

Kinematic Synthesis of Planar 4-Bar Path Generators for Finite Line Positions

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Abstract. Although the kinematic synthesis of planar function, point-path and motion generators are vastly studied in the literature, surprisingly synthesis of line-path generators is not formulized in detail. This study presents the formulization of the planar 4-bar line-path generator synthesis problem for up to 5 homologous positions. Numerical examples for 3 and 4 line positions are presented for the illustration of the formulations.

Keywords: Kinematic Synthesis, 4-Bar Mechanism, Line-Path Generation.

1 Introduction

Kinematic synthesis of planar linkages are classically studied under three main categories: motion generation, function generation, and point-path generation and path generation with prescribed timing [1-2]. There are some studies on kinematic synthesis of spatial line-path generator linkages [3-8]. Homologous lines passing through a point or tangent to a circle in plane are issued in [5]. However according to the authors' knowledge, there is no study on synthesis of planar line-path generator 4-bar linkages.

In some applications such as grinding, filing and packaging it is necessary to design a linkage, end-effector of which needs to be enveloping a given planar curve or a surface with no torsion. In other words, a line is sought on the end-effector which remains tangent to a given planar curve. If the envelope of lines can be approximately represented by finitely many lines, the problem can be cast into the design of a planar linkage, end-effector of which passes through finitely many lines. That is, a line on the coupler link instantaneously coincides with some lines attached to the fixed link at different instants. This paper presents the derivations of the necessary formulation for the kinematic synthesis of planar 4-bar linkages for specified three and four line positions. It is shown that the line generation problem can be mathematically converted into a motion generation problem and solved similarly. Numerical examples are presented to illustrate the formulations.

2 Problem Definition and Formulation

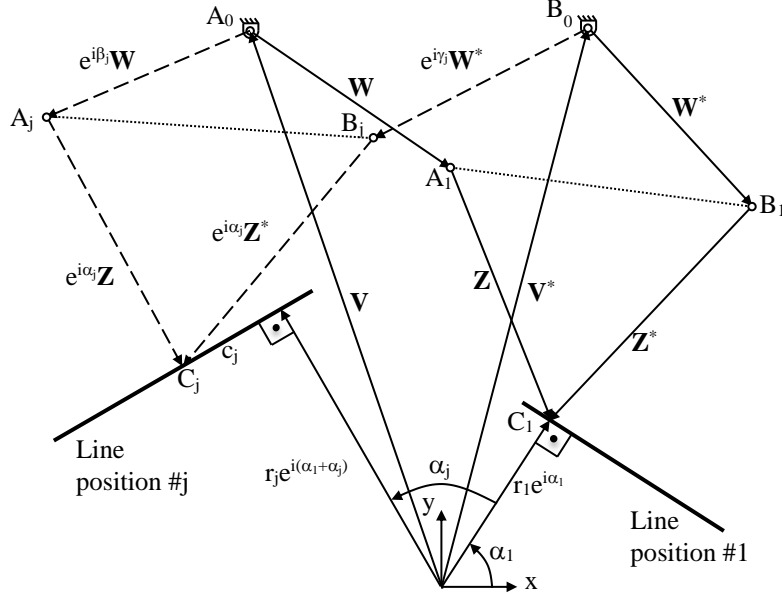


Fig. 1. Kinematic diagram for the synthesis

The homologous positions #1, #2, ..., #n of a line are represented by their distance to the origin r_1, r_2, \dots, r_n and angles $\alpha_1, \alpha_2, \dots, \alpha_n$ with respect to a selected reference frame as shown in Fig. 1. When a 4-bar linkage A_0ABB_0 is used for the task, dyads A_0AC and B_0BC can be designed separately. ABC represents the coupler link of the linkage. Point C is chosen to be a coupler point which is instantaneously coincident with the foot of the perpendicular drawn from the origin to the line at position #1. For dyad A_0AC at position #1 of the line:

$$\mathbf{V} + \mathbf{W} + \mathbf{Z} = r_1 e^{i\alpha_1} \quad (1)$$

For simplifying the solution, let the coordinate frame be set such that the first line is on the y -axis, i.e. $r_1 = 0$ and $\alpha_1 = 0$. Let link A_0A be rotated by an angle β_j from position #1 to position #j. Then for $j = 2, \dots, n$

