{(k)}}{F_1^{(k)} F_2^{(k)} \ldots F_m^{(k)}} = \frac{F_n^{(k)}!}{F_m^{(k)}! F_{n-m}^{(k)}!}, \tag{111}$$

and for $k = 1$ they reduce to Fibonomials,

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_n!}{F_{n-m}! F_m!}.$$

For arbitrary k they can be represented by mod k Fibonacci factorials (110);

$$_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_n!_{\ mod\ k}}{F_m!_{\ mod\ k}\ F_{n-m}!_{\ mod\ k}}. \tag{112}$$

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In a similar way as for Fibonomials, it is possible to derive the recursion formula for $k - th$ Fibonomials

$$_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F = \varphi'^{km} {}_{(k)}\begin{bmatrix} n-1 \\ m \end{bmatrix}_F + \varphi^{k(n-m)} {}_{(k)}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_F, \tag{113}$$

and give interpretation of it in terms of Pascal type triangle.

7 Hierarchy of Golden binomials

By $k - th$ Fibonomials we can introduce now the hierarchy of Golden binomials.

Definition 7.0.1 *The $k - th$ Golden binomial is defined by polynomial,*

$$_{(k)}\left(x - a\right)_F^n = \left(x - \varphi^{k(n-1)}a\right)\left(x - \varphi^{k(n-2)}\varphi'^k a\right) \ldots \left(x - \varphi^k \varphi'^{k(n-2)}a\right)\left(x - \varphi'^{k(n-1)}a\right)$$

if $n \geq 1$, and it is equal one if $n = 0$.

In particular case $k = 1$, it reduces to the Golden binomial [1]. The polynomial satisfies following factorization formula.

Proposition 7.0.2 *(Factorization Property)*

$$_{(k)}(x - a)_F^{n+m} = {}_{(k)}\left(x - \varphi^{km}a\right)_F^n {}_{(k)}\left(x - \varphi'^{kn}a\right)_F^m \tag{114}$$

$$= {}_{(k)}\left(x - \varphi'^{km}a\right)_F^n {}_{(k)}\left(x - \varphi^{kn}a\right)_F^m \tag{115}$$

The proof is straightforward.

Theorem 7.0.3 *The $k - th$ Golden binomial expansion is,*

$$_{(k)}(x + y)_F^n = \sum_{m=0}^n {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}} x^{n-m} y^m \tag{116}$$

In particular case $k = 1$, it reduces to the Golden binomial formula [1].

The proof can be done by induction.

Corollary 7.0.4 *From this theorem the identity follows,*

$$_{(k)}(1 + 1)_F^n = \sum_{m=0}^n {}_{(k)}\begin{bmatrix} n \\ m \end{bmatrix}_F (-1)^{k\frac{m(m-1)}{2}}. \tag{117}$$

Lemma 7.0.5 *The $k-th$ Golden derivatives are acting on $k-th$ Golden binomials as,*

$$_{(k)}D_F^x \;_{(k)}(x+y)_F^n \;=\; F_n^{(k)} \;_{(k)}(x+y)_F^{n-1}, \tag{118}$$

$$_{(k)}D_F^y \;_{(k)}(x+y)_F^n \;=\; F_n^{(k)} \;_{(k)}\left(x+(-1)^k y\right)_F^{n-1}, \tag{119}$$

$$_{(k)}D_F^y \;_{(k)}(x-y)_F^n \;=\; -F_n^{(k)} \;_{(k)}\left(x-(-1)^k y\right)_F^{n-1}. \tag{120}$$

The proof is long but straightforward.

7.1 Fibonacci divisors and Golden Taylor expansion

Here we introduce monomials,

$$P_n^{(k)} \equiv \frac{x^n}{F_n^{(k)}!}, \tag{121}$$

so that due to (64),

$$_{(k)}D_F^x \left(P_n^{(k)}\right) = P_{n-1}^{(k)}. \tag{122}$$

The monomials satisfy Theorem 2.1 in [19], and allows one to derive the Taylor type expansion for arbitrary polynomials.

