

Riemann-Eddington theory: Incorporating matter, degravitating the cosmological constant

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Here we show that Eddington's pure affine gravity, when extended with Riemann curvature, leads to gravitational field equations that incorporate matter. This Riemann-Eddington gravity outfits a setup in which matter gravitates normally with Newton's constant but vacuum gravitates differently with an independent gravitational constant. This novel setup enables degravitation of the vacuum to observed level not by any fine-tuning but by a large hierarchy between its gravitational constant and its energy density. Remarkably, degravitation of the cosmological constant is local, causal and natural yet only empirical because the requisite degravitation condition is not predicted by the theory.

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I. INTRODUCTION

Soon after the foundation of general relativity (GR) [1] as the theory of gravitation, attempts to generalize it were started (see the reviews [2] and [3]). One such attempt was the introduction of the cosmological constant [4], which initiated the modern cosmology. Another attempt of great physical and mathematical interest was Eddington's affine gravity [5,6]. His idea was to use the affine connection itself as the fundamental quantity to describe gravitation. This proposal was furthered first by Einstein [7,8] and later by Schroedinger [9] (see also the solutions in [10]).

In general, the problem with Eddington's approach is that the Einstein equations it yields involve the cosmological constant and only the cosmological constant. The equations exhibit no sensitivity to matter and radiation in the Universe. This means that matter does not gravitate, and all bodies stay weightless in this model. The model must be improved to avoid this unphysical situation. For this purpose, Banados and Ferreira [11] extended the Eddington theory by forcibly adding matter action to it (in spacetimes endowed with metric). This proposal has resulted in the revival of Eddington's theory with various cosmological and astrophysical applications. Nevertheless, this "Eddington-inspired Born-Infeld gravity" has been found to suffer from instabilities [12] (which, according to [13], might be resolved by backreaction of particles). In any case, coupling of gravity to matter is not a settled issue in Eddington formalism, and it needs a clear resolution before studying its cosmological, astrophysical and other implications.

In the present work we tackle this problem. We show that Eddington's approach leads to correct gravitational field equations if it is extended to involve the grand curvature of spacetime, encoded in the Riemann tensor. Our setup has its

roots in the Riemann-only gravitational theory constructed by the author and his collaborators in [14]. In that work, the Einstein field equations were extracted from rank-4 dynamical equations by expressing torsion contribution as the Kulkarni-Numizu product of the metric and energy-momentum tensors. In general, affine gravity provides an appropriate framework for a dynamical approach to metrical theories of gravity [3,14,15]. The setup of the present work, the "Riemann-improved Eddington theory" as we will call it from now on, completes Eddington theory by incorporating the missing matter terms properly (by inducing a general energy-momentum tensor generically valid for all kinds of matter fields). Interestingly, this improved theory not only induces matter dynamically but also enables degravitation of the cosmological constant in a local, causal and not fine-tuned way. Stating differently, in the consequent gravitational theory, matter and vacuum gravitate with different gravitational constants, and bounds on cosmological constant are satisfied not by fine-tuning but by large hierarchies. These sensible aspects do not, however, mean that the cosmological constant problem is solved. The reason is that degravitation of the vacuum energy necessitates fundamental parameters to be related to the vacuum energy in a specific way, and the model is unable to offer any dynamical mechanism that predicts the aforementioned relation. The said relation is thus empirical rather than fundamental.

In Sec. II below we briefly review the Eddington theory. In Sec. III, we study in detail construction, dynamics, and gravitational properties of matter and vacuum within the Riemann-improved Eddington theory. In Sec. IV we conclude.

II. EDDINGTON GRAVITY

Despite its unphysical results, Eddington's idea is to the point. The reason is that it is based exclusively on the affine

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connection $\Gamma_{\alpha\beta}^{\mu}$, which is a purely geometrical acceleration field that fully governs the spacetime curvature. As a matter of the fact, the Riemann tensor

$$\mathfrak{R}_{\alpha\beta}^{\mu}(\Gamma) = \partial_{\nu}\Gamma_{\beta\alpha}^{\mu} - \partial_{\beta}\Gamma_{\nu\alpha}^{\mu} + \Gamma_{\nu\lambda}^{\mu}\Gamma_{\beta\alpha}^{\lambda} - \Gamma_{\beta\lambda}^{\mu}\Gamma_{\nu\alpha}^{\lambda} \quad (1)$$

involves only connection and constitutes a unique measure of the spacetime curvature in that spacetime is flat if and only if all components of $\mathfrak{R}_{\alpha\beta}^{\mu}(\Gamma)$ vanish. It contracts to produce the Ricci tensor

$$\mathcal{R}_{\alpha\beta}(\Gamma) \equiv \mathfrak{R}_{\alpha\mu\beta}^{\mu}(\Gamma) = \partial_{\mu}\Gamma_{\beta\alpha}^{\mu} - \partial_{\beta}\Gamma_{\mu\alpha}^{\mu} + \Gamma_{\mu\lambda}^{\mu}\Gamma_{\beta\alpha}^{\lambda} - \Gamma_{\beta\lambda}^{\mu}\Gamma_{\mu\alpha}^{\lambda} \quad (2)$$

which covers only a subset of the grand curvature components in (1). Its antisymmetric part $\mathcal{R}_{[\alpha\beta]}$ is given by $\mathfrak{R}_{\mu\alpha\beta}^{\mu}(\Gamma)$. The Weyl curvature tensor $\mathfrak{B}_{\alpha\beta}^{\mu}(\Gamma)$ is completely traceless and enjoys the same symmetries as the Riemann tensor.