$$\left. \begin{aligned} \mathbf{V} + \mathbf{W} + \mathbf{Z} &= \mathbf{0} \\ \mathbf{V} + e^{i\beta_j} \mathbf{W} + e^{i\alpha_j} \mathbf{Z} &= (r_j + ic_j) e^{i\alpha_j} \end{aligned} \right\} (e^{i\beta_j} - 1) \mathbf{W} + (e^{i\alpha_j} - 1) \mathbf{Z} = (r_j + ic_j) e^{i\alpha_j} \quad (2)$$

r_j and α_j are given. If β_j values are assumed, \mathbf{W} , \mathbf{Z} and c_j 's can be solved linearly. However, note that c_j 's will appear in diad B_0BC equations as well. Let rotations of link B_0B be γ_j from position #1 to position #j. Then

$$\begin{aligned} \mathbf{V}^* + \mathbf{W}^* + \mathbf{Z}^* &= 0 \\ \mathbf{V}^* + \mathbf{W}^* e^{i\gamma_j} + \mathbf{Z}^* e^{i\alpha_j} &= (\mathbf{r}_j + i\mathbf{c}_j) e^{i\alpha_j} \end{aligned} \left\{ \begin{aligned} (e^{i\gamma_j} - 1) \mathbf{W}^* + (e^{i\alpha_j} - 1) \mathbf{Z}^* &= (\mathbf{r}_j + i\mathbf{c}_j) e^{i\alpha_j} \end{aligned} \right. \quad (3)$$

If γ_j values are assumed, \mathbf{W}^* , \mathbf{Z}^* and \mathbf{c}_j 's can be solved linearly. Keeping all the unknowns, the number of scalar equations versus the number of unknowns for different number of finite positions are listed in Table 1.

Table 1. Number of scalar equations v.s. the number of unknowns

n	Number of equations	Unknowns	Number of free parameters:
2	4	$W_x, W_y, Z_x, Z_y, c_2, \beta_2, W_x^*, W_y^*, Z_x^*, Z_y^*, \gamma_2$	7
3	8	$W_x, W_y, Z_x, Z_y, c_2, c_3, \beta_2, \beta_3, W_x^*, W_y^*, Z_x^*, Z_y^*, \gamma_2, \gamma_3$	6
4	12	$W_x, W_y, Z_x, Z_y, c_2, c_3, c_4, \beta_2, \beta_3, \beta_4, W_x^*, W_y^*, Z_x^*, Z_y^*, \gamma_2, \gamma_3, \gamma_4$	5
5	16	$W_x, W_y, Z_x, Z_y, c_2, c_3, c_4, c_5, \beta_2, \beta_3, \beta_4, \beta_5, W_x^*, W_y^*, Z_x^*, Z_y^*, \gamma_2, \gamma_3, \gamma_4, \gamma_5$	4
6	20	$W_x, W_y, Z_x, Z_y, c_2, c_3, c_4, c_5, c_6, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, W_x^*, W_y^*, Z_x^*, Z_y^*, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$	3
7	24	$W_x, W_y, Z_x, Z_y, c_2, c_3, c_4, c_5, c_6, c_7, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, W_x^*, W_y^*, Z_x^*, Z_y^*, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$	2
8	28	$W_x, W_y, Z_x, Z_y, c_2, c_3, c_4, c_5, c_6, c_7, c_8, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, W_x^*, W_y^*, Z_x^*, Z_y^*, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8$	1
9	32	$W_x, W_y, Z_x, Z_y, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, W_x^*, W_y^*, Z_x^*, Z_y^*, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9$	0

If \mathbf{c}_j 's are freely selected, the problem becomes equivalent to motion generation problem. For 2 line positions, In addition to c_2, β_2, γ_2 , any two of W_x, W_y, Z_x, Z_y for dyad A_0AC and any two of $W_x^*, W_y^*, Z_x^*, Z_y^*$ for dyad B_0BC can be freely chosen. Ground pivot specification (specification of $-\mathbf{W} - \mathbf{Z}$ and $-\mathbf{W}^* - \mathbf{Z}^*$) is also possible. For 3 line positions once $c_2, c_3, \beta_2, \beta_3, \gamma_2, \gamma_3$ are assumed, $\mathbf{W}, \mathbf{Z}, \mathbf{W}^*$ and \mathbf{Z}^* can be solved linearly. The 2- and 3-line position problems can also be solved graphically.