Theorem 7.1.1 *(k−th Golden Taylor expansion)*
    The derivative operator $_{(k)}D_F^x$ is a linear operator on the space of polynomials, and

$$P_n^{(k)}(x) \equiv \frac{x^n}{F_n^{(k)}!} \equiv \frac{x^n}{F_1^{(k)} \cdot F_2^{(k)} \dots F_n^{(k)}}$$

satisfy the following conditions:
(i) $P_0^{(k)}(0) = 1$ and $P_n^{(k)}(0) = 0$ for any $n \geq 1$;
(ii) $\deg(P_n^{(k)}) = n$;
(iii) $_{(k)}D_F^x(P_n^{(k)}(x)) = P_{n-1}^{(k)}(x)$ for any $n \geq 1$, and $_{(k)}D_F^x(1) = 0$.
Then, for any polynomial f(x) of degree N, one has the following Taylor formula;

$$f(x) = \sum_{n=0}^{N} (_{(k)}D_F^x)^n f\,(0) P_n^{(k)}(x) = \sum_{n=0}^{N} (_{(k)}D_F^x)^n f\,(0) \frac{x^n}{F_n^{(k)}!}. \tag{123}$$

As an example

$$(x+1)^3 = 24P_3^{(2)}(x) + 9P_2^{(2)}(x) + 3P_1^{(2)}(x) + P_0^{(2)}(x). \qquad (124)$$

In the limit $N \to \infty$ (if it exists) the Taylor formula (123) determines an expansion of function $f(x)$ in $P_n^{(k)}(x)$ polynomials,

$$f(x) = \sum_{n=0}^{\infty} ((k)D_F^x f)^n(0) \frac{x^n}{F_n^{(k)}!}.$$

Proposition 7.1.2 *Let,*

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$$

is an entire complex valued function of complex variable z. Then, for any integer k exists corresponding complex function $_{(k)} f_F(z)$ determined by formula,

$$_{(k)} f_F(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{F_n^{(k)}!}$$

and this function is entire.

Proof 7.1.3 *By the ratio test*

$$\rho = |z| \lim_{n\to\infty} \left| \frac{1}{n+1} \right| \left| \frac{a_{n+1}}{a_n} \right| = 0 \Rightarrow \qquad (125)$$

$$_{(k)}\rho_F = |z| \lim_{n\to\infty} \left| \frac{1}{F_{n+1}^{(k)}} \right| \left| \frac{a_{n+1}}{a_n} \right| \qquad (126)$$

$$= |z| \lim_{n\to\infty} \left| \frac{n+1}{F_{n+1}^{(k)}} \right| \left(\left| \frac{1}{n+1} \right| \left| \frac{a_{n+1}}{a_n} \right| \right)$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{F_{n+1}^{(k)}} \right| \rho = 0,$$

since $\lim_{n\to\infty} \left| \frac{n+1}{F_{n+1}^{(k)}} \right| = 0.$

As an example of entire complex functions, the following hierarchy of the pair of Golden exponential functions is introduced

Definition 7.1.4 *(k−th Golden exponentials)*

$$_{(k)}e_F^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{F_n^{(k)}!}, \tag{127}$$

$$_{(k)}E_F^x \equiv \sum_{n=0}^{\infty} (-1)^{k\frac{n(n-1)}{2}} \frac{x^n}{F_n^{(k)}!}, \tag{128}$$

where

$$F_n^{(k)}! = F_1^{(k)} \cdot F_2^{(k)} \cdot F_n^{(k)} \dots F_n^{(k)} = \frac{F_k \cdot F_{2k} \cdot F_{3k} \dots F_{nk}}{(F_k)^n}. \tag{129}$$

Proposition 7.1.5 *The $k-th$ Golden derivative of $k-th$ Golden exponentials is*

$$_{(k)}D_F^x \left(_{(k)}e_F^{\lambda x} \right) = \lambda \,_{(k)}e_F^{\lambda x}, \tag{130}$$

$$_{(k)}D_F^x \left(_{(k)}E_F^{\lambda x} \right) = \lambda \,_{(k)}E_F^{(-1)^k \lambda x}. \tag{131}$$