For a symmetric connection, $\Gamma_{\alpha\beta}^{\mu} = \Gamma_{\beta\alpha}^{\mu}$, as also assumed in Eddington's original proposal [5], torsion vanishes and curvature remains as the only tensorial object in affine spacetime [16]. Having no metric tensor, curvature tensors cannot be contracted any further to obtain scalars (like curvature scalar in GR). The absence of scalars, however, does not obstruct the construction of invariant actions because differential volume element d^4x transforms not as a scalar but as a scalar density under coordinate mappings, and therefore, all one needs is a scalar density \mathcal{L} such that the product $d^4x\mathcal{L}$ is a scalar. Generically, the said \mathcal{L} is provided by determinants of tensors. The most obvious choice is the Ricci tensor, and it leads to the Eddington action [5,6]

$$I_E[\Gamma] = \int d^4x \sqrt{\text{Det}[\mathcal{R}(\Gamma)]} \quad (3)$$

wherein the determinant of $\mathcal{R}_{\alpha\beta}$

$$\text{Det}[\mathcal{R}] = \frac{1}{4!} \epsilon^{\alpha_0\alpha_1\alpha_2\alpha_3} \epsilon^{\beta_0\beta_1\beta_2\beta_3} \mathcal{R}_{\alpha_0\beta_0} \mathcal{R}_{\alpha_1\beta_1} \mathcal{R}_{\alpha_2\beta_2} \mathcal{R}_{\alpha_3\beta_3} \quad (4)$$

involves only the antisymmetric Levi-Civita symbol $\epsilon^{\alpha_0\alpha_1\alpha_2\alpha_3}$. Stationarity of the action (3) against the variation

$$\delta\mathfrak{R}_{\alpha\beta}^{\mu} = \nabla_{\nu}(\delta\Gamma_{\beta\alpha}^{\mu}) - \nabla_{\beta}(\delta\Gamma_{\nu\alpha}^{\mu}) \quad (5)$$

gives the equation of motion

$$\nabla_{\mu}[\sqrt{\text{Det}[\mathcal{R}]}(\mathcal{R}^{-1})^{\alpha\beta}] = 0 \quad (6)$$

which is solved by

$$\sqrt{\text{Det}[\mathcal{R}]}(\mathcal{R}^{-1})^{\alpha\beta} = \Lambda\sqrt{g}g^{\alpha\beta} \quad (7)$$

provided that $g_{\alpha\beta}$ is an invertible, covariantly-constant tensor field with determinant $g = \text{Det}[g_{\alpha\beta}]$ and inverse $g^{\alpha\beta} = (g^{-1})^{\alpha\beta}$. The equality (7) fines down to

$$\mathcal{R}_{\alpha\beta}(\Gamma) = \Lambda g_{\alpha\beta} \quad (8)$$

which is nothing but the Einstein field equations with cosmological constant Λ . This is so because $g_{\alpha\beta}$ has every reason to qualify as the metric tensor on spacetime, and its covariant constancy, $\nabla_{\mu}g_{\alpha\beta} = 0$, fully determines the connection $\Gamma_{\alpha\beta}^{\lambda}$ to be the Levi-Civita connection

$$g\Gamma_{\alpha\beta}^{\lambda} = \frac{1}{2}g^{\lambda\rho}(\partial_{\alpha}g_{\beta\rho} + \partial_{\beta}g_{\rho\alpha} - \partial_{\rho}g_{\alpha\beta}) \quad (9)$$

used in GR [1]. In summary, Eddington's approach dynamically gives a GR-like setup in which (i) Eddington's affine spacetime is dynamically endowed with a metric tensor and hence with the notions of distance and angle, (ii) affine connection converts into the Levi-Civita connection in GR, and (iii) Einstein field equations in GR are only partially reproduced because the right-hand side of (8) lacks the energy-momentum tensor $T_{\alpha\beta}$ of matter. This is a disaster. In recent years, phenomenological attempts have been made to improve it by adding to (3) the matter action directly "... without insisting neither on a purely affine action nor on a theory equivalent to Einstein gravity." [11]. This model has important astrophysical and cosmological implications [17] yet it has been found [12] to suffer from certain tensor instabilities. In affine geometry, coupling of gravity to matter is a perplexing problem [18], and below we study it starting from first principles and staying parallel to Eddington's construction.

III. RIEMANNING EDDINGTON GRAVITY

In this section we show that problems with Eddington's theory can be naturally resolved if it is extended to involve the grand curvature of spacetime. To this end, we extend the Eddington action (3) as

$$I_R[\Gamma] = \int d^4x \{a\sqrt{\text{Det}[\mathcal{R}(\Gamma)]} + b\sqrt{\text{Det}[\mathfrak{R}(\Gamma)]}\} \quad (10)$$

where a, b are dimensionless constants. They are true constants since in affine spacetime there exist no curvature invariants that can promote a and b to dynamical variables. In (10), determinant of the Riemann tensor

$$\begin{aligned} \text{Det}[\mathfrak{R}] &= \frac{1}{(4!)^2} \epsilon_{\mu_0\mu_1\mu_2\mu_3} \epsilon^{\alpha_0\alpha_1\alpha_2\alpha_3} \epsilon^{\nu_0\nu_1\nu_2\nu_3} \epsilon^{\beta_0\beta_1\beta_2\beta_3} \\ &\times \mathfrak{R}_{\alpha_0\nu_0\beta_0}^{\mu_0} \mathfrak{R}_{\alpha_1\nu_1\beta_1}^{\mu_1} \mathfrak{R}_{\alpha_2\nu_2\beta_2}^{\mu_2} \mathfrak{R}_{\alpha_3\nu_3\beta_3}^{\mu_3} \end{aligned} \quad (11)$$

generalizes the usual determinant in (4) to rank-four tensors [19,20]. There are four ϵ symbols here yet the covariant-looking and contravariant-looking symbols contract as

