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Traveling Wave Solutions for Nonlinear Differential-Difference Equations of Rational Types

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Abstract *Differential-difference equations are considered to be hybrid systems because the spatial variable n is discrete while the time t is usually kept continuous. Although a considerable amount of research has been carried out in the field of nonlinear differential-difference equations, the majority of the results deal with polynomial types. Limited research has been reported regarding such equations of rational type. In this paper we present an adaptation of the (G'/G)-expansion method to solve nonlinear rational differential-difference equations. The procedure is demonstrated using two distinct equations. Our approach allows one to construct three types of exact traveling wave solutions (hyperbolic, trigonometric, and rational) by means of the simplified form of the auxiliary equation method with reduced parameters. Our analysis leads to analytic solutions in terms of topological solitons and singular periodic functions as well.*

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Key words: differential-difference equations, (G'/G)-expansion method, exact solutions, traveling wave solutions

1 Introduction

Differential-difference equations (DDEs) or lattice equations, a subject among the most refined and interdisciplinary of mathematical and physical sciences, have received considerable attention in various applied areas such as condensed matter, wave phenomena in fluids, chemical reactions, particle vibrations in lattices, currents in electrical networks, pulses in biological chains, macro phenomena, and so on. DDEs can be considered as hybrid systems for being semi-discretized with some (or all) of their spatial variables discretized while time is usually kept continuous. Moreover, it has been found that the dynamical behavior of many complex systems can be properly described by DDEs. Indeed, since the original work of Fermi, Pasta, and Ulam,[1] integrable DDEs has been extensively developed in many fields which has seen an overwhelming growth in the last three or more decades. To make mention of some; the Ablowitz–Ladik lattice equation, the Toda lattice equation, the Volterra lattice equation, the discrete (modified) KdV equation, the discrete sine-Gordon equation, the Suris lattice and so forth (see Ref. [2]). These are just a few examples, but enough to exhibit the variety and complexity of nonlinear DDEs, and consequently, the challenge of their study. It is worth to mention here that those DDEs are usually in the form $\dot{u}_n = P(u_{n-1}, u_n, u_{n+1})$ where P is a polynomial function of its arguments and the dependent variable u_n is assumed to be a function $u(n,t)$ of a lattice variable $n \in \mathbb{Z}$. While there has been considerable work done on finding exact solutions to such DDEs, as far as we could verify, little work is being done to compute exact solutions of DDEs of rational type

$$\dot{u}_n = \frac{P(u_{n-1}, u_n, u_{n+1})}{Q(u_{n-1}, u_n, u_{n+1})}, \tag{1}$$

where P and Q are rational functions of their arguments, $n \in \mathbb{Z}$, $u_n(t) = u(n,t)$ is the displacement of the n-th particle from the equilibrium position. Therefore, nonlinear DDEs we discuss in this paper are in the rational form (1). In order to fulfill the existing gaps in this direction, it is logical to focus on the equations

$$\dot{u}_n = \frac{u_{n-1} - u_{n+1} + 2u_{n-1}u_n + 2u_n u_{n+1} - 2u_{n-1}u_{n+1} - 2u_n^2}{1 + u_{n-1} - u_{n+1}}, \tag{2}$$

$$\dot{u}_n = \frac{4(u_{n-1} - u_{n+1})\, u_n^2 \left(1 - u_n^2\right)}{(u_{n-1} + u_n)(u_n + u_{n+1})}, \tag{3}$$

which appeared in Refs. [3] and [4], respectively. The equation (3) is related to discrete modified KdV equation. Our primary goal, in this work, is to analyze Eqs. (2) and (3) for exact analytic solutions using one of the most powerful methods existing in the literature. Solitary solutions of DDEs have caught much attention due to the fact that discrete spacetime may be the most radical and logical viewpoint of reality.[5] Soliton was first discovered in 1834

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by Russell,[6] who observed that a canal boat stopping suddenly gave rise to a solitary wave which traveled down the canal for several miles, without breaking up or losing strength. Russell named this phenomenon the "soliton". A soliton is a special traveling wave that after a collision with another soliton eventually emerges unscathed.

On the other hand, nonlinear evolution equations (NEEs), i.e., partial differential equations with time t as one of the independent variables, arise not only from many fields of mathematics, but also from other branches of science such as material science, mechanics, and physics.[7] Consequently, the development of mathematical methods for solving NEEs, combined with new possibilities of computational simulation, have greatly broadened the analytic study of DDEs or lattice equations as well. As a result, quite a few new modifications of the existing methods for dealing with NEEs has been put forth to solve nonlinear DDEs. The most commonly used ones are Hirota's bilinear method,[8] Jacobi elliptic function expansion method,[9] Tau method,[10] extended Jacobian elliptic function algorithm,[11] Exp-function method,[12] Riccati equation expansion method,[13] ADM-Padé technique,[14] homotopy perturbation method,[15] discrete tanh method,[16] and so on.

