THE INITIAL STAGES OF GRAVITY DRIVEN FLOWS

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ABSTRACT

THE INITIAL STAGES OF GRAVITY DRIVEN FLOWS

During the initial stage of dam breaking; the liquid flow and the free surface shape are investigated. We used small-time approximation for this investigation and derived the leading order solution of classical dam-break problem. But this solution is not valid in a small vicinity of the corner point (the intersection point between the initially vertical free surface and the horizontal rigid bottom).

The dimension of this vicinity is estimated with the help of a local analysis of the this outer solution close to the corner point. Streched local coordinates are used in this vicinity to resolve the flow singularity and to derive the leading order inner solution (which describes the formation of the jet flow along the bottom) and the correction to the leading order.

This asymptotic solution obtained is expected to be helpful in the analysis of developed gravity driven flows.

ÖZET

YERÇEKİMİ ETKİSİNDE HAREKET EDEN AKIŞLARIN İLK ANLARI

Baraj yıkılmasının ilk anlarındaki sıvı akışı ve serbest su yüzeyi şekli araştırıldı. Bu araştırma için küçük-zaman yaklaşımını kullanıldık ve klasik baraj-yıkılması probleminin birinci mertebeden çözümünü elde ettik. Ama bu çözüm köşe noktasının (başlangıçta dikey olan serbest su yüzeyi ve yatay katı zeminin kesişim noktası) küçük bir komşuluğunda geçerli değildi.

Bu komşuluğun boyutları, bu dış çözümün köşe noktasının yakınlarında bölgesel analizinin yardımı ile hesaplandı. Bu komşulukta genişletilmiş bölgesel koordinatlar kullanılarak akışın tekilliği çözüldü ve birinci mertebeden iç çözüm (zemindeki püsküren akışın oluşumunu tanımlayan) ve birinci mertebe çözüme yapılan ilave elde edildi.

Bu asimptotik çözümün elde edilmesinin yer çekimi etkisi altındaki akışların analizinde yardımcı olması bekleniyor.

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CHAPTER 1

INTRODUCTION

In this thesis, we consider the unsteady problem of gravity-driven flow, which is generated when a vertical dam in front of a liquid region is suddenly removed. Initially the liquid is at rest and lay on the region $x' > 0$, $-H < y' < 0$ (see Figure 1.1). Throughout a prime stands for dimensional variables and H is the liquid depth. The upper part of the liquid boundary, $x' > 0$, $y' = 0$, is the initial position of the liquid free surface. The side wall, $x' = 0$, $-H < y' < 0$, represents a dam. The lower horizontal boundary, $y' = -H$ represents the rigid bottom. Initially the pressure distribution in the liquid is hydrostatic, $p'(x', y', 0) = -\rho_0 g y'$, where ρ_0 is the liquid density and g is the gravitational acceleration and at the initial time instant, $t' = 0$, the dam is instantly removed and the gravity-driven flow starts. The liquid is assumed incompressible and inviscid.

Here we have two free-surfaces of the flow region, which vary in time and have to be determined as part of solution. We denote the upper part of the free-surface as $y' = \eta'(x', t')$, $x' > 0$ and for the other part of the free-surface we use the notation $x' = \xi'(y', t')$, which is initially vertical. The free-surface $y' = \eta'$ is a function of x' and t' since a liquid particle on it changes its position with time. By the same reason, $x' = \xi'$ is a function of y' and t'. The flow region is bounded by these free-surfaces and by the rigid bottom $y' = -H$.

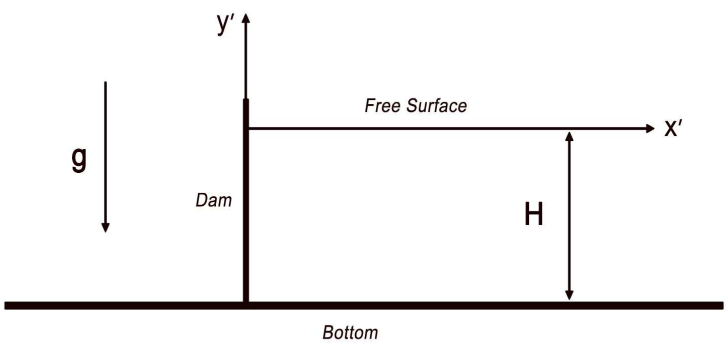

Figure 1.1. Flow region at the initial time instant $t' = 0$

In this dam-break problem, we aim to construct a uniformly valid small-time solution which holds for arbitrary parameter values by using matched asymptotic expansions. The solution in time as power series should be considered 'outer' solutions, which are needed to be corrected with 'inner' solutions near the intersection point. The outer and inner asymptotic solutions have to be matched in such a way to get a solution which is uniformly valid in the whole flow domain.