$\epsilon_{\mu\alpha\beta}\epsilon^{\mu'\alpha'\nu'\beta'} = \delta_{\mu}^{\mu'} \dots \delta_{\beta}^{\beta'}$ to leave only two contravariant-looking ϵ symbols as in the determinant of the Ricci tensor in Eq. (4). Thus, Ricci and Riemann pieces in action (10) are tensor densities of identical weights, and hence, the Riemann piece also leads to an invariant action like (3). The action (10) stays stationary against the variation in (5) provided that

$$\nabla_{\nu} [b\sqrt{\text{Det}[\mathfrak{R}]}(\mathfrak{R}^{-1})^{\beta\nu\alpha}_{\mu} + \frac{a}{2}\sqrt{\text{Det}[\mathcal{R}]}(\delta_{\mu}^{\nu}(\mathcal{R}^{-1})^{\beta\alpha} - (\mathcal{R}^{-1})^{\nu\alpha}\delta_{\mu}^{\beta})] = 0 \quad (12)$$

holds. In here, the inverse Riemann tensor [20]

$$(\mathfrak{R}^{-1})^{\beta\nu\alpha}_{\mu} = \frac{1}{(3!)^2} \frac{1}{\text{Det}[\mathfrak{R}]} \epsilon_{\mu\mu_1\mu_2\mu_3} \epsilon^{\beta\beta_1\beta_2\beta_3} \epsilon^{\nu\nu_1\nu_2\nu_3} \epsilon^{\alpha\alpha_1\alpha_2\alpha_3} \times \mathfrak{R}^{\mu_1}_{\alpha_1\nu_1\beta_1} \mathfrak{R}^{\mu_2}_{\alpha_2\nu_2\beta_2} \mathfrak{R}^{\mu_3}_{\alpha_3\nu_3\beta_3} \quad (13)$$

is direct generalization of $(\mathcal{R}^{-1})^{\alpha\beta}$ to rank-4 tensor fields. It arises in (12) for the same reason that the inverse Ricci tensor $(\mathcal{R}^{-1})^{\alpha\beta}$ arises in (6). The inverse Riemann satisfies not only the usual relations like $(\mathfrak{R}^{-1})^{\beta\nu\alpha}_{\mu} \mathfrak{R}^{\mu}_{\alpha\nu\beta} = \delta_{\mu}^{\mu'}$ which apply to rank-2 tensors as well but also the matrix multiplications like $(\mathfrak{R}^{-1})^{\beta\nu\alpha}_{\mu} \mathfrak{R}^{\mu}_{\alpha\nu\beta'} = \frac{1}{3}(\delta_{\nu'}^{\nu}\delta_{\beta'}^{\beta} - \delta_{\beta'}^{\nu}\delta_{\nu'}^{\beta})$ which arise from its higher-rank nature.

Now, we are in a position to analyze the affine equation of motion (12). It can be integrated to find

$$b\sqrt{\text{Det}[\mathfrak{R}]}(\mathfrak{R}^{-1})^{\beta\nu\alpha}_{\mu} + \frac{a}{2}\sqrt{\text{Det}[\mathcal{R}]}(\delta_{\mu}^{\nu}(\mathcal{R}^{-1})^{\beta\alpha} - (\mathcal{R}^{-1})^{\nu\alpha}\delta_{\mu}^{\beta}) = \mathfrak{Z}^{\beta\nu\alpha}_{\mu} \quad (14)$$

where $\mathfrak{Z}^{\beta\nu\alpha}_{\mu}$ is a tensor density having the same symmetries as $(\mathfrak{R}^{-1})^{\beta\nu\alpha}_{\mu}$. It is an integral of motion, and possesses the following properties:

$\mathfrak{Z}^{\beta\nu\alpha}_{\mu}$ must have vanishing divergence. This is because (14) gives back the affine equation of motion (12) provided that $\nabla_{\nu}\mathfrak{Z}^{\beta\nu\alpha}_{\mu} = 0$. (15)

Being the integral of the equation of motion (12), $\mathfrak{Z}^{\beta\nu\alpha}_{\mu}$ cannot involve the curvature tensors themselves, that is, it must be independent of $\mathfrak{R}^{\mu}_{\alpha\nu\beta}$, $\mathcal{R}_{\alpha\beta}$, $\mathfrak{W}^{\mu}_{\alpha\nu\beta}$ and their contractions. By the same token, possible parameters in $\mathfrak{Z}^{\beta\nu\alpha}_{\mu}$ must be independent of the fundamental constants a and b . It is through the observational constraints that they can have a relation. (16)

The equation of motion (12) is a vanishing divergence. This means that $\mathfrak{Z}^{\beta\nu\alpha}_{\mu}$ is not necessarily a function only of the metric tensor as in Eddington theory; it can involve novel tensor structures completely different to the metric tensor. (17)

These are the primary conditions to be fulfilled by the solution of the equations of motion. Being a rank-4 tensor equation with no clue about the content of $\mathfrak{Z}^{\beta\nu\alpha}_{\mu}$, the affine equation of motion (14) can possess a variety of solutions characterized by the spacetime structures formed and spin and mass representations propagated. We, however, specialize here to one particular solution:

The solution which leads to an equation for Ricci tensor in ways similar to the GR. In other words, we extract rank-2 portion of the rank-4 tensor equation (14) for the purpose of obtaining GR-like dynamics. In accordance with this, in light of the three conditions above and in the philosophy of (7) and (8), we structure the motion integral $\mathfrak{Z}^{\beta\nu\alpha}_{\mu}$ by proposing the ansatz

$$\mathfrak{Z}^{\beta\nu\alpha}_{\mu} = \sqrt{g} \left\{ \frac{\Lambda}{2} (\delta_{\mu}^{\nu} g^{\beta\alpha} - g^{\nu\alpha} \delta_{\mu}^{\beta}) + Q^{\beta\nu\alpha}_{\mu} + c_1 (Q^{\beta\alpha} \delta_{\mu}^{\nu} - \delta_{\mu}^{\beta} Q^{\nu\alpha}) + c_2 (g^{\beta\alpha} Q_{\mu}^{\nu} - Q_{\mu}^{\beta} g^{\nu\alpha}) + c_3 Q (g^{\beta\alpha} \delta_{\mu}^{\nu} - \delta_{\mu}^{\beta} g^{\nu\alpha}) \right\} \quad (18)$$

where $Q^{\beta\nu\alpha}_{\mu}$ contains the non-metrical tensor structure mentioned in (17). It contracts to yield the lower-rank tensor fields $Q^{\beta\alpha} = Q^{\beta\nu\alpha}_{\nu}$, $Q_{\mu}^{\nu} = g_{\beta\alpha} Q^{\beta\nu\alpha}_{\mu}$, and $Q = g_{\beta\alpha} Q^{\beta\alpha}$.