Of course, the applicability of new analytic methods (for instance, the first integral method,[17] the bifurcation method,[18–20] the modified trigonometric function series method,[21–22] the Jacobi elliptic function expansion method[23–25]) to a wide range of various problems is an important issue in nonlinear science. However, in many cases, it becomes a challenging task to extend an analytic method developed primarily for NEEs to nonlinear DDEs because series obstacles arise in searching for iterative relations from indices n to $n \pm 1$. Thus, most of the methods are usually restricted and cannot be used for numerous realistic scenarios. Recently, the (G'/G)-expansion method[26] has been shown to solve a large class of NEEs effectively, easily, and accurately. Later, the method was first adopted to solve DDEs,[27] and now it has matured into a relatively fledged theory for different kinds of nonlinear problems such as Refs. [28–30]. However, its application to nonlinear rational DDEs is rare and primary.

The rest of this study is organized as follows. In Sec. 2, we first recall the basic (G'/G)-expansion method and then demonstrate its simplification. Section 3 is devoted to the description of the simplified (G'/G)-expansion method for finding traveling wave solutions of nonlinear DDEs. Subsequently, in Secs. 4 and 5, we illustrate the method in detail with the model equations (2) and (3) which give rise to hyperbolic, trigonometric and rational solutions. Finally, some concluding remarks are given in Sec. 6.

2 Simplification of Basic (G'/G)-Expansion Method

First, to reveal the core idea of the basic (G'/G)-expansion method proposed by Wang *et al.*,[26] we consider a nonlinear evolution equation for $u = u(x,t)$ given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0\,, \tag{4}$$

where is a polynomial in its arguments and the subscripts denoting partial derivatives. By means of the transformation $u(x,t) = U(\xi)$, $\xi = kx + wt$, Eq. (4) can be reduced to an ODE of the form

$$P(U, wU',\ kU',\ w^2U'',\ kwU'',\ k^2U'', \ldots) = 0\,, \tag{5}$$

where $U = U(\xi)$ and the primes denote ordinary derivatives with respect to ξ. Then, we predict the solution of Eq. (5) as a polynomial in (G'/G) like

$$U = a_m \left(\frac{G'}{G}\right)^m + \cdots\,, \tag{6}$$

where $G = G(\xi)$ is the solution of the auxiliary linear second order ordinary differential equation

$$G'' + \lambda G' + \mu G = 0\,, \tag{7}$$

where $G' = \mathrm{d}G/\mathrm{d}\xi$, $G'' = \mathrm{d}^2G/\mathrm{d}\xi^2$, $a_1, a_0, \ldots, a_n\ (\neq 0)$, λ and μ are constants to be determined later. The unwritten part in (6) is also a polynomial in (G'/G), but the degree of which is generally equal to or less than $m-1$. The positive integer m can be determined by applying the homogeneous balancing method to the highest order derivatives and nonlinear terms appearing in Eq. (5). Then substituting (6) and its derivatives (up to the desired orders), such as

$$U' = -ma_m\left(\frac{G'}{G}\right)^{m+1} + \cdots\,,$$
$$U'' = m(m+1)a_m\left(\frac{G'}{G}\right)^{m+2} + \cdots\,, \tag{8}$$

and so on, into Eq. (5) under the consideration of Eq. (7) yields a polynomial of (G'/G) and if we set its coefficients equal to zero we obtain a system of nonlinear algebraic equations for a_i, λ, μ, k, and w. Suppose that these constants can be determined by solving the resulting algebraic equations simultaneously. On the other hand, from the linear theory, the solutions of Eq. (7) have been well known to us:

$$G = \begin{cases} \left(C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)\right)\mathrm{e}^{-(\lambda/2)\xi}, & \lambda^2 - 4\mu > 0\,, \\ \left(C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\right)\mathrm{e}^{-(\lambda/2)\xi}, & \lambda^2 - 4\mu < 0\,, \\ (C_1 + C_2\xi)\,\mathrm{e}^{-(\lambda/2)\xi}, & \lambda^2 - 4\mu = 0\,. \end{cases} \tag{9}$$

As a result, we can determine traveling wave solutions of Eq. (4) provided they exist.