For 4-line positions if c_2, c_3, c_4, β_2 , and γ_2 are assumed, solution of $\mathbf{W}, \mathbf{Z}, \mathbf{W}^*, \mathbf{Z}^*$, $\beta_3, \beta_4, \gamma_3, \gamma_4$ is the same as the 4-position motion generation problem. For dyad A_0AC

$$\begin{aligned} (e^{i\beta_2} - 1) \mathbf{W} + (e^{i\alpha_2} - 1) \mathbf{Z} &= (\mathbf{r}_2 + i\mathbf{c}_2) e^{i\alpha_2} \\ (e^{i\beta_3} - 1) \mathbf{W} + (e^{i\alpha_3} - 1) \mathbf{Z} &= (\mathbf{r}_3 + i\mathbf{c}_3) e^{i\alpha_3} \\ (e^{i\beta_4} - 1) \mathbf{W} + (e^{i\alpha_4} - 1) \mathbf{Z} &= (\mathbf{r}_4 + i\mathbf{c}_4) e^{i\alpha_4} \end{aligned} \Rightarrow \begin{bmatrix} e^{i\beta_2} - 1 & e^{i\alpha_2} - 1 \\ e^{i\beta_3} - 1 & e^{i\alpha_3} - 1 \\ e^{i\beta_4} - 1 & e^{i\alpha_4} - 1 \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} (\mathbf{r}_2 + i\mathbf{c}_2) e^{i\alpha_2} \\ (\mathbf{r}_3 + i\mathbf{c}_3) e^{i\alpha_3} \\ (\mathbf{r}_4 + i\mathbf{c}_4) e^{i\alpha_4} \end{bmatrix} \quad (4)$$

These equations have a solution provided that the augmented matrix is singular:

$$\begin{vmatrix} e^{i\beta_2} - 1 & e^{i\alpha_2} - 1 & (\mathbf{r}_2 + i\mathbf{c}_2) e^{i\alpha_2} \\ e^{i\beta_3} - 1 & e^{i\alpha_3} - 1 & (\mathbf{r}_3 + i\mathbf{c}_3) e^{i\alpha_3} \\ e^{i\beta_4} - 1 & e^{i\alpha_4} - 1 & (\mathbf{r}_4 + i\mathbf{c}_4) e^{i\alpha_4} \end{vmatrix} = \Delta_{34} (\mathbf{r}_2 + i\mathbf{c}_2) - \Delta_{24} (\mathbf{r}_3 + i\mathbf{c}_3) + \Delta_{23} (\mathbf{r}_4 + i\mathbf{c}_4) = 0 \quad (5)$$

where

$$\begin{aligned}\Delta_{34} &= e^{i\alpha_2} [(e^{i\beta_3} - 1)(e^{i\alpha_4} - 1) - (e^{i\beta_4} - 1)(e^{i\alpha_3} - 1)] \\ \Delta_{24} &= e^{i\alpha_3} [(e^{i\beta_2} - 1)(e^{i\alpha_4} - 1) - (e^{i\beta_4} - 1)(e^{i\alpha_2} - 1)] \\ \Delta_{23} &= e^{i\alpha_4} [(e^{i\beta_2} - 1)(e^{i\alpha_3} - 1) - (e^{i\beta_3} - 1)(e^{i\alpha_2} - 1)]\end{aligned}$$

The solution to this problem is well known. However, with a different set of freely chosen parameters, the problem is varied from the classical motion generation problem. Let $\beta_2, \beta_3, \beta_4, \gamma_2$ and either one of c_2, c_3 and c_4 be specified. Then, the remaining two can be linearly solved from Eq. (5). Then, \mathbf{W} and \mathbf{Z} are solved linearly from any of the three complex equations in Eq. (4). Once c_2, c_3 and c_4 are either specified or solved from the first dyad, for the second dyad, the solution of $\mathbf{W}^*, \mathbf{Z}^*, \gamma_3$ and γ_4 for given γ_2 is the same as solving the 4-position motion generation problem. There will be two sets of solutions.

For 5 line positions we can assume c_2, c_3, c_4, c_5 and solve for $W_x, W_y, Z_x, Z_y, \beta_2, \beta_3, \beta_4, \beta_5, \mathbf{W}_x^*, \mathbf{W}_y^*, \mathbf{Z}_x^*, \mathbf{Z}_y^*, \gamma_2, \gamma_3, \gamma_4$ and γ_5 in the same way of solving the 5-position motion generation problem. An alternative solution is to assume $\beta_2, \beta_3, \beta_4, \beta_5$ in the first dyad and linearly solve for c_2, c_3, c_4, c_5 from the following complex singular determinant equations:

$$\begin{vmatrix} e^{i\beta_2} - 1 & e^{i\alpha_2} - 1 & (r_2 + ic_2)e^{i\alpha_2} \\ e^{i\beta_3} - 1 & e^{i\alpha_3} - 1 & (r_3 + ic_3)e^{i\alpha_3} \\ e^{i\beta_4} - 1 & e^{i\alpha_4} - 1 & (r_4 + ic_4)e^{i\alpha_4} \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} e^{i\beta_2} - 1 & e^{i\alpha_2} - 1 & (r_2 + ic_2)e^{i\alpha_2} \\ e^{i\beta_3} - 1 & e^{i\alpha_3} - 1 & (r_3 + ic_3)e^{i\alpha_3} \\ e^{i\beta_5} - 1 & e^{i\alpha_5} - 1 & (r_4 + ic_4)e^{i\alpha_5} \end{vmatrix} = 0 \quad (6)$$