The ansatz for $\mathfrak{Z}^{\beta\nu\alpha}_{\mu}$ in (18) is composed of a gradient-free part proportional to Λ and a divergence-free part involving $Q^{\beta\nu\alpha}_{\mu}$. For this classification to make sense, it is essential that $Q^{\beta\nu\alpha}_{\mu}$ does not contain any gradient-free

part like the Λ term in (18). This constraint is a restatement of the property (17).

In regard to the property (15), the divergence of $\mathfrak{Z}^{\beta\nu\alpha}_\mu$ must vanish. This does not imply a similar condition for $Q^{\beta\nu\alpha}_\mu$. Indeed, it is possible that $\nabla_\lambda Q^{\beta\nu\alpha}_\mu \neq 0$ yet its derivatives can organize to precipitate $\nabla_\nu \mathfrak{Z}^{\beta\nu\alpha}_\mu = 0$. In fact, as emphasized in (16), $\mathfrak{Z}^{\beta\nu\alpha}_\mu$ must be independent of the curvature tensors, and this necessitates $Q^{\beta\nu\alpha}_\mu$ to be also independent of the curvature tensors. This independence is ensured by making sure that $Q^{\beta\nu\alpha}_\mu$ possesses certain properties which decisively distinguish it from curvature tensors. To this end, Bianchi identities prove pivotal in that $Q^{\beta\nu\alpha}_\mu$ is guaranteed to differ from the Riemann tensor if it enjoys, for instance, a differential identity of the form

$$\nabla_\nu Q^{\beta\nu\alpha}_\mu = c_4 \nabla_\mu Q^{\beta\alpha} + c_5 \nabla^\alpha Q_\mu^\beta + c_6 \nabla^\beta Q_\mu^\alpha \quad (19)$$

with not all of c_4, c_5, c_6 taking the values required by the Bianchi identities for curvature tensors. This Bianchi-inspired identity is an ansatz for $Q^{\beta\nu\alpha}_\mu$. It gives

$$\nabla_\nu Q_\mu^\nu = \frac{c_4}{1 - c_5 - c_6} \nabla_\mu Q \quad (20)$$

upon contraction. Then, the identities (19) and (20) facilitate the desired relation $\nabla_\nu \mathfrak{Z}^{\beta\nu\alpha}_\mu = 0$ for a generic $Q^{\beta\nu\alpha}_\mu$ if c_i satisfy the constraints

$$c_2 = -c_4 = c_5 = c_1, \quad c_3 = \frac{c_1^2}{1 - c_1}, \quad c_6 = 0 \quad (21)$$

which give a realization of the property of $\mathfrak{Z}^{\beta\nu\alpha}_\mu$ given in (15). The identity (20) ensures that $Q^{\beta\nu\alpha}_\mu$ vanishes identically unless $c_1 \neq 1$. With $c_5 = -c_4 = c_1 \neq 1$, however, the identity (19) never conforms to the Bianchi identities for curvature tensors. As a result, the motion integral $\mathfrak{Z}^{\beta\nu\alpha}_\mu$ is divergence-free as required by (15), does not involve curvature tensors as required by (16), and consists of nonmetrical structures $Q^{\beta\nu\alpha}_\mu$ as required by (17). In conclusion, the ansatz (18) for $\mathfrak{Z}^{\beta\nu\alpha}_\mu$ supplemented by the ansatz (19) for $Q^{\beta\nu\alpha}_\mu$ yields a self-consistent picture in which all three conditions (15), (16), and (17) are satisfied. This picture accommodates a divergence-free tensor field

$$K_{\alpha\beta} = Q_{\alpha\beta} + \frac{c_1}{1 - c_1} Q g_{\alpha\beta} \quad (22)$$

as directly follows from (20). This tensor field is strictly conserved, $\nabla^\alpha K_{\alpha\beta} = 0$. It is guaranteed to involve no curvature contamination by (21). Thus, as will be shown below, it acts as a nontrivial source generalizing the homogeneous source $\Lambda g_{\alpha\beta}$ in the Eddington equation (8).

Having constructed $\mathfrak{Z}^{\beta\nu\alpha}_\mu$ and established the nongeometrical nature of $Q^{\beta\nu\alpha}_\mu$, it is now time to solve for the curvature tensors themselves from (14). This is not so direct, however. The reason is that the left-hand side of (14) involves the inverse Riemann and inverse Ricci pieces together, and determining one of them necessitates a proper knowledge of the other. Thus it is with a physical ansatz and self-consistent determination of model parameters that one can extract the curvature tensors. Each of the two pieces, not a motion integral individually, violates all of (15), (16), and (17). The inverse Riemann piece, for instance, does not have to have vanishing divergence. Also, it can explicitly involve the Weyl curvature (but not the Riemann and Ricci curvatures). Then, guided by the form of $\mathfrak{Z}^{\beta\nu\alpha}_\mu$ in (18), it can be structured as

$$\begin{aligned} \sqrt{\text{Det}[\mathfrak{R}]}(\mathfrak{R}^{-1})^{\beta\nu\alpha}_\mu &= \sqrt{g} \{ \mathfrak{B}^{\beta\nu\alpha}_\mu + \tilde{\Lambda}(\delta_\mu^\nu g^{\beta\alpha} - g^{\nu\alpha} \delta_\mu^\beta) \\ &+ Q^{\beta\nu\alpha}_\mu + \tilde{c}_1(Q^{\beta\alpha} \delta_\mu^\nu - \delta_\mu^\beta Q^{\nu\alpha}) \\ &+ \tilde{c}_2(g^{\beta\alpha} Q_\mu^\nu - Q_\mu^\beta g^{\nu\alpha}) \\ &+ \tilde{c}_3 Q(g^{\beta\alpha} \delta_\mu^\nu - \delta_\mu^\beta g^{\nu\alpha}) \} \quad (23) \end{aligned}$$

where the coefficients \tilde{c}_i are different from c_i in (18) and do not have to take the values in (21). This is yet another ansatz. In reality, this ansatz does not have to depend on $Q^{\beta\nu\alpha}_\mu$; it can involve completely different tensor fields. General tensor structures as such, however, add no new physical insight in that their contributions combine with those from $Q^{\beta\nu\alpha}_\mu$ in forming the gravitational field equations. Thus, one can regard (23) as a conservative formulation and proceed to determine if it leads to a self-consistent solution for \tilde{c}_i , and from (23) one gets