In fact, as pointed out in Ref. [31], the basic (G'/G)-expansion method can be simplified further by combining the parameters λ and μ in a special way. To this end, let the function $G(\xi)$ be a solution of the auxiliary equation (7). Then the function $\psi = \psi(\xi) = G(\xi)\exp(\lambda\xi/2)$ is a solution of the differential equation

$$\psi'' + \alpha\psi = 0\,, \quad \alpha = \frac{4\mu - \lambda^2}{4}\,, \tag{10}$$

and satisfies the relation

$$\frac{\psi'}{\psi} = \frac{G'}{G} + \frac{\lambda}{2}\,. \tag{11}$$

Thus, the finite expansion (6) takes the form

$$U = b_m\left(\frac{\psi'}{\psi}\right)^m + \cdots\,, \tag{12}$$

where the disappearing parameter λ is absorbed by the coefficients b_i. Moreover, Eq. (10) assumes the solutions

$$\psi = \psi(\xi) = \begin{cases} (\sqrt{-\alpha}\xi) + C_2\sinh(\sqrt{-\alpha}\xi), & \alpha < 0\,, \\ C_1\cos(\sqrt{\alpha}\xi) + C_2\sin(\sqrt{\alpha}\xi), & \alpha > 0\,, \\ C_1 + C_2\xi, \quad \alpha = 0\,. \end{cases} \tag{13}$$

Now, it is clear that this approach is tantamount to the basic (G'/G)-expansion method in the sense that the solutions (6) obtained via the auxiliary equation (7) coincide with the solutions (12) obtained via the auxiliary equation (10). However, Eq. (10) is simpler because it minimizes the number of the parameters at the outset comparing to Eq. (7). Hence, without loss of generality, the parameter λ can be set to zero in (7), which can be refered as the simplified (G'/G)-expansion method. As a result, we have the following fact:

Observation The parameter λ of the auxiliary equation (7) is redundant, namely, it can be set to zero without loss of generality.

3 Update of Simplified (G'/G)-Expansion Method for DDEs

Now, to illustrate the basic idea of the method, we consider a system of M polynomial DDEs in the form

$$P(\boldsymbol{u}_{\boldsymbol{n}+\boldsymbol{p}_1}(\boldsymbol{x}), \ldots, \boldsymbol{u}_{\boldsymbol{n}+\boldsymbol{p}_k}(\boldsymbol{x}), \ldots, \boldsymbol{u}'_{\boldsymbol{n}+\boldsymbol{p}_1}(\boldsymbol{x}), \ldots, \boldsymbol{u}'_{\boldsymbol{n}+\boldsymbol{p}_k}(\boldsymbol{x}), \ldots, \boldsymbol{u}^{(r)}_{\boldsymbol{n}+\boldsymbol{p}_1}(\boldsymbol{x}), \ldots, \boldsymbol{u}^{(r)}_{\boldsymbol{n}+\boldsymbol{p}_k}(\boldsymbol{x})) = 0\,, \tag{14}$$

where the dependent variable $\boldsymbol{u}_{\boldsymbol{n}}$ have M components $u_{i,\boldsymbol{n}}$ and so do its shifts; the continuous variable $\boldsymbol{x}$ has N components x_i; the discrete variable $\boldsymbol{n}$ has Q components n_j; the k shift vectors $\boldsymbol{p}_i \in \mathbb{Z}^Q$; and $\boldsymbol{u}^{(r)}(\boldsymbol{x})$ denotes the collection of mixed derivative terms of order r. To search for exact solutions of Eq. (14), we rely on the following transformation

$$\boldsymbol{u}_{\boldsymbol{n}+\boldsymbol{p}_s}(\boldsymbol{x}) = \boldsymbol{U}_{\boldsymbol{n}+\boldsymbol{p}_s}(\xi_{\boldsymbol{n}})\,, \quad \xi_{\boldsymbol{n}} = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \zeta \quad (s = 1, 2, \ldots, k)\,, \tag{15}$$

where the coefficients $c_1, c_2, \ldots, c_N, d_1, d_2, \ldots, d_Q$ and the phase ζ are all constants. Then, Eq. (14) turns out to be a system of DDEs in the form

$$P(\boldsymbol{U}_{\boldsymbol{n}+\boldsymbol{p}_1}(\xi_{\boldsymbol{n}}), \ldots, \boldsymbol{U}_{\boldsymbol{n}+\boldsymbol{p}_k}(\xi_{\boldsymbol{n}}), \ldots, \boldsymbol{U}'_{\boldsymbol{n}+\boldsymbol{p}_1}(\xi_{\boldsymbol{n}}), \ldots, \boldsymbol{U}'_{\boldsymbol{n}+\boldsymbol{p}_k}(\xi_{\boldsymbol{n}}), \ldots, \boldsymbol{U}^{(r)}_{\boldsymbol{n}+\boldsymbol{p}_1}(\xi_{\boldsymbol{n}}), \ldots, \boldsymbol{U}^{(r)}_{\boldsymbol{n}+\boldsymbol{p}_k}(\xi_{\boldsymbol{n}})) = 0\,. \tag{16}$$