This two exponential functions are related by formula

$$_{(k)}E_F^x =_{(-k)} e_F^x. \tag{132}$$

The product of two exponentials is represented by series in powers of $k-$th Golden binomials

$$_{(k)}E_F^x \cdot_{(k)} e_F^y = \sum_{n=0}^{\infty} \frac{_{(k)}(x+y)_F^n}{F_n^{(k)}!} \equiv_{(k)} e_F^{(k)(x+y)_F}. \tag{133}$$

By exponential function (128) it is possible to introduce translation operator $_{(k)}E_F^{y(k)D_F^x}$ generating these binomials from monomial x^n,

$$_{(k)}E_F^{y(k)D_F^x} x^n =_{(k)} (x+y)_F^n. \tag{134}$$

Then, applying this operator to an arbitrary analytic function we get an infinite hierarchy of $k-$th Golden functions

$$_{(k)}E_F^{y(k)D_F^x} f(x) =_{(k)} E_F^{y(k)D_F^x} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \cdot_{(k)} (x+y)_F^n. \tag{135}$$

In particular, translated by this operator exponential function (127) gives equation (133),

$$_{(k)}E_F^{y_{(k)}D_F^x}\ _{(k)}e_F^x =_{(k)}\ e_F^{(k)}{}^{(x+y)_F}, \tag{136}$$

or in another form

$$_{(k)}E_F^{y_{(k)}D_F^x}\ _{(k)}e_F^x =_{(k)}\ E_F^x \cdot_{(k)}\ e_F^y. \tag{137}$$

7.2 Hierarchy of Golden analytic functions

By complex version of translation operator (134) we introduce the complex $k-$th Golden analytic binomials

$$_{(k)}E_F^{iy_{(k)}D_F^x}x^n =_{(k)}\ (x+iy)_F^n, \tag{138}$$

and the hierarchy of $k-$th Golden analytic functions

$$_{(k)}E_F^{iy_{(k)}D_F^x}f(x) = \sum_{n=0}^{\infty} a_n \cdot_{(k)}(x+iy)_F^n \equiv f(_{(k)}(x+iy)_F), \tag{139}$$

for every integer k satisfying the $\bar{\partial}$-equation

$$\frac{1}{2}(_{(k)}D_F^x + i_{(-k)}D_F^y)f(_{(k)}(x+iy)_F) = 0. \tag{140}$$

The Golden analytic functions from [13] correspond to particular value $k = 1$ of this infinite hierarchy. The real and imaginary parts of these functions

$$u(x,y) =_{(-k)}\ \cos_F(y_{(k)}D_F^x)f(x), \quad v(x,y) =_{(-k)}\ \sin_F(y_{(k)}D_F^x)f(x), \tag{141}$$

are subject to Cauchy-Riemann equations

$$_{(k)}D_F^x u(x,y) =_{(-k)}\ D_F^y v(x,y), \quad _{(-k)}D_F^y u(x,y) = -_{(k)}D_F^x v(x,y), \tag{142}$$

and are solutions of the hierarchy of Golden Laplace equations

$$(_{(k)}D_F^x)^2\phi(x,y) + (_{(-k)}D_F^y)^2\phi(x,y) = 0. \tag{143}$$

Therefore, it is natural to call these functions as the $k-$th Golden harmonic functions.

8 Hierarchy of Golden quantum oscillators

In this section we apply the quantum calculus of Fibonacci divisors or $k - th$ Golden calculus to deformed quantum oscillator problem. We define the set of creation and annihilation operators b_k and b_k^+ in the Fock basis $\{|n\rangle\}$, $n = 0, 1, 2...$, represented by infinite matrices

$$
b_k = \begin{pmatrix} 0 & \sqrt{F_1^{(k)}} & 0 & ... \\ 0 & 0 & \sqrt{F_2^{(k)}} & 0 \\ 0 & 0 & 0 & \sqrt{F_3^{(k)}} \\ ... & ... & ... & ... \end{pmatrix}, \quad b_k^+ = \begin{pmatrix} 0 & 0 & 0 & ... \\ \sqrt{F_1^{(k)}} & 0 & 0 & ... \\ 0 & \sqrt{F_2^{(k)}} & 0 & ... \\ ... & ... & ... & ... \end{pmatrix}
$$