Gravity-driven flows due to dam breaking were studied by Pohle (1950) and Stoker (1957) using the Lagrangian description. Pohle (1950) wrote "Many hydrodynamic problems consider flows in which the region occupied by the fluid is a variable function of time. The Euler representation is difficult to apply to such problems. The Lagrangian representation, however, has the far-reaching advantage that the independent space variables are the initial coordinates of the particles: the region occupied by the fluid is therefore a fixed region." This statement is true if there are no intersection points between rigid boundaries and free surfaces of the liquid. However, this is not the case for many important problems of hydrodynamics including the dam-break problem, the water-entry problem and problems of floating or free-surface piercing bodies. This is due to the fact that close to the intersection points one needs to specify which liquid particles originally belonging to the free surface may be found later on the rigid boundary. In other words, it is true that the flow region is fixed in the Lagrangian variables but the subdivision of the region boundary into "free surface" and "rigid boundary" is unknown and has to be obtained as part of the solution. This makes the analysis of the dam-break problem in the Lagrangian variables not so attractive as it was expected by Pohle (1950) and Stoker (1957).

In the paper, written by Pohle (1950), Pohle expanded the liquid displacement and the hydrodynamic pressure in power series in the time t' and analyzed only the leading-order terms. It is written in this paper (Pohle 1957) that the calculated profile of the water surface for small times can be expected to be a reasonable approximation to the physical problem except in the neighborhood of the singular point (which is the point of intersection with the rigid bottom). Thus Pohle (1957) derived a small-time solution close to the bottom in Lagrangian variables which exhibits a non-physical shape of water surface. A similar behaviour of the free surface close to the intersection point can also be calculated in Eulerian variables. In both cases this solution will be the outer solution. We are unaware of attempts to construct the inner solutions of dam-break problem either in Lagrangian or Eulerian variables.

However, such an inner solution was successfully derived in a relevant problem concerning a uniformly accelerating wavemaker by King & Needham (1994). In this thesis, we use the methodology and findings from this paper and apply them to our dam-break problem.

In Chapter 2, we give some basic concepts in fluid dynamics which will be used in the formulation of the problem and the steps of the solution.

In Chapter 3, we formulate the dam-breaking problem, derive the boundary and initial conditions and obtain dimensional non-linear boundary value problem. Then we non-dimensionalized this problem and obtained dimensionless non-linear boundary value problem.

In Chapter 4, we investigated the small-time behaviour of this dam-breaking problem. By this small-time behaviour, we obtained the leading-order linear boundary value problem and the leading order solution which has a singularity close the intersection point (corner point). Then we analyzed the behaviour of the leading order solution near the singular point and obtained the outer matching conditions.

In Chapter 5, we specified the dimensions of the inner region and wrote the problem in inner variables. Then we considered the small-time behaviour of the inner region problem and obtained the leading order and second order inner region problem. By this analysis, we obtained the exact solution to the leading-order problem and the solution of the second order inner region problem.

CHAPTER 2

BASIC CONCEPTS IN FLUID DYNAMICS

2.1. Equations of Motion

2.1.1. Euler's Equations

Let D be a region in two or three dimensional space filled with a fluid and our object is to describe the motion of such a fluid. Let $\mathbf{x}'$ is a point in D and consider the particle of fluid moving through $\mathbf{x}'$ at time t'. Relative to standard Euclidean coordinates in space, we write $\mathbf{x}' = (x', y', z')$. Imagine a particle in the fluid; this particle traverses a well-defined trajectory. Let $\mathbf{u}'((x)', t')$ denote the velocity of the particle of fluid that is moving through $\mathbf{x}'$ at time t'. Thus, for each fixed time, $\mathbf{u}'$ is a vector field on D and we call $\mathbf{u}'$ *velocity field of the fluid*. For each time t', assume that the fluid has a well-defined mass density $\rho = \rho(\mathbf{x}', t')$. Thus, if *W* is any subregion of the flow region (D), the mass of fluid in *W* at time t' is given by

$$m(W, t') = \int_W \rho(\mathbf{x}', t')\, dV \ , \tag{2.1}$$

where *dV* is the volume element in the plane or in space.

We shall assume that the functions $\mathbf{u}'$ and ρ are smooth enough so that the standard operations of calculus may be performed on them and the assumption that ρ exists is a *continuum assumption*.

Our derivation of the equations is based on three basic principles:

- mass is neither created nor destroyed (conservation of mass);
- the rate of change of momentum of a portion of the fluid equals the force applied to it (balance of momentum);
- energy is neither created nor destroyed (conservation of energy).

***i*) Conservation of Mass:**

Let *W* be a fixed subregion of D (*W* does not change with time). The rate of change of mass in *W* is

$$\frac{d}{dt'}m(W,t') = \frac{d}{dt'}\int_W \rho(\mathbf{x}',t')dV = \int_W \frac{\partial\rho}{\partial t'}(\mathbf{x}',t')dV \ . \tag{2.2}$$

Let ∂W denote the boundary of *W*, assumed to be smooth; let $\mathbf{n}'$ denote the unit outward normal defined at points of ∂W, $\mathbf{u}'$ denote the velocity vector s.t. $\mathbf{u}' = (u', v')$ and *dA* denote the area element on ∂W. The volume flow rate across ∂W per unit area is $\mathbf{u}' \cdot \mathbf{n}'$ and the mass flow rate per unit area is $\rho\mathbf{u}' \cdot \mathbf{n}'$.