$$\begin{aligned} \mathfrak{R}^\mu_{\alpha\beta}({}^g\Gamma) &= \mathfrak{B}^\mu_{\alpha\beta} + \tilde{\Lambda}(\delta_\nu^\mu g_{\alpha\beta} - g_{\alpha\nu} \delta_\beta^\mu) + Q^\mu_{\alpha\beta} \\ &+ \tilde{c}_1(Q_\nu^\mu g_{\alpha\beta} - \delta_\beta^\mu Q_{\alpha\nu}) \\ &+ \tilde{c}_2(\delta_\nu^\mu Q_{\alpha\beta} - Q_\beta^\mu g_{\alpha\nu}) + \tilde{c}_3 Q(\delta_\nu^\mu g_{\alpha\beta} - \delta_\beta^\mu g_{\alpha\nu}) \quad (24) \end{aligned}$$

as a relation between the Riemann tensor and the nongeometrical tensor field $Q^\mu_{\alpha\beta}$. Is this an equation of motion for the Riemann tensor? No, there is no such thing. The reason is that, in the presence of a metric tensor (see the earlier work [14] for nonmetrical setup) that defines the Levi-Civita connection ${}^g\Gamma$, the Riemann tensor decomposes into Weyl and Ricci parts

$$\begin{aligned} \mathfrak{R}^\mu_{\alpha\beta}({}^g\Gamma) &= \mathfrak{B}^\mu_{\alpha\beta}({}^g\Gamma) + \frac{1}{2}(\delta_\nu^\mu \mathcal{R}_{\alpha\beta}({}^g\Gamma) - \mathcal{R}_\beta^\mu({}^g\Gamma) g_{\alpha\nu}) \\ &+ \frac{1}{2}(\mathcal{R}_\nu^\mu({}^g\Gamma) g_{\alpha\beta} - \delta_\beta^\mu \mathcal{R}_{\alpha\nu}({}^g\Gamma)) \\ &- \frac{1}{6} \mathcal{R}(g, {}^g\Gamma)(\delta_\nu^\mu g_{\alpha\beta} - \delta_\beta^\mu g_{\alpha\nu}) \quad (25) \end{aligned}$$

to facilitate a dynamical equation for Ricci tensor. In this sense, the relation (24) is a rank-4 transcription of the Einstein field equations. It rests wholly on the dynamics of the Ricci tensor. Clearly, the Ricci tensor can be determined either by contracting the Riemann tensor in (24)

$$\mathcal{R}_{\alpha\beta}({}^g\Gamma) = 3\tilde{\Lambda}g_{\alpha\beta} + (1 - \tilde{c}_1 + 3\tilde{c}_2)Q_{\alpha\beta} + (\tilde{c}_1 + 3\tilde{c}_3)Qg_{\alpha\beta} \quad (26)$$

or by solving for the inverse Ricci piece in (14)

$$\begin{aligned} \mathcal{R}_{\alpha\beta}({}^g\Gamma) &= \frac{1}{a}(\Lambda - 2b\tilde{\Lambda})g_{\alpha\beta} \\ &+ \frac{2}{3a}[(1 + 3c_1 - c_2) - b(1 + 3\tilde{c}_1 - \tilde{c}_2)]Q_{\alpha\beta} \\ &+ \frac{2}{3a}[(c_2 + 3c_3) - b(\tilde{c}_2 + 3\tilde{c}_3)]Qg_{\alpha\beta} \end{aligned} \quad (27)$$

where the two solutions must agree with each other. Consequently, imposing also the Bianchi identity, the Ricci tensor is found to read as

$$\mathcal{R}_{\alpha\beta}({}^g\Gamma) = \frac{3}{3a + 2b}\Lambda g_{\alpha\beta} - \frac{(2 + 4c_1)}{(3a + 2b)}\left(K_{\alpha\beta} - \frac{1}{2}Kg_{\alpha\beta}\right) \quad (28)$$

provided that the unknown couplings in the ansatz (23) take the specific values

$$\begin{aligned} \tilde{\Lambda} &= \frac{\Lambda}{3a + 2b} \\ \tilde{c}_1 &= \frac{1}{b(12a + 8b)}(3a(3 - 2b + 6c_1) + 2b(1 - 2b - 2c_1)) \\ \tilde{c}_2 &= \frac{1}{b(12a + 8b)}(-3a(1 - 2c_1) - 2b(3 + 2c_1)) \\ \tilde{c}_3 &= \frac{1}{b(12a + 8b)(c_1 - 1)}(a(1 + 5c_1 - 6c_1^2) \\ &\quad - 2b(1 + c_1 + 2c_1^2)) \end{aligned} \quad (29)$$

where $c_1 \neq 1$ as required by (20), (21) and (22). Having determined $\tilde{\Lambda}$ and \tilde{c}_i , the ansatz for Riemann tensor in (23) gets fully fixed in terms of Λ , c_1 , and $Q^{\beta\nu\alpha}{}_{\mu}$.

The solution for Ricci tensor in (28) is the core dynamical equation governing the dynamics of the Ricci tensor. Its exegesis as gravitational field equations is dictated by the model parameters a , b and the motion integrals Λ , $Q^{\beta\nu\alpha}{}_{\mu}$. This exegesis phase or reinterpretation stage is critically important for making sense of the Riemanned Eddington theory, which is dissected below systematically for clarifying its structure in comparison to the GR.

(1) We first study the model parameters. Equation (28) contains three dimensionless parameters: a , b ,

and c_1 . Here, a and b are fundamental constants appearing in the governing action (10). They both are plain constants (not invariants) because only tensor densities can be defined in affine spacetime. The parameter c_1 is neither a fundamental constant nor a motion integral; it is just a parameter that decorates (28) to form a one-parameter family of solutions for the Ricci tensor. Excepting $c_1 = 1$ and $c_1 = \infty$, it can take any value, including $c_1 = 0$.