By introducing the Fibonacci divisor number operator

$$
F_N^{(k)} = \frac{(\varphi^k)^N - (\varphi'^k)^N}{\varphi^k - \varphi'^k}, \tag{144}
$$

where $N = a^+ a$ is the standard number operator, we find that in the Fock basis the eigenvalues of this operator are just Fibonacci divisor numbers conjugate to F_k,

$$
F_N^{(k)}|n; k\rangle = F_n^{(k)}|n; k\rangle \tag{145}
$$

and in the matrix form

$$
F_N^{(k)} = \begin{pmatrix} F_0^{(k)} & 0 & 0 & ... \\ 0 & F_1^{(k)} & 0 & 0 \\ 0 & 0 & F_2^{(k)} & 0 \\ ... & ... & ... & ... \end{pmatrix}, \quad F_{N+I}^{(k)} = \begin{pmatrix} F_1^{(k)} & 0 & 0 & ... \\ 0 & F_2^{(k)} & 0 & 0 \\ 0 & 0 & F_3^{(k)} & 0 \\ ... & ... & ... & ... \end{pmatrix}. \tag{146}
$$

The operator satisfies recursion rule

$$
F_{N+I}^{(k)} = L_k F_N^{(k)} + (-1)^{k-1} F_{N-I}^{(k)}
$$

and is expressible as

$$
b_k b_k^+ = F_{N+I}^{(k)}, \quad b_k^+ b_k = F_N^{(k)}, \tag{147}
$$

giving the commutator

$$
[b_k, b_k^+] = F_{N+I}^{(k)} - F_N^{(k)}. \tag{148}
$$

From definition (144) for $F_N^{(k)}$ follows matrix identity

$$(\varphi^k)^N = \varphi^k F_N^{(k)} + (-1)^{k+1} F_{N-I}^{(k)}, \tag{149}$$

where

$$(\varphi^k)^N = \begin{pmatrix} 1 & 0 & 0 & ... \\ 0 & \varphi^k & 0 & 0 \\ 0 & 0 & \varphi^{2k} & 0 \\ ... & ... & ... & ... \end{pmatrix}, \tag{150}$$

and deformed commutation relations

$$b_k b_k^+ - \varphi^k b_k^+ b_k = \varphi'^{kN}, \qquad b_k b_k^+ - \varphi'^k b_k^+ b_k = \varphi^{kN}. \tag{151}$$

The hierarchy of Golden deformed bosonic Hamiltonians, defined as

$$H_k = \frac{\hbar\omega}{2}(b_k b_k^+ + b_k^+ b_k) = \frac{\hbar\omega}{2}(F_N^{(k)} + F_{N+I}^{(k)}) \tag{152}$$

is diagonal and gives the energy spectrum in terms of Fibonacci divisors

$$E_n^{(k)} = \frac{\hbar\omega}{2}(F_n^{(k)} + F_{n+1}^{(k)}). \tag{153}$$

For even k it becomes

$$E_n^{(k)} = \frac{\hbar\omega}{2}\frac{\sinh[(n+\frac{1}{2})k\ln\varphi]}{\sinh[\frac{k}{2}\ln\varphi]}. \tag{154}$$

In the limit $k \to 0$ this spectrum gives the one for linear harmonic oscillator

$$E_n^{(0)} = \hbar\omega\left(n + \frac{1}{2}\right) \tag{155}$$

and for $k \to \infty$ it is exponentially growing as powers of Golden ratio

$$E_n^{(k)} \approx \frac{\hbar\omega}{2}\varphi^{nk}. \tag{156}$$

Following [13] we can derive the "semiclassical expansion" of energy levels in powers of $\ln\varphi < 1$, giving nonlinear corrections to the harmonic oscillator

$$E_n^{(k)} = \hbar\omega \left(n + \frac{1}{2} \right) + 2 \sum_{s=1}^{\infty} B_{2s+1}(n+1) \frac{k^{2s}(\ln \varphi)^{2s}}{(2s+1)!}, \qquad (157)$$

where $B_m(x)$ are Bernoulli polynomials.

