

**QUANTUM WALKS: ENTANGLEMENT BETWEEN SPATIAL
DEGREES OF FREEDOM AND INTERFERENCE IN
MULTI-PHOTON WALKS**

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**by
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ABSTRACT

QUANTUM WALKS: ENTANGLEMENT BETWEEN SPATIAL DEGREES OF FREEDOM AND INTERFERENCE IN MULTI-PHOTON WALKS

Quantum walks can be described as quantum analogues of classical random walks. In quantum walks, the direction of the walker is dictated by the quantum state of a coin in a coherent fashion. Unlike classical random walk with a fair coin, quantum walk has non-Markovian property. First, we studied 2-D quantum walk analytically and numerically with one-walker and two entangled coins to investigate the transfer of the entanglement in initial coins state to spatial degrees of freedom. The coins are Hadamard Coin, Fourier Coin, among which the Fourier coin generates entanglement, thus increase entanglement between spatial degrees of freedom. Here we calculated the amount of entanglement using negativity. In the second part we studied average photon number correlations for 1-D quantum walk with many body bosonic walkers, like different light sources, to investigate quantum interference effects and we showed the second-order intensity correlations function ($g^{(2)}(\tau)$) in terms of the probability amplitudes of the 1-D quantum walk with Hadamard coin. We compared the resulting correlations for various initial many photon states.

Keywords: Quantum, Entanglement, Interference, Photon.

ÖZET

KUANTUM YÜRÜYÜŞÜ: UZAY SERBESTLİK DERECELERİ ARASINDA DOLANIKLIK VE ÇOK FOTONLU YÜRÜYÜŞLERDE GİRİŞİM

Kuantum yürüyüşleri, klasik rastgele yürüyüşünün kuantum benzeri olarak tanımlanabilir. Kuantum yürüyüşlerinde yürüyücünün gideceği yön bir kuantum parasının kuantum durumuna göre belirlenir. Klasik rastgele yürüyüşlerin aksine kuantum yürüyüşleri Markov özelliklerine sahip değildir. Bu tezde ilk bölümde 2 boyuttaki kuantum yürüyüşlerde, başlangıçta kuantum paralar arasında bulunan dolanıklığın uzay serbestlik derecelerine transferini inceledik. Kullanılan kuantum paralar Hadamard ve Fourier paralar olarak seçildi. Hadamard parasının aksine Fourier parası her adımda dolanıklık yaratmaktadır. Dolanıklık miktarı "Negatiflik" metodu kullanılarak hesaplandı. İkinci bölümde ortalama foton sayıları üzerinden 1 boyutta çok fotonlu kuantum yürüyüşlerdeki kuantum girişim etkilerini ve 2. mertebe yoğunluk korelasyonlarını Hadamard para ile yapılan 1 boyutta kuantum yürüyüşünün olasılık genlikleri üstünden gösterdik ve farklı başlangıç durumlarına sahip sonuçları birbirleriyle karşılaştırdık.

Anahtar Kelimeler: Kuantum, Dolanıklık, Girişim, Foton.

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CHAPTER 1

INTRODUCTION

A random walk is the process involving of a walker who moves randomly away from the initial starting point. The basic example of a random walk is the classical motion of a particle(walker) on a line, the direction is determined randomly by tossing a fair coin which has two degrees of freedom. Quantum walk is the analog of classical random walk. However, it is important to clearly define quantum walk, randomness is the result of the quantum mechanical superposition principle and the measurement postulate. Quantum walk was defined first in 1993 by Y. Aharonov, L. Davidovich, and N. Zagury. Quantum walks were studied both on Discrete and Continuous models. The first quantum algorithm for discrete quantum walk was studied by Y. Aharonov(2001) and Ambainis(2001). The first quantum search algorithms based on quantum walk which proved that quantum walk can be used to make faster search on graphs, (A. Ambainis, J. Kempe, A. Rivosh, 2005)(Aaronson, Ambainis, 2005)(Childs, 2004).

1.1. Classical Random Walk

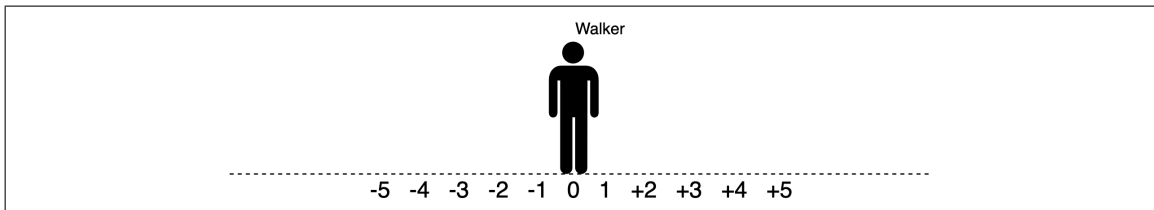


Figure 1.1.: A walker is positioned at $x=0$ and $t=0$. where t represents the instant preceding the tossing of a coin.

Classical random walk on a line can be defined as random movements on a space that has one spatial degree of freedom. This process can be imagined as walking where each step is determined by the tossing of a coin and if the coin gives head the walker makes one steps to left, if it gives tails the walker makes a step in the opposite direction. If we consider that the particle (or walker)is initially at $x=0$ and $t=0$, then after t steps, the position of walker becomes probabilistic. At $t=1$, the walker can be at $x=1$ with a probability of $\frac{1}{2}$ or at $x=-1$ with a probability of $\frac{1}{2}$. Long time probability distribution of

Table 1.1. Probability of the walker, that is initially positioned at $x=0$, after time t .

| $t \backslash x$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|------------------|------|-----|-----|-----|-----|-----|-----|-----|------|
| $t=0$ | | | | | 1 | | | | |
| $t=1$ | | | | 1/2 | | 1/2 | | | |
| $t=2$ | | | 1/4 | | 1/2 | | 1/4 | | |
| $t=3$ | | 1/8 | | 3/8 | | 3/8 | | 1/8 | |
| $t=4$ | 1/16 | | 1/4 | | 3/8 | | 1/4 | | 1/16 |

the walker can be described by factorial representation of Pascal's Triangle and Stirling's formula. When $t + x$ is odd or $x > t$, the probability is zero. Where $\binom{x}{y} = \frac{x!}{(x-y)!y!}$ and $x! \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$ due to Stirling's Formula.

$$P_{t,x} = 2^{-t} \binom{t}{\frac{t+x}{2}} \quad (1.1)$$

Where $\binom{x}{y} = \frac{x!}{(x-y)!y!}$ and $x! \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$ due to Stirling's Formula.

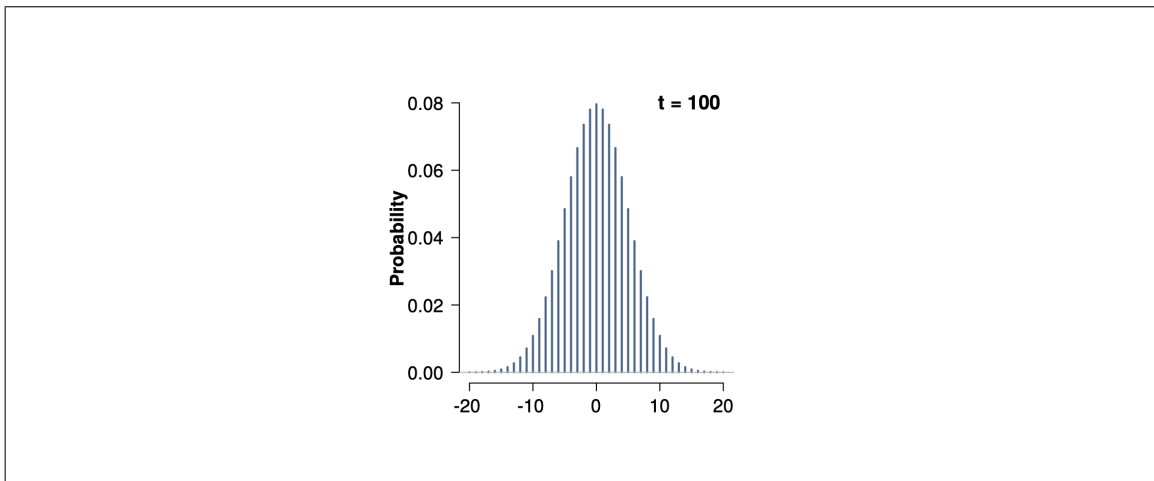


Figure 1.2.: Probability distribution after t step later

1.2. Quantum Walk

To introduce the quantum version of the classical random walk, first we should start by defining the Hilbert spaces for the coin and the spatial degrees of freedom. The coin Hilbert space \mathcal{H}_C has 2 dimensions with orthonormal basis states ($|0\rangle$ and $|1\rangle$) the analog of the classical coin states being heads/tail or rightward/leftward. The Hilbert space (\mathcal{H}_P) of the spatial part has infinite dimension with the discrete basis ($|x\rangle : x \in \mathbb{Z}$). Total Hilbert space of the system becomes $\mathcal{H}_C \otimes \mathcal{H}_P$. The state of coins can be defined as follows:

$$|0\rangle_c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.2)$$

The position of the walker is represented by computational basis. For instance, if the walker is at the point $x=0$, the spatial part of the state can be described as follows:

$$|x = 0\rangle = |0\rangle_p \quad (1.3)$$

Hadamard coin is the most natural quantum analog of the classical coin operator, it mixes the coin and maps the coin states $|0\rangle$ and $|1\rangle$ to superposition states with equal weight.

$$H_c |0\rangle_c = \frac{1}{\sqrt{2}}(|0\rangle_c + |1\rangle_c) \quad (1.4)$$

$$H_c |1\rangle_c = \frac{1}{\sqrt{2}}(|0\rangle_c - |1\rangle_c) \quad (1.5)$$

The shift operator S changes the walker's position depending on the state of the coin. If the coin state is $|0\rangle$, the shift operator moves the walker's position from $|x\rangle$ to $|x + 1\rangle$. If the coin state is $|1\rangle$, the position of the walker becomes $|x - 1\rangle$.

$$S |0\rangle_c |0\rangle_p = |0\rangle_c |1\rangle_p \quad (1.6)$$

$$S |1\rangle_c |0\rangle_p = |1\rangle_c |-1\rangle_p \quad (1.7)$$

S can be written as follows:

$$S = |0\rangle_c \langle 0|_c \otimes \sum_{x=-\infty}^{\infty} |x + 1\rangle \langle x| + |1\rangle_c \langle 1|_c \otimes \sum_{x=-\infty}^{\infty} |x - 1\rangle \langle x| \quad (1.8)$$

The coin operator should act only on the coin degree of freedom and the shift operator(S) should act on all degrees of freedom. Therefore the total evolution operator U can be defined as follows:

$$U = S(C \otimes I) \quad (1.9)$$

Where I represents the identity operator which has the same dimension with spatial degree of freedom. The total state of the walker one step later can be described as follows:

$$|\psi\rangle_{t+1} = U |\psi\rangle_t \quad (1.10)$$

If the walker is positioned at $x = 0$ and the coin state is $|0\rangle$. The initial state of the walker can be written as follows

$$|\psi\rangle_0 = |0\rangle_c |0\rangle_p \quad (1.11)$$

One step later, the total state becomes as follows:

$$|\psi\rangle_1 = U |\psi\rangle_0 \quad (1.12)$$

The total state of the walker after t steps can be described as follows:

$$|\psi\rangle_t = U^t |\psi\rangle_0 \quad (1.13)$$

To better understand, we can use σ symbols to represent the coin state. Thus the shift operator S can be written as follows:

$$S = \sum_{\sigma=0}^1 \sum_{x=-\infty}^{\infty} |\sigma, x + (-1)^\sigma\rangle \langle \sigma, x| \quad (1.14)$$

The quantum walk is a non-markovian process which means the evolution of the walk is dependent on the initial coin state. The state of the walker after t steps later can be described in terms of the probability amplitudes as follows:

$$|\psi_t^{\sigma'}\rangle = \sum_{x,\sigma} A_{x\sigma}^{\sigma'}(t) |x\rangle |\sigma\rangle \quad (1.15)$$

Where $A_{x\sigma}^{\sigma'}(t)$ is the probability amplitude. σ is the coin state and σ' is the initial coin state. The probability amplitudes should satisfy the normalization for a chosen initial state σ' .

$$\sum_x \left| A_{x0}^{\sigma'}(t) \right|^2 + \left| A_{x1}^{\sigma'}(t) \right|^2 = 1 \quad (1.16)$$

The probability of the walker being at position x at time t can be defined as follows:

$$P_x^{\sigma'}(t) = \sum_{\sigma} \left| A_{x\sigma}^{\sigma'}(t) \right|^2 \quad (1.17)$$

The unitary evolution operator U^t can be represented in a diagonal form in momentum space. Analytical solution of unitary operator for initial coin state $\sigma = 0$ was shown by

R. Portugal. Portugal (2013) To find an analytical solution of initial coin state $\sigma = 1$, The total state after $t + 1$ time later can be written as follows:

$$|\psi_{t+1}\rangle = U |\psi_t\rangle \quad (1.18)$$

$$|\psi_{t+1}\rangle = \sum_x S (A_{x,0}(t)C |0, x\rangle + A_{x,1}(t)C |1, x\rangle) \quad (1.19)$$

Where C is the coin operator. In this case it's Hadamard transform. C operator only acts on coin states and maps them to equalweight superposition states of $|0\rangle$ and $|1\rangle$.

$$\sum_x \frac{1}{\sqrt{2}}(A_{x,0}(t) + A_{x,1}(t))S |0, x\rangle + \frac{1}{\sqrt{2}}(A_{x,0}(t) - A_{x,1}(t))S |1, x\rangle \quad (1.20)$$

Using the Equation 1.14, S operator moves the walker depending the coin state.

$$\sum_x \frac{1}{\sqrt{2}}(A_{x,0}(t) + A_{x,1}(t)) |0, x + 1\rangle + \frac{1}{\sqrt{2}}(A_{x,0}(t) - A_{x,1}(t)) |1, x - 1\rangle \quad (1.21)$$

If we can regroup terms with the same indices, we can write the probability amplitudes above for $x+1$ as follows:

$$A_{x,0}(t + 1) = \frac{1}{\sqrt{2}} (A_{x-1,0}(t) + A_{x-1,1}(t)) \quad (1.22)$$

$$A_{x,0}(t + 1) = \frac{1}{\sqrt{2}} (A_{x+1,0}(t) - A_{x+1,1}(t)) \quad (1.23)$$

To find the analytical solutions for the evolution of the probability amplitudes, we will write the state of the walker in the discrete momentum space using Fourier transform.

$$\tilde{A}_{k,\sigma}(t) = \sum_{x=-\infty}^{\infty} e^{-ikx} A_{x,\sigma}(t) \quad (1.24)$$

Then we can define $|k\rangle$ vector in terms of $|x\rangle$ in momentum space.

$$|k\rangle = \sum_x e^{ikx} |x\rangle \quad (1.25)$$

Thus, the total state of the walker at time t can be written as follows:

$$|\psi_t\rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{\sigma} \tilde{A}_{k,\sigma} |\sigma, k\rangle \quad (1.26)$$

Now we can apply U operator to the total state in the momentum space. First we will apply S operator to the state $|\sigma, k\rangle$.

$$S |\sigma, k\rangle = \sum_x e^{ikx} S |\sigma, x\rangle \quad (1.27)$$

We defined the action of S operator in the equation 1.14 . Thus the equation above becomes

$$S |\sigma, k\rangle = \sum_x e^{ikx} |\sigma, x + (-1)^\sigma\rangle. \quad (1.28)$$

We can rename indices as $x' = x + (-1)^\sigma$. Thus we can write $x = x' - (-1)^\sigma$

$$S |\sigma, k\rangle = \sum_{x'} e^{ik(x' - (-1)^\sigma)} S |\sigma, x'\rangle \quad (1.29)$$

We can write $e^{ikx'} |x'\rangle$ in momentum space as $|k\rangle$

$$S |\sigma, k\rangle = \sum_{x'} e^{ik(-(-1)^\sigma)} S |\sigma, k\rangle \quad (1.30)$$

The equation above shows us the shift operator changes phase of the total state in momentum space. So the evolution operator in momentum space for each k can be defined as follows:

$$\tilde{U}_k = \begin{pmatrix} e^{-ik} & 0 \\ 0 & e^{ik} \end{pmatrix} C_H \quad (1.31)$$

Where C_H is Hadamard coin.

$$\tilde{U}_k = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-ik} & e^{-ik} \\ e^{ik} & -e^{ik} \end{pmatrix} \quad (1.32)$$

To diagonalize \tilde{U}_k , we can write characteristic polynomial as follows:

$$= \begin{pmatrix} \frac{e^{-ik}}{\sqrt{2}} - \lambda & \frac{e^{-ik}}{\sqrt{2}} \\ \frac{e^{ik}}{\sqrt{2}} & -\frac{e^{ik}}{\sqrt{2}} - \lambda \end{pmatrix} \quad (1.33)$$

$$= \left(\frac{e^{-ik}}{\sqrt{2}} - \lambda \right) \left(-\frac{e^{ik}}{\sqrt{2}} - \lambda \right) - \frac{e^{-ik}}{\sqrt{2}} \frac{e^{ik}}{\sqrt{2}} \quad (1.34)$$

$$= \left(-\frac{1}{2} - \lambda \frac{e^{-ik}}{\sqrt{2}} + \lambda \frac{e^{ik}}{\sqrt{2}} + \lambda^2 - \frac{1}{2} \right) \quad (1.35)$$

Using the exponential form of $\sin k = \frac{e^{ik} - e^{-ik}}{2i}$, the characteristic polynomial becomes as follows:

$$\lambda^2 + \sqrt{2}\lambda i \sin k - 1 \quad (1.36)$$

And the eigenvalues a_i and eigenvectors $|a_i\rangle$ can be found as follows:

$$a_1 = e^{-i\omega_k}, a_2 = e^{i(\pi + \omega_k)} \quad (1.37)$$

$$\begin{aligned}
|a_1\rangle &= \frac{1}{\sqrt{c^-}} \begin{pmatrix} e^{-ik} \\ \sqrt{2}e^{-i\omega_k} - e^{-ik} \end{pmatrix} \\
|a_2\rangle &= \frac{1}{\sqrt{c^+}} \begin{pmatrix} e^{-ik} \\ -\sqrt{2}e^{i\omega_k} - e^{-ik} \end{pmatrix}
\end{aligned} \tag{1.38}$$

With the following definitions:

$$\omega_k = \arcsin\left(\frac{1}{\sqrt{2}}\sin k\right) \tag{1.39}$$

$$\frac{1}{c^\pm} = \frac{1}{2} \left(1 \mp \frac{\cos k}{\sqrt{1 + \cos^2 k}}\right) \tag{1.40}$$

Therefore the unitary evolution operator can be shown as follows:

$$U^t = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega_k t} |a_1\rangle |k\rangle \langle k| \langle a_1| + e^{i(\pi+\omega_k)t} |a_2\rangle |k\rangle \langle k| \langle a_2| \tag{1.41}$$

Where $|k\rangle$ is,

$$|k\rangle = \sum_{x=-\infty}^{\infty} e^{ikx} |x\rangle \tag{1.42}$$

If we write the total state after t time later.

$$|\psi_t\rangle = U^t |\psi_i\rangle \tag{1.43}$$

For the initial coin state $\sigma = 0$, initial state can be written as $|\psi_i\rangle = |\sigma = 0\rangle |x = 0\rangle$.

$$|\psi_t\rangle = \left[\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega_k t} |a_1\rangle |k\rangle \langle k| \langle a_1| + e^{i(\pi+\omega_k)t} |a_2\rangle |k\rangle \langle k| \langle a_2| \right] |\sigma = 0\rangle |x = 0\rangle \tag{1.44}$$

Using the eigenvectors in the Equation 1.38.

$$\begin{aligned}
&\langle k| \langle a_1| \sigma = 0\rangle |x = 0\rangle \\
&\langle k|x = 0\rangle = 1 \\
&\langle a_1|\sigma = 0\rangle = \frac{e^{ik}}{\sqrt{c^-}}
\end{aligned} \tag{1.45}$$

$$\begin{aligned}
&\langle k| \langle a_2| \sigma = 0\rangle |x = 0\rangle \\
&\langle k|x = 0\rangle = 1 \\
&\langle a_2|\sigma = 0\rangle = \frac{e^{ik}}{\sqrt{c^+}}
\end{aligned} \tag{1.46}$$

Thus $|\psi_t\rangle$ becomes as follows:

$$|\psi_t\rangle = \left[\int_{-\pi}^{\pi} e^{-i\omega_k t} \frac{e^{ik}}{\sqrt{c^-}} |a_1\rangle + e^{i(\pi+\omega_k)t} \frac{e^{ik}}{\sqrt{c^+}} |a_2\rangle \right] |k\rangle \frac{dk}{2\pi} \tag{1.47}$$

$$\begin{aligned}
|\psi_t\rangle = & \left[\int_{-\pi}^{\pi} e^{-i\omega_k t + ik} \frac{1}{c^-} \begin{pmatrix} e^{-ik} \\ \sqrt{2}e^{-i\omega_k} - e^{-ik} \end{pmatrix} \right. \\
& \left. + e^{i(\pi+\omega_k)t + ik} \frac{1}{c^+} \begin{pmatrix} e^{-ik} \\ -\sqrt{2}e^{i\omega_k} - e^{-ik} \end{pmatrix} \right] |k\rangle \frac{dk}{2\pi}
\end{aligned} \tag{1.48}$$

For the initial coin state $\sigma = 0$, the probability amplitudes can be found in momentum space as follows:

$$A_{k0}^0(t) = \frac{1}{c^-} e^{-i\omega_k t} + \frac{1}{c^+} e^{i(\pi+\omega_k)t} \tag{1.49}$$

$$A_{k1}^0(t) = \frac{1}{c^-} e^{-i\omega_k t + ik} (\sqrt{2}e^{-i\omega_k} - e^{-ik}) - \frac{1}{c^+} e^{i(\pi+\omega_k)t + ik} (\sqrt{2}e^{i\omega_k} + e^{-ik}) \tag{1.50}$$

For the initial coin state $\sigma = 1$, the initial state can be written as

$$|\psi_i\rangle = |\sigma = 1\rangle |x = 0\rangle.$$

$$|\psi_t\rangle = \left[\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega_k t} |a_1\rangle |k\rangle \langle k| \langle a_1| + e^{i(\pi+\omega_k)t} |a_2\rangle |k\rangle \langle k| \langle a_2| \right] |\sigma = 1\rangle |x = 0\rangle \tag{1.51}$$

$$\langle k| \langle a_1| |\sigma = 1\rangle |x = 0\rangle$$

$$\langle k| |x = 0\rangle = 1 \tag{1.52}$$

$$\langle a_1| |\sigma = 1\rangle = \frac{\sqrt{2}e^{i\omega_k} - e^{ik}}{\sqrt{c^-}}$$

$$\langle k| \langle a_2| |\sigma = 1\rangle |x = 0\rangle$$

$$\langle k| |x = 0\rangle = 1 \tag{1.53}$$

$$\langle a_2| |\sigma = 1\rangle = \frac{-\sqrt{2}e^{-i\omega_k} - e^{ik}}{\sqrt{c^+}}$$

The equation 1.51 can be written as follows:

$$|\psi_t\rangle = \left[\int_{-\pi}^{\pi} e^{-i\omega_k t} \left(\frac{\sqrt{2}e^{i\omega_k} - e^{ik}}{\sqrt{c^-}} \right) |a_1\rangle + e^{i(\pi+\omega_k)t} \left(\frac{-\sqrt{2}e^{-i\omega_k} - e^{ik}}{\sqrt{c^+}} \right) |a_2\rangle \right] |k\rangle \frac{dk}{2\pi} \tag{1.54}$$

For initial coin state $\sigma = 1$, the probability amplitudes can be found in the same way in momentum space as follows:

$$A_{k0}^1(t) = \frac{1}{c^-} e^{-(i\omega_k t + ik)} (\sqrt{2}e^{i\omega_k} - e^{ik}) + \frac{1}{c^+} e^{i(\pi+\omega_k)t - ik} (-\sqrt{2}e^{-i\omega_k} - e^{ik}) \tag{1.55}$$

$$\begin{aligned}
A_{k1}^1(t) = & \left[\frac{1}{c^-} e^{-i\omega_k t} (\sqrt{2}e^{i\omega_k} - e^{ik}) (\sqrt{2}e^{-i\omega_k} - e^{-ik}) \right] \\
& + \left[\frac{1}{c^+} e^{i(\pi+\omega_k)t} (-\sqrt{2}e^{-i\omega_k} - e^{ik}) (-\sqrt{2}e^{i\omega_k} - e^{-ik}) \right]
\end{aligned} \tag{1.56}$$

The probability amplitudes in real space can be found by taking inverse Fourier transform as follows:

$$A_{x\sigma}^{\sigma'}(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikx} A_{k\sigma}^{\sigma'}(t) \quad (1.57)$$

Using the equation 1.17, the probability distributions of quantum walk with initial coin state $\sigma' = 0$ and $\sigma' = 1$ after 100 steps later can be plotted by using the analytical result at the equation 1.56. The Matlab code that is used to evaluate the plotted results can be found in the Appendix ...

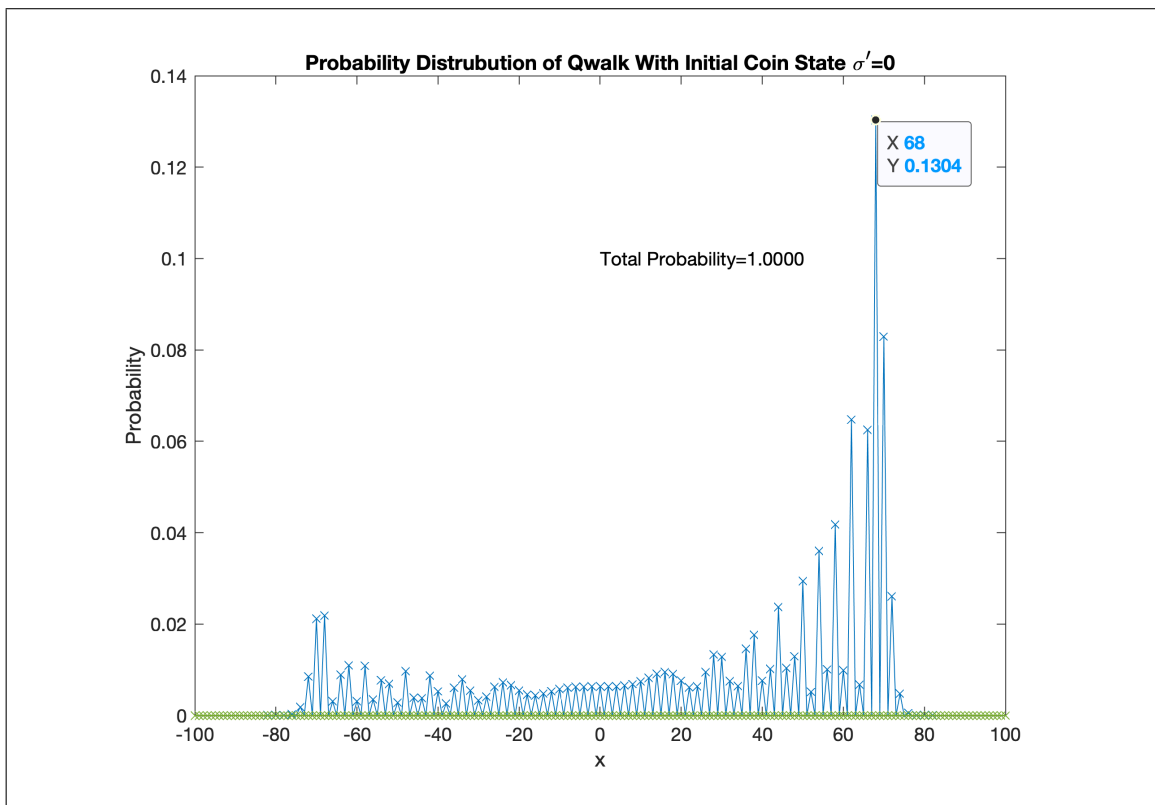


Figure 1.3.: Probability distribution of 1-D quantum walk (with initial coin state zero) is plotted by using the following formula, $P_x^0(100) = |A_{x0}^0(100)|^2 + |A_{x1}^0(100)|^2$. It can be clearly said that the quantum walk with the hadamard coin is much faster compared to the probability distribution of the classical random walk in the Figure 1.2. The most probable position for the walker to be at is the position $x = 68$. The direction of the walk is dictated to the right by the initial coin state.