The model parameters cannot take arbitrary values. Their ranges must ensure that the Eddington theory is transcended to involve nongeometrical sources like $K_{\alpha\beta}$. This is crucial because, physically, the Riemanned Eddington theory should reduce to the original Eddington theory not because of some parameter takes a bad value but because nongeometrical sources like $K_{\alpha\beta}$ vanish. Below we determine the admissible ranges of model parameters:

- (a) The $b \rightarrow 0$ limit. In this case, as follows from (29), all three $\tilde{c}_i \rightarrow \infty$ ($i = 1, 2, 3$) and it becomes obligatory to set $Q^{\mu}{}_{\alpha\nu\beta} \rightarrow 0$ to keep the Riemann tensor in (24) finite. This, however, reduces the whole dynamics to Eddington setup because both (24) and (28) give the Eddington solution (7). In accordance with the action (10), $b \rightarrow 0$ limit necessarily yields the Eddington result.
- (b) The $a \rightarrow 0$ limit. In this limit, the Riemann piece itself is forced to satisfy the equation of motion (12), and Bianchi identities require $c_1 = -1/2$. However, this particular c_1 value kills the $K_{\alpha\beta}$ contribution in (28), leaving behind precisely the Eddington solution (7).
- (c) The $c_1 \rightarrow -1/2$ limit. This limit is reached through also $a \rightarrow 0$ limit, as described just above. It completely erases the matter part of (28), leaving behind again the Eddington solution.

In consequence, Eddington theory is transcended to incorporate nongeometrical sources like $K_{\alpha\beta}$ if $a \neq 0$ and $b \neq 0$ and $c_1 \neq -1/2$ simultaneously, and it is attained back only when $K_{\alpha\beta}$ itself vanishes.

(2) Having dealt with the model parameters, we now turn to a detailed discussion of the motion integrals Λ and $K_{\alpha\beta}$. In general, they fall in two physically distinct categories depending on whether they are independent or not, and the gravitational field equations they lead to read as follows.

- (a) Λ and $K_{\alpha\beta}$ are independent quantities. By this we mean that, they are related neither parametrically nor dynamically. In this case, in (28), one is free to perform the rescalings $\frac{3\Lambda}{(3a+2b)} \rightarrow \Lambda$ and $-\frac{(2+4c_1)}{(3a+2b)}K_{\alpha\beta} \rightarrow K_{\alpha\beta}$ to eliminate a , b , c_1 from

equations. This simplifies the right-hand side of (28) to $\Lambda g_{\alpha\beta} + K_{\alpha\beta}$, in which while the homogeneous part Λ sets the energy-momentum tensor for empty space

$$T_{\alpha\beta}^{(\text{vac})} = -M_{\text{Pl}}^2 \Lambda g_{\alpha\beta} \quad (30)$$

the nongeometrical part $K_{\alpha\beta}$ sets the energy-momentum tensor of matter

$$T_{\alpha\beta}^{(\text{matt})} = M_{\text{Pl}}^2 K_{\alpha\beta} \quad (31)$$

with $M_{\text{Pl}}^2 = (8\pi G_N)^{-1/2}$ being the fundamental scale of gravity. These associations between curvature sources and energy-momentum tensors make sense as long as the two energy-momentum tensors do not convert into each other. In other words, the material system must be genuinely classical and must undergo no phase transition. This is because phase transitions induce constant curvatures that add to Λ —an effect that makes Λ depend explicitly on the matter sector parameters whereby disrupting independent rescalings of Λ and $K_{\alpha\beta}$. Consequently, the model parameters a , b , and c_1 can be embedded into motion integrals to obtain exactly the gravitational field equations in GR provided that the energy-momentum tensors of matter and empty space are independent of each other. This can happen particularly in classical systems undergoing no phase transitions.

(b) Λ and $K_{\alpha\beta}$ are not independent quantities. By this we mean that, they can have common parameters. In fact, all realistic systems fall in this category. Now, because of their mutual dependence, Λ and $K_{\alpha\beta}$ cannot be rescaled freely simply because the rescaled Λ cannot be guaranteed to correspond to the Λ which would be induced by the rescaled $K_{\alpha\beta}$. In general, Λ and $K_{\alpha\beta}$ develop nontrivial correlations through phase transitions and quantum corrections because these effects give contributions to Λ in a way explicitly involving the matter sector parameters along with the momentum cutoffs applied [21]. Consequently, the vacuum energy-momentum tensor in (30) and the matter energy-momentum tensor in (31) cannot be rescaled independently to eliminate the parameters a , b , and c_1 in (28). This means that the Ricci tensor there wholly sets the sought gravitational field equations

$$\mathcal{R}_{\alpha\beta}(g\Gamma) = \frac{1}{M_{\text{Pl}}^2} \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) \quad (32)$$

wherein the energy-momentum tensor

$$T_{\alpha\beta} = \frac{3}{(3a+2b)} T_{\alpha\beta}^{(\text{vac})} - \frac{(2+4c_1)}{(3a+2b)} T_{\alpha\beta}^{(\text{matt})} \quad (33)$$

combines the vacuum and matter energy-momentum tensors with a , b , and c_1 staying as additional parameters not found in GR. In the language of effective field theory, one first constructs quantum effective action by integrating out all quantum fluctuations beyond the matching scale and then extracts from it $T_{\alpha\beta}^{(\text{vac})}$ and $T_{\alpha\beta}^{(\text{matt})}$ [21]. They are thus intimately correlated. The way they contribute to (33) involves not just $1/M_{\text{Pl}}^2$ as in GR but also a , b , and c_1 as extra degrees of freedom. In this sense, vacuum and matter can be made to gravitate differently—a novel feature not found in GR. Nevertheless, if matter is to gravitate as experimented and observed it is necessary to set

$$3a+2b = -(2+4c_1) \quad (34)$$

so that the material part of (33) becomes $+T_{\alpha\beta}^{(\text{matt})}/M_{\text{Pl}}^2$. This is precisely the way matter gravitates in the GR.