The energy levels (153) satisfy the three term recurrence relation

$$E_{n+1}^{(k)} = L_k E_n^{(k)} + (-1)^{k-1} E_{n-1}^{(k)}, \qquad (158)$$

where L_k are Lucas numbers. The first few energy levels for different values of k are:

$$k = 1 : E_n^{(1)} = 2\frac{\hbar\omega}{2}, \, 3\frac{\hbar\omega}{2}, \, 5\frac{\hbar\omega}{2}, \, 8\frac{\hbar\omega}{2}, \, ... \qquad (159)$$

$$k = 2 : E_n^{(2)} = 2\frac{\hbar\omega}{2}, \, 21\frac{\hbar\omega}{2}, \, 29\frac{\hbar\omega}{2}, \, 38\frac{\hbar\omega}{2}, \, ... \qquad (160)$$

$$k = 3 : E_n^{(3)} = 5\frac{\hbar\omega}{2}, \, 21\frac{\hbar\omega}{2}, \, 89\frac{\hbar\omega}{2}, \, 377\frac{\hbar\omega}{2}, \, ... \qquad (161)$$

$$k = 4 : E_n^{(4)} = 8\frac{\hbar\omega}{2}, \, 55\frac{\hbar\omega}{2}, \, 377\frac{\hbar\omega}{2}, \, 2584\frac{\hbar\omega}{2}, \, ... \qquad (162)$$

$$k = 5 : E_n^{(5)} = 12\frac{\hbar\omega}{2}, \, 133\frac{\hbar\omega}{2}, \, 1475\frac{\hbar\omega}{2}, \, 16358\frac{\hbar\omega}{2}, \, ... \qquad (163)$$

The difference between levels for odd $k = 2l + 1$ is growing as

$$\Delta E_n^{(k)} = E_{n+1} - E_n = \frac{\hbar\omega}{2} L_k F_{n+1}^{(k)}, \qquad (164)$$

while for even $k = 2l$ it is

$$\Delta E_n^{(k)} = \frac{\hbar\omega}{2} (L_k F_{n+1}^{(k)} - 2F_n^{(k)}). \qquad (165)$$

Then, the relative distance

$$\frac{\Delta E_n^{(k)}}{E_n^{(k)}} = \frac{F_{n+2}^{(k)} - F_n^{(k)}}{F_{n+1}^{(k)} + F_n^{(k)}} \qquad (166)$$

for asymptotic states $n \to \infty$ is given by the k-th power of Golden ratio

$$\lim_{n\to\infty} \frac{\Delta E_n^{(k)}}{E_n^{(k)}} = \varphi^k - 1. \qquad (167)$$

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For odd k we define the hierarchy of Golden deformed fermionic oscillators by Hamiltonians

$$H_k = \frac{\hbar\omega}{2}(b_k^+ b_k - b_k b_k^+) = \frac{\hbar\omega}{2}(F_N^{(k)} - F_{N+I}^{(k)}),$$ (168)

with integer spectrum

$$E_n^{(k)} = \frac{\hbar\omega}{2}(F_n^{(k)} - F_{n+1}^{(k)}).$$ (169)

In the limit $k \to 0$ (though it is not odd number) it becomes the usual fermionic two level system with energy $E_n^{(0)} = \frac{\hbar\omega}{2}$ for even n, and $E_n^{(0)} = -\frac{\hbar\omega}{2}$ for odd n. In this form it can be applied for description of the qubit. Then, the deformed case opens possibility to study modifications of qubits as units of quantum information, depending on k.

The first few members of fermionic hierarchy are follows. For $k = 1$ the spectrum is determined just by Fibonacci numbers $E_n^{(1)} = \frac{\hbar\omega}{2}F_{n-1}$. For $k = 3$ and $k = 5$ we have an infinite number of states with energy:

$$k = 3 : E_n^{(3)} = 3\frac{\hbar\omega}{2}, 13\frac{\hbar\omega}{2}, 55\frac{\hbar\omega}{2}, 233\frac{\hbar\omega}{2}, ...$$ (170)

$$k = 5 : E_n^{(5)} = 10\frac{\hbar\omega}{2}, 111\frac{\hbar\omega}{2}, 1231\frac{\hbar\omega}{2}, 13652\frac{\hbar\omega}{2}, ...$$ (171)