The principle of conservation of mass can be more precisely stated as follows: The rate of increase of mass in *W* equals the rate at which mass is crossing ∂W in the inward direction; *i.e.*,

$$\int_W \frac{\partial\rho}{\partial t'}dV = -\int_{\partial W} \rho\mathbf{u}' \cdot \mathbf{n}'dA \ . \tag{2.3}$$

This is the *integral form of the law of conservation of mass*. By the divergence theorem, this statement is equivalent to

$$\int_W \left[\frac{\partial\rho}{\partial t'} + div(\rho\mathbf{u}')\right]dV \quad = \quad 0 \ . \tag{2.4}$$

Because this is to hold for all *W*, it is equivalent to

$$\frac{\partial\rho}{\partial t'} + div(\rho\mathbf{u}') = 0 \tag{2.5}$$

The last equation is the *differential form of the law of conservation of mass*, also known as the *continuity equation*.

***ii*) Balance of Momentum:**

If *W* is a region in the fluid at a particular instant of time t', the total force exerted on the fluid inside *W* by means of stress on its boundary is

$$\mathbf{S}_{\partial W} = force\ on\ W = -\int_{\partial W} p'\mathbf{n}' dA \tag{2.6}$$

where p' is the pressure. Then, the divergence theorem gives

$$\mathbf{S}_{\partial W} = -\int_{W} \nabla' p' dV. \tag{2.7}$$

If $\mathbf{b}(\mathbf{x}', t')$ denotes the given body force *per unit mass*, the total body force is

$$\mathbf{B} = \int_{W} \rho \mathbf{b} dV. \tag{2.8}$$

Thus, on any piece of fluid material,

$$\text{force per unit volume} = -\nabla' p' + \rho \mathbf{b} \tag{2.9}$$

By Newton's second law $(force = mass \times acceleration)$ we are led to the differential form of the law of *balance of momentum*:

$$\rho \frac{D\mathbf{u}'}{Dt'} \quad = \quad -\nabla' p' + \rho \mathbf{b}\ . \tag{2.10}$$

where $\frac{D}{Dt'} = \frac{\partial}{\partial t'} + \mathbf{u}' \cdot \nabla$ is *material derivative*; the rate of change of *'following the fluid'*.

***iii*) Conservation of Energy:**

For fluid moving in a domain D, with velocity field $\mathbf{u}'$, the kinetic energy contained in a region $W \subset D$ is

$$E_{kinetic} = \frac{1}{2} \int_{W} \rho \parallel \mathbf{u}' \parallel^{2} dV. \tag{2.11}$$

We assume that total energy of the fluid can be written as

$$E_{total} = E_{kinetic} + E_{internal}, \tag{2.12}$$

$$\frac{1}{2}\frac{D}{Dt'} \parallel \mathbf{u}' \parallel^2 = \mathbf{u}'\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}'(\mathbf{u}' \cdot \nabla)\mathbf{u}'. \tag{2.13}$$

2.1.2. Incompressibility

Consider a fixed closed surface *S* drawn in the fluid, with unit outward normal $\mathbf{n}'$. Fluid will be entering the enclosed region *V* at some places on *S*, and leaving it at others. The velocity component along the outward normal is $\mathbf{u}' \cdot \mathbf{n}'$, so the volume of fluid leaving through a small surface element δS in unit time is $\mathbf{u}' \cdot \mathbf{n}'\delta S$. The net volume rate at which fluid is leaving *V* is therefore

$$\int_S \mathbf{u}' \cdot \mathbf{n}' dS \ .$$

But this must be zero for an incompressible fluid, and on using the divergence theorem we find that

$$\int_V \nabla' \cdot \mathbf{u}' dV \ = 0.$$

Since this must be true for all regions *V* in the fluid, we conclude that

$$\nabla' \cdot \mathbf{u}' = 0 \tag{2.14}$$

everywhere in the fluid, which is known as *incompressibility condition*.

2.1.3. Euler's Equations for Incompressible Flows

The equations

$$\rho \frac{D\mathbf{u}'}{Dt'} = -\nabla' p' + \rho \mathbf{g} \,, \tag{2.15}$$

$$\nabla' \cdot \mathbf{u}' = 0 \tag{2.16}$$

are known as *Euler's equation of motion for an incompressible fluid*. The gravitational force, being conservative, can be written as the gradient of a potential s.t. $\mathbf{g} = -\nabla' \chi$. Using the definition of material derivative, we may rewrite equation (2.15) in the form