To solve (16), we assume the solution(s) of (16) can be written as a finite power series in $G'(\xi_{\boldsymbol{n}})/G(\xi_{\boldsymbol{n}})$ like

$$\boldsymbol{U}_{\boldsymbol{n}}(\xi_{\boldsymbol{n}}) = \sum_{l=0}^{m} a_l\left(\frac{G'(\xi_{\boldsymbol{n}})}{G(\xi_{\boldsymbol{n}})}\right)^l, \quad a_m \neq 0\,, \tag{17}$$

where m is a positive integer, which is determined by balancing the highest order nonlinear term(s) and the highest-order derivative term(s) in Eq. (16), a_i's are constants to be specified through the solution process, and the function $G(\xi_{\boldsymbol{n}})$ is a solution of the second-order linear auxiliary equation

$$G''(\xi_{\boldsymbol{n}}) + \mu G(\xi_{\boldsymbol{n}}) = 0\,, \tag{18}$$

where μ is an arbitrary parameter and prime denotes derivative with respect to $\xi_{\boldsymbol{n}}$. Now, let $G(\xi_{\boldsymbol{n}})$ be a solution of Eq. (18). From the linear theory of ODEs, the general solution of Eq. (18) is well known to us. Thus, the following expressions hold true:

$$\frac{G'(\xi_{\boldsymbol{n}})}{G(\xi_{\boldsymbol{n}})} = \sqrt{-\mu}\left(\frac{C_1\sinh(\sqrt{-\mu}\xi_{\boldsymbol{n}}) + C_2\cosh(\sqrt{-\mu}\xi_{\boldsymbol{n}})}{C_1\cosh(\sqrt{-\mu}\xi_{\boldsymbol{n}}) + C_2\sinh(\sqrt{-\mu}\xi_{\boldsymbol{n}})}\right), \quad \mu < 0\,, \tag{19}$$

$$\frac{G'(\xi_{\boldsymbol{n}})}{G(\xi_{\boldsymbol{n}})} = \sqrt{\mu}\left(\frac{-C_1\sin(\sqrt{\mu}\xi_{\boldsymbol{n}}) + C_2\cos(\sqrt{\mu}\xi_{\boldsymbol{n}})}{C_1\cos(\sqrt{\mu}\xi_{\boldsymbol{n}}) + C_2\sin(\sqrt{\mu}\xi_{\boldsymbol{n}})}\right), \quad \mu > 0\,, \tag{20}$$

$$\frac{G'(\xi_{\boldsymbol{n}})}{G(\xi_{\boldsymbol{n}})} = \frac{C_1}{C_1 + C_2\xi_{\boldsymbol{n}}}\,, \quad \mu = 0\,, \tag{21}$$

where C_1 and C_2 are arbitrary constants. In addition, we have the following shift formula:

$$\frac{G'(\xi_{\boldsymbol{n}\pm\boldsymbol{p}_s})}{G(\xi_{\boldsymbol{n}\pm\boldsymbol{p}_s})} = \frac{\frac{G'(\xi_{\boldsymbol{n}})}{G(\xi_{\boldsymbol{n}})} \pm \varepsilon\frac{\sqrt{\delta-4\varepsilon\mu}}{2} f\left(\frac{\sqrt{\delta-4\varepsilon\mu}}{2}\varphi_s\right)}{1 \pm \frac{2}{\sqrt{\delta-4\varepsilon\mu}}\frac{G'(\xi_{\boldsymbol{n}})}{G(\xi_{\boldsymbol{n}})} f\left(\frac{\sqrt{\delta-4\varepsilon\mu}}{2}\varphi_s\right)}\,, \tag{22}$$

where $\varepsilon \in \{0, \pm 1\}$, $\delta \in \{0, 4\}$, $\xi_{\boldsymbol{n}+\boldsymbol{p}_s} = \xi_{\boldsymbol{n}} + \varphi_s$, $\varphi_s = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q$, p_{sj} is the j-th component of the shift vector $\boldsymbol{p}_s$, and

$$f\Big(\frac{\sqrt{\delta - 4\varepsilon\mu}}{2}\varphi_s\Big) = \begin{cases} \tanh(\sqrt{-\mu}\varphi_s)\,, & \text{if } \varepsilon = 1,\ \delta = 0,\ \mu < 0, \\ \tan(\sqrt{\mu}\varphi_s)\,, & \text{if } \varepsilon = -1,\ \delta = 0,\ \mu > 0, \\ \varphi_s\,, & \text{if } \varepsilon = 0,\ \delta = 4,\ \mu = 0. \end{cases} \tag{23}$$