8.1 Hierarchy of Golden coherent states

By transformation

$$b_k = a\sqrt{\frac{F_N^{(k)}}{N}} = \sqrt{\frac{F_{N+I}^{(k)}}{N+I}}\, a, \quad b_k^+ = \sqrt{\frac{F_N^{(k)}}{N}}\, a^+ = a^+ \sqrt{\frac{F_{N+I}^{(k)}}{N+I}},$$ (172)

we introduce the states

$$|n; k\rangle_F = \frac{(b_k^+)^n}{\sqrt{F_n^{(k)}!}}|0; k\rangle_F,$$ (173)

where $b_k|0; k\rangle_F = 0$, coinciding with the Fock states $\{|n\rangle\}$ and satisfying

$$b_k^+|n; k\rangle_F = \sqrt{F_{n+1}^{(k)}}\,|n+1, k\rangle_F, \quad b_k|n; k\rangle_F = \sqrt{F_n^{(k)}}\,|n-1; k\rangle_F.$$ (174)

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The hierarchy of Golden coherent states is defined by eigenstates of annihilation operator

$$b_k \, |\beta_k\rangle_F = \beta_k \, |\beta_k\rangle_F. \tag{175}$$

Expanding in basis (173), the normalized coherent states are found as

$$|\beta_k\rangle_F = \left({}_{(k)}e_F^{|\beta_k|^2} \right)^{-1/2} \sum_{n=0}^{\infty} \frac{\beta_k^n}{\sqrt{F_n^{(k)}!}} |n; k\rangle_F, \tag{176}$$

were the scalar product of two states is

$$_F\langle\alpha_k|\beta_k\rangle_F = \frac{{}_{(k)}e_F^{\bar{\alpha}\beta}}{\left({}_{(k)}e_F^{|\alpha|^2} \right)^{1/2} \left({}_{(k)}e_F^{|\beta|^2} \right)^{1/2}}. \tag{177}$$

The exponential functions in the last two equations are the k-th Golden exponentials, defined in (127).

8.2 Hierarchy of Golden Fock-Bargman representations

An arbitrary state from the Hilbert space (173),

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n; k\rangle_F$$

by the inner product with coherent state (176)

$$\langle\beta_k|\psi\rangle = \left({}_{(k)}e_F^{|\beta_k|^2} \right)^{-1/2} \sum_{n=0}^{\infty} c_n \frac{\bar{\beta}_k^n}{\sqrt{F_n^{(k)}!}} = \left({}_{(k)}e_F^{|\beta_k|^2} \right)^{-1/2} \psi_k(\bar{\beta}) \tag{178}$$

determines the complex analytic wave function

$$\psi_k(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{\sqrt{F_n^{(k)}!}} \tag{179}$$

in the corresponding k-th Golden Fock-Bargman representation. The operators b_k and b_k^+ in this representation are given by

$$b_k \to {}_{(k)}D_F^z, \quad b_k^+ \to z, \tag{180}$$

were the complex k-th Golden derivative $_{(k)}D_F^z$ was defined in (60). Then, the higher order Fibonacci number operator (144) is represented as

$$F_N^{(k)} \rightarrow z \,_{(k)}D_F^z. \tag{181}$$

This way a link of our Fibonacci divisors Golden calculus with Fock-Bargman representation for hierarchy of deformed quantum oscillators is established.

As an example we consider the scale invariant analytic functions. If function $\psi(z)$ is the scale invariant $\psi_s(\lambda z) = \lambda^s \psi_s(z)$, then it satisfies equation

$$_{(k)}D_F^z \psi_s(z) = \frac{\psi_s(\varphi^k z) - \psi_s(\varphi'^k z)}{(\varphi^k - \varphi'^k)z} = \frac{(\varphi^k)^s - (\varphi'^k)^s}{(\varphi^k - \varphi'^k)z} \psi_s(z) \tag{182}$$

or

$$z \,_{(k)}D_F^z \psi_s(z) = F_s^{(k)} \psi_s(z). \tag{183}$$

This eigenvalue problem is just the k-th Golden Fock-Bargman representation of the Fibonacci divisor operator eigenvalue problem (145), where eigenfunctions

$$\psi_s(z) = \frac{z^s}{\sqrt{F_s^{(k)}!}} \tag{184}$$

are scale invariant. Then, the general solution of (183) is of the form

$$\psi_s(z) = z^s f_k(z), \tag{185}$$

where $f_k(z)$ is an arbitrary k-th Golden-periodic analytic function $f_k(\varphi^k z) = f_k(\varphi'^k z)$, which has been introduced in Section 3.1. It was noted that such self-similar wave function characterizes the quantum fractals [16],[11].