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}' \cdot \nabla')\mathbf{u}' = -\nabla' \big(\frac{p'}{\rho} + \chi\big), \tag{2.17}$$

where we assume that ρ is constant. Furthermore by using the identity

$$(\mathbf{u}' \cdot \nabla')\mathbf{u}' = (\nabla' \wedge \mathbf{u}') \wedge \mathbf{u}' + \nabla(\frac{1}{2}\mathbf{u'^2}) \tag{2.18}$$

the momentum equation takes the form

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\nabla' \wedge \mathbf{u}') \wedge \mathbf{u}' = -\nabla' \big(\frac{p'}{\rho} + \frac{1}{2}\mathbf{u'^2} + \chi\big). \tag{2.19}$$

2.2. Vorticity: Irrotational Flow

2.2.1. Vorticity

Definition 2.1 *The vorticity ω is defined as*

$$\omega = \nabla' \wedge \mathbf{u}'$$

and it is a concept of central importance in fluid dynamics. The vorticity is, by definition, zero for an irrotational flow.
We consider vorticity first in the context of two-dimensional flow, for if

$$\mathbf{u}' = [u'(x', y', t'), v'(x', y', t'), 0]$$

then ω *is* $(0, 0, \omega)$*, where*

$$\omega = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}. \tag{2.20}$$

2.2.2. Potential Flow

Definition 2.2 ***(Velocity Potential)***
The velocity potential ϕ' *is something that exists only if* $\nabla' \wedge \mathbf{u}' = 0$ *and it is defined at any point P by*

$$\phi' = \int_O^P \mathbf{u}' \cdot d\mathbf{x}' \tag{2.21}$$

where O is some arbitrary fixed point. In a simply connected fluid region ϕ' *is independent of the path between O and P, thus a single-valued function of position. Partial differentiation of the equation (2.21) gives*

$$\mathbf{u}' = \nabla' \phi', \tag{2.22}$$

and the vector identity $\nabla' \wedge \nabla' \phi' = 0$ *confirms that this flow is irrotational, as desired.*

Such an inviscid, irrotational flow is called a *potential flow*.

2.2.3. Bernoulli's Equation for Unsteady Irrotational Flow

If the flow is irrotational, so that $\mathbf{u}' = \nabla'\phi'$, the second term of the Euler's equation (2.19) vanishes and we are left with

$$\frac{\partial}{\partial t'}\nabla'\phi' = -\nabla'\big(\frac{p'}{\rho} + \frac{1}{2}\mathbf{u'^2} + \chi\big) \tag{2.23}$$

where $\chi = gy'$ in the present context. Integration gives

$$\frac{\partial \phi'}{\partial t'} + \frac{p'}{\rho} + \frac{1}{2}\mathbf{u'^2} + \chi = G(t), \tag{2.24}$$

where G(t) is an arbitrary function of time alone and equation (2.24) is called *Bernoulli's equation for unsteady irrotational flow*.

CHAPTER 3

DAM-BREAKING PROBLEM

3.1. Nonlinear Problem

3.1.1. Equations of Motion (2D)

For the formulation of the problem, we use Euler's equations of motion for incompressible flows (2.15) and (2.16). Since the problem we consider is two-dimensional, the equation (2.15) becomes

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\frac{1}{\rho} \nabla' p' + \vec{g}$$

in vector notation and

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} = -\frac{1}{\rho} \frac{\partial p'}{\partial x'} \ , \tag{3.1}$$

$$\frac{\partial v'}{\partial t'} + u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} = -\frac{1}{\rho} \frac{\partial p'}{\partial y'} - g \tag{3.2}$$

in component form. The equation of continuity (2.16) becomes

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0. \tag{3.3}$$

Hence we have three main equations; (3.3), (3.1), (3.2) for the formulation of the problem

$$\begin{aligned} \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} &= 0 \ , \\ \frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} &= -\frac{1}{\rho} \frac{\partial p'}{\partial x'} \ , \\ \frac{\partial v'}{\partial t'} + u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} &= -\frac{1}{\rho} \frac{\partial p'}{\partial y'} - g \ . \end{aligned}$$

3.1.2. Boundary and Initial Conditions

At the initial time instant the liquid is at rest. Thus the velocity components and the free surfaces at the initial time $(t' = 0)$ are zero and pressure distribution is hydrostatic. Then we have three initial conditions for this problem such that;

$$\left.\begin{array}{c} u'(x',y',0) = v'(x',y',0) = 0, \quad \eta'(x',0) = \xi'(y',0) = 0 \\ \text{and} \quad p'(x',y',0) = -\rho g y' \ . \end{array}\right\} \tag{3.4}$$