Moreover, (22) leads to the uniform shift function

$$\boldsymbol{U}_{\boldsymbol{n}\pm\boldsymbol{p}_s}(\xi_{\boldsymbol{n}}) = \sum_{l=0}^{m} a_l \left(\frac{\frac{G'(\xi_{\boldsymbol{n}})}{G(\xi_{\boldsymbol{n}})} \pm \varepsilon \frac{\sqrt{\delta-4\varepsilon\mu}}{2} f\Big(\frac{\sqrt{\delta-4\varepsilon\mu}}{2}\varphi_s\Big)}{1 \pm \frac{2}{\sqrt{\delta-4\varepsilon\mu}} \frac{G'(\xi_{\boldsymbol{n}})}{G(\xi_{\boldsymbol{n}})} f\Big(\frac{\sqrt{\delta-4\varepsilon\mu}}{2}\varphi_s\Big)} \right)^l , \quad a_m \neq 0 . \tag{24}$$

Indeed, the following well-known identities

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

and the Lemma along with the identity $\xi_{\boldsymbol{n}\pm\boldsymbol{p}_s} = \xi_{\boldsymbol{n}} \pm \varphi_s$ provides (22). Then, substituting (22) into (17) gives (24).

On the other hand, by means of (17), we define the degree of $\boldsymbol{U}_{\boldsymbol{n}}(\xi_{\boldsymbol{n}})$ as $D[\boldsymbol{U}_{\boldsymbol{n}}(\xi_{\boldsymbol{n}})] = m$ which gives rise to the degree of other expressions such as $D[\boldsymbol{U}_{\boldsymbol{n}}^{(r)}(\xi_{\boldsymbol{n}})] = m + r$, $D[(\boldsymbol{U}_{\boldsymbol{n}}^{(r)}(\xi_{\boldsymbol{n}}))^\beta] = \beta(m + r)$, $D[(\boldsymbol{U}_{\boldsymbol{n}}(\xi_{\boldsymbol{n}}))^\alpha(\boldsymbol{U}_{\boldsymbol{n}}^{(r)}(\xi_{\boldsymbol{n}}))^\beta] = \alpha m+\beta\,(m + r)$. Then, balancing the highest-order derivative term and the highest order nonlinear term(s) in $\boldsymbol{U}_{\boldsymbol{n}}(\xi_{\boldsymbol{n}})$ as in the continuous case, the degree of (17) and (24) can be easily determined from (16). Of course, the leading terms of $\boldsymbol{U}_{\boldsymbol{n}+\boldsymbol{p}_s}(\boldsymbol{p}_s \neq \boldsymbol{0})$ will not have any effect on the balancing procedure because $\boldsymbol{U}_{\boldsymbol{n}+\boldsymbol{p}_s}$ can be regarded as being of degree zero in $G'(\xi_{\boldsymbol{n}})/G(\xi_{\boldsymbol{n}})$. Substituting (17) and (24) together with (18) into (16), equating the coefficients of $(G'(\xi_{\boldsymbol{n}})/G(\xi_{\boldsymbol{n}}))^l$ $(l = 0, 1, 2, \ldots)$ to zero, we obtain a system of nonlinear algebraic equations from which the unspecified constants a_i, d_i, c_j and μ can be explicitly found. Finally, substituting these results into (17) will lead to varies kind of traveling wave solutions for (14).

4 Application to Eq. (2)

It should be obvious that, by means of the wave transformation

$$u_n = U_n\left(\xi_n\right)\,, \quad \xi_n = dn + kt + \chi\,, \tag{25}$$

where d and k are real parameters to be specified, while χ denotes the phase shift, Eq. (2) reduces to the equation

$$\begin{aligned} &kU_n{}'\left(1 + U_{n-1} - U_{n+1}\right) \\ &- (U_{n-1} - U_{n+1} + 2U_{n-1}U_n + 2U_nU_{n+1} \\ &-2U_{n-1}U_{n+1} - 2U_n^2) = 0\,, \end{aligned} \tag{26}$$

where prime denotes ordinary derivative with respect to the new independent variable ξ_n. Then, according to the method described in Sec. 3, suppose that Eq. (26) assumes a solution in the form

$$U_n = a_0 + a_1\left(\frac{G'}{G}\right)\,, \quad a_1 \neq 0\,, \tag{27}$$

where $G = G\left(\xi_n\right)$ is a solution of Eq. (18), while a_0 and a_1 remain arbitrary to be determined through the solution procedure. Now, we discuss the solutions in three cases as follows.