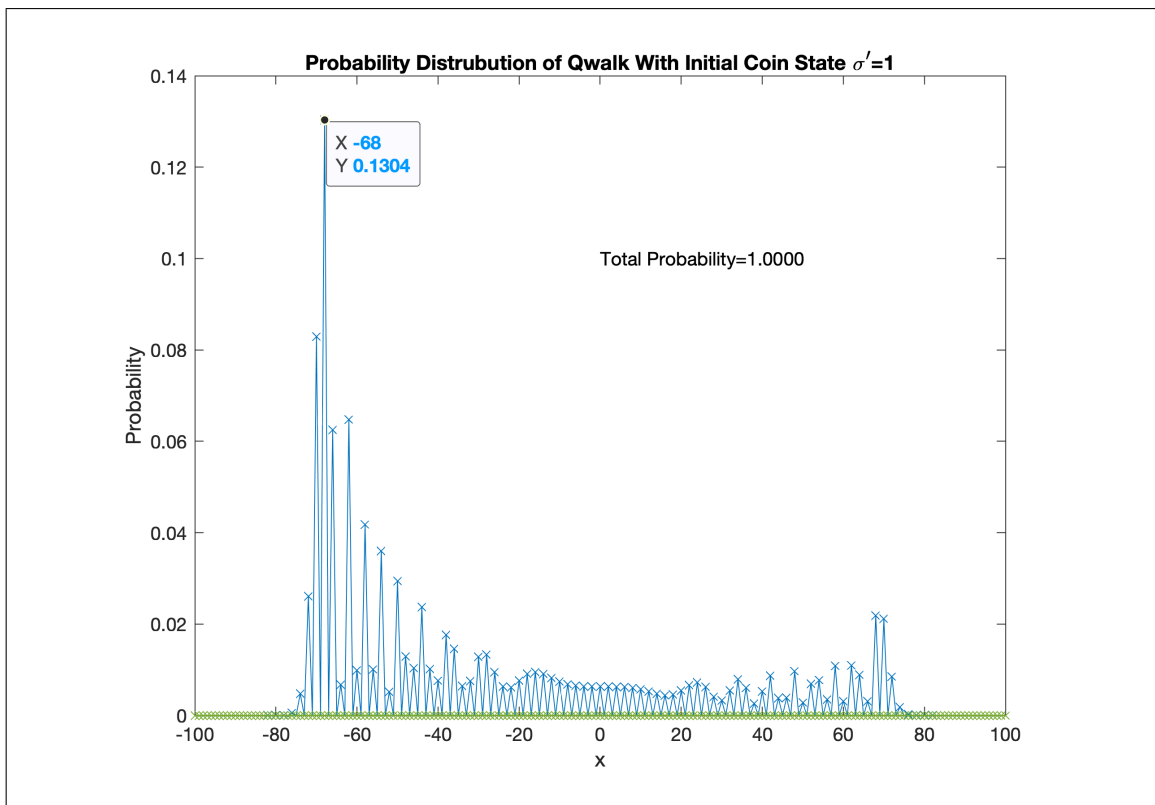


Figure 1.4.: Probability distribution of 1-D quantum walk (with initial coin state is one) is plotted with the following formula, $P_x^1(100) = |A_{x0}^1(100)|^2 + |A_{x1}^1(100)|^2$. Unless the case initial coin state is zero, the direction of the walk is dictated to the left by the initial coin state.

CHAPTER 2

ENTANGLEMENT TRANSFER IN 2-D QUANTUM WALK

2.1. 2-D Quantum Walk with Entangled Coins

2-D quantum walk with two coins can be described as one walker walking on the x direction depending on the first coin state and the y direction depending on the second coin state. 2-D quantum walk with entangled coins is the analog of 2-D classical walk but the coins are initially entangled. The state of the coins can be chosen as one of the Bell States. As it is well known, Bell states are maximally entangled pure bipartite states.

$$|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \quad (2.1)$$

$$|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \quad (2.2)$$

Where the coin state $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Also $|00\rangle$ represents the tensor product of two $|0\rangle$ states. It can be shown as tensor product $|00\rangle = |0\rangle |0\rangle = |0\rangle \otimes |0\rangle$

As an example of a coin in 2 dimensions, The Hadamard Coin in 2-D is defined as the tensor product of two single coin operators.

$$C = H \otimes H \quad (2.3)$$

As well the shift operator can be chosen for 2-Dimensions.

$$\begin{aligned} S = & |00\rangle \langle 00| \otimes \sum_{x,y=-\infty}^{\infty} |x+1, y+1\rangle \langle x, y| \\ & + |01\rangle \langle 01| \otimes \sum_{x,y=-\infty}^{\infty} |x+1, y-1\rangle \langle x, y| \\ & + |10\rangle \langle 10| \otimes \sum_{x,y=-\infty}^{\infty} |x-1, y+1\rangle \langle x, y| \\ & + |11\rangle \langle 11| \otimes \sum_{x,y=-\infty}^{\infty} |x-1, y-1\rangle \langle x, y| \end{aligned} \quad (2.4)$$

Using S and C operators, the evolution operator can be written. First we should multiply with the coin operator by using the tensor product, since the coin operator only acts on the coin subspace, the action on the two spatial subspaces are denoted by identity operators and therefore the unitary that describe the evolution becomes

$$U = S(C \otimes I \otimes I). \quad (2.5)$$

Therefore, like 1-D quantum walk, the state of the walker can be written in terms of the evolution operator and the initial state

$$|\psi_t\rangle = U^t |\psi_0\rangle. \quad (2.6)$$

In this thesis $|\Psi_+\rangle$ is chosen as the initial coin state and the walker is positioned at $x=0$ and $y=0$. Thus the initial state of the walker becomes:

$$|\psi_0\rangle = |\text{coin}\rangle_0 \otimes |x\rangle_0 \otimes |y\rangle_0 \quad (2.7)$$

$$|\psi_0\rangle = |\Psi_+\rangle \otimes |x=0\rangle \otimes |y=0\rangle \quad (2.8)$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |x=0\rangle \otimes |y=0\rangle \quad (2.9)$$

2.1.1. Hadamard Coin in 2-Dimensions

Coins are the vital part of the fairness of the evolution operators. Hadamard coin represents Hadamard Transform that makes the superposition of $|0\rangle$ and $|1\rangle$ with equal weight. Therefore it can be said that Hadamard Coin is a fair coin and it can be shown in Dirac notation as follows.

$$H = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1| \quad (2.10)$$

$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ can be also shown as $|+\rangle$ and $|-\rangle$ respectively. It also can be shown in matrix form as follows.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.11)$$

Hadamard transform in 2 Dimensions can be defined as the tensor product of Hadamard Coin in 1-D.

$$C = H \otimes H \quad (2.12)$$

In matrix form:

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.13)$$

$$C = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad (2.14)$$

In general, Hadamard transform in N dimensions can be defined as follows:

$$H_N = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{N-1} & H_{N-1} \\ H_{N-1} & -H_{N-1} \end{pmatrix} \quad (2.15)$$

Where $H_0 = 1$ and $N > 0$. Aгаian et al. (2011)

Two dimensional Hadamard coin in Dirac notation can be defined with the following equation. (Venegas-Andraca (2012))

$$C = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1| \right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1| \right) \quad (2.16)$$

$$C = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| \right) + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1| \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| \right) \\ + \frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1| \right) + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 0| \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1| \right) \quad (2.17)$$

$$C = \frac{1}{2} [(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \langle 00| + (|00\rangle + |01\rangle - |10\rangle - |11\rangle) \langle 10| + \\ (|00\rangle - |01\rangle + |10\rangle - |11\rangle) \langle 01| + (|00\rangle - |01\rangle - |10\rangle + |11\rangle) \langle 11|] \quad (2.18)$$

$$C = \frac{1}{2} [|00\rangle \langle 00| + |01\rangle \langle 00| + |10\rangle \langle 00| + |11\rangle \langle 00| \\ + |00\rangle \langle 10| + |01\rangle \langle 10| - |10\rangle \langle 10| - |11\rangle \langle 10| \\ + |00\rangle \langle 01| - |01\rangle \langle 01| + |10\rangle \langle 01| - |11\rangle \langle 01| \\ + |00\rangle \langle 11| - |01\rangle \langle 11| - |10\rangle \langle 11| + |11\rangle \langle 11|] \quad (2.19)$$

The state of the walker after one step later can be written as follows:

$$|\psi_1\rangle = U |\psi_0\rangle \quad (2.20)$$

$$|\psi_1\rangle = S(C \otimes I \otimes I) \left(\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{\text{coin}} \otimes |0\rangle_p |0\rangle_p \right) \quad (2.21)$$

2.1.2. Fourier Coin in 2-Dimensions

The second coin which is used in the simulations is Fourier coin. Fourier coin can be defined as a DFT(Discrete Fourier Transform) matrix. DFT matrix is a unitary transform, it preserves the total probability. The general formula of DFT matrix can be given as follows:

$$F_{DFT} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ 1 & w^2 & w^4 & \dots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \dots & w^{(N-1)(N-1)} \end{pmatrix} \quad (2.22)$$

Where $\omega = e^{\frac{-2\pi}{N}}$.

Unlike the Hadamard transform, the DFT matrix mixes the coins and also generates entanglement at every step. The DFT matrix in 1-Dimension is the same with the Hadamard Transform. DFT matrix in 2-Dimensions is defined as Fourier Coin as follows:

$$F_c = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \quad (2.23)$$

2.1.3. Measuring Entanglement

In this thesis we used Negativity to measure entanglement. It is based on positive partial transpose(also known Peres-Horodecki criterion). [Horodecki et al. (1996)] A density matrix ρ Negativity is defined as follows [Simon (2000)]

$$\mathcal{N}(\rho) \equiv \frac{\|\rho^{\Gamma_A}\| - 1}{2} \quad (2.24)$$

Where $\|\rho^{\Gamma_A}\| = \text{Tr}|\rho^{\Gamma_A}| = \text{Tr}\sqrt{\rho^{\Gamma_A\dagger}\rho^{\Gamma_A}}$. It can be described as the sum of the absolute eigenvalues of ρ^{Γ_A} . [Simon (2000)] [Horodecki et al. (1996)]

$$\mathcal{N}(\rho) = \sum_{\sigma_i < 0} \frac{|\sigma_i| - \sigma_i}{2} \quad (2.25)$$

Where σ_i are eigenvalues of ρ^{Γ_A} . Therefore negativity gives us a computable way to measure entanglement of the density matrix which is constituted by the 2-D quantum walk simulations.

2.2. Simulations

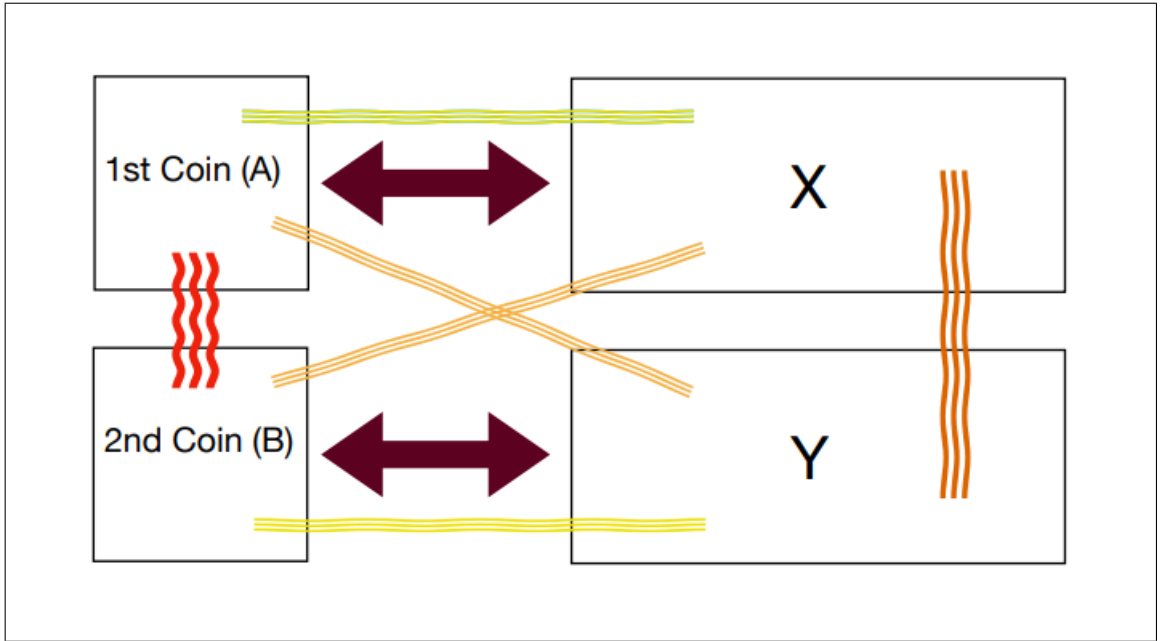


Figure 2.1.: Entanglement Transfer

The first coin A is responsible of the walking on the X degree of freedom, and the second coin B is responsible of the walking on the Y degree of freedom. The red wavy lines between A and B describe the initial entanglement. The other wavy lines describe the entanglement between different subsystems after t time later. Therefore after t time later, we have a density matrix ρ_{ABXY} describing the total system ABXY. For example if we want to measure entanglement between X and Y degree of freedom, we trace out AB subsystems from the total density matrix and we obtain the reduced density matrix $\rho^{XY} = Tr_{AB}(\rho^{ABXY})$. For numerical simulations we used Matlab with the QLib (Machnes (2007)). The codes can be found at Appendix A.

2.2.1. Hadamard Coin

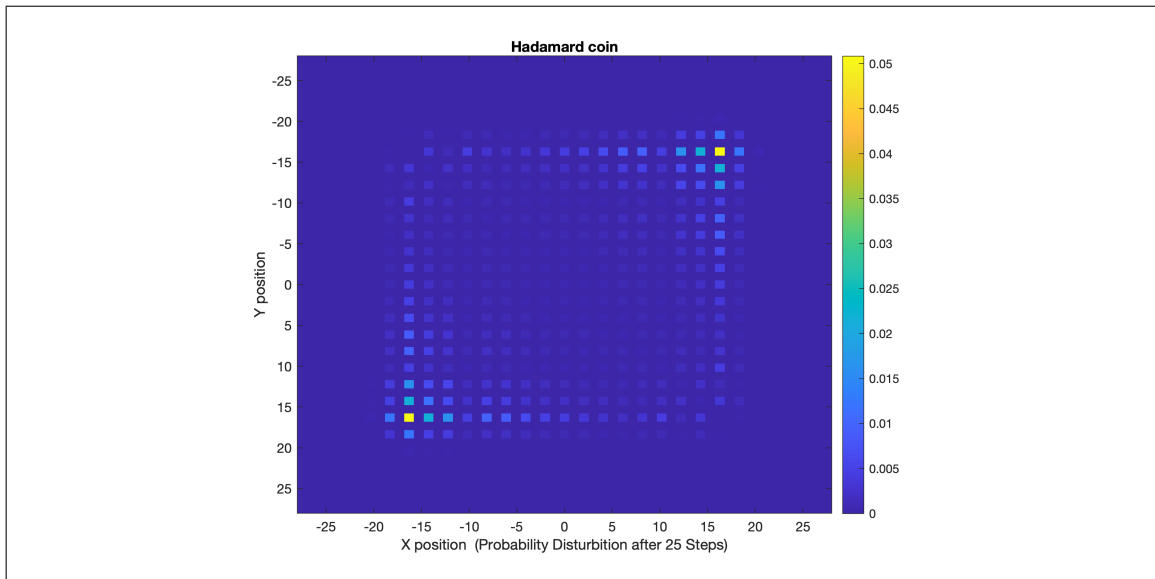


Figure 2.2.: The probability distribution of the walker after 25 steps by using Hadamard coin

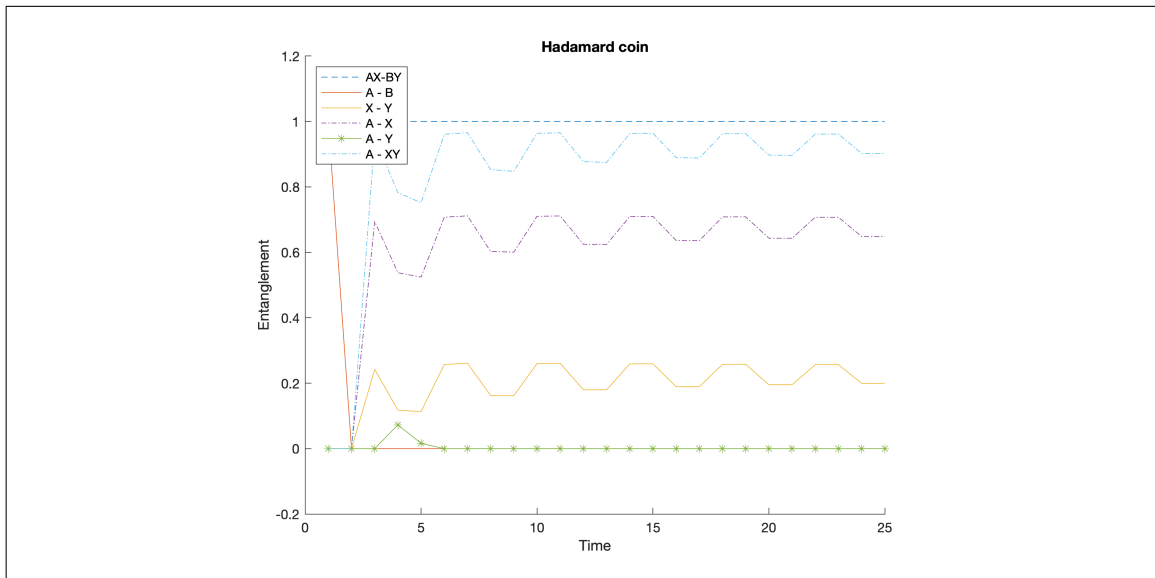


Figure 2.3.: Hadamard Coin - Entanglement Transfer in 2-D Quantum Walk with Initially Entangled Coins. Lines with different colors represent different subsystems.

2.2.2. Fourier Coin

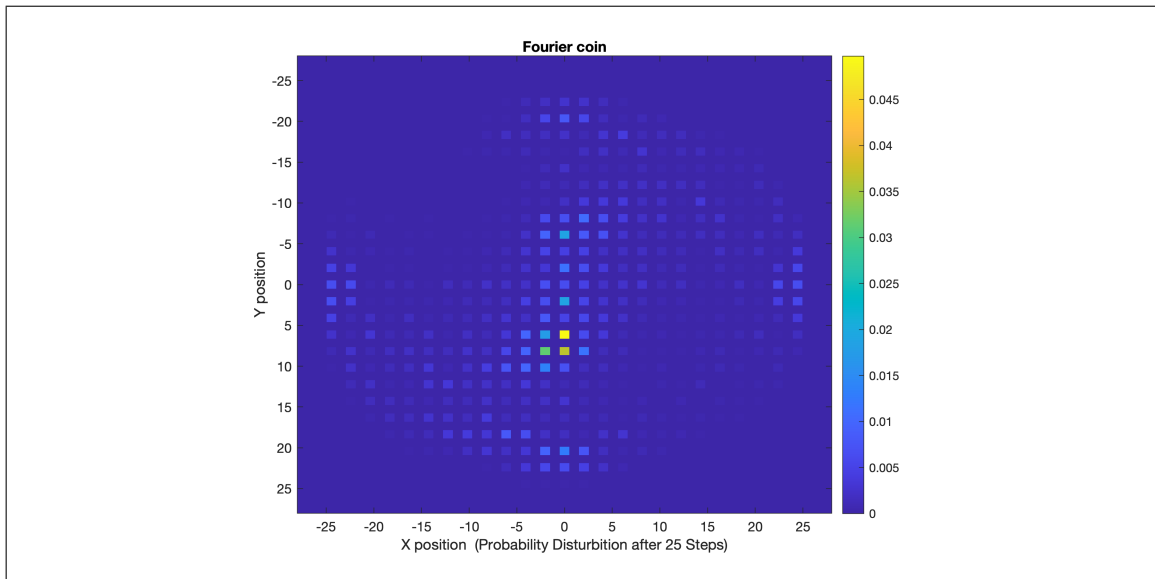


Figure 2.4.: The probability distribution of the walker after 25 steps

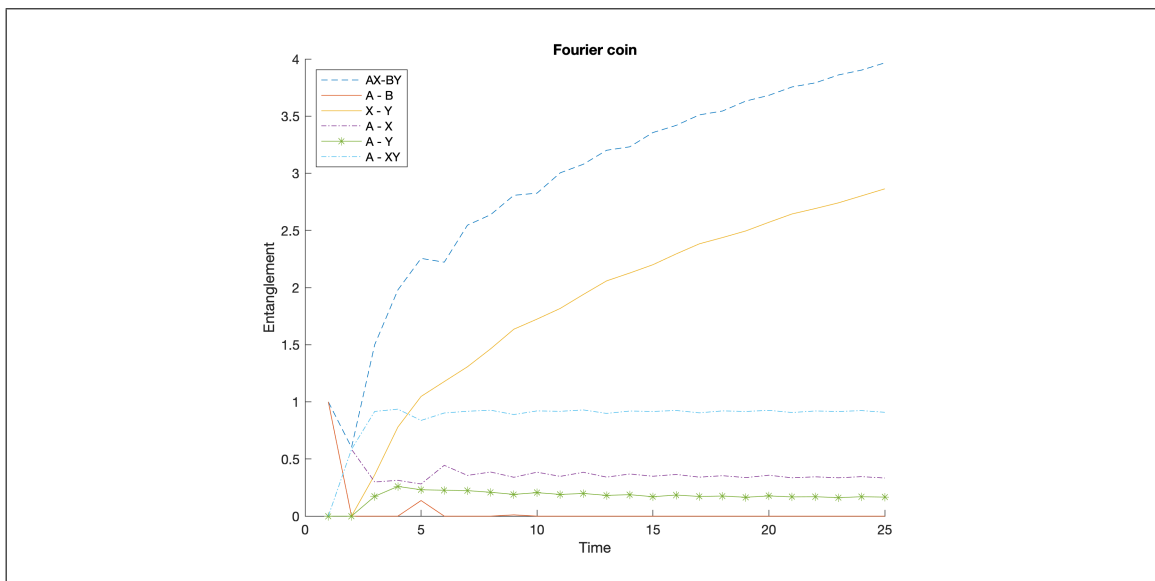


Figure 2.5.: Fourier Coin - Entanglement Transfer in 2-D Quantum Walk with Initially Entangled Coins. Lines with different colors represent different subsystems.

2.2.3. Discussion

The Figure 2.2 shows the probability distribution of the walker after 25 steps by using Hadamard coin. The range goes from blue being the minimum probability to yellow being the maximum. The variance is similar to the 1D quantum walk in both spatial degrees of freedom. If we look at the Figure 2.3 It can be clearly seen that at the beginning entanglement was between coins, after one step it transfers to the different subsystems. Hadamard transform conserves the entanglement so we can measure that the entanglement between AX and BY subsystems is 1. We measure more entanglement in the subsystem that has more degrees of freedom.

Second simulation is done with the Fourier Coin. The Figure 2.4 shows the probability distribution of the walker after 25 steps by using Fourier coin. Due to the Figure 2.5, we initially have entanglement between coins. After every step entanglement is created by Fourier coin and transfers to the subsystem that with the most degrees of freedom.

According to the results of the simulations, it can be said that the entanglement of the system always transfers to the subsystem which has more degrees of freedom. The phenomenon of entanglement transfer might be explained analytically by quantum topology.

CHAPTER 3

INTERFERENCE IN MULTI-PHOTON WALKS

3.1. Beyond the Idea

In the 3rd chapter of the thesis, the correlations of multi particle quantum walks with different light sources on photonic systems is discussed. The main idea of this chapter is to calculate the average photon number correlations on two arbitrary beam-splitters by using probability amplitudes of 1-D quantum walk.

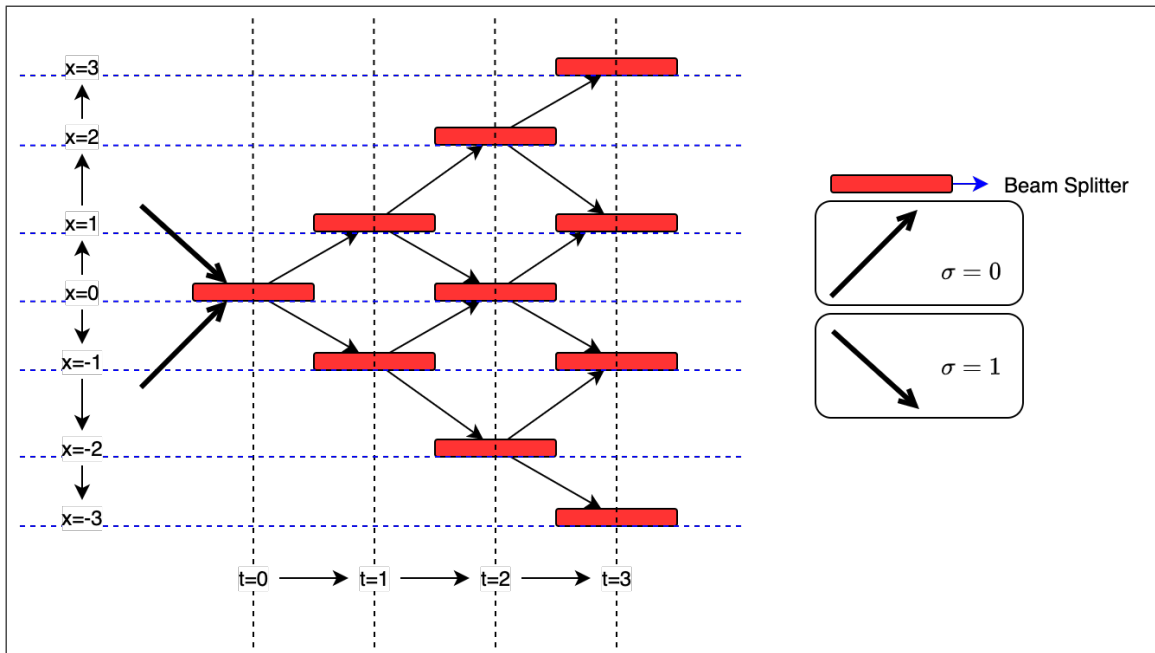


Figure 3.1.: Demonstration of Quantum Walk on Photonic Systems

The Figure 3.1 simply demonstrates quantum walk on a photonic systems . Vertical axis shows spatial degree of freedom. Horizontal axis shows time evolutions. Red rectangles represent 50/50 beam splitters. Upwards and downwards arrows describe two spins degree of freedoms, $\sigma = 0$ and $\sigma = 1$ respectively, the meaning direction of the photon. Beam splitter is the analog of the evolution operator, $U = S(C \otimes I)$, in 1-D quantum walk. When a photon goes to the beam splitter at $x=0$ and $t=0$, it reflects to the beam splitter at $x = 1$ or transmits to the beam splitter at $x = -1$. The arrows before the first beam splitter are the analog of the initial coin state in 1-D quantum walk. The photons

going downwards have $\sigma = 1$ spin state and the photons going upwards have $\sigma = 0$ spin state.