Furthermore we have four free-surface conditions (two for each surface), two of them are kinematic boundary conditions and the others are dynamic boundary conditions. For the kinematic boundary condition at the free-surfaces we know that fluid particles on the surface must remain on the surface. Thus if we define the quantities $F(x',y',t') = y' - \eta'(x',t')$ and $G(x',y',t') = x' - \xi'(y',t')$ we may then claim that $F(x',y',t')$ and $G(x',y',t')$ remain constant (in fact, zero) for any particular fluid particle on the free surface. It follows that $\dfrac{DF}{Dt'} = 0$ and $\dfrac{DG}{Dt'} = 0$ on the free-surfaces $y' = \eta'(x',t')$ and $x' = \xi'(y',t')$ respectively, *i.e.*

$$\begin{aligned} \frac{\partial F}{\partial t'} + (\mathbf{u}' \cdot \nabla')F &= 0 \qquad \text{on } y' = \eta'(x',t') \ , \\ \frac{\partial G}{\partial t'} + (\mathbf{u}' \cdot \nabla')G &= 0 \qquad \text{on } x' = \xi'(y',t') \end{aligned}$$

which are equivalent to

$$\frac{\partial y'}{\partial t'} = \frac{\partial \eta'}{\partial t'} + \frac{\partial \eta'}{\partial x'}\frac{\partial x'}{\partial t'} \qquad \text{on } y' = \eta'(x',t') \ , \tag{3.5}$$

$$\frac{\partial x'}{\partial t'} = \frac{\partial \xi'}{\partial t'} + \frac{\partial \xi'}{\partial y'}\frac{\partial y'}{\partial t'} \qquad \text{on } x' = \xi'(y',t') \ . \tag{3.6}$$

The dynamic condition is that, the pressure on the free-surfaces is atmospheric. These conditions can be written as

1) on $y' = \eta'(x', t')$

$$v' = \frac{\partial \eta'}{\partial t'} + u' \frac{\partial \eta'}{\partial x'} \Bigg\} \qquad \text{kinematic condition ,} \tag{3.7}$$

$$p'(x', \eta', t') = 0 \quad \Big\} \qquad \text{dynamic condition .} \tag{3.8}$$

2) on $x' = \xi'(y', t')$

$$u' = v' \frac{\partial \xi'}{\partial y'} + \frac{\partial \xi'}{\partial t'} \Bigg\} \qquad \text{kinematic condition ,} \tag{3.9}$$

$$p'(\xi', y', t') = 0 \quad \Big\} \qquad \text{dynamic condition .} \tag{3.10}$$

Hence we get the mathematical statement of the problem (dimensional, non-linear) such that

$$u'_{x'} + v'_{y'} = 0 \ ,$$

$$u'_{t'} + u'u'_{x'} + v'u'_{y'} = -\frac{1}{\rho} p'_{x'} \ ,$$

$$v'_{t'} + u'v'_{x'} + v'v'_{y'} = -\frac{1}{\rho} p'_{y'} - g \ ,$$

$$v' = \eta'_{t'} + u'\eta'_{x'} \ , \quad p' = 0 \qquad \text{on } y' = \eta'(x', t'),$$

$$u' = \xi'_{t'} + v'\xi'_{y'} \ , \quad p' = 0 \qquad \text{on } x' = \xi'(y', t'),$$

$$v'(x', -H, t') = 0,$$

$$\eta'(x', 0) = \xi'(y', 0) = u'(x', y', 0) = v'(x', y', 0) = 0,$$

$$\text{as } x' \to \infty, \ u', v' \to 0 \text{ and } p' \to -\rho g y'.$$

3.2. Non-Dimensionalizing the Equations

In asymptotical analysis, dimensionless equations are used. Thus we use the following transformations to non-dimensionalize the equations (3.1) - (3.10),

$$x' = Hx \quad , \quad \eta' = H\eta \, , \tag{3.11}$$

$$y' = Hy \quad , \quad \xi' = H\xi \, , \tag{3.12}$$

$$t' = Tt \quad , \quad p' = p\rho g H \, . \tag{3.13}$$

Then the velocity components, the derivatives of the velocity with respect to time and coordinate axes, the derivatives of pressure and free-surfaces with respect to coordinate axes become as follows,

$$\begin{aligned}
u' &= \frac{\partial x'}{\partial t'} = \frac{\partial (xH)}{\partial (tT)} = \frac{H}{T}\frac{\partial x}{\partial t} = \frac{H}{T}u \, , \\
v' &= \frac{\partial y'}{\partial t'} = \frac{\partial (yH)}{\partial (tT)} = \frac{H}{T}\frac{\partial y}{\partial t} = \frac{H}{T}v \, , \\
\frac{\partial u'}{\partial x'} &= \frac{\partial (\frac{H}{T}u)}{\partial (xH)} = \frac{1}{T}\frac{\partial u}{\partial x} \, .
\end{aligned}$$