The empirical condition in (34) calibrates weight of matter. It, however, leaves the weight of vacuum undetermined. In exact GR limit, one sets $3a+2b=3$ so that cosmological constant Λ gravitates with the same strength as matter. Then, the gravitational field equation (32) gives a maximally symmetric background geometry with $\mathcal{R}(g, g\Gamma) = 4\Lambda$. The cosmological observations [22] require this curvature to acquire a tiny value $\mathcal{R}^{(\text{obs})} \sim m_\nu^4/M_{\text{Pl}}^2$, m_ν being the neutrino mass scale. This vanishingly small curvature, given the diversity and enormity of contributions to vacuum energy [23], is perplexingly difficult to realize. This is the well-known cosmological constant problem [23]—the severest naturalness problem plaguing both particle physics and cosmology.

It is clear that the cosmological constant problem is special to GR because it is in GR that a bound on curvature directly translates into a bound on Λ . Fortunately, in the Riemann-improved Eddington theory being developed, this impasse metamorphoses into a tractable problem simply because there is no physical reason as in (34) that enforces $3a+2b=3$ (or $c_1 = -5/4$). As a matter of fact, these fundamental constants must be fixed by using observational data on the cosmological constant itself [22]. Then, extreme smallness of $\mathcal{R}^{(\text{obs})}$ necessitates at least one of a , b to be a large number. Pertaining to a physical theory, in the philosophy of Dirac's large number hypothesis, a and b can be interpreted as the ratio of two hierarchically different mass scales. By force of the gravitational nature of the setup, one of the mass scales is M_{Pl} . The other one, denoted hereon by M_{Co} ,

is plausibly a cosmological scale. Letting $a \propto (M_{\text{Co}}/M_{\text{Pl}})^{n_a}$, $b \propto (M_{\text{Co}}/M_{\text{Pl}})^{n_b}$, and fixing their proportionality constants appropriately one can write

$$a + \frac{2}{3}b = \left(\frac{M_{\text{Co}}}{M_{\text{Pl}}}\right)^n \quad (35)$$

where $n = \max\{n_a, n_b\}$. Replacement of this parametrization in (32) gives $\mathcal{R}(g, {}^g\Gamma) = 4\Lambda(M_{\text{Pl}}/M_{\text{Co}})^2$ for $n = 2$. This prediction complies with existing bounds [22] without constraining Λ itself provided that it is possible to realize

$$(M_{\text{Co}}^{(\text{obs})})^2 \simeq \Lambda \left(\frac{M_{\text{Pl}}}{m_\nu}\right)^4 \quad (36)$$

for $\Lambda > 0$ (signs of a and/or b are reversed for $\Lambda < 0$). For instance, for $\Lambda \sim M_{\text{Pl}}^2$ —the worst case [23] in GR, one gets $M_{\text{Co}}^{(\text{obs})} \sim M_{\text{Pl}}(M_{\text{Pl}}/m_\nu)^2 \sim 10^{80}$ GeV. This is the mass of the Universe (which consists of some 10^{80} hydrogen atoms). In general, smaller the Λ closer the M_{Co} to M_{Pl} . It is astonishing that this whole procedure involves no fine-tuning. Indeed, the observed curvature $\mathcal{R}^{(\text{obs})} \sim m_\nu^4/M_{\text{Pl}}^2$ is reproduced not by finely tuning parameters but by hierarchically splitting M_{Co}^2 and Λ via (36). Consequently, the Riemann-improved Eddington theory, or simply the Riemann Eddington theory, proves firmly natural. It is able to cover the 120 orders of magnitude discrepancy between the observed curvature $\mathcal{R}^{(\text{obs})} \sim m_\nu^4/M_{\text{Pl}}^2$ [22] and the theoretical expectation $\mathcal{R}^{(\text{thr})} \sim M_{\text{Pl}}^2$ [23] without enforcing any fine adjustments of model parameters. This crucial feature is manifestly reflected in the refined gravitational field equations

$$\begin{aligned} \mathcal{R}_{\alpha\beta}({}^g\Gamma) &= \frac{1}{M_{\text{Co}}^2} (V_0 g_{\alpha\beta}) \\ &+ \frac{1}{M_{\text{Pl}}^2} \left(T_{\alpha\beta}^{(\text{matt})} - \frac{1}{2} T^{(\text{matt})} g_{\alpha\beta} \right) \end{aligned} \quad (37)$$

following from (32) after imposing (31), (34), (35), and defining the vacuum energy $V_0 = M_{\text{Pl}}^2 \Lambda$. This dynamics ensures that, in the Riemann Eddington theory matter and vacuum gravitate not universally with Newton's constant but individually with their own gravitational constants. In particular, the vacuum energy V_0 gravitates not with Newtonian strength M_{Pl}^{-2} but with cosmological strength M_{Co}^{-2} . It is this difference between gravitational constants that enables a local observer to distinguish between vacuum energy and temporary blimps in matter energy-momentum tensor. Consequently, in contrast to nonlocal, acausal modifications of Einstein gravity which degravitate cosmological constant like

(37) without fine-tuning [24,25], the Riemann-improved Eddington theory facilitates a local, causal and natural setup that degravitates the cosmological constant.

It is timely to ask: Does the Riemann-extended Eddington theory solve the cosmological constant problem? No! There is no such thing. The reason is that the cosmological scale M_{Co} (equivalently, the fundamental constants a and b) must be a rather specific function of Λ , as given in (36). The model, however, offers no dynamics for such a functional relation. The relation (36) is an empirical relation not a fundamental one derived from dynamics. In fact, M_{Co} is a fundamental constant yet Λ is a motion integral, and the two do not have to have a predefined relation like (36). Thus, degravitation of the cosmological constant in Riemann Eddington theory is local, causal and not fine-tuned; however, the gravitational constant of the vacuum, M_{Co} , cannot be fixed as a fundamental constant unless Λ is known to take a fixed, predefined value. In other words, Λ must be known in terms of the model parameters (gravitational constant, particle masses and the like) as a universal quantity. More specifically, given matter spectrum and underlying dynamics, one must be able to compute Λ to fix M_{Co} in the fundamental action (10). Could this be done? Yes and no. The answer is yes because the zero-point energies of quantum fields (see [23] as well as explicit predictions in [21]) give a vacuum energy density $V_0 \simeq k_{\text{UV}}^4$, k_{UV} being the ultraviolet momentum cutoff. The answer is no because it is possible to know neither k_{UV} nor particle spectrum beyond the Fermi scale. It is with certain plausible assumptions that one can have an idea about Λ . For instance, if supersymmetry is not a symmetry of Nature at short distances, one necessarily takes $k_{\text{UV}} \simeq M_{\text{Pl}}$ as the ultimate ultraviolet scale for any quantum field theory, and the resultant vacuum energy does not curve spacetime overly if M_{Co} is set in (10) as the mass of the Universe. Summarizing, having no information about the physics beyond the Fermi scale, it is not possible to determine Λ from the outset, and hence, the relation (36) cannot facilitate a solution for the cosmological constant problem. In view of (37), one immediately notes that all that M_{Co} can do is to provide a rationale, not a mechanism, for degravitating the cosmological constant.