4.1 *Solutions in Terms of Hyperbolic Functions*

In case $\mu < 0$, based on (24), we derive the shift formula

$$U_{n\pm1} = a_0+a_1\Big(\frac{G'/G \pm \sqrt{-\mu}\tanh\left(\sqrt{-\mu}d\right)}{1 \pm (1/\sqrt{-\mu})\,(G'/G)\tanh\left(\sqrt{-\mu}d\right)}\Big). \tag{28}$$

Then, substituting (27) and (28) along with (18) into Eq. (26), clearing the denominator, setting the coefficients of $(G'/G)^l$ $(l = 0, 2, 4)$ to zero, we obtain a system of nonlinear algebraic equations for a_0, a_1, d, k and μ. It is observed that a solution of the resulting system is

$$a_0 = a_0,\ a_1 = \frac{\tanh\left(2d\sqrt{-\mu}\right)}{2\sqrt{-\mu}},\ k = -\frac{\sinh\left(2d\sqrt{-\mu}\right)}{\sqrt{-\mu}}, \tag{29}$$

which provides a hyperbolic function solution to Eq. (2) in the form

$$\begin{aligned} u_n(t) &= a_0 + \frac{1}{2}\tanh(2d\sqrt{-\mu}) \\ &\times \left(\frac{C_1\cosh\left(\sqrt{-\mu}\xi_n\right) + C_2\,\sinh\left(\sqrt{-\mu}\xi_n\right)}{C_1\sinh\left(\sqrt{-\mu}\xi_n\right) + C_2\cosh\left(\sqrt{-\mu}\xi_n\right)}\right), \end{aligned} \tag{30}$$

where $\xi_n = dn - [\sinh\left(2d\sqrt{-\mu}\right)/\sqrt{-\mu}]t + \chi$, while a_0, d, χ, $\mu(< 0)$, C_1 and C_2 remain arbitrary.

4.2 *Solutions in Terms of Trigonometric Functions*

In case $\mu > 0$, based on (24), the following shift formula

$$U_{n\pm1} = a_0 + a_1\left(\frac{G'/G \mp \sqrt{\mu}\tan\left(\sqrt{\mu}d\right)}{1 \pm (1/\sqrt{\mu})\,(G'/G)\tan\left(\sqrt{\mu}d\right)}\right) \tag{31}$$

is derived. Then substituting (27) and (31) along with (18) into Eq. (26), clearing the denominator, setting the coefficients of $(G'/G)^l$ $(l = 0, 2, 4)$ to zero, one obtains a system of nonlinear algebraic equations for a_0, a_1, d, k and μ. A solution of the resulting system turns out to be

$$\begin{aligned} a_0 &= a_0\,, \quad a_1 = \frac{\tan\left(2d\sqrt{\mu}\right)}{2\sqrt{\mu}}\,, \\ k &= -\frac{\sin\left(2d\sqrt{\mu}\right)}{\sqrt{\mu}}\,, \end{aligned} \tag{32}$$

which gives a trigonometric function solution to Eq. (2) in the form

$$\begin{aligned} u_n\left(t\right) &= a_0 + \frac{1}{2}\tan\left(2d\sqrt{\mu}\right) \\ &\times \left(\frac{-C_1\sin\left(\sqrt{\mu}\xi_n\right) + C_2\cos\left(\sqrt{\mu}\xi_n\right)}{C_1\cos\left(\sqrt{\mu}\xi_n\right) + C_2\sin\left(\sqrt{\mu}\xi_n\right)}\right), \end{aligned} \tag{33}$$

where $\xi_n = dn - (\sin(2d\sqrt{\mu})/\sqrt{\mu})t + \chi$, while a_0, d, χ, $\mu(>0)$, C_1 and C_2 remain arbitrary.

4.3 *Rational Solutions*

In case $\mu = 0$, based on (24), the following shift formula

$$U_{n\pm1} = a_0 + a_1 \left(\frac{G'/G}{1 \pm (G'/G)\, d}\right) \tag{34}$$

is obtained in accordance with (24). Then substituting (27) and (34) along with (18) into Eq. (26), clearing the denominator, setting the coefficients of $(G'/G)^l$ $(l = 2, 4)$ to zero, results in a system of nonlinear algebraic equations for a_0, a_1, d, and k . A solution of the resulting system is

$$a_0 = a_0\,, \quad a_1 = d\,, \quad k = -2d\,, \tag{35}$$

which yields a rational function solution to Eq. (2) in the form

$$u_n(t) = a_0 + d\left(\frac{C_1}{C_1 + C_2(dn - 2dt + \chi)}\right), \tag{36}$$

where a_0, d, χ, C_1, and C_2 remain arbitrary.

Remark 1 If we set "$\mu = -1$ and $C_2 = 0$" or "$\mu = -1$ and $C_1 = 0$", respectively, the solution (30) becomes

$$u_n(t) = a_0 + \frac{1}{2}\tanh(2d)\tanh(dn - \sinh(2d)t + \chi)\,, \tag{37}$$

$$u_n(t) = a_0 + \frac{1}{2}\tanh(2d)\coth(dn - \sinh(2d)t + \chi)\,, \tag{38}$$

where a_0, d, and χ remain arbitrary. Observe that (37) is an antikink-type solitary wave solution while (38) is a singular traveling wave solution.