These singular determinants yield two complex equations which are linear in c_2, c_3, c_4, c_5 . Once c_2, c_3, c_4, c_5 are solved, the second dyad can be determined as in the 5-position motion generation problem.

For $n > 5$ line positions, there are $9-n$ free parameters to choose, $2(n+1)$ unknowns for each dyad, where $n-1$ many c_j 's are common for both dyads. Since there are $2(n-1)$ equations for each dyad, once $9-n < 4$ parameters are specified, the remaining parameters cannot be solved for a dyad without taking the other dyad into considerations. That is, the dyad equations become coupled and are highly nonlinear. An analytical solution is unlikely.

3 Examples

As an example, three and four positions of a line enveloping a parabola is worked out. Consider the parabola $y = x^2 + x/2$ and three tangent lines at points $(0, 0), (1, 1.5)$ and $(3, 10.5)$ (all units are mm). The corresponding distance r_j and angle α_j values for $j = 2, 3$ are listed in Table 2. An Excel simulation environment is created. After several trials $c_2, c_3, \beta_2, \beta_3, \gamma_2, \gamma_3$ values are chosen as listed in Table 2. Using Eqs. (2) and (3), $\mathbf{W}, \mathbf{Z}, \mathbf{W}^*, \mathbf{Z}^*$ vector coordinates and link lengths $a_1 = |\mathbf{W} + \mathbf{Z} - \mathbf{W}^* - \mathbf{Z}^*|$,

$a_2 = |\mathbf{W}|$, $a_3 = |\mathbf{Z} - \mathbf{Z}^*|$, $a_4 = |\mathbf{W}^*|$, $b_3 = |\mathbf{Z}|$ and $c_3 = |\mathbf{Z}^*|$ are determined and listed in Table 2. The resulting 4-bar linkage and the family of coupler lines are depicted in Fig. 2.

Table 2. Parameters for the 3-line positions example

r_2	r_3	α_2	α_3	c_2	c_3	β_2	β_3	γ_2	γ_3
0.371	1.369	41.64°	54.69°	0.6	1	70°	140°	30°	60°
\mathbf{W}	\mathbf{Z}	\mathbf{W}^*	\mathbf{Z}^*	a_1	a_2	a_3	a_4	b_3	c_3
-1.405 -i1.969	1.431 +i0.367	-0.878 -i0.321	1.768 -i1.112	1.186	0.935	1.516	2.419	2.088	1.477

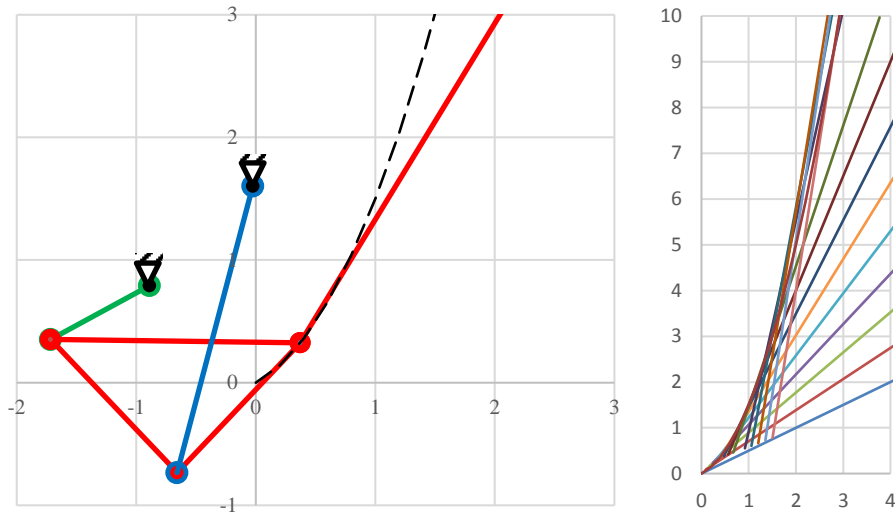


Fig. 2. The designed 4-bar linkage for 3-line positions and the family of coupler lines