Similarly,

$$\frac{\partial u'}{\partial y'} = \frac{1}{T}\frac{\partial u}{\partial y} \, , \qquad \frac{\partial v'}{\partial x'} = \frac{1}{T}\frac{\partial v}{\partial x} \, , \qquad \frac{\partial v'}{\partial y'} = \frac{1}{T}\frac{\partial v}{\partial y} \, ,$$

$$\frac{\partial u'}{\partial t'} = \frac{\partial (\frac{H}{T}u)}{\partial (tT)} = \frac{H}{T^2}\frac{\partial u}{\partial t} \quad , \quad \frac{\partial v'}{\partial t'} = \frac{H}{T^2}\frac{\partial v}{\partial t}$$

$$\frac{\partial p'}{\partial x'} = \frac{\partial (p\rho g H)}{\partial (xH)} = \rho g \frac{\partial p}{\partial x} \, ,$$

$$\frac{\partial \eta'}{\partial x'} = \frac{\partial (\eta H)}{\partial (xH)} = \frac{\partial \eta}{\partial x} \quad , \quad \frac{\partial \xi'}{\partial y'} = \frac{\partial \xi}{\partial y}$$

$$\frac{\partial \eta'}{\partial t'} = \frac{\partial(\eta H)}{\partial(tT)} = \frac{H}{T}\frac{\partial \eta}{\partial t} \qquad , \qquad \frac{\partial \xi'}{\partial t'} = \frac{H}{T}\frac{\partial \xi}{\partial t} \ .$$

For simplicity let us use the notation u_x instead of $\dfrac{\partial u}{\partial x}$ and so on. Then the incompressibility condition (3.3) becomes

$$\begin{aligned} \frac{1}{T}u_x + \frac{1}{T}v_y &= 0, \\ u_x + v_y &= 0. \end{aligned} \tag{3.14}$$

First of Euler's equations (3.1) becomes

$$\begin{aligned} \frac{H}{T^2}u_t + \frac{H}{T}u\frac{1}{T}u_x + \frac{H}{T}v\frac{1}{T}u_y &= -\frac{1}{\rho}\rho g p_x \ , \\ u_t + uu_x + vu_y &= -\frac{T^2}{H}gp_x \ , \end{aligned}$$

and choosing $T^2 = \dfrac{H}{g}$,

$$u_t + uu_x + vu_y = -p_x \ . \tag{3.15}$$

Similarly second of Euler's equations (3.2) becomes

$$\begin{aligned} \frac{H}{T^2}v_t + \frac{H}{T}u\frac{1}{T}v_x + \frac{H}{T}v\frac{1}{T}v_y &= -\frac{1}{\rho}\rho g p_y - g \ , \\ v_t + uv_x + vv_y &= -\frac{T^2}{H}gp_y - \frac{T^2}{H}g \ , \\ v_t + uv_x + vv_y &= -p_y - 1 \ . \end{aligned} \tag{3.16}$$

And the initial conditions (3.17) become

$$\left.\begin{array}{c} u(x,y,0) = v(x,y,0) = 0, \quad \eta(x,0) = \xi(y,0) = 0 \\ \text{and} \quad p(x,y,0) = -y \ . \end{array}\right\} \tag{3.17}$$

The kinematic and dynamic free-surface conditions in (3.7)-(3.8) and (3.9)-(3.10) become,

on $y = \eta(x,t)$,

$$\frac{H}{T}v = \frac{H}{T}\eta_t + \frac{H}{T}u\eta_x \ ,$$

$$\left. v = \eta_t + u\eta_x \right\} \qquad \text{kinematic condition ,} \tag{3.18}$$

$$\left. p(x,\eta,t) = 0 \right\} \qquad \text{dynamic condition .} \tag{3.19}$$

on $x = \xi(y,t)$,

$$\frac{H}{T}u = \frac{H}{T}\xi_t + \frac{H}{T}v\xi_y \ ,$$

$$\left. u = \xi_t + v\xi_y \right\} \qquad \text{kinematic condition ,} \tag{3.20}$$

$$\left. p(\xi,y,t) = 0 \right\} \qquad \text{dynamic condition .} \tag{3.21}$$

3.3. Boundary Value Problem

A mathematical statement of the problem can now be written as a dimensionless nonlinear boundary value problem in the form

$$
\begin{aligned}
u_x + v_y &= 0 \, , &(3.22)\\
u_t + uu_x + vu_y &= -p_x \, , &(3.23)\\
v_t + uv_x + vv_y &= -p_y - 1 \, , &(3.24)\\
v = \eta_t + u\eta_x \, , \quad p &= 0 \quad \text{on } y = \eta(x,t), &(3.25)\\
u = \xi_t + v\xi_y \, , \quad p &= 0 \quad \text{on } x = \xi(y,t), &(3.26)\\
v(x,-1,t) &= 0 &(3.27)\\
\eta(x,0) = \xi(y,0) = 0, &\qquad u(x,y,0) = v(x,y,0) = 0, &(3.28)
\end{aligned}
$$

$$\text{as } x \to \infty, \, u, v \to 0 \text{ and } p \to -y \qquad (3.29)$$

The solution domain for this set of equations is described as

$$D(t) = \{(x,y) : -1 \le y \le \eta(x,t) \, , \, \xi(y,t) \le x \le \infty\}.$$

There are three governing equations (3.22-3.24), six boundary conditions (3.25-3.27 and 3.29) and four initial conditions (3.28). The unknowns are u, v, p, ξ, η.