By the same token, setting "$\mu = 1$ and $C_2 = 0$" or "$\mu = 1$ and $C_1 = 0$" respectively, the solution (33) becomes singular periodic wave solutions in the form

$$u_n(t) = a_0 - \frac{1}{2}\tan(2d)\tan(dn - \sin(2d)t + \chi)\,, \tag{39}$$

$$u_n(t) = a_0 + \frac{1}{2}\tan(2d)\cot(dn - \sin(2d)t + \chi)\,, \tag{40}$$

where a_0, d, and χ remain arbitrary.

5 Application to Eq. (3)

Like before, the wave transformation

$$u_n = U_n(\xi_n)\,, \quad \xi_n = dn + kt + \chi\,, \tag{41}$$

where d and k are real parameters to be specified, while χ denotes the phase shift, converts Eq. (3) to the equation

$$kU_n{}'(U_{n-1} + U_n)(U_n + U_{n+1}) - 4(U_{n-1} - U_{n+1})U_n^2(1 - U_n^2) = 0\,, \tag{42}$$

where prime denotes ordinary derivative with respect to the new independent variable ξ_n. According to the procedure of Sec. 3, we look for special solutions of Eq. (42) in the form

$$U_n = a_0 + a_1\left(\frac{G'}{G}\right), \ a_1 \neq 0\,, \tag{43}$$

where $G = G(\xi_n)$ satisfies Eq. (18), while a_0 and a_1 are arbitrary constants to be specified. In the next three subsections, some details will be omitted for the sake of brevity because the procedure is similar to that of Sec. 4.

5.1 *Solutions in Terms of Hyperbolic Functions*

In case $\mu < 0$, substituting (43) and $U_{n\pm1}$ together with (18) into (42), clearing the denominator, setting the coefficients of to zero $(G'/G)^l$ $(l = 0, 1, \ldots, 6)$, we derive a system of nonlinear algebraic equations for a_0, a_1, d, k, and μ. From the obtained system, we get the solutions

$$a_0 = -\frac{1}{2}\tanh(d\sqrt{-\mu})\,, \quad a_1 = \pm\frac{\tanh(d\sqrt{-\mu})}{2\sqrt{-\mu}}\,, \quad k = -\frac{2\tanh(d\sqrt{-\mu})}{\sqrt{-\mu}}\,, \tag{44}$$

$$a_0 = \frac{1}{2}\tanh(d\sqrt{-\mu})\,, \quad a_1 = \pm\frac{\tanh(d\sqrt{-\mu})}{2\sqrt{-\mu}}\,, \quad k = -\frac{2\tanh(d\sqrt{-\mu})}{\sqrt{-\mu}}\,. \tag{45}$$

Here and henceforth, we order the signs in a vertical manner. Setting the parameter values (44) and (45) into the expression (43) in accordance with (19), some solutions to Eq. (3) written in terms of hyperbolic functions can be constructed like

$$u_n(t) = -\frac{1}{2}\tanh(d\sqrt{-\mu}) \pm \frac{1}{2}\tanh(d\sqrt{-\mu})\left(\frac{C_1\sinh(\sqrt{-\mu}\xi_{\rm n}) + C_2\cosh(\sqrt{-\mu}\xi_{\rm n})}{C_1\cosh(\sqrt{-\mu}\xi_{\rm n}) + C_2\sinh(\sqrt{-\mu}\xi_{\rm n})}\right), \tag{46}$$

$$u_n(t) = \frac{1}{2}\tanh(d\sqrt{-\mu}) \pm \frac{1}{2}\tanh(d\sqrt{-\mu})\left(\frac{C_1\sinh(\sqrt{-\mu}\xi_{\rm n}) + C_2\cosh(\sqrt{-\mu}\xi_{\rm n})}{C_1\cosh(\sqrt{-\mu}\xi_{\rm n}) + C_2\sinh(\sqrt{-\mu}\xi_{\rm n})}\right), \tag{47}$$

where $\xi_n = dn - (2\tanh(d\sqrt{-\mu})/\sqrt{-\mu}t) + \chi$, while d, χ, $\mu\,(<0)$, C_1 and C_2 remain arbitrary.

Remark 2 Of course, as in Sec. 4, one can assign special values to the arbitrary parameters involved in the expressions (46) and (47) leading to kink-type solitary wave and singular traveling wave solutions. However, we skip the procedure for brevity.