To better understand, let's assume that a photon goes upward to the first beamsplitter. The state of the photon can be defined in terms of the creation operator as follows:

$$|\psi_i\rangle = \hat{a}_{00}^\dagger |0\rangle \quad (3.1)$$

The first 0 of the lower indice of the creation operator represents the position of the photon and the second 0 represents the direction (spin) of the photon. The state of the photon after the first beamsplitter, meaning the state of the photon at $t=1$, can be written as follows:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \left(\hat{a}_{10}^\dagger + \hat{a}_{-11}^\dagger \right) |0\rangle \quad (3.2)$$

The state of the photon above is the analog of the state of a walker after 1-step later in 1-D quantum walk with Hadamard coin. Now let's assume that one photon goes downwards and one photon goes upwards to the first beamsplitter. The total state of the photons at the first beamsplitter can be written as follow:

$$|\psi_i\rangle = \hat{a}_{01}^\dagger \hat{a}_{00}^\dagger |0\rangle \quad (3.3)$$

The total state of the photon after the first beamsplitter becomes:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \left(\hat{a}_{10}^\dagger - \hat{a}_{-11}^\dagger \right) \frac{1}{\sqrt{2}} \left(\hat{a}_{10}^\dagger + \hat{a}_{-11}^\dagger \right) |0\rangle \quad (3.4)$$

Similarly to the 1-D quantum walk, the Hadamard coin maps the state of the photon going downward from \hat{a}_{01}^\dagger to $\frac{1}{\sqrt{2}} \left(\hat{a}_{10}^\dagger - \hat{a}_{-11}^\dagger \right)$. After the multiplication of the terms in the parentheses, the equation above becomes:

$$|\psi_1\rangle = \frac{1}{2} \left(\hat{a}_{10}^{\dagger 2} + \hat{a}_{10}^\dagger \hat{a}_{-11}^\dagger - \hat{a}_{-11}^\dagger \hat{a}_{10}^\dagger - \hat{a}_{-11}^{\dagger 2} \right) |0\rangle \quad (3.5)$$

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0 \quad (3.6)$$

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}$$

Due to the bosonic commutation relation below, \hat{a}_{10}^\dagger and \hat{a}_{-11}^\dagger operators commute thus they can exchange positions and cancel each others.

$$|\psi_1\rangle = \frac{1}{2} \left(\hat{a}_{10}^{\dagger 2} - \hat{a}_{-11}^{\dagger 2} \right) |0\rangle \quad (3.7)$$

The mathematical expression above describes two photons going either upwards or downwards. This is a well known phenomenon and it was found in 1987 by C. K. Hong, Z. Y. Ou and L. Mandel Hong et al. (1987).

Hadamard coin definition is given in Eq. 3.8 However, the beamsplitter can not represent Hadamard coin by itself.

$$H = -ie^{i\sigma_z\pi/2}e^{i\sigma_y\pi/4} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (3.8)$$

The beamsplitter operation on two inputs can be defined as follows:

$$U_{BS} = U_p H U_p = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad (3.9)$$

Where U_p is the phase operator acting on one of the inputs, $U_p = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Therefore to use the analytical solutions of the probability amplitudes of Hadamard coin, the probability amplitudes should be changed as follows:

$$A_{x\sigma}^{\sigma'}(t) = (i)^{\sigma+\sigma'} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikx} A_{k\sigma}^{\sigma'}(t) \quad (3.10)$$

3.1.1. Average Photon Number Correlation

In this subsection, average photon number correlation for two arbitrary beamsplitters will be defined in terms of the probability amplitudes in the Heisenberg Picture. The state of the walker after t time later in 1-D quantum walk is described in Chapter 1. It can also be written to describe the state of the photons after t time later.

$$|\psi_t^{\sigma'}\rangle = \sum_{\sigma,x} A_{x\sigma}^{\sigma'}(t) |\sigma, x\rangle \quad (3.11)$$

Where $\sigma = 0, 1$ represents the photons going downward and upward photons, respectively. Since the evolution of the probability amplitudes depends on the initial spin(direction) state, σ' represents initial spin state, 0, 1 respectively downward and upward. $A_{x\sigma}^{\sigma'}(t)$ are the probability amplitudes.

Average photon number correlation for two arbitrary beamsplitters is defined as follow:

$$C_{y,y'} = \langle \hat{n}_y \hat{n}_{y'} \rangle - \langle \hat{n}_y \rangle \langle \hat{n}_{y'} \rangle \quad (3.12)$$

To calculate the correlation between y^{th} and y'^{th} beamsplitters, the number operators \hat{n}_y and $\hat{n}_{y'}$ can be written as follows:

$$\hat{n}_y = \sum \hat{a}_{y\sigma}^\dagger \hat{a}_{y\sigma} \quad (3.13)$$

$$\hat{n}_{y'} = \sum \hat{a}_{y'\beta}^\dagger \hat{a}_{y'\beta} \quad (3.14)$$

Using 3.13 and 3.14, the expectation value of \hat{n}_y can be written as follows:

$$\langle \hat{n}_y \rangle = \langle \psi_t | \hat{a}_{y\sigma}^\dagger \hat{a}_{y\sigma} | \psi_t \rangle \quad (3.15)$$

$$\langle \hat{n}_y \rangle = \langle \psi_i | U(t)^\dagger \hat{a}_{y\sigma}^\dagger \hat{a}_{y\sigma} U(t) | \psi_i \rangle \quad (3.16)$$

The identity operator UU^\dagger can be added between $\hat{a}_{y\sigma}^\dagger$ and $\hat{a}_{y\sigma}$. Then the equation becomes:

$$\langle \hat{n}_y \rangle = \langle \psi_i | U(t)^\dagger \hat{a}_{y\sigma}^\dagger U(t) U(t)^\dagger \hat{a}_{y\sigma} U(t) | \psi_i \rangle \quad (3.17)$$

To find the correlation, $U(t)^\dagger \hat{a}_{y\sigma} U(t)$ and $U(t)^\dagger \hat{a}_{y\sigma}^\dagger U(t)$ should be written in terms of the probability amplitudes.

3.1.2. Heisenberg Picture

As a general formula, the probability amplitude of a quantum walk after t time later which is started at arbitrary y^{th} beam splitter with σ spin can be described as follows:

$$\langle x\sigma' | U(t) | \sigma y \rangle = A_{x-y,\sigma'}^\sigma(t) \quad (3.18)$$

Where $\langle x\sigma' |$ describes the measurement at x^{th} beam splitter with σ' spin.

The evolution operator was described as follows:

$$U(t) = [S(C \otimes I)]^t \quad (3.19)$$

$$U(t)^\dagger = [(C^\dagger \otimes I)S^\dagger]^t \quad (3.20)$$

Where $C^\dagger = C^{-1}$ and $S^\dagger = S^{-1}$ because of their unitarity. Thus the equation above can be written as follows:

$$U(t)^\dagger = [(C^{-1} \otimes I)S^{-1}]^t \quad (3.21)$$

$$U(t)^\dagger = [(C \otimes I)S]^{-t} \quad (3.22)$$

$$U(t)^\dagger = U(-t) = [S(C \otimes I)]^{-t} \quad (3.23)$$

If we take time reversal of equation 3.18, the probability amplitude of a quantum walk after t time later which started at arbitrary x^{th} beam splitter with σ' spin can be written as follows:

$$\langle y\sigma | U(-t) | x\sigma' \rangle = A_{y-x,\sigma}^{\sigma'^*}(t) \quad (3.24)$$

To solve $U(t)^\dagger \hat{a}_{y\sigma} U(t)$ and $U(t)^\dagger \hat{a}_{y\sigma}^\dagger U(t)$, we can start with writing the final state after t time later.

$$|\psi_f\rangle = U(t) |\psi_i\rangle \quad (3.25)$$

if we describe $|\psi_i\rangle$ in terms of creation operator for a photon which is at y^{th} beam-splitter and σ spin, Equation 3.25 becomes:

$$|\psi_f\rangle = U(t) \hat{a}_{y\sigma}^\dagger |0\rangle \quad (3.26)$$

$U(t)^\dagger U(t)$ can be added before $|0\rangle$.

$$|\psi_f\rangle = U(t) \hat{a}_{y\sigma}^\dagger U(t)^\dagger U(t) |0\rangle \quad (3.27)$$

Applying operator $U(t)$ to vacuum state $|0\rangle$ gives vacuum state $|0\rangle$. Thus the equation becomes:

$$|\psi_f\rangle = U(t) \hat{a}_{y\sigma}^\dagger U(t)^\dagger |0\rangle \quad (3.28)$$

From equation $|\psi_t\rangle = \sum_{\sigma,x} A_{x\sigma}^{\sigma'}(t) |\sigma, x\rangle$, we can write $U(t) \hat{a}_{y\sigma}^\dagger U(t)^\dagger$ in terms of probability amplitudes as follows:

$$U(t) \hat{a}_{y\sigma}^\dagger U(t)^\dagger = \sum A_{x-y,\sigma'}^\sigma(t) \hat{a}_{x\sigma'}^\dagger \quad (3.29)$$

The equation above describes the probability amplitude of the photon being at the x^{th} beam-splitter with σ' spin after t time later which starts the quantum walk at arbitrary y^{th} beam splitter with σ spin. The time reversal of the equation above describes the probability amplitude of the photon being at the y^{th} beam-splitter with σ spin after t time later which starts the quantum walk at arbitrary x^{th} beam splitter with σ' spin.

$$U(t)^\dagger \hat{a}_{y\sigma}^\dagger U(t) = \sum A_{y-x,\sigma}^{\sigma'^*} \hat{a}_{x\sigma'}^\dagger \quad (3.30)$$

The above formula is one of the formulas that we need to calculate expectation value of \hat{n}_y . To simplify, we will not write the time dependency of the probability amplitudes (t) at each step. To find the second formula, we take the conjugate transpose of 3.30.

$$U(t)^\dagger \hat{a}_{y\sigma} U(t) = \sum A_{y-x,\sigma}^{\sigma'} \hat{a}_{x\sigma'} \quad (3.31)$$

Using 3.31 and 3.30, expectation value of \hat{n}_y can be written as follows:

$$\langle \hat{n}_y \rangle = \sum \langle \psi_i | A_{y-x,\sigma}^{\sigma'*} \hat{a}_{x\sigma'}^\dagger A_{y-x',\sigma}^{\sigma''} \hat{a}_{x'\sigma''} | \psi_i \rangle \quad (3.32)$$

The probability amplitudes can be written out of the bracket.

$$\langle \hat{n}_y \rangle = \sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} \langle \psi_i | \hat{a}_{x\sigma'}^\dagger \hat{a}_{x'\sigma''} | \psi_i \rangle \quad (3.33)$$

Similar to $\langle \hat{n}_y \rangle$, $\langle \hat{n}_y \hat{n}_{y'} \rangle$ can be written as follows:

$$\begin{aligned} \langle \hat{n}_y \hat{n}_{y'} \rangle &= \sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y'-x''',\beta}^{\beta''} \\ &\langle \psi_i | \hat{a}_{x\sigma'}^\dagger \hat{a}_{x'\sigma''} \hat{a}_{x''\beta'}^\dagger \hat{a}_{x'''\beta''} | \psi_i \rangle \end{aligned} \quad (3.34)$$

To understand the general result, annihilation operator $\hat{a}_{x'\sigma''}$ should be moved to the right of the creation operator $\hat{a}_{x''\beta'}^\dagger$. Using the Bosonic commutation relation, we can write

$$\hat{a}_{x'\sigma''} \hat{a}_{x''\beta'}^\dagger = \hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}. \quad (3.35)$$

$$\begin{aligned} \langle \hat{n}_y \hat{n}_{y'} \rangle &= \sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y'-x''',\beta}^{\beta''} \\ &\langle \psi_i | \hat{a}_{x\sigma'}^\dagger (\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) \hat{a}_{x'''\beta''} | \psi_i \rangle \end{aligned} \quad (3.36)$$

3.2. Number State

Number state is also known as Fock state. Applying the annihilation operator \hat{a} to number states gives

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad (3.37)$$

Number states should obey the normalization condition $\langle n|n\rangle = 1$. Any number states can be written as follows:

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle \quad (3.38)$$

Where $\frac{1}{\sqrt{n!}}$ is the normalization constant and $\langle n'|n\rangle = \delta_{nn'}$. Thus the initial state of n photons going upwards and m photons going downwards can be written as follows:

$$|\psi_i\rangle = \frac{1}{\sqrt{n!m!}} \left(\hat{a}_{00}^\dagger\right)^n \left(\hat{a}_{01}^\dagger\right)^m |0\rangle \quad (3.39)$$

Where $\frac{1}{\sqrt{n!m!}}$ is the normalization. Clearly, the excitation only occurs at 0^{th} point. Then Equation 3.33 can be written as:

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle \psi_i | \hat{a}_{0\sigma'}^\dagger \hat{a}_{0\sigma''} | \psi_i \rangle \quad (3.40)$$

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \frac{1}{(n!m!)} \langle 0 | (\hat{a}_{00})^n (\hat{a}_{01})^m \hat{a}_{0\sigma'}^\dagger \hat{a}_{0\sigma''} (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^m | 0 \rangle \quad (3.41)$$

To solve the equation above, the annihilation operator should be moved to the right by using the commutation relations below.

$$\begin{aligned} [\hat{a}_k, \hat{a}_{k'}] &= [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0 \\ [\hat{a}_k, \hat{a}_{k'}^\dagger] &= \delta_{kk'} \\ [\hat{a}_k, f(\hat{a}_{k'}^\dagger)] &= \delta_{kk'} f'(\hat{a}_{k'}^\dagger) \end{aligned} \quad (3.42)$$

Using the relations above, $\hat{a}_{0\sigma''}$ can be moved to the right.

$$\rightarrow \hat{a}_{0\sigma''} (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^m |0\rangle \quad (3.43)$$

The equation becomes

$$\rightarrow \left(n (\hat{a}_{00}^\dagger)^{n-1} \delta_{\sigma''0} (\hat{a}_{01}^\dagger)^m + (\hat{a}_{00}^\dagger)^n \hat{a}_{0\sigma''} (\hat{a}_{01}^\dagger)^m \right) |0\rangle. \quad (3.44)$$

$\hat{a}_{0\sigma''}$ should be moved to the right again. After moving annihilation operator to the right, the equation becomes

$$\rightarrow \left(n (\hat{a}_{00}^\dagger)^{n-1} (\hat{a}_{01}^\dagger)^m \delta_{\sigma''0} + m (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^{m-1} \delta_{\sigma''1} + (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^m \hat{a}_{0\sigma''} \right) |0\rangle. \quad (3.45)$$

Using this relation $\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$, the term below is equal to zero.

$$(\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^m \hat{a}_{0\sigma''} |0\rangle = 0 \quad (3.46)$$

Equation 3.45 becomes as:

$$\rightarrow \left(n (\hat{a}_{00}^\dagger)^{n-1} (\hat{a}_{01}^\dagger)^m \delta_{\sigma''0} + m (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^{m-1} \delta_{\sigma''1} \right) |0\rangle \quad (3.47)$$

On the other hand, the creation operator can be moved to the right by using the same method.

$$\langle 0 | \left((\hat{a}_{00})^{n-1} (\hat{a}_{01})^m n \delta_{\sigma''0} + (\hat{a}_{00})^n (\hat{a}_{01})^{m-1} m \delta_{\sigma''1} \right) \leftarrow \quad (3.48)$$

Using 3.44 and 3.48, Equation 3.41 can be written as follows:

$$\begin{aligned} \rightarrow \langle 0 | [& [(\hat{a}_{00})^{n-1}(\hat{a}_{01})^m n \delta_{\sigma'0} + (\hat{a}_{00})^n (\hat{a}_{01})^{m-1} m \delta_{\sigma'1}] \\ & [n(\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^m \delta_{\sigma''0} + m(\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^{m-1} \delta_{\sigma''1}] | 0 \rangle \end{aligned} \quad (3.49)$$

Where \rightarrow (right arrow) represents the probability amplitudes, normalization and the summation symbol before the formula. After multiplication, four terms can be written as:

$$\begin{aligned} \rightarrow n^2 \langle 0 | & (\hat{a}_{00})^{n-1} (\hat{a}_{01})^m (\hat{a}_{00}^\dagger)^{n-1} (\hat{a}_{01}^\dagger)^m | 0 \rangle \\ & + \\ nm \langle 0 | & (\hat{a}_{00})^{n-1} (\hat{a}_{01})^m (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^{m-1} | 0 \rangle \\ & + \\ mn \langle 0 | & (\hat{a}_{00})^n (\hat{a}_{01})^{m-1} (\hat{a}_{00}^\dagger)^{n-1} (\hat{a}_{01}^\dagger)^m | 0 \rangle \\ & + \\ m^2 \langle 0 | & (\hat{a}_{00})^n (\hat{a}_{01})^{m-1} (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^{m-1} | 0 \rangle \end{aligned} \quad (3.50)$$

The inner products of the states in the second and the third rows are orthogonal states thus the second and the third terms are equal to zero. The inner products of the states in the first and the last terms are equal to norm of themselves. The first term becomes:

$$\delta_{\sigma''0} \delta_{\sigma'0} n^2 \langle 0 | (\hat{a}_{00})^{n-1} (\hat{a}_{01})^m (\hat{a}_{00}^\dagger)^{n-1} (\hat{a}_{01}^\dagger)^m | 0 \rangle = n^2 (n-1)! m! \delta_{\sigma''0} \delta_{\sigma'0} \quad (3.51)$$

And the last term becomes:

$$\delta_{\sigma''1} \delta_{\sigma'1} m^2 \langle 0 | (\hat{a}_{00})^n (\hat{a}_{01})^{m-1} (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^{m-1} | 0 \rangle = m^2 n! (m-1)! \delta_{\sigma''1} \delta_{\sigma'1} \quad (3.52)$$

Using Equation 3.51 and 3.52 the expectation value of \hat{n}_y can be written with amplitudes and normalization as:

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \frac{1}{(n!m!)} (n^2 (n-1)! m! \delta_{\sigma''0} \delta_{\sigma'0} + m^2 n! (m-1)! \delta_{\sigma''1} \delta_{\sigma'1}) \quad (3.53)$$

After simplification, the final equation for $\langle \hat{n}_y \rangle$ can be written as:

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} (n \delta_{\sigma''0} \delta_{\sigma'0} + m \delta_{\sigma''1} \delta_{\sigma'1}) \quad (3.54)$$

The expectation value of $\hat{n}_{y'}$ can be found by using the same steps with different raising and lower indices. Similarly, $\langle \hat{n}_{y'} \rangle$ can be written as:

$$\langle \hat{n}_{y'} \rangle = \sum A_{y',\beta}^{\beta'*} A_{y',\beta}^{\beta''} (n \delta_{\beta''0} \delta_{\beta'0} + m \delta_{\beta''1} \delta_{\beta'1}) \quad (3.55)$$

To find $\langle \hat{n}_y \hat{n}_{y'} \rangle$, we can start with writing the general form of $\langle \hat{n}_y \hat{n}_{y'} \rangle$ which is given in Eq. 3.36.

$$\begin{aligned} \langle \hat{n}_y \hat{n}_{y'} \rangle = & \sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y'-x''',\beta}^{\beta''} \\ & \langle \psi_i | \hat{a}_{x\sigma'}^\dagger (\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) \hat{a}_{x'''\beta''} | \psi_i \rangle \end{aligned} \quad (3.56)$$

After writing initial state, 3.88 becomes:

$$\begin{aligned} \langle \hat{n}_y \hat{n}_{y'} \rangle = & \sum \frac{\sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y'-x''',\beta}^{\beta''}}{(n!m!)} \\ & \langle 0 | (\hat{a}_{00})^n (\hat{a}_{01})^m \hat{a}_{x\sigma'}^\dagger (\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) \hat{a}_{x'''\beta''} (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^m | 0 \rangle \end{aligned} \quad (3.57)$$

To better understand, the summation symbol, the normalization and the probability amplitudes will be represented by an arrow "→".

$$\rightarrow \langle 0 | (\hat{a}_{00})^n (\hat{a}_{01})^m \hat{a}_{x\sigma'}^\dagger (\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) \hat{a}_{x'''\beta''} (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^m | 0 \rangle \quad (3.58)$$

The annihilation operators should be moved to the right and the creation operators should be moved to the left. To simplify the calculations, the result of Equation 3.43 can be used here with different raising and lower indices.

$$\begin{aligned} & \langle 0 | (\hat{a}_{00})^n (\hat{a}_{01})^m \hat{a}_{x\sigma'}^\dagger \leftarrow \\ & \rightarrow \hat{a}_{x'''\beta''} (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^m | 0 \rangle \end{aligned} \quad (3.59)$$

After putting the results of above equation to 3.58, Equation 3.58 becomes:

$$\begin{aligned} \rightarrow \langle 0 | & [(\hat{a}_{00})^{n-1} (\hat{a}_{01})^m n \delta_{\sigma'0} + (\hat{a}_{00})^n (\hat{a}_{01})^{m-1} m \delta_{\sigma'1}] (\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) \\ & [n (\hat{a}_{00}^\dagger)^{n-1} (\hat{a}_{01}^\dagger)^m \delta_{\beta''0} + m (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^{m-1} \delta_{\beta''1}] | 0 \rangle \end{aligned} \quad (3.60)$$

Equation 3.60 can be written as follows:

$$\begin{aligned} \rightarrow \langle 0 | & ((\hat{a}_{00})^{n-1} (\hat{a}_{01})^m n \delta_{\sigma'0} + (\hat{a}_{00})^n (\hat{a}_{01})^{m-1} m \delta_{\sigma'1}) (\delta_{\sigma''\beta'} \delta_{x''x'}) \\ & \left(n (\hat{a}_{00}^\dagger)^{n-1} (\hat{a}_{01}^\dagger)^m \delta_{\beta''0} + m (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^{m-1} \delta_{\beta''1} \right) | 0 \rangle \\ & + \\ & \langle 0 | ((\hat{a}_{00})^{n-1} (\hat{a}_{01})^m n \delta_{\sigma'0} + (\hat{a}_{00})^n (\hat{a}_{01})^{m-1} m \delta_{\sigma'1}) \left(\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} \right) \\ & \left(n (\hat{a}_{00}^\dagger)^{n-1} (\hat{a}_{01}^\dagger)^m \delta_{\beta''0} + m (\hat{a}_{00}^\dagger)^n (\hat{a}_{01}^\dagger)^{m-1} \delta_{\beta''1} \right) | 0 \rangle \end{aligned} \quad (3.61)$$

The first terms can be written easily by using the result of $\langle \hat{n}_y \rangle$ as:

$$(n \delta_{\beta''0} \delta_{\sigma'0} + m \delta_{\beta''1} \delta_{\sigma'1}) (\delta_{\sigma''\beta'} \delta_{x''x'}) \quad (3.62)$$

To solve the second term of Equation 3.61, the annihilation operator and the creation operator should be moved to the right and left, respectively. Since the steps for moving to the left are the same with the steps for moving to the right, only the steps for $\hat{a}_{0\sigma''}$ will be shown.

$$\rightarrow [n\hat{a}_{0\sigma''}(\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^m\delta_{\beta''0} + m\hat{a}_{0\sigma''}(\hat{a}_{00}^\dagger)^n(\hat{a}_{01}^\dagger)^{m-1}\delta_{\beta''1}] |0\rangle \quad (3.63)$$

After moving $\hat{a}_{0\sigma''}$ to the right, Equation 3.63 becomes:

$$\begin{aligned} &\rightarrow (n(n-1)(\hat{a}_{00}^\dagger)^{n-2}(\hat{a}_{01}^\dagger)^m\delta_{\beta''0}\delta_{\sigma''0} + nm(\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^{m-1}\delta_{\beta''0}\delta_{\sigma''1} \\ &\quad + n(\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^m\hat{a}_{0\sigma''}) |0\rangle \\ &\quad + \\ &\quad (mn(\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^{m-1}\delta_{\beta''1}\delta_{\sigma''0} + m(m-1)(\hat{a}_{00}^\dagger)^n(\hat{a}_{01}^\dagger)^{m-2}\delta_{\beta''1}\delta_{\sigma''1} \\ &\quad + m(\hat{a}_{00}^\dagger)^n(\hat{a}_{01}^\dagger)^{m-1}\hat{a}_{0\sigma''}) |0\rangle \end{aligned} \quad (3.64)$$

The terms before $|0\rangle$ give zero. Equation 3.64 becomes:

$$\begin{aligned} &\rightarrow (n(n-1)(\hat{a}_{00}^\dagger)^{n-2}(\hat{a}_{01}^\dagger)^m\delta_{\beta''0}\delta_{\sigma''0} + nm(\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^{m-1}\delta_{\beta''0}\delta_{\sigma''1}) |0\rangle \\ &\quad + \\ &\quad (mn(\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^{m-1}\delta_{\beta''1}\delta_{\sigma''0} + m(m-1)(\hat{a}_{00}^\dagger)^n(\hat{a}_{01}^\dagger)^{m-2}\delta_{\beta''1}\delta_{\sigma''1}) |0\rangle \end{aligned} \quad (3.65)$$

Similarly, the left part of the equation can be written by using the same steps.

$$\langle 0 | ((\hat{a}_{00})^{n-1}(\hat{a}_{01})^m n \delta_{\sigma'0} + (\hat{a}_{00})^n (\hat{a}_{01})^{m-1} m \delta_{\sigma'1}) \hat{a}_{0\beta'}^\dagger \quad (3.66)$$

Since the steps are shown, Equation 3.66 can be written as follows:

$$\begin{aligned} &\langle 0 | [(\hat{a}_{00})^{n-2}(\hat{a}_{01})^m (n-1)n \delta_{\sigma'0} \delta_{\beta'0} + (\hat{a}_{00})^{n-1}(\hat{a}_{01})^{m-1} nm \delta_{\sigma'0} \delta_{\beta'1} \\ &\quad + (\hat{a}_{00})^{n-1}(\hat{a}_{01})^{m-1} mn \delta_{\sigma'1} \delta_{\beta'0} + (\hat{a}_{00})^n (\hat{a}_{01})^{m-2} m(m-1) \delta_{\sigma'1} \delta_{\beta'1}] \leftarrow \end{aligned} \quad (3.67)$$

After multiplication of Equations 3.67 and 3.65, only six terms remain because of the orthonormality. Thus:

$$\begin{aligned} &n^2(n-1)^2 \langle 0 | (\hat{a}_{00})^{n-2}(\hat{a}_{01})^m (\hat{a}_{00}^\dagger)^{n-2}(\hat{a}_{01}^\dagger)^m |0\rangle + \\ &n^2 m^2 \langle 0 | (\hat{a}_{00})^{n-1}(\hat{a}_{01})^{m-1} (\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^{m-1} |0\rangle + \\ &n^2 m^2 \langle 0 | (\hat{a}_{00})^{n-1}(\hat{a}_{01})^{m-1} (\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^{m-1} |0\rangle + \\ &m^2 n^2 \langle 0 | (\hat{a}_{00})^{n-1}(\hat{a}_{01})^{m-1} (\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^{m-1} |0\rangle + \\ &m^2 n^2 \langle 0 | (\hat{a}_{00})^{n-1}(\hat{a}_{01})^{m-1} (\hat{a}_{00}^\dagger)^{n-1}(\hat{a}_{01}^\dagger)^{m-1} |0\rangle + \end{aligned}$$

$$m^2(m-1)^2 \langle 0 | (\hat{a}_{00})^{m-2} (\hat{a}_{01})^n (\hat{a}_{00}^\dagger)^{m-2} (\hat{a}_{01}^\dagger)^n | 0 \rangle \quad (3.68)$$

All terms above are equal to their norms. It can be written explicitly as follows:

$$\begin{aligned} \rightarrow & (n^2(n-1)^2(n-2)! m! \delta_{\sigma'0} \delta_{\beta'0} \delta_{\beta''0} \delta_{\sigma''0} \\ & + n^2 m^2 (n-1)! (m-1)! \delta_{\sigma'0} \delta_{\beta'1} \delta_{\beta''0} \delta_{\sigma''1} \\ & + n^2 m^2 (n-1)! (m-1)! \delta_{\sigma'0} \delta_{\beta'1} \delta_{\beta''1} \delta_{\sigma''0} \\ & + n^2 m^2 (n-1)! (m-1)! \delta_{\sigma'1} \delta_{\beta'0} \delta_{\beta''0} \delta_{\sigma''1} \\ & + n^2 m^2 (n-1)! (m-1)! \delta_{\sigma'1} \delta_{\beta'0} \delta_{\beta''1} \delta_{\sigma''0} \\ & + m^2 (m-1)^2 (m-2)! n! \delta_{\sigma'1} \delta_{\beta'1} \delta_{\beta''1} \delta_{\sigma''1}) \end{aligned} \quad (3.69)$$

