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ABSTRACT

We study the eigenvalue problems for the buckling of a clamped plate. The previous upper bound on low eigenvalues due to Payne, Pólya, and Weinberger, and Hile and Yeh are reviewed. Using methods similar to those used in bounding ratios of eigenvalues of the membrance problem, bounds for ratios of eigenvalues are found for the buckling of a clamped plate.

1. INTRODUCTION

In the present paper, we consider the buckling eigenvalue problem

$$\left.\begin{aligned} &\Delta^2 u + \upsilon \Delta u = 0 \quad \text{in } D \\ &u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial D \end{aligned}\right\} \tag{1.1}$$

where D is bounded domain in $\mathbb{R}^n$ and Δ denotes the Laplace operator, ∂D denotes the boundary of D, n denotes the outward normal to ∂D, and υ denotes eigenvalues of (1.1) and u denotes the corresponding eigenfunction.

Payne, Pólya, and Weinberger [2] considered the problem (1.1) on a bounded domain in $\mathbb{R}^2$ and showed that for domains in the plane,

$$\upsilon_2 \leq 3\upsilon_1 . \tag{1.2}$$

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Hile and Yeh [1] improved on this and extended it to higher dimensions as

$$\upsilon_2 \leq \frac{n^2 + 8n + 20}{(n+2)^2} \upsilon_1 . \tag{1.3}$$

Thus, for $n = 2$, this gives the better bound

$$\upsilon_2 \leq 2.5\upsilon_1 . \tag{1.4}$$

Using methods similar to those used in bounding ratios of eigenvalues of the membrane problem, bounds for ratios of eigenvalues are found for the buckling of a clamped plate.

LEMMA 1. *The quantities* $I_l = \int_D \Psi_l \Delta u_{x_l}$ $(1 \leq l \leq n)$ *satisfy*:

(i) $\sum_{l=1}^{n} I_l = \frac{n+2}{2}$.

(ii) $\left(\sum_{l=1}^{n} I_l \right)^2 \leq \left(\sum_{l=1}^{n} \int_D |\nabla \Psi_l|^2 \right) \int_D (\Delta u)^2$.

(iii) $\int_D u^2 \geq \frac{1}{\upsilon_1}$.

For the vibrating clamped plate problem, Payne, Pólya, and Weinberger and Hile and Yeh use trial functions $\Psi_i = xu_i - \sum_{j=1}^{k} a_{ij} u_j$. With the appropriate choice of a_{ij}, the orthogonality conditions necessary to use the Rayleigh-Ritz quotient are met. They both use rotations later in their proofs to simplify the Rayleigh-Ritz inequality. For Theorem 1.2, we would like to use just $\Psi_i = x_i u_1$ as trial functions for υ_{i+1} for $i = 1, \ldots, n$. However, given an arbitrary choice of Cartesian coordinates x_i, we have no guarantee that the appropriate orthogonalities (i.e., $\langle \Psi_i, u_j \rangle = 0$ for all $j \leq i$, where $\langle \cdot, \cdot \rangle$ denotes the appropriate inner product) will hold. To remedy this situation, we argue that we can always find a suitable rotation of axes to a new system of Cartesian coordinates x_i' so that the desired orthogonalites are obtained (i.e., $\langle \widetilde{\Psi}_i, u_j \rangle = 0$ for all $j \leq i$ and $i = 1, \ldots, n$ where $\widetilde{\Psi}_i = x_i' u_1$). Thus the necessary orthogonality conditions will hold for the trial functions $\widetilde{\Psi}_i = x_i' u_1$, where the new Cartesian variables x_i' are obtained via a rotation from our original variables. In other words, there exists a real orthogonal matrix S such that $x_i' = \sum_{j=1}^{n} S_{ij}\, x_j$, for $i = 1, \dots, n$, and the following theorem holds.

THEOREM 1.1. *There exists a set of Cartesian coordinates* x_i' *such that the functions* $\widetilde{\Psi}_i = x_i' u_1$ *are suitable trial functions for* v_{i+1} *in the corresponding Rayleigh quotient. That is, we have*

$$\widetilde{\Psi}_i = \frac{\partial \widetilde{\Psi}_i}{\partial n} = 0 \quad \text{on } \partial D \tag{1.5}$$

$$<\widetilde{\Psi}_i, u_j> = 0, \quad \text{for all} \quad 1 \le j \le i \le n. \tag{1.6}$$

PROOF. We start from the arbitrary of Cartesian coordinates x_i and let $\Psi_i = x_i u_1$. We can assume that these obey $<x_i u_1, u_1> = 0$, for, if not, we can simply translate each x_i by $a_i = \frac{<x_i u_1, u_1>}{<u_1, u_1>}\cdot$ Assuming this has been done, we find that $\widetilde{\Psi}_i = \widetilde{x}_i u_1$ satisfies (1.5) for all i and satisfies (1.6) for $j = 1$ and all i, $1 \le i \le n$. To prove (1.6) for $j = 2, 3, \dots, i$, let C be an $n \times n$ matrix such that $C = [c_{ij}]_{1 \le i \le j \le n}$ where $c_{ij} = <\Psi_i, u_{j+1}> = <x_i u_1, u_{j+1}>$.