5.2 *Solutions in Terms of Trigonometric Functions*

In case $\mu > 0$, substituting (43) and $U_{n\pm1}$ together with (18) into (42), clearing the denominator, setting the coefficients of $(G'/G)^l$ $(l = 0, 1, \ldots, 6)$ to zero, we derive a system of nonlinear algebraic equations for a_0, a_1, d, k, and

μ. Solving the resulting system, we get the relations

$$a_0 = \mp\frac{1}{2}\mathrm{i}\tan(d\sqrt{\mu})\,,\quad a_1 = -\frac{\tan(d\sqrt{\mu})}{2\sqrt{\mu}}\,,\quad k = -\frac{2\tan(d\sqrt{\mu})}{\sqrt{\mu}}\,, \tag{48}$$

$$a_0 = \mp\frac{1}{2}\mathrm{i}\tan(d\sqrt{\mu})\,,\quad a_1 = \frac{\tan(d\sqrt{\mu})}{2\sqrt{\mu}}\,,\quad k = -\frac{2\tan(d\sqrt{\mu})}{\sqrt{\mu}}\,. \tag{49}$$

Setting the parameter values (48) and (49) into the expression (43) in accordance with (20), some solutions of Eq. (3) written in terms of trigonometric functions can be constructed such as

$$u_n(t) = \mp\frac{1}{2}\mathrm{i}\tan(d\sqrt{\mu}) - \frac{1}{2}\tan(d\sqrt{\mu})\left(\frac{-C_1\sin(\sqrt{\mu}\xi_n) + C_2\cos(\sqrt{\mu}\xi_n)}{C_1\cos(\sqrt{\mu}\xi_n) + C_2\sin(\sqrt{\mu}\xi_n)}\right)\,, \tag{50}$$

$$u_n(t) = \mp\frac{1}{2}\mathrm{i}\tan(d\sqrt{\mu}) + \frac{1}{2}\tan(d\sqrt{\mu})\left(\frac{-C_1\sin(\sqrt{\mu}\xi_n) + C_2\cos(\sqrt{\mu}\xi_n)}{C_1\cos(\sqrt{\mu}\xi_n) + C_2\sin(\sqrt{\mu}\xi_n)}\right)\,, \tag{51}$$

where $\xi_n = dn - (2\tan(d\sqrt{\mu})/\sqrt{\mu}t) + \chi$, while d, χ, μ (> 0), C_1, and C_2 remain arbitrary.

Remark 3 If we pay a special attention to the solutions (50) and (51), we recognize that they are complex-valued functions. Thus, it can be concluded that Eq. (4) assumes periodic wave solutions, which mathematically exist but physically unrealistic. Naturally, one wonders under what circumstances the complex-valued solutions (50) and (51) would have a practical application because of their non-physical nature.

5.3 *Rational Solutions*

In case $\mu = 0$, substituting (43) and $U_{n\pm1}$ together with (18) into (42), clearing the denominator, setting the coefficients of $(G'/G)^l$ $(l = 2, 3, \ldots, 6)$ to zero, we derive a system of nonlinear algebraic equations for a_0, a_1, d and k. The resulting system leads the solution

$$a_0 = 0\,,\quad a_1 = \pm\frac{d}{2}\,,\quad k = -2d\,. \tag{52}$$

Inserting the parameter values (52) into the expression (43) in accordance with (21), we arrive at a rational function solution of Eq. (3) such as

$$u_n(t) = \pm\frac{d}{2}\left(\frac{C_1}{C_1 + C_2(dn - 2dt + \chi)}\right)\,, \tag{53}$$

where d, χ, C_1, and C_2 remain arbitrary.

6 Conclusions

Complexity of nonlinear DDEs and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences. Recently the application of the basic (G'/G)-expansion method has had much attention and it still leaves much space for further development. In this study, we first simplified the method and then presented an updated version for solving nonlinear DDEs. It has been shown that the parameter is redundant in the basic (G'/G)-expansion method. Besides, we primarily focused on nonlinear DDEs of rational type because of their rare treatment by the well-known methods. Hence, the present work has made an attempt to fill the gap in the related literature. Our successfull analysis of the model equations lead to three kinds of traveling wave solutions; hyperbolic, trigonometric and rational including antikink-type solitary and singular solutions. The obtained results with free parameters may be important to explain some physical phenomena. Of course, the availability of computer algebra systems (such as *MATHEMATICA*) has made a valuable contribution to the current study. We are aware of the fact that not all rational DDEs can be treated with our approach presented here. Thus, as a future task, our plan will be the investigation of exact solutions of such equations via some recent powerful techniques (for example, the modified (G'/G)-expansion method,[32] the extended tanh-coth expansion method,[33] the so-called new method,[34] auxiliary ordinary differential equation method,[35] the extended (G'/G)-expansion method[35]), which have been proven to be useful in solving problems of applied mathematical and physical sciences.

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