If we multiply Equation 3.69 with normalization $\frac{1}{n!m!}$. Equation 3.69 becomes:

$$\begin{aligned} \rightarrow & (n(n-1) \delta_{\sigma'0} \delta_{\beta'0} \delta_{\beta''0} \delta_{\sigma''0} \\ & + nm \delta_{\sigma'0} \delta_{\beta'1} \delta_{\beta''0} \delta_{\sigma''1} \\ & + nm \delta_{\sigma'0} \delta_{\beta'1} \delta_{\beta''1} \delta_{\sigma''0} \\ & + nm \delta_{\sigma'1} \delta_{\beta'0} \delta_{\beta''0} \delta_{\sigma''1} \\ & + nm \delta_{\sigma'1} \delta_{\beta'0} \delta_{\beta''1} \delta_{\sigma''0} \\ & + m(m-1) \delta_{\sigma'1} \delta_{\beta'1} \delta_{\beta''1} \delta_{\sigma''1}) \end{aligned} \quad (3.70)$$

Summing Equation 3.70 with Equation 3.62, $(n\delta_{\beta''0}\delta_{\sigma'0} + m\delta_{\beta''1}\delta_{\sigma'1}) (\delta_{\sigma''\beta'}\delta_{x''x'})$, the expectation value of $\langle \hat{n}_y \hat{n}_{y'} \rangle$ can be written with the probability amplitudes as:

$$\begin{aligned} \langle \hat{n}_y \hat{n}_{y'} \rangle = & \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} [\\ & (n(n-1) \delta_{\sigma'0} \delta_{\beta'0} \delta_{\beta''0} \delta_{\sigma''0} + nm \delta_{\sigma'0} \delta_{\beta'1} \delta_{\beta''0} \delta_{\sigma''1} \\ & + nm \delta_{\sigma'1} \delta_{\beta'0} \delta_{\beta''1} \delta_{\sigma''0} + nm \delta_{\sigma'0} \delta_{\beta'1} \delta_{\beta''1} \delta_{\sigma''0} \\ & + nm \delta_{\sigma'1} \delta_{\beta'0} \delta_{\beta''0} \delta_{\sigma''1} + m(m-1) \delta_{\sigma'1} \delta_{\beta'1} \delta_{\beta''1} \delta_{\sigma''1}) \delta_{x''0} \delta_{x'0} \\ & + (n\delta_{\beta''0}\delta_{\sigma'0} + m\delta_{\beta''1}\delta_{\sigma'1}) (\delta_{\sigma''\beta'}\delta_{x''x'})] \end{aligned} \quad (3.71)$$

Now we have all the results to write the correlation function. $\langle \hat{n}_y \rangle$, $\langle \hat{n}_{y'} \rangle$ and $\langle \hat{n}_y \hat{n}_{y'} \rangle$ are calculated in terms of the probability amplitudes. To better see, we can write the result of $\langle \hat{n}_y \rangle \langle \hat{n}_{y'} \rangle$ explicitly.

$$\begin{aligned} \langle \hat{n}_y \rangle \langle \hat{n}_{y'} \rangle = & \\ & \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} A_{y',\beta}^{\beta'*} A_{y',\beta}^{\beta''} (n\delta_{\sigma''0}\delta_{\sigma'0} + m\delta_{\sigma''1}\delta_{\sigma'1}) (n\delta_{\beta''0}\delta_{\beta'0} + m\delta_{\beta''1}\delta_{\beta'1}) \end{aligned} \quad (3.72)$$

Therefore the correlation function is described as:

$$C_{y,y'} = \langle \hat{n}_y \hat{n}_{y'} \rangle - \langle \hat{n}_y \rangle \langle \hat{n}_{y'} \rangle \quad (3.73)$$

Finally the correlation function $C_{y,y'}$ can be written using Equations 3.71 and 3.72 as follows:

$$\begin{aligned} C_{y,y'} = & \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} [\\ & (n(n-1) \delta_{\sigma'0} \delta_{\beta'0} \delta_{\beta''0} \delta_{\sigma''0} + nm \delta_{\sigma'0} \delta_{\beta'1} \delta_{\beta''0} \delta_{\sigma''1} \\ & + nm \delta_{\sigma'1} \delta_{\beta'0} \delta_{\beta''1} \delta_{\sigma''0} + nm \delta_{\sigma'0} \delta_{\beta'1} \delta_{\beta''1} \delta_{\sigma''0} \\ & + nm \delta_{\sigma'1} \delta_{\beta'0} \delta_{\beta''0} \delta_{\sigma''1} + m(m-1) \delta_{\sigma'1} \delta_{\beta'1} \delta_{\beta''1} \delta_{\sigma''1}) \delta_{x''0} \delta_{x'0} \\ & + (n \delta_{\beta''0} \delta_{\sigma'0} + m \delta_{\beta''1} \delta_{\sigma'1}) (\delta_{\sigma''\beta'} \delta_{x''x'}) - \\ & ((n \delta_{\sigma''0} \delta_{\sigma'0} + m \delta_{\sigma''1} \delta_{\sigma'1}) (n \delta_{\beta''0} \delta_{\beta'0} + m \delta_{\beta''1} \delta_{\beta'1})) \delta_{x''0} \delta_{x'0}] \end{aligned} \quad (3.74)$$

3.2.1. Results For Number States

We used Matlab to plot the correlation functions. The code which is used for plotting the function at Equation 3.74 can be found at the appendix.

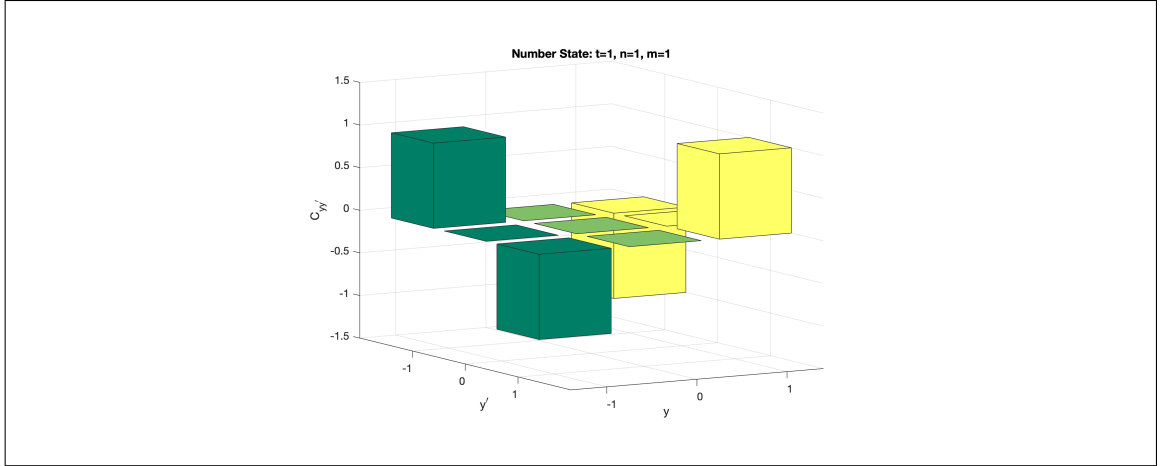


Figure 3.2.: The figure above shows the correlations for $n=1$ and $m=1$

Figure 3.2 shows us the correlation result of number state (with $n=1$, $m=1$) after one step later. The negative correlation can be seen at $y = 1$ and $y' = -1$ or $y = -1$ and $y' = -1$. It can be said that the result is coherent to Hong-Ou-Mandel effect.

Figure 3.3 show us the correlation result of number state (with $n=1$, $m=1$) after 10 steps later. We have maximum correlations at $y = 6$ and $y' = 6$ or $y = -6$ and $y' =$

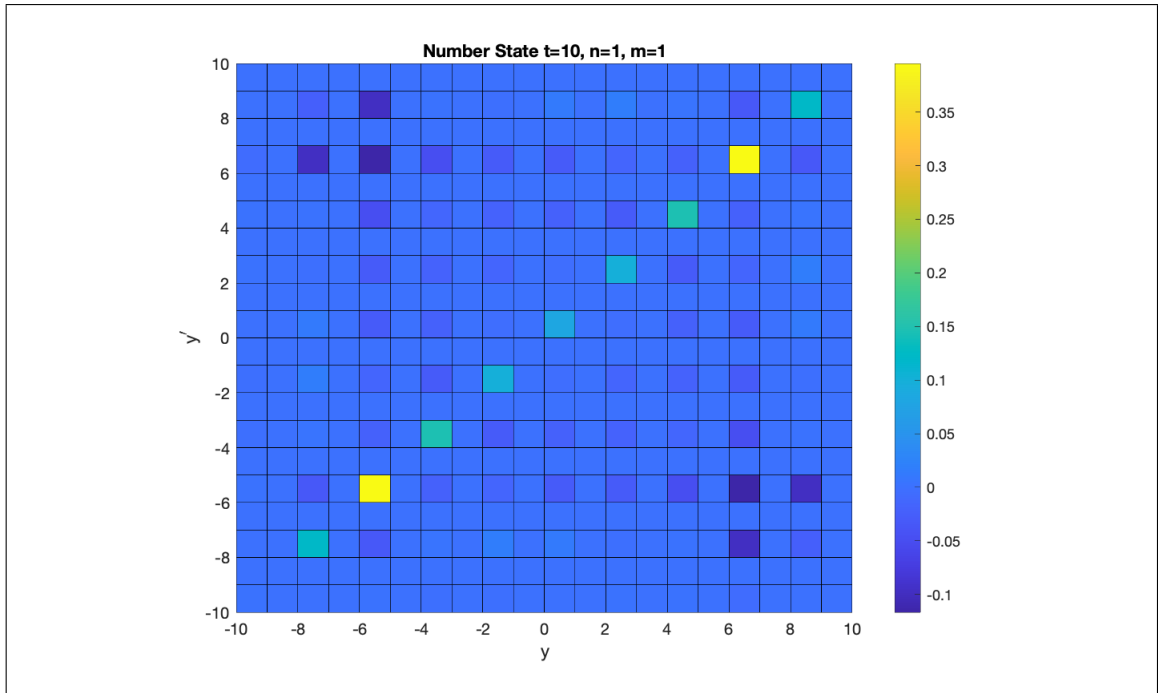


Figure 3.3.: The figure shows the correlations for $n=1$ and $m=1$ after 10 steps later

–6. Correlations at these points ($y = y'$ mean variance of the walk). Thus the maximum correlations are coherent to 1-D quantum walk variance in the introduction part. It can be seen as follows:

$$C_{yy} = \langle \hat{n}_y \hat{n}_y \rangle - \langle \hat{n}_y \rangle \langle \hat{n}_y \rangle \quad (3.75)$$

$$C_{yy} = \langle \hat{n}_y^2 \rangle - \langle \hat{n}_y \rangle^2 \quad (3.76)$$

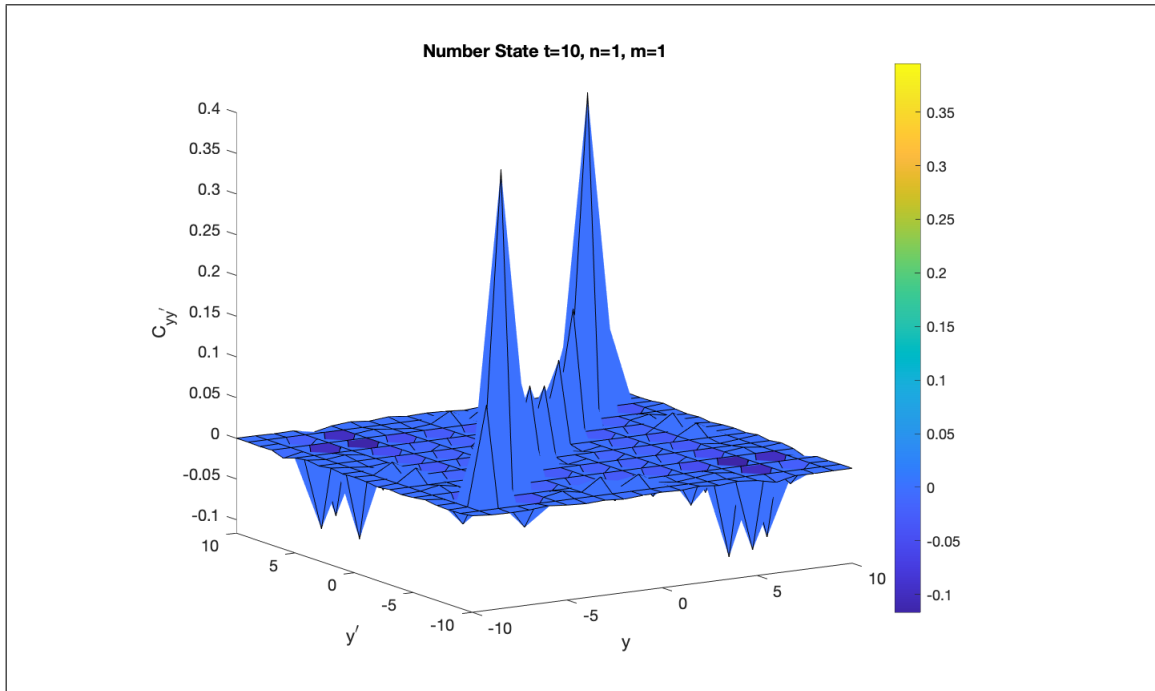


Figure 3.4.: The figure shows the correlations for $n=1$ and $m=1$ after 10 steps later

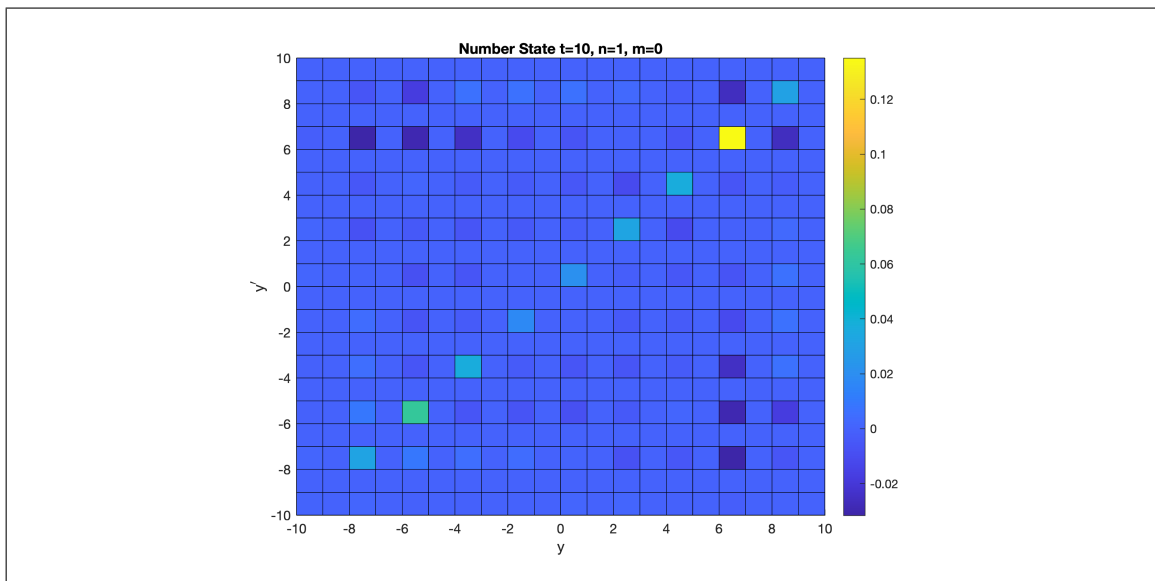


Figure 3.5.: The figure shows the correlations for $n=1$ and $m=0$ after 10 steps later

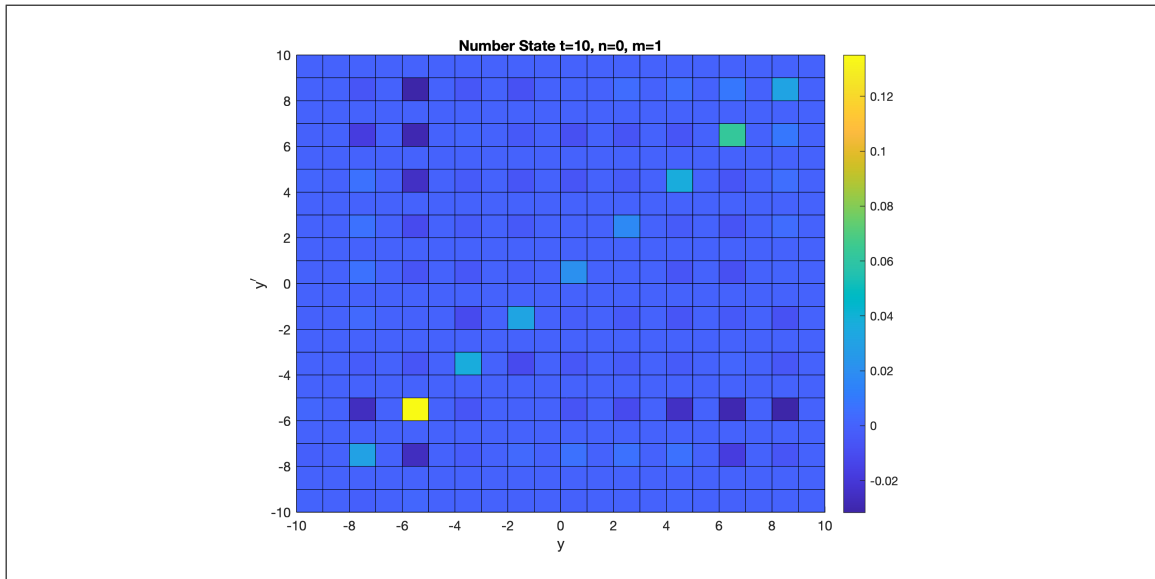


Figure 3.6.: The figure shows the correlations for $n=1$ and $m=0$ after 10 steps later

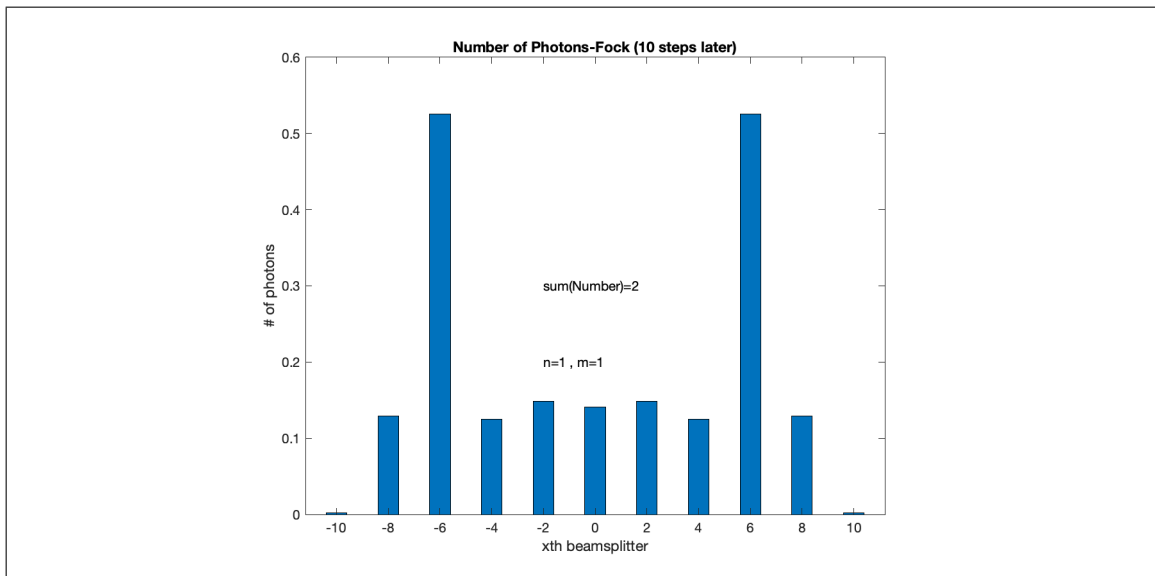


Figure 3.7.: The figure shows $\langle n_y \rangle$ distribution for $n=1$ and $m=1$ after 10 steps later. It can be clearly seen that total number of photons is conserved.

3.3. Coherent States

Coherent states (also known as Glauber states) can be defined in phase space as circular symmetric uncertainty and displaced from the origin with $|\alpha|^2$. When $\alpha = 0$, it defines vacuum states.

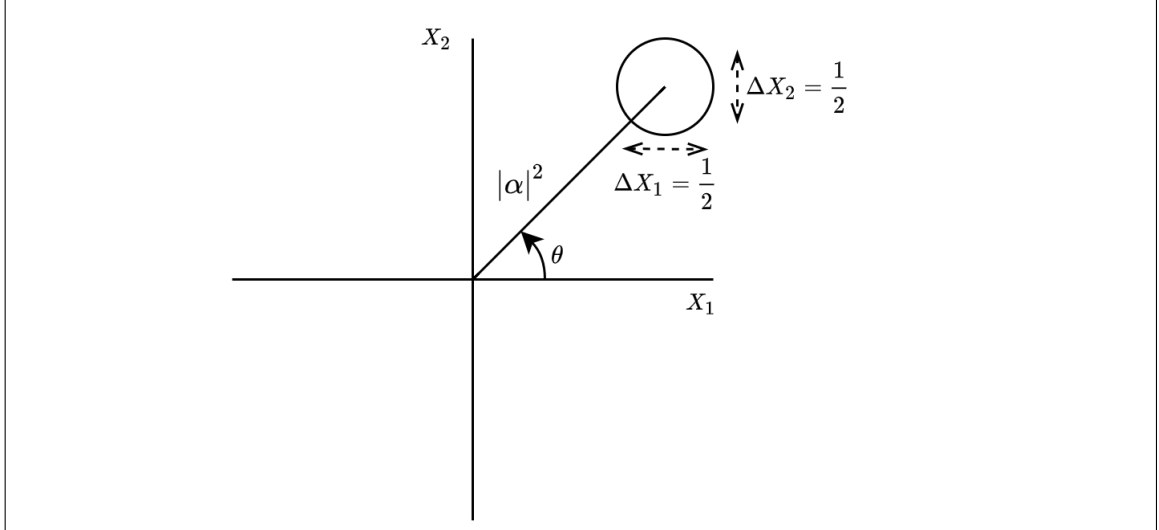


Figure 3.8.: Coherent states in phase space. The circle describes the uncertainty of the states.

We can describe coherent states as initial state:

$$|\psi_i\rangle = |\alpha_{00}, \alpha_{01}\rangle = |\vec{\alpha}\rangle \quad (3.77)$$

As a specific example of a coherent state α going upwards to the first beams splitter can be written as:

$$|\alpha_{00}\rangle = \sum_n e^{-\frac{1}{2}|\alpha_{00}|^2} \frac{\alpha_{00}^n}{\sqrt{n!}} (\hat{a}_{00}^\dagger)^n |0\rangle \quad (3.78)$$

On the other hand, a coherent state α going downwards to the first beams splitter can be written as:

$$|\alpha_{01}\rangle = \sum_n e^{-\frac{1}{2}|\alpha_{01}|^2} \frac{\alpha_{01}^n}{\sqrt{n!}} (\hat{a}_{01}^\dagger)^n |0\rangle \quad (3.79)$$

Applying annihilation operator \hat{a} to $|\alpha\rangle$ state gives the relation below.

$$\hat{a}_{x\sigma} |\vec{\alpha}\rangle = \vec{\alpha}_{x\sigma} |\vec{\alpha}\rangle \quad (3.80)$$

$$\langle \vec{\alpha} | \hat{a}_{x\sigma}^\dagger = \langle \vec{\alpha} | \vec{\alpha}_{x\sigma}^* \quad (3.81)$$

Where $\vec{\alpha}(0) = \alpha_{00}$ and $\vec{\alpha}(1) = \alpha_{01}$. To calculate the correlation function of $|\vec{\alpha}\rangle$ state, we will use the relation 3.80.

$$C_{y,y'} = \langle \hat{n}_y \hat{n}_{y'} \rangle - \langle \hat{n}_y \rangle \langle \hat{n}_{y'} \rangle \quad (3.82)$$

Similarly to the number state, first we will find the expectation values of \hat{n}_y and $\hat{n}_{y'}$ by using the formula below.

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle \psi_i | \hat{a}_{0\sigma}^\dagger \hat{a}_{0\sigma''} | \psi_i \rangle \quad (3.83)$$

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle \hat{\alpha} | \hat{a}_{0\sigma}^\dagger \hat{a}_{0\sigma''} | \hat{\alpha} \rangle \quad (3.84)$$

Using Equations 3.80 and 3.81, we can simply find the result of $\langle \hat{n}_y \rangle$ and $\langle \hat{n}_{y'} \rangle$

$$\Rightarrow \langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle \hat{\alpha} | \hat{a}_{0\sigma}^\dagger \hat{a}_{0\sigma''} | \hat{\alpha} \rangle \quad (3.85)$$

$$\Rightarrow \langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \vec{\alpha}_{\sigma'}^* \vec{\alpha}_{\sigma''}$$

The expectation value of $\hat{n}_{y'}$ can be written by using same steps with different indices.

$$\langle \hat{n}_{y'} \rangle = \sum A_{y',\beta}^{\beta'*} A_{y',\beta}^{\beta''} \langle \vec{\alpha} | \hat{a}_{0\beta'}^\dagger \hat{a}_{0\beta''} | \vec{\alpha} \rangle \quad (3.86)$$

$$\Rightarrow \langle \hat{n}_{y'} \rangle = \sum A_{y',\beta}^{\beta'*} A_{y',\beta}^{\beta''} \langle \vec{\alpha} | \hat{a}_{0\beta'}^\dagger \hat{a}_{0\beta''} | \vec{\alpha} \rangle \quad (3.87)$$

$$\Rightarrow \langle \hat{n}_{y'} \rangle = \sum A_{y',\beta}^{\beta'*} A_{y',\beta}^{\beta''} \vec{\alpha}_{\beta'}^* \vec{\alpha}_{\beta''}$$

To find the expectation value of $\hat{n}_y \hat{n}_{y'}$, we can start by writing the following formula.

$$\begin{aligned} \langle \hat{n}_y \hat{n}_{y'} \rangle &= \sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y'-x''',\beta}^{\beta''} \\ &\langle \alpha | \hat{a}_{x\sigma}^\dagger (\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) \hat{a}_{x'''\beta''} | \alpha \rangle \end{aligned} \quad (3.88)$$

$$\Rightarrow \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} \vec{\alpha}_{\sigma'}^* \vec{\alpha}_{\beta''} \langle \vec{\alpha} | (\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) | \vec{\alpha} \rangle \quad (3.89)$$

To better see, we will write just the following part of the equation above.

$$\Rightarrow \vec{\alpha}_{\sigma'}^* \vec{\alpha}_{\beta''} \langle \vec{\alpha} | (\hat{a}_{0\beta'}^\dagger \hat{a}_{0\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) | \vec{\alpha} \rangle \quad (3.90)$$

$$\Rightarrow \vec{\alpha}_{\sigma'}^* \vec{\alpha}_{\beta''} \left(\langle \vec{\alpha} | \hat{a}_{0\beta'}^\dagger \hat{a}_{0\sigma''} | \vec{\alpha} \rangle + \langle \vec{\alpha} | \delta_{\sigma''\beta'} \delta_{x''x'} | \vec{\alpha} \rangle \right) \quad (3.91)$$

Using Equations 3.80 and 3.81, the expectation value of $\hat{n}_y \hat{n}_{y'}$ can be written as follows:

$$\begin{aligned} \langle \hat{n}_y \hat{n}_{y'} \rangle &= \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} \\ &(\vec{\alpha}_{\sigma'}^* \vec{\alpha}_{\sigma''} \vec{\alpha}_{\beta'}^* \vec{\alpha}_{\beta''} \delta_{x''0} \delta_{x'0} + \vec{\alpha}_{\sigma'}^* \vec{\alpha}_{\beta''} \delta_{\sigma''\beta'} \delta_{x''x'}) \end{aligned} \quad (3.92)$$

Finally using the results of 3.85, 3.87 and 3.92 and after simplification, the correlation function can be written as follows:

$$C_{y,y'} = \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} (\vec{\alpha}_{\sigma'}^* \vec{\alpha}_{\beta''} \delta_{\sigma''\beta'} \delta_{x''x'}) \quad (3.93)$$

3.3.1. Results For Coherent States

The figure 3.9 shows us the result of coherent states ($\alpha = 1$ and $\alpha' = 1$) after one step and after 10 steps, respectively. The figure 3.5 shows that there are no negative correlations compared to the result of the number state($n=1,m=1$). If we look at the figure 3.6, it shows the correlations after ten steps later, the variance satisfies the quantum walk variance and there are still no negative correlations. It can be said that the correlation analysis gives clear differences due to different kind of photon sources.