Then

$$C = \begin{bmatrix} <\Psi_i, u_2> & <\Psi_i, u_3> & \dots & <\Psi_i, u_{n+1}> \\ <\Psi_i, u_2> & <\Psi_i, u_3> & \dots & <\Psi_i, u_{n+1}> \\ \vdots & \vdots & & \vdots \\ <\Psi_i, u_2> & <\Psi_i, u_3> & \dots & <\Psi_i, u_{n+1}> \end{bmatrix} = [\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n]$$

where the $\vec{c}_j$'s are the columns of C. Now using the Gram-Schmidt procedure we can ortogonalize the $\vec{c}_j$'s in order, followed by the standard basis vectors $\vec{e}_i$ as needed, to get n independent, orthonormal column vectors $\vec{r}_j$. Let R be the matrix with columns $\vec{r}_j$, $1 \le j \le n$. Then we have $C = RT$ where R is an $n \times n$ real orthogonal matrix and T is an $n \times n$ upper triangular matrix. Therefore, $R^T C = T$ and each entry in the matrix T, denoted by T_{ij}, can be represented as follows:

$$T_{ij} = \sum_{k=1}^{n} (R^T)_{ik} c_{kj} = \sum_{k=1}^{n} (R^T)_{ik} <x_k u_1, u_{j+1}>$$

$$= \left\langle \left(\sum_{k=1}^{n} (R^T)_{ik} x_k u_1 \right), u_{j+1} \right\rangle = <x_k' u_1, u_{j+1}>.$$

Thus we identify S from our discussion leading up to this theorem as R^T. Since T is an upper triangular matrix, we have $<x_k' u_1, u_{j+1}> = 0$ for $1 \le j \le i$, $i = 2, \ldots, n$. So $<x_k' u_1, u_j> = 0$ for $2 \le j \le i$, and thus $\widetilde{\Psi}_i = x_i' u_1 \perp u_2, u_3, \ldots, u_i$ for $i = 2, 3, \ldots, n$. We note also that since $<x_k u_1, u_1> = 0$ for $k = 1, \ldots, n$, $<x_k' u_1, u_1> = \sum_{k=1}^{n} (R^T)_{ik} <x_k u_1, u_1> = 0$ for each $i = 1, \ldots, n$. We therefore have $<\widetilde{\Psi}_i, u_j> = <x_k' u_1, u_j> = 0$ for $1 \le j \le i$ and $i = 1, \ldots, n$, which shows that (1.6) is satisfied. □

Remark 1. Having established the existence of a suitable system of Cartesian co-ordinates, we revert to our usual notaion (x_i and Ψ_i) in denoting these objects everywhere aside from this theorem, that is, when using this orthogonality condition outside this theorem, by x_i and Ψ_i we shall mean x_k' and $\widetilde{\Psi}_i$, respectively, of Theorem 1.1. Also, this orthogonality argument works for the buckling of a clamped plate and vibrating clamped plate problems since the specific inner products of each problems are represented by a general inner product in this theorem.

THEOREM 1.2. (The Main Theorem) *Let* $v_1, v_2, \ldots$ *denote the successive eigenvalues of the eigenvalue problem below on a bounded domain* $D \subset \mathbb{R}^n$ $(n \ge 2)$.

$$\Delta^2 u + v\Delta u = 0 \quad in\ D \tag{1.7}$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad on\ \partial D. \tag{1.8}$$

Let $u = u_1$ *denote the eigenfunction corresponding to* v_1 *normalized so that* $\int_D |\nabla u|^2 = 1$. *Then*

$$\frac{v_2 + v_3 + \ldots + v_{n+1}}{v_1} < n + 4. \tag{1.9}$$

PROOF. The Rayleigh-Ritz inequality states that

$$v_{k+1} \le \frac{\int_D \Psi_k \Delta^2 \Psi_k}{\int_D |\nabla \Psi_k|^2} \tag{1.10}$$

where Ψ_k is any sufficient smooth nontrivial function Ψ_k, such that

$$\Psi_k = \frac{\partial \Psi_k}{\partial n} = 0 \quad \text{on } \partial D \tag{1.11}$$

and

$$\int_D \nabla \Psi_k \cdot \nabla u_j = 0, \quad j \le k, \quad k = 1, \ldots, n. \tag{1.12}$$

Let $\Psi_k = x_k u$. Then conditions (1.11) are met by virtue of (1.8), and conditions (1.12) are met by an appropriate translation and rotation of the coordinate axes, as detailed in the Theorem 1.1.