CHAPTER 4

SMALL-TIME BEHAVIOUR OF THE DAM-BREAKING PROBLEM

4.1. Small-Time Behaviour ($t \to 0$)

A small-time solution to (3.22) - (3.28) may be developed by posing the expansions

$$
\begin{aligned}
u &= u_0(x,y) + tu_1(x,y) + O(t^2), \qquad v = v_0(x,y) + tv_1(x,y) + O(t^2),\\
\eta &= \eta_0(x) + t\eta_1(x) + t^2\eta_2(x) + O(t^3), \qquad \xi = \xi_0(y) + t\xi_1(y) + t^2\xi_2(y) + O(t^3),\\
p &= p_0(x,y) + tp_1(x,y) + O(t^2)
\end{aligned}
$$

as $t \to 0$ with $\mathbf{x} = O(1)$.

Since $u(x,y,0) = 0$ and $v(x,y,0) = 0$, we conclude that $u_0 = 0$ and $v_0 = 0$.Similarly since $\eta(x,0) = 0$ and $\xi(y,0) = 0$, we conclude that $\eta_0 = 0$ and $\xi_0 = 0$. Now the small-time solution expansions to (3.22) - (3.28) can be written as

$$
\begin{aligned}
u &= tu_1(x,y) + O(t^2), \qquad v = tv_1(x,y) + O(t^2)\ , && (4.1)\\
\eta &= t\eta_1(x) + t^2\eta_2(x) + O(t^3), \qquad \xi = t\xi_1(y) + t^2\xi_2(y) + O(t^3)\ , && (4.2)\\
p &= p_0(x,y) + tp_1(x,y) + O(t^2) && (4.3)
\end{aligned}
$$

as $t \to 0$ with $x = O(1)$. Then substitute the above expansions into the equations (3.22) - (3.28).
From the incompressibility condition (3.22) we have

$$tu_{1,x} + tv_{1,y} + t^2u_{2,x} + t^2v_{2,y} + \ldots = 0$$

which gives,

$$
\begin{aligned}
u_{1,x} + v_{1,y} &= 0 \qquad \text{(of order t)} , && (4.4)\\
u_{2,x} + v_{2,y} &= 0 \qquad \text{(of order } t^2\text{)} . && (4.5)
\end{aligned}
$$

From the first of Euler's equations (3.23) we have

$$
\begin{gathered}
u_1 + 2tu_2 + \ldots + (tu_1 + t^2u_2 + \ldots)(tu_{1,x} + t^2u_{2,x} + \ldots) + \\
(tv_1 + t^2v_2 + \ldots)(tu_{1,y} + t^2u_{2,y} + \ldots) = -p_{0,x} - tp_{1,x} - \ldots
\end{gathered}
$$

which gives,

$$
\begin{aligned}
u_1 &= -p_{0,x} \qquad \text{(of order 1)} , && (4.6)\\
u_2 &= -\frac{1}{2}p_{1,x} \qquad \text{(of order t)} . && (4.7)
\end{aligned}
$$

From the second of Euler's equations (3.24) we have

$$
\begin{gathered}
v_1 + 2tv_2 + \ldots + (tu_1 + t^2u_2 + \ldots)(tv_{1,x} + t^2v_{2,x} + \ldots) + \\
(tv_1 + t^2v_2 + \ldots)(tv_{1,y} + t^2v_{2,y} + \ldots) = -p_{0,y} - tp_{1,y} - \ldots - 1
\end{gathered}
$$

which gives,

$$
\begin{aligned}
v_1 &= -p_{0,y} - 1 \qquad \text{(of order 1)} , && (4.8)\\
v_2 &= -\frac{1}{2}p_{1,y} \qquad \text{(of order t)} . && (4.9)
\end{aligned}
$$

From the kinematic boundary condition on $y = \eta(x,t)$ (3.25) we have

$$
\begin{gathered}
tv_1(x,\eta(x,t)) + t^2v_2(x,\eta(x,t)) + \ldots = (\eta_1 + 2t\eta_2 + 3t^2\eta_3 + \ldots) \\
+(tu_1(x,\eta(x,t)) + t^2u_2(x,\eta(x,t)) + \ldots)(t\eta_{1,x} + t^2\eta_{2,x} + \ldots) ,
\end{gathered}
$$

by using the Taylor series expansions of $v_i(x,\eta(x,t))$ and $u_i(x,\eta(x,t))$ for $i=1,2,\ldots$ about $y=\eta(x,t)=0$, we have

$$t\big[v_1(x,0)+v_{1,y}(x,0)\eta(x,t)+\ldots\big]+t^2\big[v_2(x,0)+v_{2,y}(x,0)\eta(x,t)+\ldots\big]+\ldots=$$
$$(\eta_1+2t\eta_2+3t^2\eta_3+\ldots)+\big[t(u_1(x,0)+u_{1,y}(x,0)\eta(x,t)+\ldots)+$$
$$t^2(u_2(x,0)+u_{2,y}(x,0)\eta(x,t)+\ldots)+\ldots\big](t\eta_{1,x}+t^2\eta_{2,x}+\ldots)$$

where $\eta(x,t)=t\eta_1+t^2\eta_2+O(t^3)$ and we get,

$$\eta_1(x) = 0 \qquad \text{(of order 1)} \quad , \quad x>0\,, \tag{4.10}$$
$$v_1(x,0) = 2\eta_2(x) \qquad \text{(of order t)} \quad , \quad x>0\,. \tag{4.11}$$