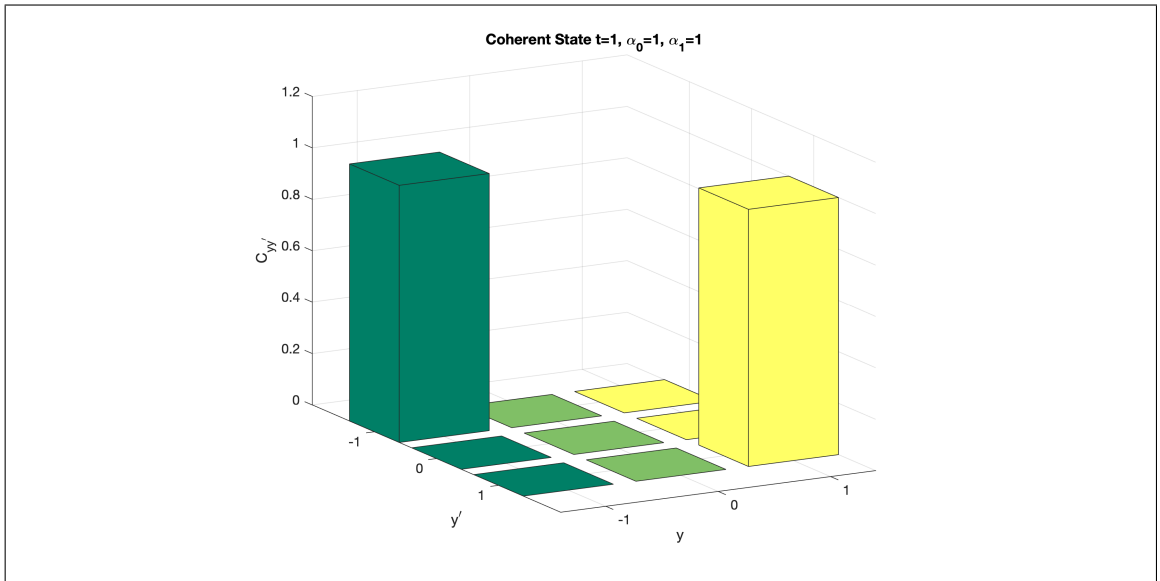


Figure 3.9.: Coherent State: $t=1, \alpha_0 = 1 \alpha_1 = 1$

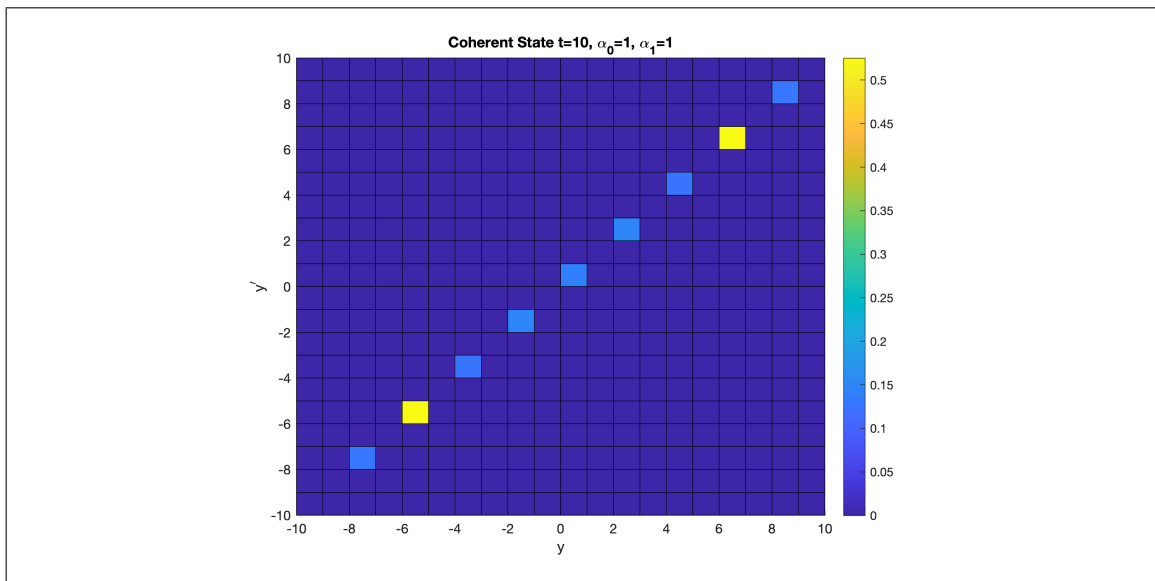


Figure 3.10.: Coherent State: $t=10, \alpha_{00} = 1, \alpha_{01} = 1$

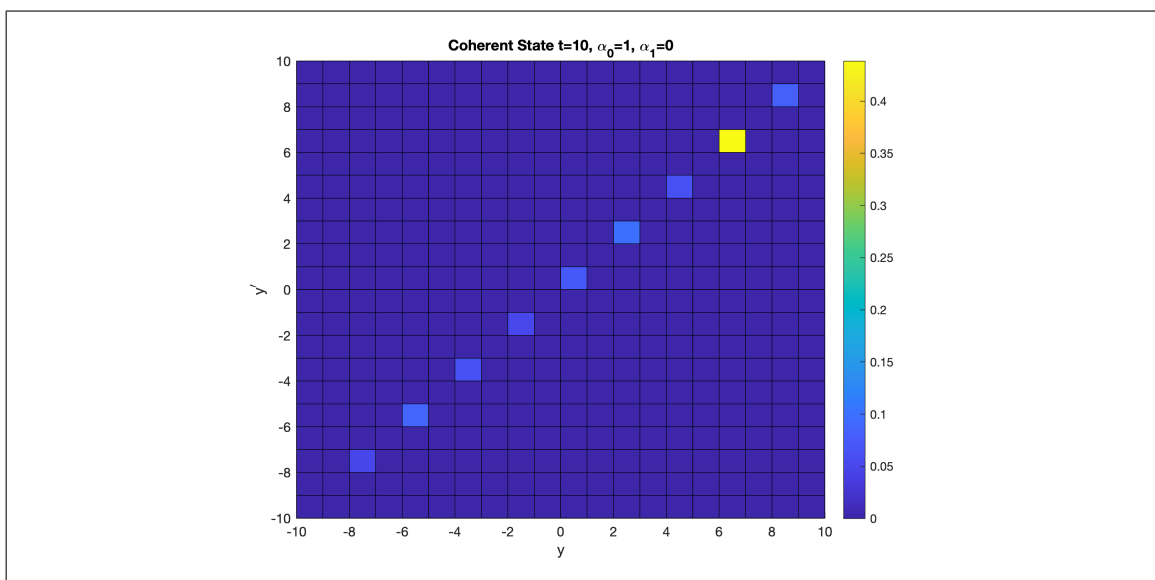


Figure 3.11.: Coherent State: $t=10, \alpha_{00} = 1, \alpha_{01} = 0$

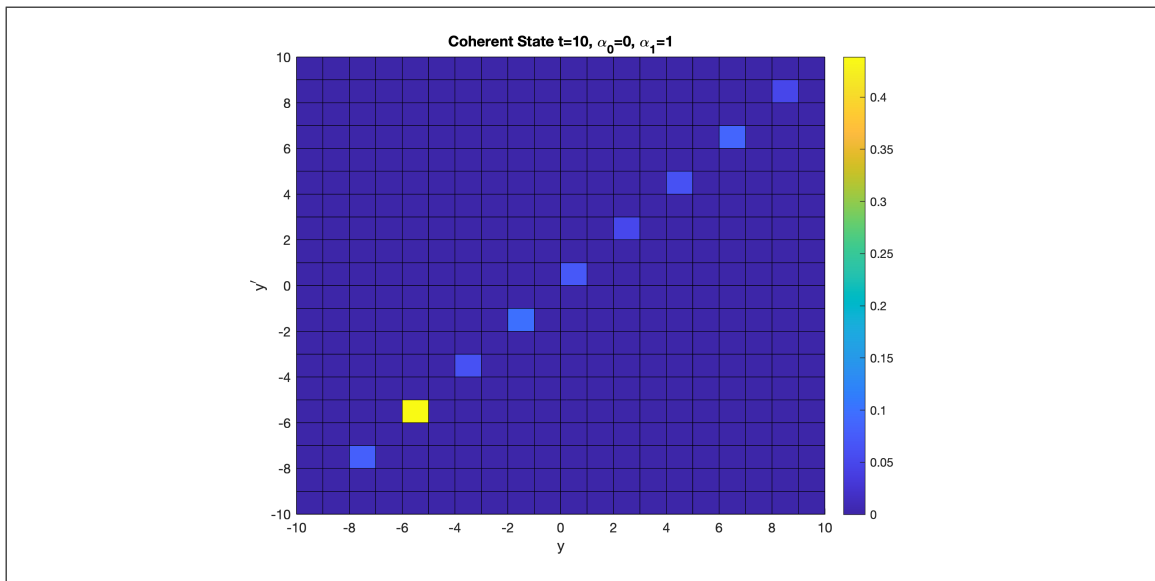


Figure 3.12.: Coherent State: $t=10, \alpha_{00} = 0, \alpha_{01} = 1$

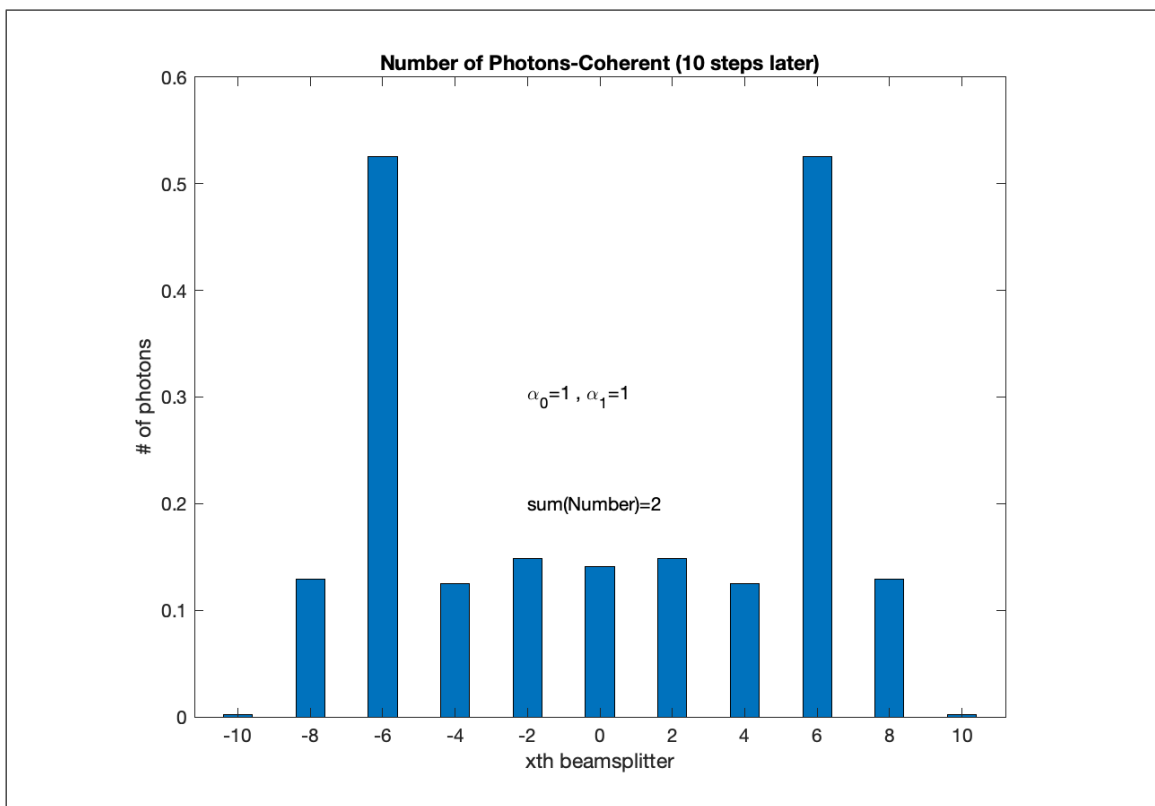


Figure 3.13.: The figure shows $\langle n_y \rangle$ distribution for $\alpha_0 = 1$ and $\alpha_1 = 1$ after 10 steps later. It can be clearly seen that total number of photons is conserved.

3.4. Vacuum Squeezed State

Squeezed states of light can be defined as one of the non-classical light. Unlike the coherent states, the uncertainty of squeezed states is not circularly symmetric.

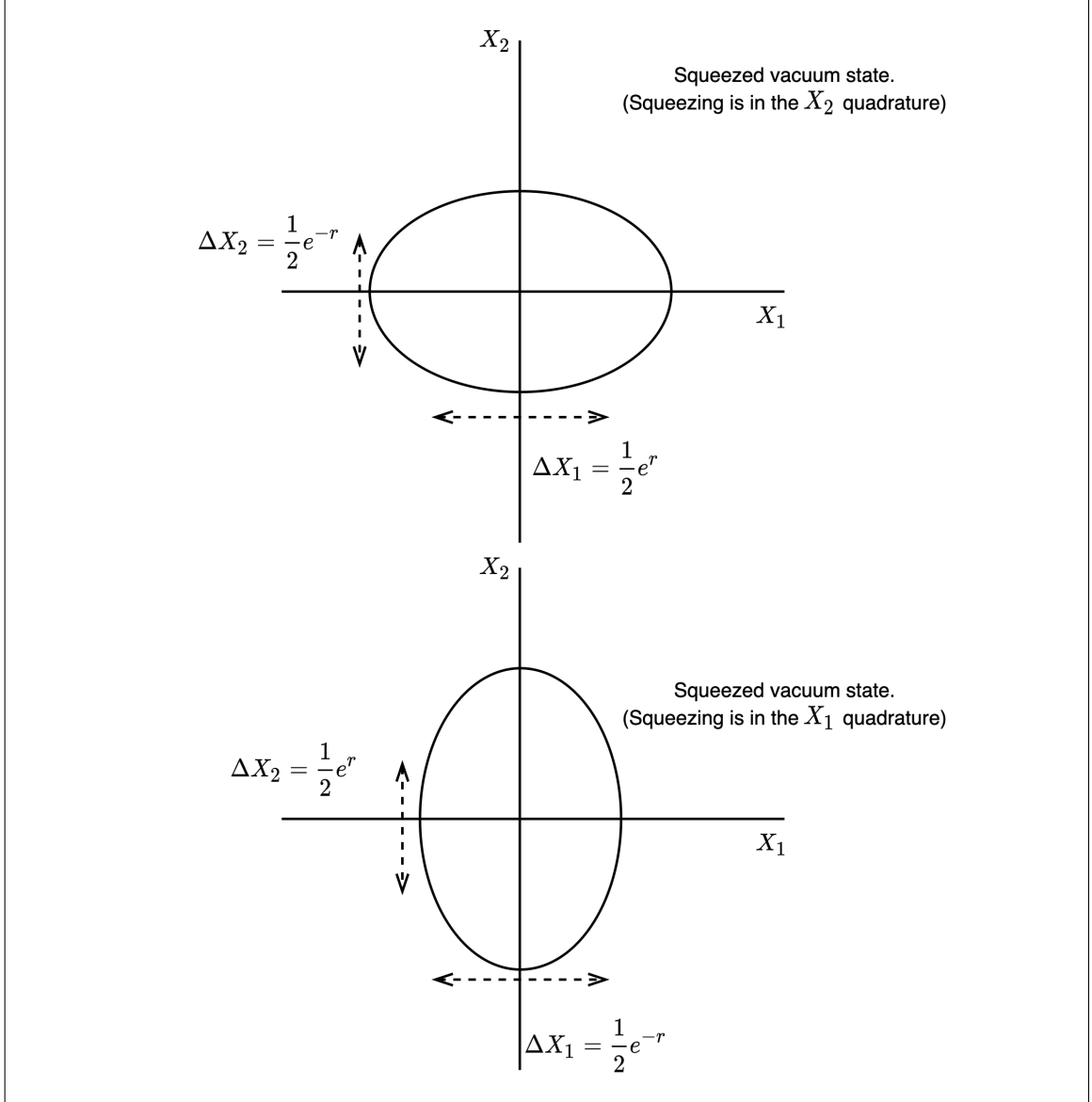


Figure 3.14.: Squeezed vacuum states in phase space. The ellipses describe uncertainty of the states.

Vacuum squeezed states can be described by the following formula.

$$|\xi\rangle = S(\xi) |0\rangle \quad (3.94)$$

where $\xi = r e^{i\theta}$. The relation between squeezed operators and annihilation/creation operators can be defined as follows:

$$S^\dagger(\xi) \hat{a}_{0\sigma''} S(\xi) = \hat{a}_{0\sigma''} \cosh r_{0\sigma''} - \hat{a}_{0\sigma''}^\dagger e^{i\theta_{0\sigma''}} \sinh r_{0\sigma''} \quad (3.95)$$

$$S^\dagger(\xi) \hat{a}_{0\sigma'}^\dagger S(\xi) = \hat{a}_{0\sigma'}^\dagger \cosh r_{0\sigma'} - \hat{a}_{0\sigma'} e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} \quad (3.96)$$

To find the expectation value of \hat{n}_y and $\hat{n}_{y'}$ we will take the initial state as follows:

$$|\psi_i\rangle = |\xi, \xi\rangle = |\xi' \xi\rangle \quad (3.97)$$

Where $\xi = re^{i\theta}$.

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle \xi, \xi' | \hat{a}_{0\sigma'}^\dagger \hat{a}_{0\sigma''} | \xi' \xi \rangle \quad (3.98)$$

where $|\xi\rangle = S(\xi) |0\rangle$ and $|\xi'\rangle = S(\xi') |0\rangle$. The initial state can be written as $S(\xi)S(\xi') |0\rangle$.

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle 0 | S(\xi')^\dagger S(\xi)^\dagger \hat{a}_{0\sigma'}^\dagger \hat{a}_{0\sigma''} S(\xi') S(\xi) |0\rangle \quad (3.99)$$

$S(\xi')S(\xi)$ can be taken as one \hat{S} operator.

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle 0 | \hat{S}^\dagger \hat{a}_{0\sigma'}^\dagger \hat{a}_{0\sigma''} \hat{S} |0\rangle \quad (3.100)$$

Since S is unitary we can insert SS' between $\hat{a}_{0\sigma'}^\dagger$ and $\hat{a}_{0\sigma''}$.

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle 0 | \hat{S}^\dagger \hat{a}_{0\sigma'}^\dagger \hat{S} \hat{S}^\dagger \hat{a}_{0\sigma''} \hat{S} |0\rangle \quad (3.101)$$

Thus we can use the relations 3.95 and 3.96 to represent the equation above in terms of the annihilation and the creation operators.

$$\left(\hat{S}^\dagger \hat{a}_{0\sigma''} \hat{S} \right) \rightarrow \hat{a}_{0\sigma''} \cosh r_{0\sigma''} - \hat{a}_{0\sigma''}^\dagger e^{i\theta_{0\sigma''}} \sinh r_{0\sigma''} \quad (3.102)$$

$$\left(\hat{S}^\dagger \hat{a}_{0\sigma'}^\dagger \hat{S} \right) \rightarrow \hat{a}_{0\sigma'}^\dagger \cosh r_{0\sigma'} - \hat{a}_{0\sigma'} e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} \quad (3.103)$$

If we write Equation 3.101 using the relations above. Equation 3.101 becomes as follows:

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle 0 | \left(\hat{a}_{0\sigma'}^\dagger \cosh r_{0\sigma'} - \hat{a}_{0\sigma'} e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} \right) \left(\hat{a}_{0\sigma''} \cosh r_{0\sigma''} - \hat{a}_{0\sigma''}^\dagger e^{i\theta_{0\sigma''}} \sinh r_{0\sigma''} \right) |0\rangle \quad (3.104)$$

After the multiplication of the terms above, only the terms aa^\dagger will remain.

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle 0 | \left(\hat{a}_{0\sigma'} \hat{a}_{0\sigma''}^\dagger e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} e^{i\theta_{0\sigma''}} \sinh r_{0\sigma''} \right) |0\rangle \quad (3.105)$$

After taking the inner product, the expectation value of \hat{n}_y becomes:

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \left(e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} e^{i\theta_{0\sigma''}} \sinh r_{0\sigma''} \right) \delta_{\sigma'\sigma''} \quad (3.106)$$

To find the expectation value of $\hat{n}_{y'}$, we can follow the same steps for the expectation value of \hat{n}_y .

$$\langle \hat{n}_{y'} \rangle = \sum A_{y',\beta}^{\beta'*} A_{y',\beta}^{\beta''} \left(e^{-i\theta_{0\beta'}} \sinh r_{0\beta'} e^{i\theta_{0\beta''}} \sinh r_{0\beta''} \right) \delta_{\beta'\beta''} \quad (3.107)$$

To find the expectation value of $\hat{n}_y \hat{n}_{y'}$ we will follow the same steps. First we write the $\langle \hat{n}_y \hat{n}_{y'} \rangle$.

$$\langle \hat{n}_y \hat{n}_{y'} \rangle = \sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y'-x''',\beta}^{\beta''} \langle 0 | \hat{S}^\dagger \hat{a}_{x\sigma'}^\dagger (\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) \hat{a}_{x'''\beta''} \hat{S} | 0 \rangle \quad (3.108)$$

$$\langle \hat{n}_y \hat{n}_{y'} \rangle = \sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y'-x''',\beta}^{\beta''} \langle 0 | \hat{S}^\dagger \hat{a}_{x\sigma'}^\dagger \hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} \hat{a}_{x'''\beta''} \hat{S} | 0 \rangle + \langle 0 | \hat{S}^\dagger \hat{a}_{x\sigma'}^\dagger \delta_{\sigma''\beta'} \delta_{x''x'} \hat{a}_{x'''\beta''} \hat{S} | 0 \rangle \quad (3.109)$$

The identity operator $\hat{S}\hat{S}^\dagger$ can be written between all annihilation and creation operators.

$$\Rightarrow \langle 0 | \hat{S}^\dagger \hat{a}_{0\sigma'}^\dagger \hat{S} \hat{S}^\dagger \hat{a}_{0\beta'}^\dagger \hat{S} \hat{S}^\dagger \hat{a}_{0\sigma''} \hat{S} \hat{S}^\dagger \hat{a}_{0\beta''} \hat{S} | 0 \rangle + \langle 0 | \hat{S}^\dagger \hat{a}_{x\sigma'}^\dagger \delta_{\sigma''\beta'} \delta_{x''x'} \hat{a}_{x'''\beta''} \hat{S} | 0 \rangle \quad (3.110)$$

Using the relations 3.95 and 3.96 we can write the following transformations in terms of the annihilation and the creation operators.

$$\Rightarrow \left(\hat{S}^\dagger \hat{a}_{0\beta''} \hat{S} \right) \rightarrow \hat{a}_{0\beta''} \cosh r_{0\beta''} - \hat{a}_{0\beta''}^\dagger e^{i\theta_{0\beta''}} \sinh r_{0\beta''} \quad (3.111)$$

$$\Rightarrow \left(\hat{S}^\dagger \hat{a}_{0\beta'}^\dagger \hat{S} \right) \rightarrow \hat{a}_{0\beta'}^\dagger \cosh r_{0\beta'} - \hat{a}_{0\beta'} e^{-i\theta_{0\beta'}} \sinh r_{0\beta'} \quad (3.112)$$

$$\Rightarrow \left(\hat{S}^\dagger \hat{a}_{0\sigma''} \hat{S} \right) \rightarrow \hat{a}_{0\sigma''} \cosh r_{0\sigma''} - \hat{a}_{0\sigma''}^\dagger e^{i\theta_{0\sigma''}} \sinh r_{0\sigma''} \quad (3.113)$$

$$\Rightarrow \left(\hat{S}^\dagger \hat{a}_{0\sigma'}^\dagger \hat{S} \right) \rightarrow \hat{a}_{0\sigma'}^\dagger \cosh r_{0\sigma'} - \hat{a}_{0\sigma'} e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} \quad (3.114)$$

To simplify, we will not write the summation symbol and the probability amplitudes until the final result.

$$\begin{aligned} & \langle 0 | [(\hat{a}_{0\sigma'}^\dagger \cosh r_{0\sigma'} - \hat{a}_{0\sigma'} e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'}) \\ & (\hat{a}_{0\beta'}^\dagger \cosh r_{0\beta'} - \hat{a}_{0\beta'} e^{-i\theta_{0\beta'}} \sinh r_{0\beta'}) \\ & (\hat{a}_{0\sigma''} \cosh r_{0\sigma''} - \hat{a}_{0\sigma''}^\dagger e^{i\theta_{0\sigma''}} \sinh r_{0\sigma''}) \\ & (\hat{a}_{0\beta''} \cosh r_{0\beta''} - \hat{a}_{0\beta''}^\dagger e^{i\theta_{0\beta''}} \sinh r_{0\beta''})] | 0 \rangle \end{aligned} \quad (3.115)$$

After the multiplication of the all terms, only the following terms will remain.

$$\begin{aligned}
& \langle 0 | \hat{a}_{0\sigma'} \hat{a}_{0\beta'}^\dagger \hat{a}_{0\sigma''} \hat{a}_{0\beta''}^\dagger | 0 \rangle e^{-i\theta_{0\sigma'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \cosh r_{0\sigma''} \cosh r_{0\beta'} + \\
& \langle 0 | \hat{a}_{0\sigma'} \hat{a}_{0\beta'}^\dagger \hat{a}_{0\sigma''} \hat{a}_{0\beta''}^\dagger | 0 \rangle e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} e^{-i\theta_{0\beta'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\sigma''} \sinh r_{0\beta'} \sinh r_{0\beta''} + \\
& e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \delta_{\sigma''\beta'} \delta_{\sigma'\beta''} \delta_{x''x'}
\end{aligned} \tag{3.116}$$

The third term can be easily solved by using number operator unlike the first term. We will use the commutation relation to move the annihilation $\hat{a}_{0\sigma''}$ operator to the right of the creation operator $\hat{a}_{0\beta'}^\dagger$. After using the commutation relations, Equation 3.117 becomes:

$$\begin{aligned}
& (\delta_{\sigma'\beta'} \delta_{\sigma''\beta''}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \cosh r_{0\sigma''} \cosh r_{0\beta'} + \\
& (\delta_{\sigma'\sigma''} \delta_{\beta'\beta''} + \delta_{\sigma'\beta''} \delta_{\sigma''\beta'}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} e^{-i\theta_{0\beta'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\sigma''} \sinh r_{0\beta'} \sinh r_{0\beta''} + \\
& e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \delta_{\sigma''\beta'} \delta_{\sigma'\beta''} \delta_{x''x'}
\end{aligned} \tag{3.117}$$

Thus the expectation value of $\hat{n}_y \hat{n}_{y'}$ can be written as:

$$\begin{aligned}
& \langle \hat{n}_y \hat{n}_{y'} \rangle = \\
& \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} [\\
& (\delta_{\sigma'\beta'} \delta_{\sigma''\beta''}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \cosh r_{0\sigma''} \cosh r_{0\beta'} + \\
& (\delta_{\sigma'\sigma''} \delta_{\beta'\beta''} + \delta_{\sigma'\beta''} \delta_{\sigma''\beta'}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} e^{-i\theta_{0\beta'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\sigma''} \sinh r_{0\beta'} \sinh r_{0\beta''} + \\
& e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \delta_{\sigma''\beta'} \delta_{\sigma'\beta''} \delta_{x''x'}]
\end{aligned} \tag{3.118}$$

Finally, the correlation function of the squeezed state can be written.