Next, let's consider the same parabola and four tangent lines at points (0, 0), (1, 1.5), (3, 10.5) and (4, 18). In addition to the r_j and α_j values in Table 2, r_4 and α_4 are given in Table 3. This time, a Mathcad simulation environment is created. After several trials c_2 , β_2 , β_3 , β_4 , γ_2 values are chosen as listed in Table 2. Using Eqs. (4) and (5), c_3 , c_4 , γ_3 , γ_4 , \mathbf{W} , \mathbf{Z} , \mathbf{W}^* , \mathbf{Z}^* vector coordinates and link lengths $a_1 = |\mathbf{W} + \mathbf{Z} - \mathbf{W}^* - \mathbf{Z}^*|$, $a_2 = |\mathbf{W}|$, $a_3 = |\mathbf{Z} - \mathbf{Z}^*|$, $a_4 = |\mathbf{W}^*|$, $b_3 = |\mathbf{Z}|$, $c_3 = |\mathbf{Z}^*|$ are determined and listed in Table 3. The resulting 4-bar linkage is presented in Fig. 3.

Table 3. Specified, chosen and computed parameters for the 4-line positions example

r_4	α_4	β_2	β_3	β_4	c_2	γ_2	c_3	c_4	γ_3	γ_4
0.371	54.69°	30°	70°	100°	0.8	90°	0.444	0.279	142.03°	176.22°
\mathbf{W}	\mathbf{Z}	\mathbf{W}^*	\mathbf{Z}^*	a_1	a_2	a_3	a_4	b_3	c_3	
-0.910 -i0.500	1.096 -i1.583	0.331 +i0.790	0.518 -i2.852	1.190	1.043	1.394	0.858	1.926	2.899	

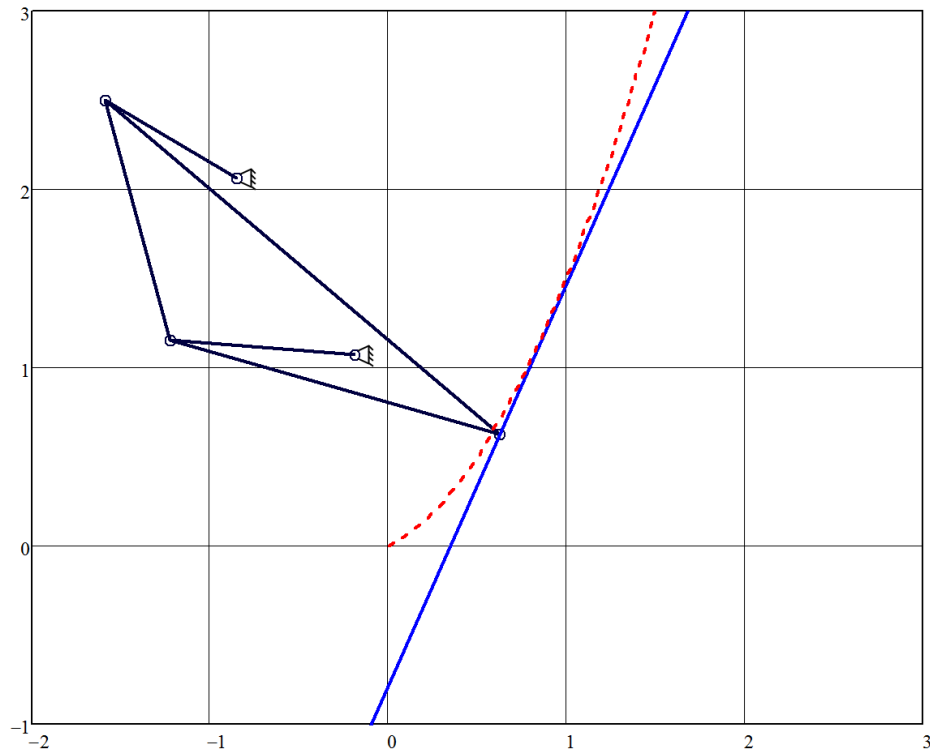


Fig. 3. The designed 4-bar linkage for 4-line positions

4 Conclusions

In this study the planar 4-bar line-path generator synthesis problem is formulated for up to 5 homologous positions. As in function and point-path generation problems, the line-path generation problem also can be mathematically cast into motion generation problem. Numerical examples for 3 and 4 line positions are provided.

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