Calculating the numerator of (1.10), since $\int_D \Psi_k \Delta^2 \Psi_k = -\int_D \upsilon_1 \Psi_k \Delta \Psi_k = -\upsilon_1 \int_D \Psi_k \nabla^2 \Psi_k = \upsilon_1 D[\Psi_k]$, we have

$$\int_D \Psi_k \Delta^2 \Psi_k = \int_D \Psi_k (x_k \Delta^2 u + 4 \Delta u_{x_k})$$

$$= -\upsilon_1 \int_D \Psi_k x_k \Delta u + 4 \int_D \Psi_k \Delta u_{x_k}. \tag{1.13}$$

Using integration by parts and the fact that $D[\Psi_k] = \int_D x_k^2 \,|\nabla \Psi_k|^2$ ($D[\ ]$ is Dirichlet integral), we have

$$\int_D \Psi_k x_k \Delta u = \int_D |\nabla \Psi_k|^2 + \int_D u^2. \tag{1.14}$$

Subtitution of (1.13) and (1.14) into (1.10) yields

$$\upsilon_{k+1} \le \frac{\upsilon_1 \int_D |\nabla^2 \Psi_k| - \upsilon_1 \int_D u^2 + 4 \int_D \Psi_k \Delta u_{x_k}}{\int_D |\nabla \Psi_k|^2} \tag{1.15}$$

and hence

$$\upsilon_{k+1} - \upsilon_1 \le \frac{-\upsilon_1 \int_D u^2 + 4 \int_D \Psi_k \Delta u_{x_k}}{D[x_k u]}. \tag{1.16}$$

Using (iii) of Lemma 1.1, we have

$$\upsilon_{k+1} - \upsilon_1 \le \frac{4 \int_D x_k u \Delta u_{x_k} - 1}{D[x_k u]} \quad \text{for } k = 1, \ldots, n. \tag{1.17}$$

We need the following:

$$2\int_D x_k u\Delta u_{x_k} = 2\int_D \nabla\cdot(x_k u\Delta u_{x_k}) - 2\int_D \nabla(x_k u)\cdot\nabla u_{x_k}$$

$$= -2\int_D x_k\nabla u\cdot\nabla u_{x_k} - 2\int_D u u_{x_k x_k}$$

$$= \int_D |\nabla u|^2 + 2\int_D u_{x_k} \tag{1.18}$$

and

$$\left(\int_D x_k u\nabla u_{x_k}\right)^2 = (D[x_k u, u_{x_k}])^2 \le D[x_k u]D[u_{x_k}]$$

which implies

$$\frac{1}{D[x_k u]} \le \frac{D[u_{x_k}]}{\left(\int_D x_k u\Delta u_{x_k}\right)^2}. \tag{1.19}$$

Substitution of (1.19) into (1.17) yields

$$v_{k+1} - v_1 \le \frac{2\left(\int_D |\nabla u|^2 + 2\int_D u_{x_k}^2\right) - 1}{\left(\int_D x_k u\Delta u_{x_k}\right)^2} D[u_{x_k}]$$

$$= \frac{4\left(2\int_D |\nabla u|^2 + 4\int_D u_{x_k}^2 - 1\right)}{\left(\int_D |\nabla u|^2 + 2\int_D u_{x_k}^2\right)^2} D[u_{x_k}]$$

$$= \frac{4\left(4\int_D u_{x_k}^2 + 1\right)}{\left(1 + 2\int_D u_{x_k}^2\right)^2} D[u_{x_k}]. \tag{1.20}$$

Let $a_k = \int_D u_{x_k}^2$, then $\sum_{k=1}^{n} a_{k_k} = 1$ and $0 < a_k < 1$ for $k = 1, \ldots, n$. Since we have $\frac{4a_k + 1}{(1 + 2a_k)^2} < 1$, (1.20) implies

$$v_{k+1} - v_1 < 4D[u_{x_k}]. \tag{1.21}$$

Summing over k, we have

$$v_2 + v_3 + \ldots + v_{n+1} - nv_1 < 4 \int_D (\Delta u)^2 \tag{1.22}$$

and thus

$$\frac{v_2 + v_3 + \ldots + v_{n+1}}{v_1} < n + 4,$$

which is (1.9). □

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