We close this section by configuring the matter action. Though its energy-momentum tensor $T_{\alpha\beta}^{(\text{matt})}$ is the only attribute needed for gravitational field equations, the matter action itself is needed for quantization of matter and determination of conserved charges. In affine spacetime, however, one cannot write a matter action like (10) because there is no such determinant in matter sector [26] (see also

[15] in this respect). It is actually not needed at all. The reason is that the matter action emerges dynamically along with the metric tensor, the Levi-Civita connection and the energy-momentum tensor. In other words, once metric and energy-momentum tensor are induced, the matter action is constructed as that functional of matter fields whose variation with respect to the metric tensor gives the energy-momentum tensor in (33) [3,14,15,27]. This association of energy-momentum tensor (see [28] for a detailed discussion of energy-momentum current) leads to a dynamical equivalence $I_R[\Gamma] \rightsquigarrow I[g, \psi]$, where

$$I[g, \psi] = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} M_{\text{Pl}}^2 g^{\alpha\beta} \mathcal{R}_{\alpha\beta}(^g\Gamma) - \frac{M_{\text{Pl}}^2}{M_{\text{Co}}^2} V_0 + \mathcal{L}_{(\text{matt})}(g, \psi) \right\} \quad (38)$$

is the matter and gravity action in spacetimes endowed with metric tensor $g_{\alpha\beta}$. It gives the Einstein field equations (37) through $\frac{\delta I}{\delta g^{\alpha\beta}} = 0$ where $T_{\alpha\beta}^{(\text{matt})} = -2 \frac{\delta \mathcal{L}_{(\text{matt})}}{\delta g^{\alpha\beta}} + g_{\alpha\beta} \mathcal{L}_{(\text{matt})}$ is the matter energy-momentum tensor. The equations of motion of the matter fields ψ follow from $\frac{\delta I}{\delta \psi} = 0$, with $\mathcal{L}_{(\text{matt})}(g, \psi)$ being the matter Lagrangian. The metrical framework of (38), dynamically equivalent to (10), precisely gives the GR plus matter except for one single term, which is the cosmological constant. The Λ term in GR is replaced by $(\Lambda M_{\text{Pl}}^2)/M_{\text{Co}}^2$, which can degravitate Λ for sufficiently large M_{Co} (or a and/or b). Moreover, dynamical equivalence on material dynamics ensures that there exists actually no reason to expect instabilities in tensor or other modes (They are known to occur when Eddington action (3) is directly added the matter action [11,12]).

IV. DISCUSSIONS AND CONCLUSION

In summary, the findings and implications of the present work are as follows:

- (i) The Riemann-improved Eddington theory, under reasonable ansätze, gives a dynamical equation for Ricci tensor. In view of its decomposition into Weyl and Schouten tensors, however, the Riemann tensor cannot possess an independent dynamical equation by itself. Gravitational waves are accommodated by the presence of the Weyl curvature in the ansatz for Riemann tensor.
- (ii) The resulting dynamical equation for Ricci tensor can be put in exact GR form. However, parameter space is wide enough to allow for novel solutions where Ricci curvature is sourced by cosmological constant and matter not directly as in GR but through nontrivial dressings by fundamental constants. Nevertheless, its coupling to the energy-momentum tensor of matter can be brought to observed strength by correlating model parameters, and this ensures that matter gravitates just as in the GR.

- (iii) The dressing factor for cosmological constant is set also by observations. Interestingly, this determination is accomplished not by finely tuning model parameters as in GR but by hierarchically splitting the cosmological constant and the cosmological scale introduced by the model. The cosmological scale evaluates around the mass of the Universe when the cosmological constant lies at the Planck scale.
- (iv) The cosmological scale must be born in the theory with right value to suppress the cosmological constant. The model, however, does not offer a dynamical basis for such a relation so that the said relation is empirical rather than fundamental.
- (v) The metric and energy-momentum tensors facilitate introduction of an effective matter action, and this establishes a dynamical equivalence between the Riemann-Eddington theory and a GR-like metrical theory of gravity. The latter differs from GR plus matter setup only by the degravitation factor in front of the cosmological constant. The vacuum energy receives contributions from all zero-point energies and phase transitions yet it weighs too small to cause cosmological constant problem, at least at the empirical level.
- (vi) It must be kept in mind that the GR-like gravitational dynamics found in the present work is based on a specific ansatz which solves the affine equations of motion. There can exist other solutions as well. In this sense, the Riemann-Eddington theory can give rise to alternative gravitational theories which may or may not reduce to GR.

The Riemann-improved Eddington theory can be further generalized, for instance, by considering determinants of higher-rank curvature tensors, by switching on torsion, or by going to higher-dimensions. In addition, one can search for other possible solutions of the affine equations of motion. If any, it would be interesting to see if there are specific solutions that can provide a fundamental resolution for the cosmological constant problem. Also interesting would be a proper understanding of the flat rotation curves of galaxies *a la* Milgrom's modified gravitational dynamics (cold dark matter being a competing alternative). These attempts, if prove positive, would further strengthen Eddington's affine gravity program.

In summary, the Riemann-improved Eddington theory, inherently formulated in affine space, leads to a novel framework in which one preserves all the successes of the GR while fathoming the observed size of the Universe in a local, causal, natural but, alas, empirical way.

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