9   Acknowledgements

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10   Appendix

Here we proof Theorem 2.2.4. as equation (51) and Theorem 2.2.5. as equation (52). First we prove Theorem 2.2.5.

$$
\begin{aligned}
F_{kn+k+\alpha} &= \frac{1}{\varphi - \varphi'}\left[\varphi^{kn+k+\alpha} - \varphi'^{kn+k+\alpha}\right] \\
&= \frac{1}{\varphi - \varphi'}\left[\varphi^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= \frac{1}{\varphi - \varphi'}\left[\varphi^{kn+\alpha}\varphi^{k} + (-\varphi'^{kn+\alpha}\varphi^{k} + \varphi'^{kn+\alpha}\varphi^{k}) - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= \frac{1}{\varphi - \varphi'}\left[(\varphi^{kn+\alpha} - \varphi'^{kn+\alpha})\varphi^{k} + \varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= F_{kn+\alpha}\varphi^{k} + \frac{1}{\varphi - \varphi'}\left[\varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= F_{kn+\alpha}\left[\varphi^{k} + (-\varphi'^{k} + \varphi'^{k})\right] + \frac{1}{\varphi - \varphi'}\left[\varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= F_{kn+\alpha}(\varphi^{k} + \varphi'^{k}) - F_{kn+\alpha}\varphi'^{k} + \frac{1}{\varphi - \varphi'}\left[\varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right] \\
&= F_{kn+\alpha}L_{k} + \frac{1}{\varphi - \varphi'}\left[\varphi'^{kn+\alpha}\varphi'^{k} - \varphi^{kn+\alpha}\varphi'^{k} + \varphi'^{kn+\alpha}\varphi^{k} - \varphi'^{kn+\alpha}\varphi'^{k}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= L_k F_{kn+\alpha} + \frac{1}{\varphi - \varphi'} \left[\varphi'^{kn+\alpha} \varphi^k - \varphi^{kn+\alpha} \varphi'^k \right] \\
&= L_k F_{kn+\alpha} + \frac{\varphi^k \varphi'^k}{\varphi - \varphi'} \left[\varphi'^{kn-k+\alpha} - \varphi^{kn-k+\alpha} \right] \\
&= L_k F_{kn+\alpha} - \frac{\varphi^k \varphi'^k}{\varphi - \varphi'} \left[\varphi^{kn-k+\alpha} - \varphi'^{kn-k+\alpha} \right] \\
&= L_k F_{kn+\alpha} - \frac{(\varphi \varphi')^k}{\varphi - \varphi'} \left[\varphi^{kn-k+\alpha} - \varphi'^{kn-k+\alpha} \right] \quad \text{since} \; (\varphi \varphi')^k = (-1)^k, \\
&= L_k F_{kn+\alpha} - (-1)^k \left[\frac{\varphi^{kn-k+\alpha} - \varphi'^{kn-k+\alpha}}{\varphi - \varphi'} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= L_k F_{kn+\alpha} + (-1)^{k+1} F_{kn-k+\alpha} \; \text{ and since} \; (-1)^{k+1}(-1)^{-2} = (-1)^{k-1}, \\
&= L_k F_{kn+\alpha} + (-1)^{k-1} F_{kn-k+\alpha}
\end{aligned}
$$

Therefore, equation (52)

$$
F_{k(n+1)+\alpha} = L_k F_{kn+\alpha} + (-1)^{k-1} F_{k(n-1)+\alpha}
$$

is obtained. By choosing $\alpha = 0$ and dividing both sides of the equation with F_k gives us the desired recursion formula (51).