$$\begin{aligned}
& C_{yy'} = \\
& \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} [\\
& (\delta_{\sigma'\beta'} \delta_{\sigma''\beta''}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \cosh r_{0\sigma''} \cosh r_{0\beta'} + \\
& (\delta_{\sigma'\sigma''} \delta_{\beta'\beta''} + \delta_{\sigma'\beta''} \delta_{\sigma''\beta'}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} e^{-i\theta_{0\beta'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\sigma''} \sinh r_{0\beta'} \sinh r_{0\beta''} + \\
& e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \delta_{\sigma''\beta'} \delta_{\sigma'\beta''} \delta_{x''x'} - \\
& e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} e^{-i\theta_{0\beta'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\sigma''} \sinh r_{0\beta'} \sinh r_{0\beta''} (\delta_{\sigma'\sigma''} \delta_{\beta'\beta''})]
\end{aligned} \tag{3.119}$$

After simplifying the cancelling terms, we have the following formula for the correlation

function of the squeezed state.

$$\begin{aligned}
C_{y,y'} = & \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} [\\
& (\delta_{\sigma'\beta'} \delta_{\sigma''\beta''}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} e^{-i\theta_{0\beta'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\sigma''} \sinh r_{0\beta'} \sinh r_{0\beta''} + \\
& (\delta_{\sigma'\beta'} \delta_{\sigma''\beta''}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \cosh r_{0\sigma''} \cosh r_{0\beta'} + \\
& e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \delta_{\sigma''\beta'} \delta_{\sigma'\beta''} \delta_{x''x'}]
\end{aligned} \tag{3.120}$$

3.4.1. Results For Vacuum Squeezed States

Figure 3.15 shows us the result of vacuum squeezed state results($r_0 = 0.2$ $r_1 = 0.2$ $\theta_0 = 0$ $\theta_1 = \pi$)

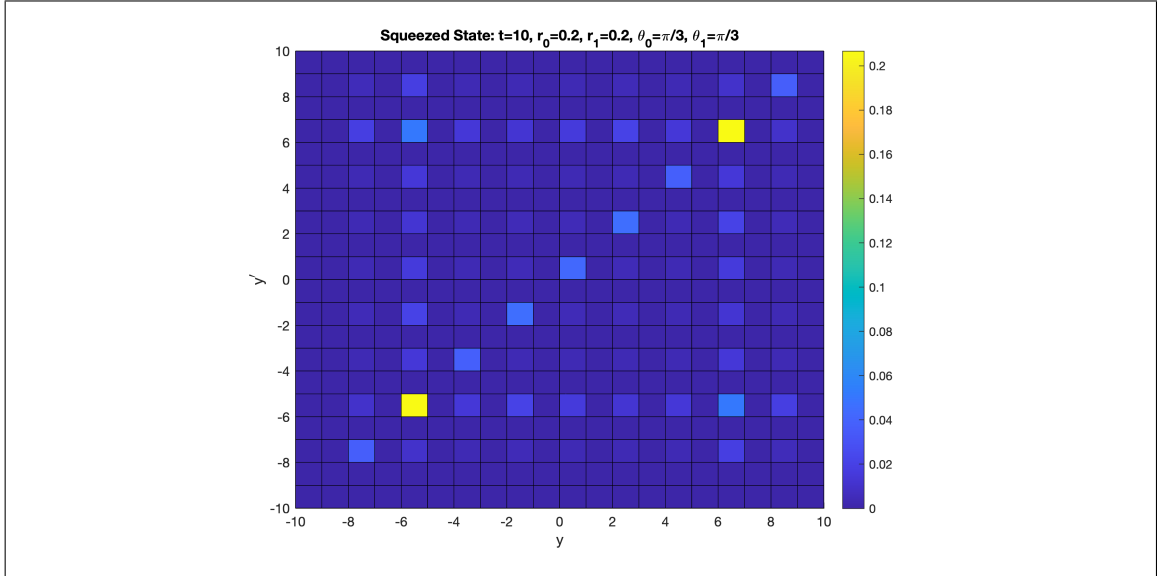


Figure 3.15.: Squeezed State: $t=10$ $r_0 = 0.2$ $r_1 = 0.2$ $\theta_0 = \frac{\pi}{3}$ $\theta_1 = \frac{\pi}{3}$

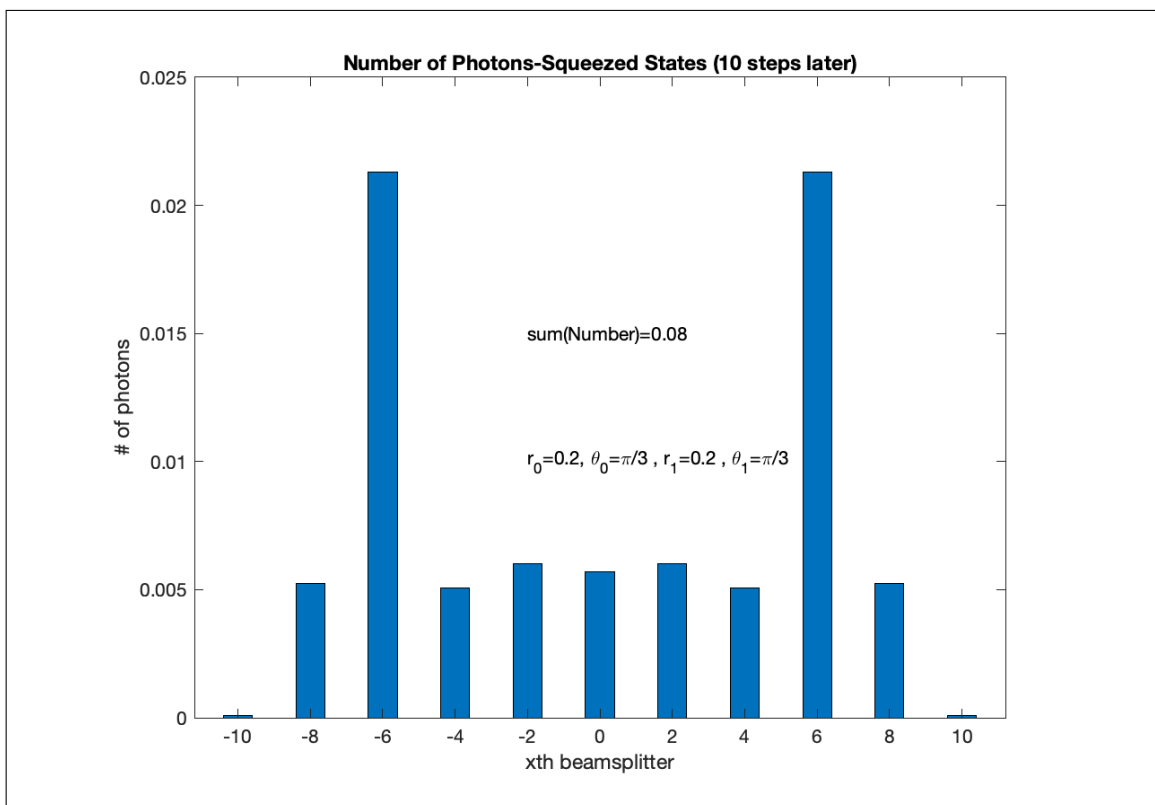


Figure 3.16.: The figure $\langle n_y \rangle$ distribution for $r_0 = 0.2$ and $r_1 = 0.2$ after 10 steps later

3.5. Squeezed Coherent States

Squeezed coherent states (also called displaced squeezed states) can be defined as displaced squeezed vacuum states with $|\alpha|^2$ in phase space.

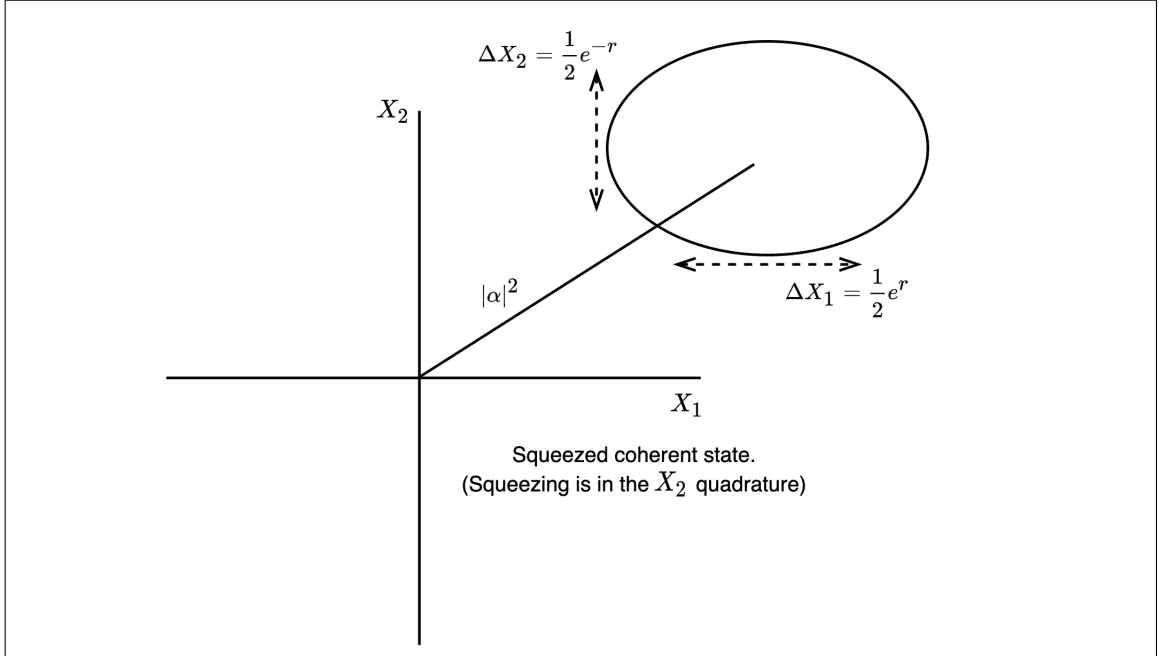


Figure 3.17.: Squeezed coherent states in phase space. The ellipse describes the uncertainty of the states

$$|\psi_i\rangle = D(\alpha)S(\xi)|0\rangle \quad (3.121)$$

The relation between displacement operators and annihilation/creation operators can be defined as follows:

$$D(\alpha)^\dagger \hat{a}_{x,\sigma} D(\alpha) = \hat{a}_{x,\sigma} + \alpha_{x,\sigma} \quad (3.122)$$

$$D(\alpha)^\dagger \hat{a}_{x,\sigma}^\dagger D(\alpha) = \hat{a}_{x,\sigma}^\dagger + \alpha_{x,\sigma}^* \quad (3.123)$$

To find the expectation value of \hat{n}_y ,

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} \langle 0 | S(\xi)^\dagger D(\alpha)^\dagger \hat{a}_{0\sigma'}^\dagger \hat{a}_{0\sigma''} D(\alpha) S(\xi) | 0 \rangle \quad (3.124)$$

The displacement operator is unitary operator thus DD^\dagger is an identity operator.

$$\rightarrow \langle 0 | \hat{S}^\dagger \hat{D}^\dagger \hat{a}_{0\sigma'}^\dagger \hat{D} \hat{D}^\dagger \hat{a}_{0\sigma''} \hat{D} \hat{S} | 0 \rangle \quad (3.125)$$

$$\rightarrow \langle 0 | \hat{S}^\dagger (\hat{a}_{0,\sigma'}^\dagger + \alpha_{0,\sigma'}^*) (\hat{a}_{0,\sigma''} + \alpha_{0,\sigma''}) \hat{S} | 0 \rangle \quad (3.126)$$

Only the following terms will remain.

$$\rightarrow \langle 0 | \hat{S}^\dagger (\hat{a}_{0,\sigma'}^\dagger \hat{a}_{0,\sigma''} + \alpha_{0,\sigma'}^* \alpha_{0,\sigma''}) \hat{S} | 0 \rangle \quad (3.127)$$

$$\rightarrow \langle 0 | \hat{S}^\dagger \hat{a}_{0,\sigma'}^\dagger \hat{a}_{0,\sigma''} \hat{S} | 0 \rangle + \langle 0 | \hat{S}^\dagger \alpha_{0,\sigma'}^* \alpha_{0,\sigma''} \hat{S} | 0 \rangle \quad (3.128)$$

The first term of the equation above was solved in Section ???. The second part gives us one. Then the expectation value of \hat{n}_y becomes

$$\langle \hat{n}_y \rangle = \sum A_{y,\sigma'}^{\sigma'*} A_{y,\sigma}^{\sigma''} \left((e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} e^{i\theta_{0\sigma''}} \sinh r_{0\sigma''}) \delta_{\sigma'\sigma''} + \alpha_{0,\sigma'}^* \alpha_{0,\sigma''} \right) \quad (3.129)$$

Similar to \hat{n}_y , the expectation values of $\hat{n}_{y'}$ can be written as follows:

$$\langle \hat{n}_{y'} \rangle = \sum A_{y',\beta}^{\beta'*} A_{y',\beta}^{\beta''} \left((e^{-i\theta_{0\beta'}} \sinh r_{0\beta'} e^{i\theta_{0\beta''}} \sinh r_{0\beta''}) \delta_{\beta'\beta''} + \alpha_{0,\beta'}^* \alpha_{0,\beta''} \right) \quad (3.130)$$

To calculate $\langle \hat{n}_y \hat{n}_{y'} \rangle$, we can write the general form of the $\langle \hat{n}_y \hat{n}_{y'} \rangle$.

$$\begin{aligned} \langle \hat{n}_y \hat{n}_{y'} \rangle = & \sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y'-x''',\beta}^{\beta''} \\ & \langle \psi_i | \hat{a}_{x\sigma'}^\dagger (\hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} + \delta_{\sigma''\beta'} \delta_{x''x'}) \hat{a}_{x'''\beta''} | \psi_i \rangle \end{aligned} \quad (3.131)$$

After writing the initial state as squeezed coherent state, $\langle \hat{n}_y \hat{n}_{y'} \rangle$ becomes

$$\begin{aligned} \langle \hat{n}_y \hat{n}_{y'} \rangle = & \sum A_{y-x,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y'-x''',\beta}^{\beta''} [\\ & \langle 0 | S(\xi)^\dagger D(\alpha)^\dagger \hat{a}_{x\sigma'}^\dagger \hat{a}_{x''\beta'}^\dagger \hat{a}_{x'\sigma''} \hat{a}_{x'''\beta''} D(\alpha) S(\xi) | 0 \rangle + \\ & \langle 0 | S(\xi)^\dagger D(\alpha)^\dagger \hat{a}_{x\sigma'}^\dagger \hat{a}_{x'''\beta''} D(\alpha) S(\xi) | 0 \rangle \delta_{\sigma''\beta'} \delta_{x''x'}] \end{aligned} \quad (3.132)$$

The second term can be easily calculated unlike the first term. We will apply DD^\dagger operator between all annihilation and creator operators.

$$\begin{aligned} \rightarrow & \langle 0 | S(\xi)^\dagger (\hat{a}_{0,\sigma'}^\dagger + \alpha_{0,\sigma'}^*) (\hat{a}_{0,\beta'}^\dagger + \alpha_{0,\beta'}^*) (\hat{a}_{0,\sigma''} + \alpha_{0,\sigma''}) (\hat{a}_{0,\beta''} + \alpha_{0,\beta''}) S(\xi) | 0 \rangle \\ & + (e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} e^{i\theta_{0\beta''}} \sinh r_{0\beta''} \delta_{\sigma'\beta''} + \alpha_{0,\sigma'}^* \alpha_{0,\beta''}) \delta_{\sigma''\beta'} \delta_{x''x'} \end{aligned} \quad (3.133)$$

Only the following terms survive.

$$\begin{aligned}
\langle \hat{n}_y \hat{n}_{y'} \rangle = & \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} [\\
& (\delta_{\sigma'\beta'} \delta_{\sigma''\beta''}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \cosh r_{0\sigma''} \cosh r_{0\beta'} + \\
& (\delta_{\sigma'\sigma''} \delta_{\beta'\beta''} + \delta_{\sigma'\beta''} \delta_{\sigma''\beta'}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} e^{-i\theta_{0\beta'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\sigma''} \sinh r_{0\beta'} \sinh r_{0\beta''} \\
& - \alpha_{0,\sigma'}^* \alpha_{0,\beta'}^* e^{i\theta_{0\beta''}} \sinh r_{\beta''} \cosh r_{\sigma''} \delta_{\sigma''\beta''} + \alpha_{0,\beta'}^* \alpha_{0,\sigma''} e^{-i\theta_{0\sigma'}} e^{i\theta_{0\beta''}} \sinh r_{\beta''} \sinh r_{\sigma''} \delta_{\sigma'\beta''} \\
& + \alpha_{0,\beta'}^* \alpha_{0,\beta''} e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} \sinh r_{\sigma'} \sinh r_{\sigma''} \delta_{\sigma'\sigma''} - \alpha_{0,\sigma''} \alpha_{0,\beta''} e^{-i\theta_{0\sigma'}} \sinh r_{\sigma'} \cosh r_{\beta'} \delta_{\sigma'\beta'} \\
& + \alpha_{0,\sigma'}^* \alpha_{0,\beta''} e^{-i\theta_{0\beta'}} e^{i\theta_{0\sigma''}} \sinh r_{\beta'} \sinh r_{\sigma''} \delta_{\beta'\sigma''} \\
& + \alpha_{0,\sigma'}^* \alpha_{0,\sigma''} e^{-i\theta_{0\beta'}} e^{i\theta_{0\beta''}} \sinh r_{\beta'} \sinh r_{\beta''} \delta_{\beta'\beta''} \\
& + \alpha_{0,\sigma'}^* \alpha_{0,\beta'}^* \alpha_{0,\sigma''} \alpha_{0,\beta''} + (e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} e^{i\theta_{0\beta''}} \sinh r_{0\beta''} \delta_{\sigma'\beta''} + \alpha_{0,\sigma'}^* \alpha_{0,\beta''}) \delta_{\sigma''\beta'} \delta_{x''x'}]
\end{aligned} \tag{3.134}$$

The multiplication of $\langle \hat{n}_y \rangle$ and $\langle \hat{n}_{y'} \rangle$ can be written

$$\begin{aligned}
\langle \hat{n}_y \rangle \langle \hat{n}_{y'} \rangle = & \sum A_{y,\sigma}^{\sigma'*} A_{y,\sigma}^{\sigma''} A_{y',\beta}^{\beta'*} A_{y',\beta}^{\beta''} \\
& (e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} e^{i\theta_{0\sigma''}} \sinh r_{0\sigma''} \delta_{\sigma'\sigma''} + \alpha_{0,\sigma'}^* \alpha_{0,\sigma''}) \\
& (e^{-i\theta_{0\beta'}} \sinh r_{0\beta'} e^{i\theta_{0\beta''}} \sinh r_{0\beta''} \delta_{\beta'\beta''} + \alpha_{0,\beta'}^* \alpha_{0,\beta''})
\end{aligned} \tag{3.135}$$

Therefore the correlation function can be written as follows:

$$\begin{aligned}
C_{yy'} = & \sum A_{y,\sigma}^{\sigma'*} A_{y-x',\sigma}^{\sigma''} A_{y'-x'',\beta}^{\beta'*} A_{y',\beta}^{\beta''} [\\
& (\delta_{\sigma'\beta'} \delta_{\sigma''\beta''}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\beta''} \cosh r_{0\sigma''} \cosh r_{0\beta'} + \\
& (\delta_{\sigma'\beta''} \delta_{\sigma''\beta'}) e^{-i\theta_{0\sigma'}} e^{i\theta_{0\sigma''}} e^{-i\theta_{0\beta'}} e^{i\theta_{0\beta''}} \sinh r_{0\sigma'} \sinh r_{0\sigma''} \sinh r_{0\beta'} \sinh r_{0\beta''} \\
& - \alpha_{0,\sigma'}^* \alpha_{0,\beta'}^* e^{i\theta_{0\beta''}} \sinh r_{\beta''} \cosh r_{\sigma''} \delta_{\sigma''\beta''} - \alpha_{0,\sigma''} \alpha_{0,\beta''} e^{-i\theta_{0\sigma'}} \sinh r_{\sigma'} \cosh r_{\beta'} \delta_{\sigma'\beta'} \\
& + \alpha_{0,\sigma'}^* \alpha_{0,\beta''} e^{-i\theta_{0\beta'}} e^{i\theta_{0\sigma''}} \sinh r_{\beta'} \sinh r_{\sigma''} \delta_{\beta'\sigma''} \\
& + \alpha_{0,\beta'}^* \alpha_{0,\sigma''} e^{-i\theta_{0\sigma'}} e^{i\theta_{0\beta''}} \sinh r_{\beta''} \sinh r_{\sigma''} \delta_{\sigma'\beta''} \\
& + (e^{-i\theta_{0\sigma'}} \sinh r_{0\sigma'} e^{i\theta_{0\beta''}} \sinh r_{0\beta''} \delta_{\sigma'\beta''} + \alpha_{0,\sigma'}^* \alpha_{0,\beta''}) \delta_{\sigma''\beta'} \delta_{x''x'}]
\end{aligned} \tag{3.136}$$

For $r_0 = 0$ and $r_1 = 0$, the correlation function becomes only for coherent state, thus it is obvious that we have the same results with Equation 3.93. Moreover, for $\alpha_0 = 0$ and $\alpha_1 = 0$ we have the correlation function for the vacuum squeezed state calculated in Equation 3.120

3.5.1. Results for Squeezed Coherent States

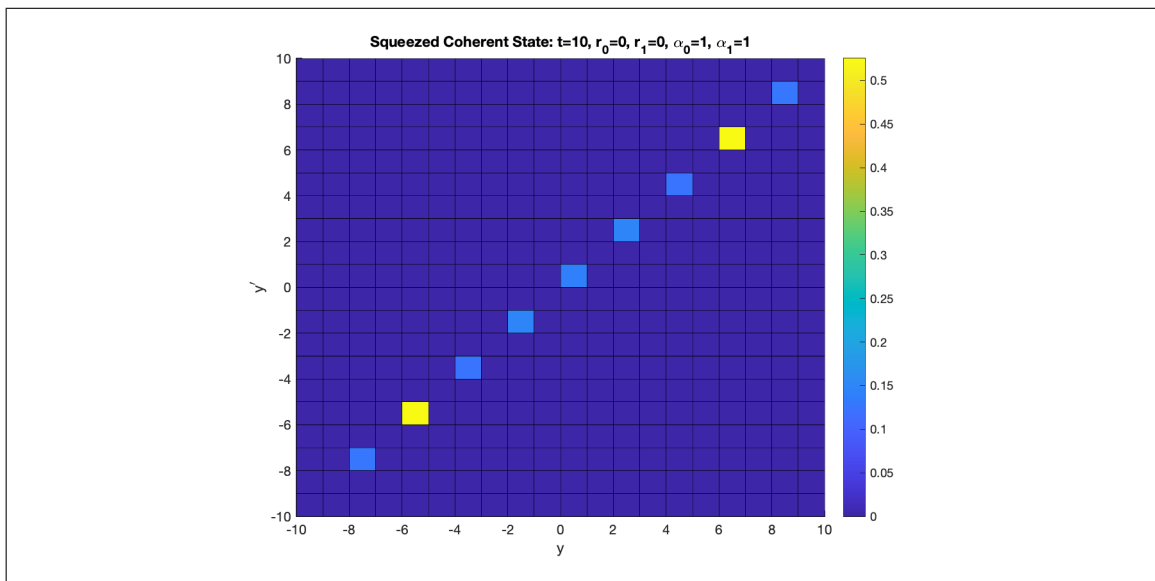


Figure 3.18.: When the squeezed parameters are chosen zero and $\alpha_{0,1} = 1$, the correlation figure has no off-diagonal entries. The graph is the same with the correlation graph of coherent states.

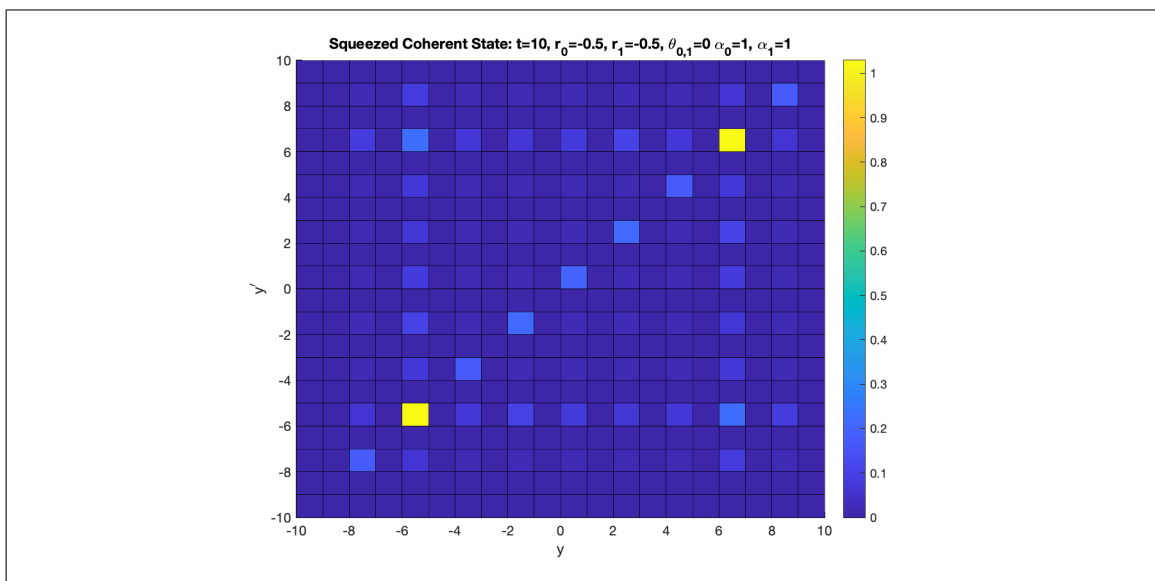


Figure 3.19.: When the squeezed parameters are chosen negative and the $\theta_{0,1} = 0$ and $\alpha_{0,1} = 1$, the off-diagonal entries appear with positive values.

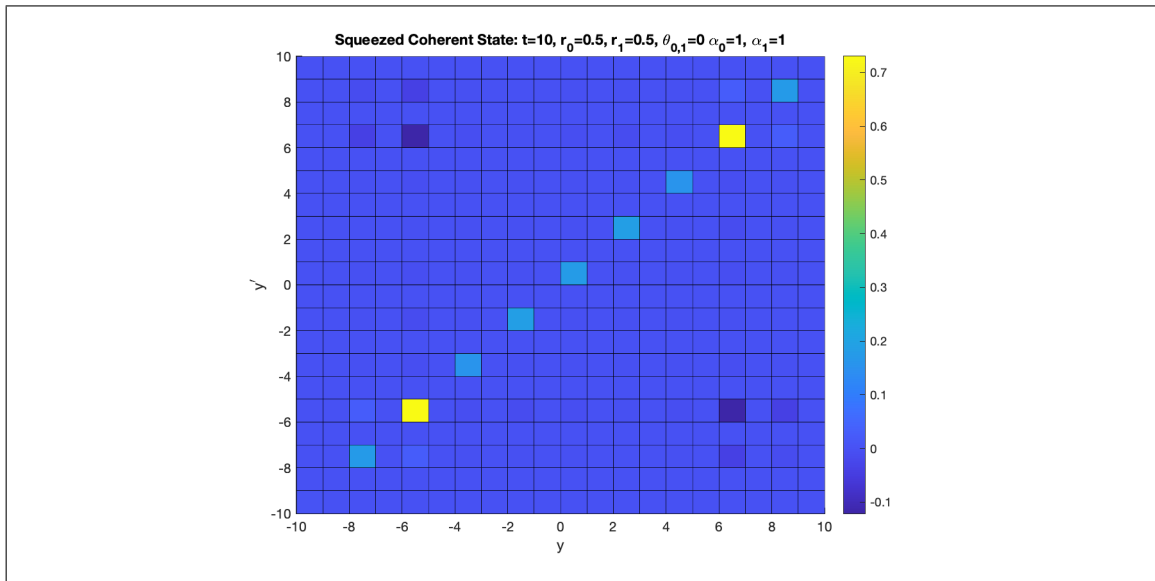


Figure 3.20.: When the squeezed parameters are chosen negative and the $\theta_{0,1} = 0$ and $\alpha_{0,1} = 1$, the off-diagonal entries appear with negative values.

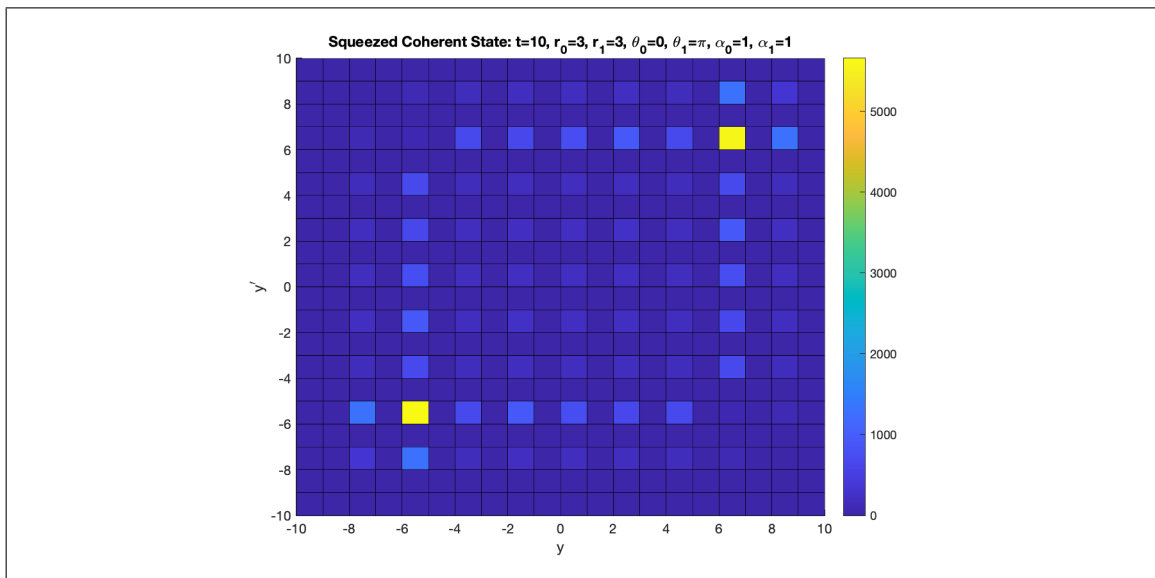


Figure 3.21.: When the squeezed parameters are chosen $r_{0,1} = 3$ and the $\theta_{0,1} = 0, \pi$ (respectively) and $\alpha_{0,1} = 1$. The correlations increases dramatically due to r value.

3.6. Thermal States

The correlation function of the thermal states can be written by using the correlation function of the number states.

$$C_{yy'}(Thermal) = \sum_{n,m=0}^{\infty} \frac{e^{\frac{-n\hbar\omega}{k_B T}}}{Z} \frac{e^{\frac{-m\hbar\omega}{k_B T'}}}{Z'} C_{yy'}(Number\ State) \quad (3.137)$$

$$Z = \frac{1}{1 - \exp(-\hbar\omega/k_B T)} \quad (3.138)$$

$$Z' = \frac{1}{1 - \exp(-\hbar\omega/k_B T')} \quad (3.139)$$

Where Z and Z' are the partition functions. Gerry and Knight (2004) \hbar is reduced Planck constant. ω is the angular frequency. T is temperature in terms of Kelvin and k_B is the Boltzmann constant.

3.6.1. Results For Thermal States

Figure 3.22 shows us the correlations of the thermal state for W is close to 1 where W is defined as $\frac{\hbar\omega}{k_B T}$.

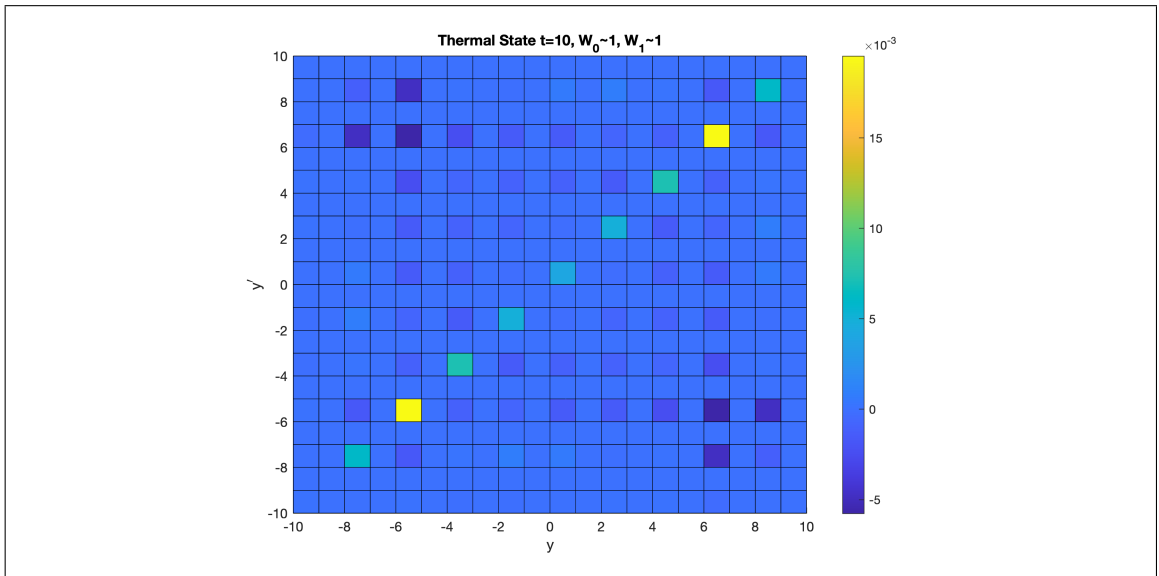


Figure 3.22.: The figure above shows the correlations for $W_0 = 0.99$ and $W_1 = 0.99$ after 10 steps later

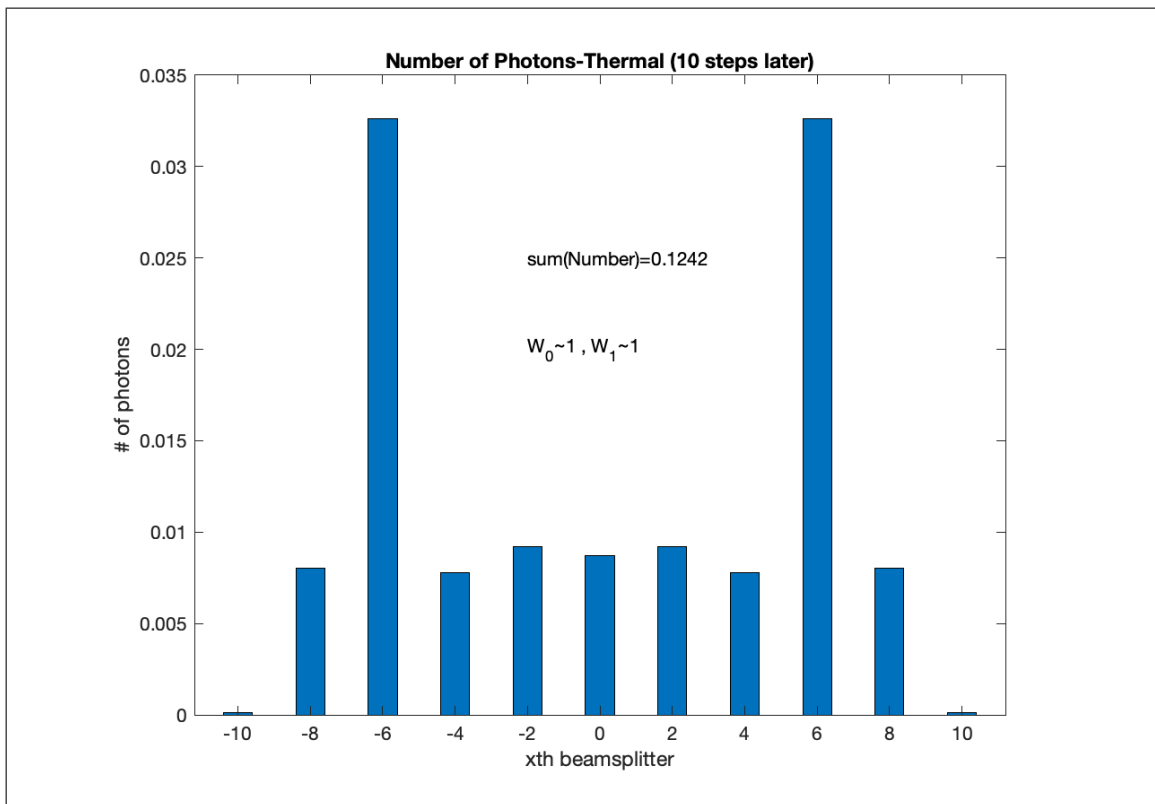


Figure 3.23.: The figure $\langle n_y \rangle$ distribution for $W_0 = 0.99$ and $W_1 = 0.99$ after 10 steps later

3.7. Second Order Quantum Coherence Function

Second order quantum coherence function at a fixed position can be defined with the following equation. Armstrong and Rohrlich (1966)

$$g^{(2)}(\tau) = \frac{\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t) \rangle}{\langle \hat{E}^{(-)}(t)\hat{E}^{(+)}(t) \rangle \langle \hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau) \rangle} \quad (3.140)$$

Where $\tau = t_2 - t_1$ is the time difference. When $\tau \rightarrow 0$, $g^{(2)}(\tau)$ of a mode k field can be reduced to the following form. (Gerry and Knight, 2004, p. 435)

$$g^{(2)}(\tau \rightarrow 0) = \frac{\langle \hat{a}_k^\dagger \hat{a}_k^\dagger \hat{a}_k \hat{a}_k \rangle}{\langle \hat{a}_k^\dagger \hat{a}_k \rangle^2} = \frac{\langle \hat{n}(\hat{n} - 1) \rangle}{\langle \hat{n} \rangle^2} \quad (3.141)$$

$$g^{(2)}(\tau \rightarrow 0) = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle^2} \quad (3.142)$$

$$g^{(2)}(\tau \rightarrow 0) = \frac{(\Delta n)^2 + \langle n \rangle^2 - \langle n \rangle^2}{\langle n \rangle^2} \quad (3.143)$$

$$g^{(2)}(\tau \rightarrow 0) = 1 + \frac{(\Delta n)^2 - \langle n \rangle^2}{\langle n \rangle^2} \quad (3.144)$$

$(\Delta n)^2$ for different light sources can be calculated as follows:

$$\begin{aligned} (\Delta n)^2 \text{ for number state} &= 0 \\ (\Delta n)^2 \text{ for coherent state} &= \langle n \rangle \\ (\Delta n)^2 \text{ for thermal state} &= \langle n^2 \rangle + \langle n \rangle \end{aligned} \quad (3.145)$$

Thus the $g^{(2)}(\tau \rightarrow 0)$ value can be calculated as

$$\begin{aligned} g^{(2)}(\tau \rightarrow 0)_{\text{for number state}} &= 1 - \frac{1}{n} \\ g^{(2)}(\tau \rightarrow 0)_{\text{for coherent state}} &= 1 \\ g^{(2)}(\tau \rightarrow 0)_{\text{for thermal state}} &= 2 \end{aligned} \quad (3.146)$$

It can be clearly seen that for $n = 1$, $g^{(2)}(\tau \rightarrow 0)$ is zero. For coherent states $g^{(2)}(\tau \rightarrow 0)$ is always one and independent from τ . Figure 3.24 shows $g^{(2)}(\tau)$ graphs of two different light sources. Gerry and Knight (2004)

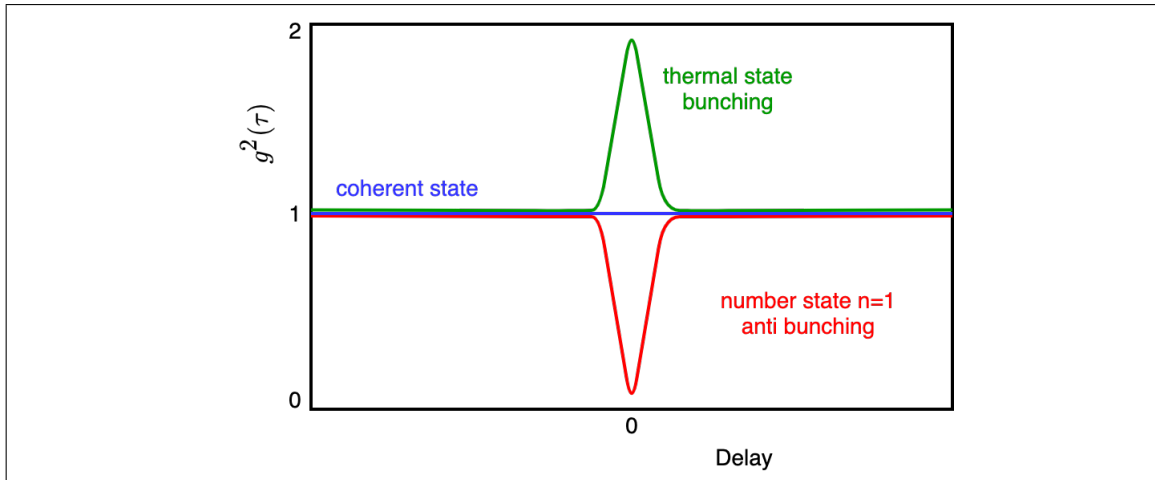


Figure 3.24.: $g^{(2)}(\tau)$ measurement of different light sources. Blue, red and green lines represent coherent, number and thermal states, respectively.

In the second chapter we already calculated $\langle \hat{a}_k^\dagger \hat{a}_k^\dagger \hat{a}_k \hat{a}_k \rangle$ and $\langle \hat{a}_k^\dagger \hat{a}_k \rangle$. Thus we can calculate $g^{(2)}(0)$ values for different light sources in terms of the probability amplitudes of 1-D quantum walk with Hadamard coin.

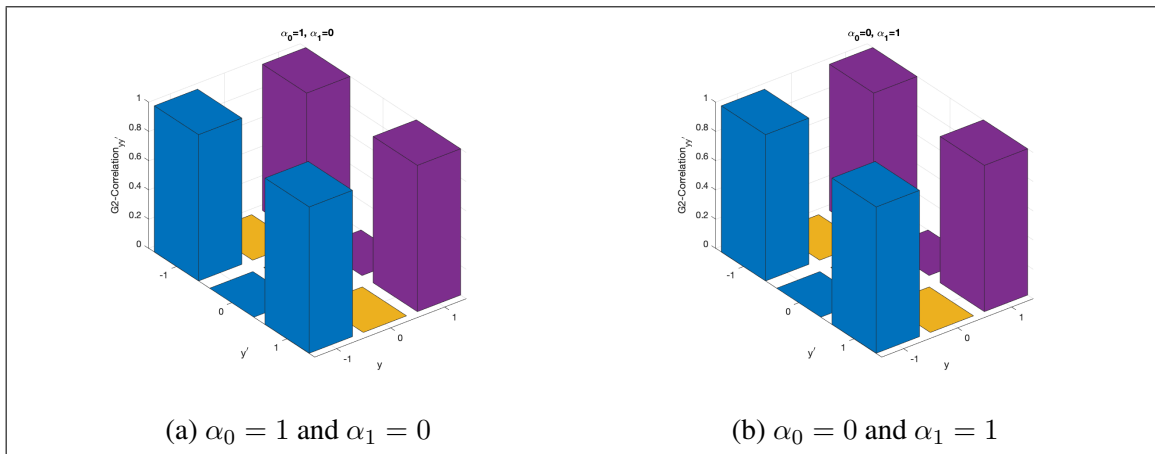


Figure 3.25.: $g^{(2)}(0)$ analysis of coherent states for different initial conditions. $g^{(2)}(0)$ is always one as expected.

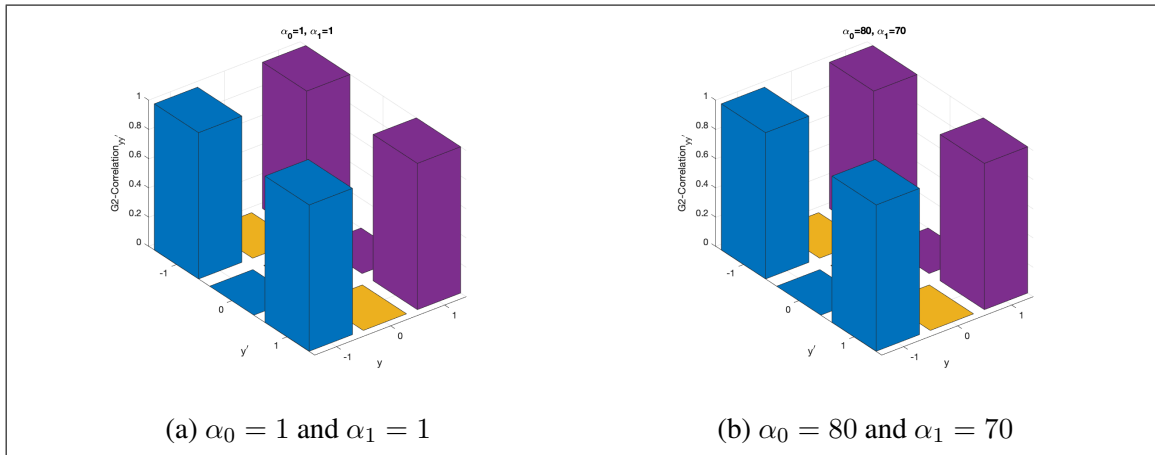


Figure 3.26.: $g^{(2)}(0)$ analysis of coherent states for different initial conditions. $g^{(2)}(0)$ is always one as expected.

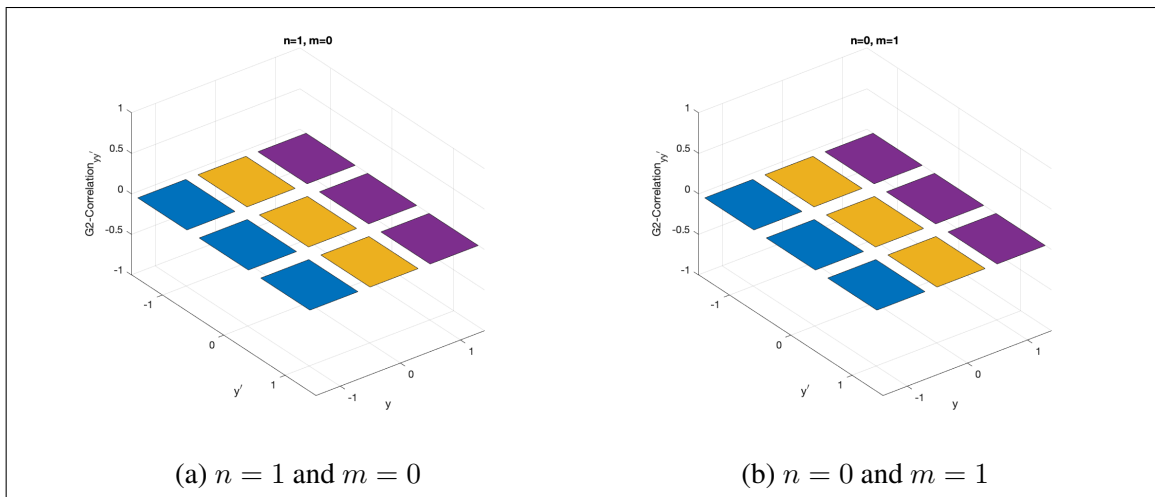


Figure 3.27.: $g^{(2)}(0)$ analysis of the number states for different initial conditions of n and m . a) shows $g^{(2)}(0)$ for $n=1$ and $m=0$. b) shows $g^{(2)}(0)$ for $n=0$ and $m=1$. The both initial conditions satisfy the result of $g^{(2)}(0)$ measurement for single photon sources.

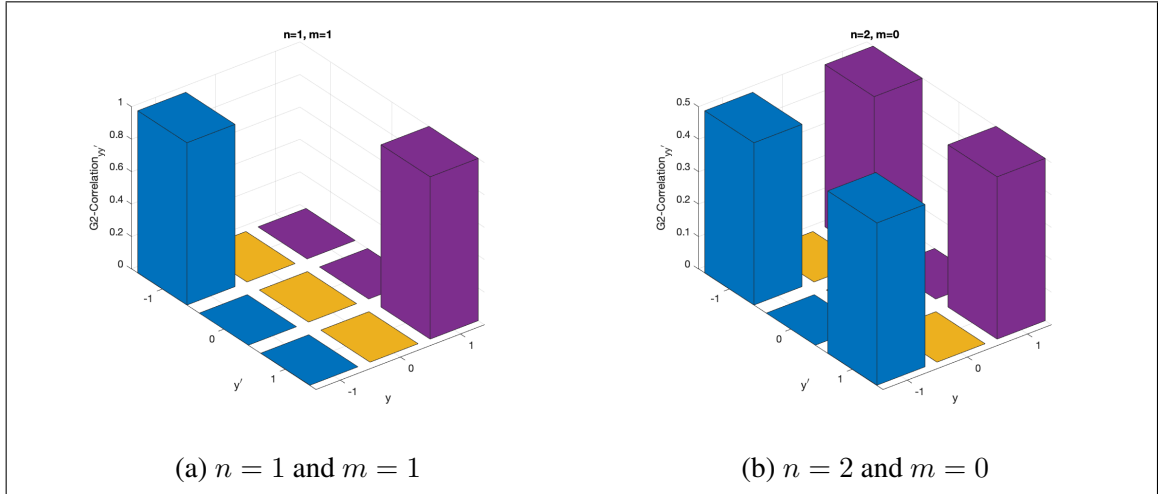


Figure 3.28.: $g^{(2)}(0)$ analysis of the number states for different initial conditions of n and m . a) The figure shows $g^{(2)}(0)$ for $n = 1$ and $m = 1$. This special initial condition is known as two-photon interference effect in quantum optics (Hong-Ou-Mandel Effect). According to the Hong-Ou-Mandel Effect, Hong et al. (1987) $g^{(2)}(0)$ value should be zero for the measurement at the beamsplitters at $y = 1$ and $y' = -1$ at the time $t = 1$ or vice versa. Thus our results satisfy the Hong-Ou-Mandel Effect. b) The figure shows $g^{(2)}(0)$ for $n = 2$ and $m = 0$. This initial condition can be thought as a number state source with two photons. According to Equation 3.146, $g^{(2)}(0)$ values should be 0.5 for the measurement at the beamsplitters at $y = 1$ and $y' = -1$ at the time $t = 1$. Thus our results satisfy the $g^{(2)}(0)$ value.

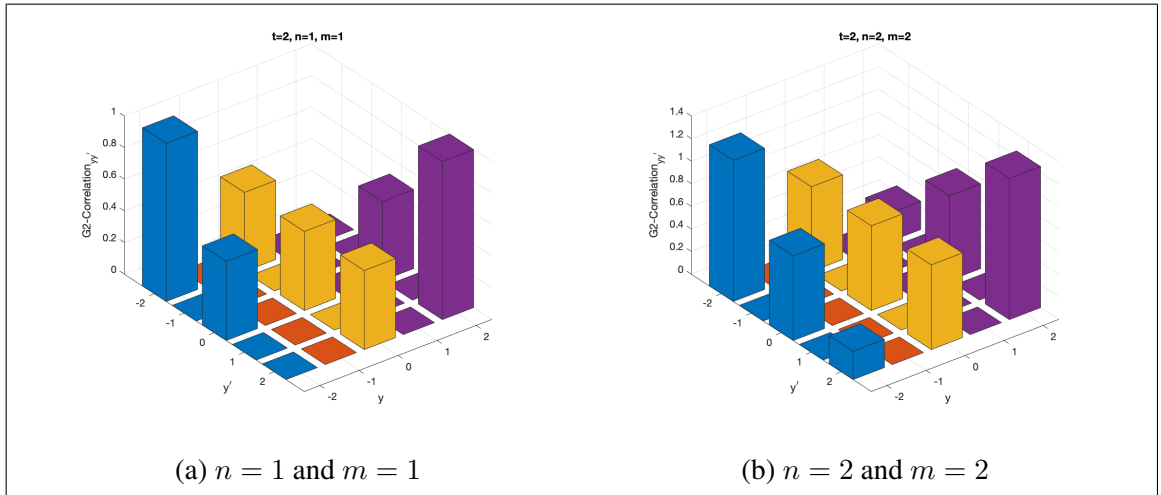


Figure 3.29.: $g^{(2)}(0)$ analysis of the number states at time $t = 2$ for different initial conditions of n and m . a) $g^{(2)}(0)$ values between different beamsplitters are shown. We can observe the Hong-Ou-Mandel Effect at the beamsplitters $y = 2$ and $y' = -2$ at the time $t = 2$. $g^{(2)}(0)$ value is 0.5 at the beamsplitters $y = 2$ and $y' = 0$ or $y = -2$ and $y' = 0$. b) For $n = 2$ and $m = 2$ $g^{(2)}(0)$ value is 0.25 at the beamsplitters $y = 2$ and $y' = -2$ at the time $t = 2$ and 0.75 at the beamsplitters $y = 2$ and $y' = 0$ or $y = -2$ and $y' = 0$

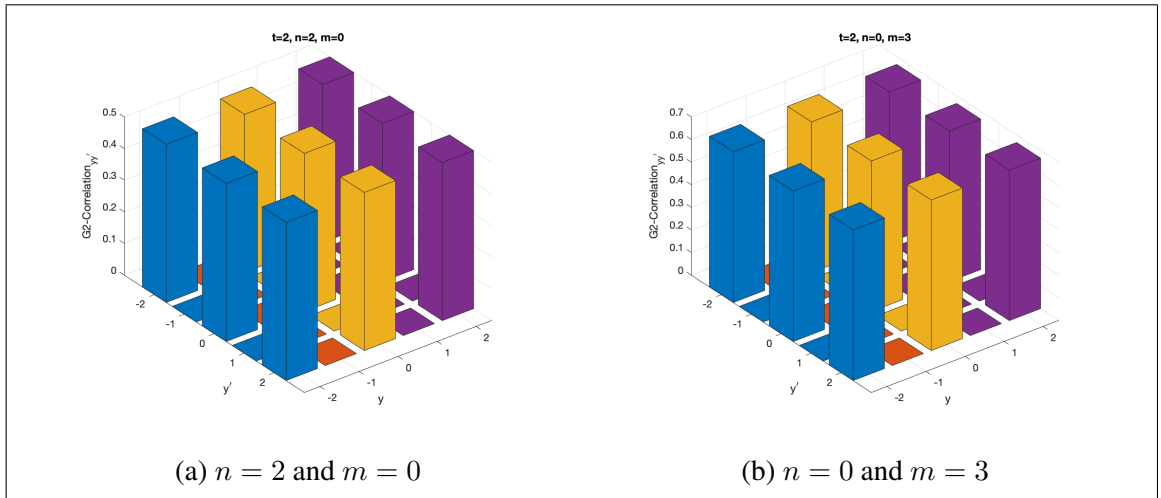


Figure 3.30.: $g^{(2)}(0)$ analysis of the number states at time $t = 2$ for different initial conditions, $n = 2, m = 0$ (left) and $n = 0, m = 3$ (right). In both cases $g^{(2)}(0)$ values satisfy Equation 3.146.

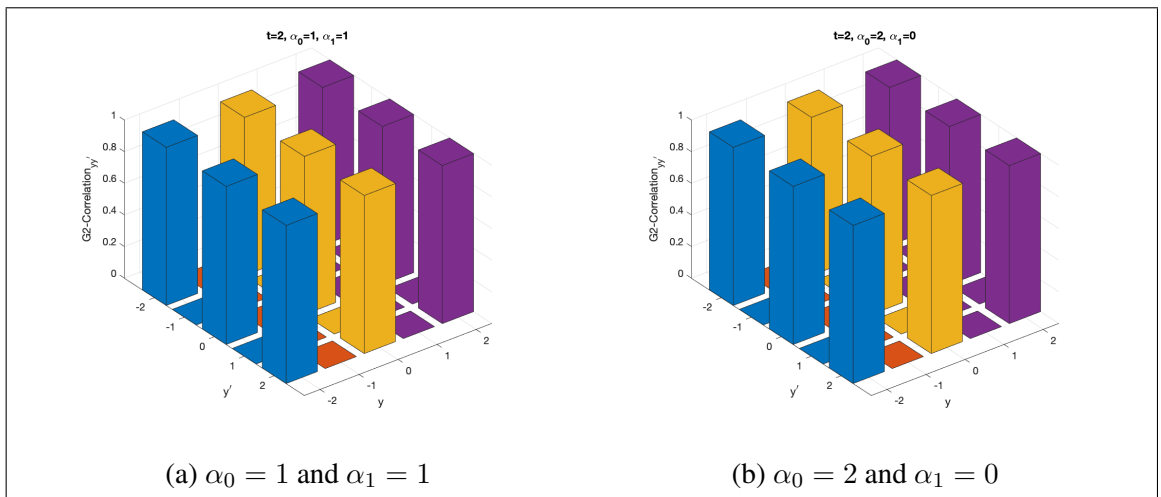


Figure 3.31.: $g^{(2)}(0)$ analysis of the coherent states at time $t = 2$ for different initial conditions of α_0 and α_1 . $g^{(2)}(0)$ values of all different scenarios are always one. This result is expected because, $g^{(2)}(0)$ value for any coherent states is always one and independent from the τ .

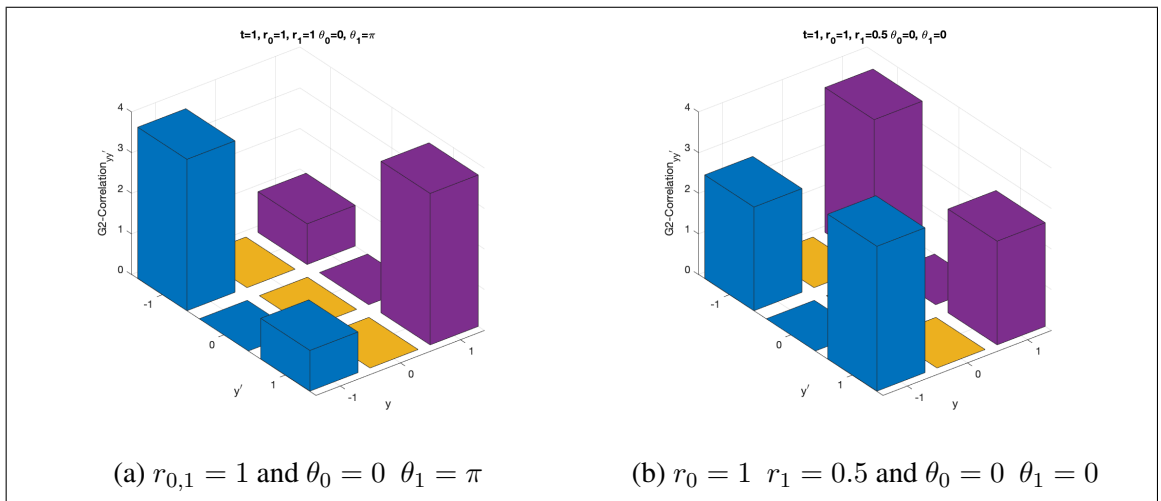


Figure 3.32.: $g^{(2)}(0)$ analysis of the vacuum squeezed states at time $t = 2$ for different initial conditions of $r_{0,1}$ and $\theta_{0,1}$. a) The figure shows us the result of a specific initial condition of vacuum squeezed states. When $r_0 = -r_1$ and $\theta_{0,1} = 0$ or $r_0 = r_1$ and $\theta_1 - \theta_0 = \pi$, $g^{(2)}(0)$ value is calculated one at the beamsplitters $y = 1$ and $y' = -1$ or vice versa. b) The figure shows us the result of a arbitrary r_0 and r_1 with the same $\theta_{0,1}$.

3.8. Discussion

According to the results of this thesis, we can say that in a photonic quantum walk system, the statistics of the initial photons can be defined by the correlation analysis which is calculated by using the analytical solution of the 1-D quantum walk with Hadamard coin. Besides, we explained the second order quantum coherence function in terms of the probability amplitudes of the 1-D quantum walk with Hadamard coin. It can be say that the $g^{(2)}(0)$ measurement can be the experimental approach of the correlation function $C_{y,y'}$.

$$C_{y,y'} = \langle \hat{n}_y \hat{n}_{y'} \rangle - \langle \hat{n}_y \rangle \langle \hat{n}_{y'} \rangle$$

$$g^{(2)}(\tau \rightarrow 0) = \frac{\langle \hat{a}_k^\dagger \hat{a}_k^\dagger \hat{a}_k \hat{a}_k \rangle}{\langle \hat{a}_k^\dagger \hat{a}_k \rangle^2} \quad (3.147)$$

If we consider $g^{(2)}(0)$ value of the coherent states

$$g^{(2)}(\tau \rightarrow 0)_{\text{for coherent state}} = 1 \quad (3.148)$$

$$1 = \frac{\langle \hat{a}_k^\dagger \hat{a}_k^\dagger \hat{a}_k \hat{a}_k \rangle}{\langle \hat{a}_k^\dagger \hat{a}_k \rangle^2} \quad (3.149)$$

$$\langle \hat{a}_k^\dagger \hat{a}_k \rangle^2 = \langle \hat{a}_k^\dagger \hat{a}_k^\dagger \hat{a}_k \hat{a}_k \rangle \quad (3.150)$$

$$\langle \hat{a}_k^\dagger \hat{a}_k^\dagger \hat{a}_k \hat{a}_k \rangle - \langle \hat{a}_k^\dagger \hat{a}_k \rangle^2 = 0$$

For the different beamsplitters ($y \neq y'$) the correlation function becomes $g^{(2)}(0)$ measurement. If we look the Figures 3.10, 3.11 and 3.12, it can be clearly seen that the values are zero for different y and y' values. However, the correlation function can give us information about the variance of the walk. For $y = y'$ the correlation function becomes

$$C_{y,y} = Var(y) = \langle \hat{n}_y^2 \rangle - \langle \hat{n}_y \rangle^2 \quad (3.151)$$

Therefore we can see the variance of the walk and it satisfies the ballistic behaviour of the 1-D quantum walk. In the classical case, the standart deviation of the walk is $\sigma(t) = \sqrt{t}$ but in the quantum case the standart deviation is about $\sigma(t) = 0.54t$ Portugal (2013) Venegas-Andraca (2012). The Figure 3.33 shows us the comparison of the probability distribution of 1-D quantum walk with initial coin state is zero and the correlation function of the coherent state with $\alpha_0 = 1$ and $\alpha_1 = 0$. If we look the correlation function of the coherent state with $\alpha_0 = 1$ and $\alpha_1 = 0$, the maximum value of the variance is about $\frac{t}{\sqrt{2}}$

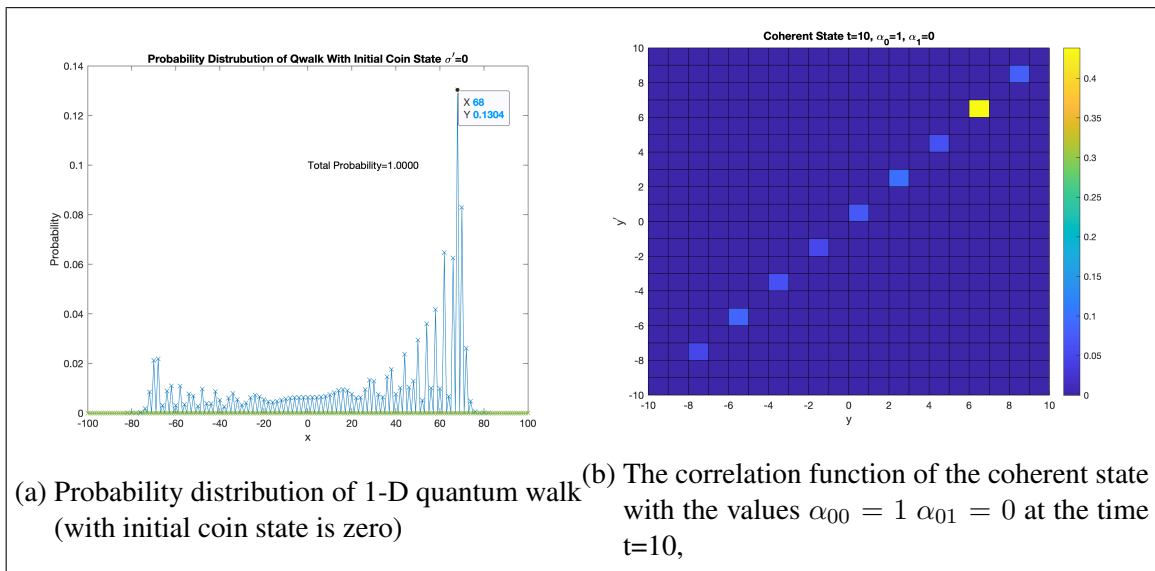


Figure 3.33.: Comparison of the variances

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APPENDIX A

CODES FOR 2ND CHAPTER

A.1. Simulations

```
1 tic
2 clear
3 close all
4 up = anyS(0,2);
5 dn = anyS(1,2);
6 Nstart = 7;
7 Nstep = 6;
8     coin = 'Grover';
9     initCoinState = Bell(0);
10    tMax = 2;
11
12    initCoinPure = isvector(initCoinState);    % flag indicating ...
13    %     pureness of initial coin state
14    switch coin
15        case 'Hadamard'
16            csco = hadamard(4)/2;
17        case 'Kempe'
18            csco = kron([1 1i; 1i 1],[1 1i; 1i 1])/2;
19        case 'Grover'
20            csco = .5*( ones(4,4) - 2*diag(ones(4,1)) );
21        case 'Fourier'
22            csco = dftmtx(4)/2;
23        otherwise
24            error('Coin operator can be Hadamard, Grover or Fourier');
25    end
26    figure
27    lText = {'AX-BY', 'A - B', 'X - Y', 'A - X', 'A - Y', 'A - XY'};
28    slt = size(lText);
29    tText = '          step ';
```

```

29     for tt = 1:slt(2)
30         tText = sprintf('%s%9s ',tText,lText{tt});
31     end
32     fprintf('%s\n',tText);
33     eMax = slt(2);
34     vl = zeros(4,tMax);
35     vc = zeros(4,4,tMax);
36     E = zeros(tMax,eMax);
37     Ccs = zeros(tMax,1);
38
39     N = Nstart;
40     initPositionState = kron( anyS(round(N/2)-1,N), ...
41         anyS(round(N/2)-1,N) );
42     if initCoinPure
43         icDM = initCoinState*initCoinState';
44         ps = kron( initCoinState, initPositionState );
45     else
46         icDM = initCoinState;
47     end
48     ipDM = initPositionState*initPositionState';
49     rhoABXY = kron( icDM, ipDM );
50
51     C = kron3( cscs, eye(N), eye(N) );
52     WL = diag(ones(1,N-1),-1);
53     WL(1,N) = 1;
54     WD = WL;
55     WR = diag(ones(1,N-1),1);
56     WR(N,1) = 1;
57     WU = WR;
58     W = kron3( kron(up,up)*kron(up,up)', WU, WR )...
59         + kron3( kron(dn,dn)*kron(dn,dn)', WD, WL )...
60         + kron3( kron(up,dn)*kron(up,dn)', WU, WL )...
61         + kron3( kron(dn,up)*kron(dn,up)', WD, WR );
62     U = W * C;
63
64     for tt = 1:tMax
65         rhoAB = partial_trace( rhoABXY , [ 1 1 0 0 ], [ 2 2 N N ] );
66         rhoAX = partial_trace( rhoABXY , [ 1 0 1 0 ], [ 2 2 N N ] );
67         rhoAY = partial_trace( rhoABXY , [ 1 0 0 1 ], [ 2 2 N N ] );

```

```

67     rhoXY = partial_trace( rhoABXY , [ 0 0 1 1], [ 2 2 N N ] );
68     rhoAXY = partial_trace( rhoABXY , [ 1 0 1 1], [ 2 2 N N ] );
69
70     [ vc(:, :, tt), vl(:, tt) ] = eig(rhoAB, 'vector');
71
72     E(tt,1) = S_Von_Neumann(rhoXY);
73     Ccs(tt) = concurrence(rhoAB);
74     E(tt,2) = logarithmic_negativity(rhoAB, [1 0], [ 2 2 ] );
75     E(tt,3) = logarithmic_negativity(rhoXY, [1 0], [ N N ] );
76     E(tt,4) = logarithmic_negativity(rhoAX, [1 0], [ 2 N ] );
77     E(tt,5) = logarithmic_negativity(rhoAY, [1 0], [ 2 N ] );
78     E(tt,6) = logarithmic_negativity(rhoAXY, [1 0], [ 2 N*N ] );
79
80     pDist = reshape(diag(rhoXY), N, N)';
81     imagesc([-N/2 N/2], [+N/2 -N/2], pDist);
82     xlabel('X position (Probability Disturbition after 25 ...
           Steps)');
83     ylabel('Y position');
84     title(sprintf('%s coin', coin));
85     drawnow;
86
87     disp( [ tt, E(tt,:) ] );
88
89     if initCoinPure
90         ps = U*ps;
91         rhoABXY = ps*ps';
92     else
93         rhoABXY = U*rhoABXY*U';
94     end
95
96     if tt >= (N-1)/2
97         if initCoinPure
98             tOld = pure2tensor(ps, [ 2 2 N N ] );
99         else
100             tOld = dm2tensor(rhoABXY, [ 2 2 N N ] );
101         end
102
103         N = N + Nstep;
104         a = Nstep/2 + 1;

```

```

105         b = N - Nstep/2;
106
107         if initCoinPure
108             tNew = zeros(2,2,N,N);
109             tNew(:, :, a:b, a:b) = tOld;
110             ps = tensor2pure(tNew);
111             rhoABXY = ps*ps';
112         else
113             tNew = zeros(2,2,2,2,N,N,N,N);
114             tNew(:, :, :, :, a:b, a:b, a:b, a:b) = tOld;
115             rhoABXY = tensor2dm(tNew);
116         end
117
118         C = kron3( csc0, eye(N), eye(N) );
119         WL = diag(ones(1,N-1),-1);
120         WL(1,N) = 1;
121         WD = WL;
122         WR = diag(ones(1,N-1),1);
123         WR(N,1) = 1;
124         WU = WR;
125         W = kron3( kron(up,up)*kron(up,up)', WU, WR ) ...
126             + kron3( kron(dn,dn)*kron(dn,dn)', WD, WL ) ...
127             + kron3( kron(up,dn)*kron(up,dn)', WU, WL ) ...
128             + kron3( kron(dn,up)*kron(dn,up)', WD, WR );
129         U = W * C;
130
131     end
132
133 end
134
135 figStyles = {'--', '-', '-', '-.', '-*', '-.'};
136 figure
137 hold on
138 for tt = 1:eMax
139     plot(1:tMax,E(:,tt),figStyles{tt});
140 end
141 hold off
142 title(sprintf('%s coin',coin));
143 xlabel('Time');

```

```
144 ylabel('Entanglement');
145 legend(lText, 'Location', 'northwest');
146 toc
```

APPENDIX B

CODES FOR 3RD CHAPTER

B.1. Number States

```
1 tic
2 clear
3 close all
4 t=1; %number of steps, meaning time.
5 x = linspace (-t,t,2*t+1);
6 Plx_t = zeros(length(x));
7 P0x_t = zeros(length(x));
8 Psi00 = [];
9 Psi01 = [];
10 Psi10 = [];
11 Psi11 = [];
12 Total0 =0; %calculate total probability of probablity amplitude
13 Total1 =0; %calculate total probability of probablity amplitude
14 Total2 =0; %calculate total probability of probablity amplitude
15
16 funcminus = @(kk) (1/2)*(1 + cos(kk)/(sqrt(1+(cos(kk))^2)));
17 funcplus = @(kk) (1/2)*(1 - cos(kk)/(sqrt(1+(cos(kk))^2)));
18
19 %the probability amplitudes in momentum space
20 A00 = @(kk,tt,xx,wk) (1/(2*pi))*(1 + cos(kk)/(sqrt(1+(cos(kk))^2)))...
21     *exp(-1i*(wk*tt - kk*xx));
22 A01 = @(kk,tt,xx,wk) (1/(2*pi))*(exp(1i*kk)/(sqrt(1+(cos(kk))^2)))...
23     *exp(-1i*(wk*tt - kk*xx));
24 A10 = @(kk,tt,xx,wk) (1/(2*pi))*(exp(1i*kk*xx))* ((funcminus(kk)*...
25     ((sqrt(2)*exp(-1i*wk*(tt-1)-1i*kk))-exp(-1i*wk*tt)) + ...
26     (funcplus(kk)...
27     *exp(1i*pi*tt+1i*wk*tt-1i*kk))*(-sqrt(2)*exp(-1i*wk)-exp(1i*kk))));
28 A11 = @(kk,tt,xx,wk) (1/(2*pi))*(exp(1i*kk*xx))* ((funcminus(kk)...
29     *exp(-1i*wk*tt))*(sqrt(2))...
```

```

29     *exp(1i*wk)-exp(1i*kk))*(sqrt(2)*exp(-1i*wk)...
30     -exp(-1i*kk))) + (funccplus(kk)*exp(1i*pi*tt+1i*wk*tt)*(-sqrt(2)...
31     *exp(-1i*wk)-exp(1i*kk))*(-sqrt(2)*exp(1i*wk)-exp(-1i*kk)));
32
33 A=zeros(2,2,length(x));
34 for j = 1:2:length(x)
35     %inverse fourier transforms of the results in momentum space
36     A_00_x = integral ...
37         (@(k)A00(k,t,x(j),w_k(k)),-pi,pi,'ArrayValued',true);
38     A_01_x = integral ...
39         (@(k)A01(k,t,x(j),w_k(k)),-pi,pi,'ArrayValued',true);
40     Psi00(j)= A_00_x;
41     Psi01(j)= A_01_x;
42     P0x_t(j) = (abs(A_00_x))^2 + (abs(A_01_x))^2;
43     Total0 = Total0 + P0x_t(j);
44
45     A_10_x = integral ...
46         (@(k)A10(k,t,x(j),w_k(k)),-pi,pi,'ArrayValued',true);
47     A_11_x = integral ...
48         (@(k)A11(k,t,x(j),w_k(k)),-pi,pi,'ArrayValued',true);
49     Psi10(j)= A_10_x;
50     Psi11(j)= A_11_x;
51     P1x_t(j) = (abs(A_10_x))^2 + (abs(A_11_x))^2;
52     Total1 = Total1 + P1x_t(j);
53 end
54
55 A(1,1,1:end)=Psi00; %mapping probability amplitudes to the 3D array
56 A(1,2,1:end)=Psi01;
57 A(2,1,1:end)=Psi10;
58 A(2,2,1:end)=Psi11;
59
60 %Number States Parameters
61 n=1;
62 m=1;
63 C=zeros(length(x),length(x));
64 for i = -t:t
65     for l = -t:t
66         for j= max(i,l)-t:min(i,l)+t
67             for s=1:2

```

```

64     for b=1:2
65         for sp=1:2
66             for spp=1:2
67                 for bp=1:2
68                     for bpp=1:2
69
70 fi=i+t+1;
71 fl=l+t+1;
72         %%%---- correlation-number state ----%%%
73 C(fi,fl)=C(fi,fl)+ conj(A(sp,s,fi))*A(spp,s,f_i(i,j,t))*...
74     conj(A(bp,b,f_i(l,j,t)))*A(bpp,b,fl)*((li)^(sp-1)*(-li)^(spp-1)...
75     *(li)^(bp-1)*(-li)^(bpp-1) )*(...
76 ( n*(n-1)*(sp==1)*(spp==1)*(bp==1)*(bpp==1) +...
77 n*m* ( (sp==1)*(bp==2)+(sp==2)*(bp==1) )*( (spp==1)*(bpp==2)+(spp==2)...
78 *(bpp==1) ) +m*(m-1)*(sp==2)*(spp==2)*(bp==2)*(bpp==2) -...
79 ( (n*(sp==1)*(spp==1)+m*(sp==2)*(spp==2) )*(n*(bp==1)*(bpp==1)+m...
80 *(bp==2)*(bpp==2) ) )*(j==0) +...
81 ( n*(sp==1)*(bpp==1)*(spp==bp) + m*(sp==2)*(bpp==2)*(spp==bp) ) ) ;
82
83     end
84     end
85     end
86     end
87 end
88 end
89 end
90 end
91 end
92
93 Total0
94 Total1
95
96 figure
97 bar3(real(C));
98 colormap summer;
99 xlabel y;
100 ylabel y^\prime;
101 zlabel C_{yy^\prime};
102 xticklabels({'-1' , '0', '1'});

```



```
103 yticklabels({'-1' , '0' , '1'});  
104 toc
```

B.2. Coherent States

The codes for the probability amplitudes are same with number states. Therefore only correlation function part will be shown.

```
1 %Coherent States Parameters
2 alpha(1)=1;
3 alpha(2)=1;
4 C=zeros(length(x),length(x));
5 for i = -t:t
6 for l = -t:t
7 for j= max(i,l)-t:min(i,l)+t
8     for s=1:2
9     for b=1:2
10        for sp=1:2
11        for spp=1:2
12        for bp=1:2
13        for bpp=1:2
14            %%%---- correlation-coherent states ----%%%
15 C(fi,fl)=C(fi,fl) + ...
            ((1i)^(sp-1)*(-1i)^(spp-1)*(1i)^(bp-1)*(-1i)^(bpp-1) ) ...
16 *conj(A(sp,s,fi))*A(spp,s,f_i(i,j,t))...
17 *conj(A(bp,b,f_i(l,j,t)))*A(bpp,b,fl)...
18 *( conj(alpha(sp)) * alpha(bpp)*(spp==bp) );
19
20     end
21     end
22     end
23     end
24 end
25 end
26 end
27 end
28 end
```

B.3. Squeezed States and Squeezed Coherent States

The codes for the probability amplitudes are same with number states. Therefore only correlation function part will be shown.

```
1 % Squeezed Coherent States Parameters
2 theta(1)= 0 ;
3 theta(2)= pi/3 ;
4 r(1)=0.2;
5 r(2)=0.2;
6 alpha(1)=1;
7 alpha(2)=1;
8 C=zeros(length(x),length(x));
9 for i = -t:t
10 for l = -t:t
11 for j= max(i,l)-t:min(i,l)+t
12     for s=1:2
13         for b=1:2
14             for sp=1:2
15                 for spp=1:2
16                     for bp=1:2
17                         for bpp=1:2
18
19 fi=i+t+1;
20 fl=l+t+1;
21         %%%---- correlation-vacuum squeezed states ----%%%
22 C(fi,fl)=C(fi,fl)+ ...
23 conj(A(sp,s,fi))*A(spp,s,f_i(i,j,t))...
24 *conj(A(bp,b,f_i(l,j,t)))*A(bpp,b,fl)*...
25 (-li)^(sp-1)*(li)^(spp-1)*(-li)^(bp-1)*(li)^(bpp-1)*(...
26 (exp(li*theta(bpp))*exp(-li*theta(sp))*exp(-li*theta(bp))*...
27 exp(li*theta(spp))*sinh(r(sp))...
28 *sinh(r(bpp))*sinh(r(spp))*sinh(r(bp))*...
29 (sp==bpp)*(spp==bp)*(j==0)) +...
30 (exp(li*theta(bpp))*exp(-li*theta(sp))*sinh(r(sp))*sinh(r(bpp))*...
31 cosh(r(spp))*cosh(r(bp))* (sp==bp)*(spp==bpp)*(j==0)) +...
32 (exp(-li*theta(sp))*exp(li*theta(bpp))*sinh(r(sp))*sinh(r(bpp))*...
33 (sp==bpp)*(spp==bp) ) );
```

```

34         %%%---- correlation-displacement squeezed states ----%%%
35 C(fi,fl)=C(fi,fl)+ ...
36 conj(A(sp,s,fi))*A(spp,s,f_i(i,j,t))...
37 *conj(A(bp,b,f_i(l,j,t)))*A(bpp,b,fl)*...
38 (-li)^(sp-1)*(li)^(spp-1)*(-li)^(bp-1)*(li)^(bpp-1)*(...
39 (exp(li*theta(bpp))*exp(-li*theta(sp))...
40 *exp(-li*theta(bp))*exp(li*theta(spp))...
41 *sinh(r(sp))*sinh(r(bpp))*sinh(r(spp))...
42 *sinh(r(bp))* (sp==bpp)*(spp==bp)*(j==0) +...
43 (exp(li*theta(bpp))*exp(-li*theta(sp))...
44 *sinh(r(sp))*sinh(r(bpp))*cosh(r(spp))...
45 *cosh(r(bp))* (sp==bp)*(spp==bpp)*(j==0) +...
46 (exp(-li*theta(sp))*exp(li*theta(bpp))...
47 *sinh(r(sp))*sinh(r(bpp))*(sp==bpp)*...
48 (spp==bp) + conj(alpha(sp))*alpha(bpp)*(spp==bp))...
49 - alpha(spp)*alpha(bpp)*cosh(r(bp))...
50 *sinh(r(sp))*exp(-li*theta(sp))*(sp==bp)*(j==0) ...
51 + conj(alpha(sp))*alpha(bpp)...
52 *sinh(r(bp))*sinh(r(spp))*exp(-li*theta(bp))*...
53 exp(li*theta(spp))*(bp==spp)*(j==0)...
54 -conj(alpha(sp))*conj(alpha(bp))...
55 *cosh(r(spp))*sinh(r(bpp))*exp(li*theta(bpp))...
56 *(spp==bpp)*(j==0)...
57 + conj(alpha(bp))*alpha(spp)...
58 *sinh(r(sp))*sinh(r(bpp))*exp(-li*theta(sp))*...
59 exp(li*theta(bpp))*(sp==bpp)*(j==0) );
60
61
62     end
63
64     end
65
66     end
67
68 end
69 end
70 end

```

B.4. Thermal States

The codes for the probability amplitudes are same with number states. Therefore only correlation function part will be shown.

```
1 % Thermal States Parameters
2 T = 5000;
3 T1= 5000;
4 w=2*pi*1.042*10^(14);
5 w1=2*pi*600000*10^(9);
6 k = physconst('Boltzmann');
7 h_b =1.054571*10^(-34);
8 beta=1/(k*T);
9 beta1=1/(k*T1);
10 W=(h_b*w)/(k*T);
11 W1=(h_b*w)/(k*T);
12 % W =0.99;
13 % W1=0.99;
14 Z=exp(1/(1-exp(-1*W)));
15 Z1=exp((1)/(1-exp(-1*W1)));
16
17 for nn=0:20
18 for mm=0:20
19 for i = -t:t
20 for l = -t:t
21 for j= max(i,l)-t:min(i,l)+t
22     for s=1:2
23         for b=1:2
24             for sp=1:2
25                 for spp=1:2
26                     for bp=1:2
27                         for bpp=1:2
28
29 fi=i+t+1;
30 fl=l+t+1;
31         %%%---- Thermal State ----%%%
32 C(fi,fl)=C(fi,fl)+ ...
33         ( (exp(-W*nn)/Z) * (exp(-W1*mm)/Z1) ) ...
```

```

34     *conj(A(sp,s,fi))*A(spp,s,f_i(i,j,t))...
35     *conj(A(bp,b,f_i(l,j,t)))*A(bpp,b,fl)...
36     *(-1i)^(sp-1)*(1i)^(spp-1)*(-1i)^(bp-1)*(1i)^(bpp-1)*...
37 ( ( nn*mm*(sp==1)*(spp==1)*(bp==2)*(bpp==2)...
38 + nn*mm*(sp==2)*(spp==2)*(bp==1)*(bpp==1)...
39 + nn*mm*(sp==1)*(spp==2)*(bp==2)*(bpp==1)...
40 + nn*mm*(sp==2)*(spp==1)*(bp==1)*(bpp==2)...
41 - nn*mm*(sp==1)*(spp==1)*(bp==2)*(bpp==2)...
42 - nn*mm*(sp==2)*(spp==2)*(bp==1)*(bpp==1)...
43 - nn*(sp==1)*(spp==1)*(bp==1)*(bpp==1)...
44 - mm*(sp==2)*(spp==2)*(bp==2)*(bpp==2)*(j==0)...
45     + (nn*(sp==1)*(bpp==1)*(spp==bp)...
46     + mm*(sp==2)*(bpp==2)*(spp==bp) ) ) ;
47
48     end
49     end
50     end
51     end
52     end
53     end
54     end
55 end
56 end
57 end
58 end

```