On the other hand since $p=0$ by the dynamic boundary condition on $y=\eta(x,t)$ we have

$$p=p_0(x,\eta(x,t))+tp_1(x,\eta(x,t))+t^2p_2(x,\eta(x,t))+\ldots=0$$

by using the Taylor series expansion of $p_i(x,\eta(x,t))$ for $i=0,1,\ldots$ about $y=\eta(x,t)=0$, we have

$$\big[p_0(x,0)+p_{0,y}(x,0)\eta(x,t)+\ldots\big]+t\big[p_1(x,0)+p_{1,y}(x,0)\eta(x,t)+\ldots\big]+$$
$$t^2\big[(p_2(x,0)+p_{2,y}(x,0)\eta(x,t)+\ldots)\big]+\ldots=0$$

where $\eta(x,t)=t\eta_1+t^2\eta_2+O(t^3)$ and we get,

$$p_0(x,0) = 0 \qquad \text{(of order 1)} \quad , \quad x>0\,, \tag{4.12}$$
$$p_1(x,0) = 0 \qquad \text{(of order t)} \quad , \quad x>0\,, \tag{4.13}$$
$$p_2(x,0) = -p_{0,y}(x,0)\eta_2 \qquad \text{(of order } t^2\text{)} \quad , \quad x>0\,. \tag{4.14}$$

From the kinematic boundary condition on $x = \xi(y,t)$ (3.26) we have

$$
\begin{aligned}
tu_1(\xi(y,t),y) + t^2u_2(\xi(y,t),y) + \ldots &= (\xi_1 + 2t\xi_2 + 3t^2\xi_3 + \ldots) \\
&+ (tv_1(\xi(y,t),y) + t^2v_2(\xi(y,t),y) + \ldots)(t\xi_{1,y} + t^2\xi_{2,y} + \ldots)
\end{aligned}
$$

by using the Taylor series expansions of $v_i(\xi(y,t),y)$ and $u_i(\xi(y,t),y)$ for $i = 1, 2, \ldots$ about $x = \xi(y,t) = 0$, we have

$$
\begin{aligned}
t\big[u_1(0,y) + u_{1,x}(0,y)\xi(y,t) + \ldots\big] + t^2\big[u_2(0,y) + u_{2,x}(0,y)\xi(y,t) + \ldots\big] + \ldots = \\
(\xi_1 + 2t\xi_2 + 3t^2\xi_3 + \ldots) + \big[t(v_1(0,y) + v_{1,x}(0,y)\xi(y,t) + \ldots) + \\
t^2(v_2(0,y) + v_{2,x}(0,y)\xi(y,t) + \ldots) + \ldots\big](t\xi_{1,y} + t^2\xi_{2,y} + \ldots)
\end{aligned}
$$

where $\xi(y,t) = t\xi_1 + t^2\xi_2 + O(t^3)$ and we get,

$$
\begin{aligned}
\xi_1(y) &= 0 \qquad \text{(of order 1)} , && (4.15) \\
u_1(0,y) &= 2\xi_2(y) \qquad \text{(of order t)} . && (4.16)
\end{aligned}
$$

On the other hand since $p = 0$ by the dynamic boundary condition on $x = \xi(y,t)$ we have

$$
p = p_0(\xi(y,t),y) + tp_1(\xi(y,t),y) + t^2p_2(\xi(y,t),y) + \ldots = 0
$$

by using the Taylor series expansion of $p_i(\xi(y,t),y)$ for $i = 0, 1, \ldots$ about $x = \xi(y,t) = 0$, we have

$$
\begin{aligned}
\big[p_0(0,y) + p_{0,x}(0,y)\xi(y,t) + \ldots\big] + t\big[p_1(0,y) + p_{1,x}(0,y)\xi(y,t)\big] + \\
t^2\big[p_2(0,y) + p_{2,x}(0,y)\xi(y,t) + \ldots\big] + \ldots = 0
\end{aligned}
$$

where $\xi(y,t) = t\xi_1 + t^2\xi_2 + O(t^3)$ and we get,

$$
\begin{aligned}
p_0(0,y) &= 0 \qquad \text{(of order 1)}\,, &(4.17)\\
p_1(0,y) &= 0 \qquad \text{(of order t)}\,, &(4.18)\\
p_2(0,y) &= -p_{0,x}(0,y)\xi_2 \qquad \text{(of order } t^2\text{)}\,. &(4.19)
\end{aligned}
$$

Therefore using the equations (4.10) and (4.15), small-time solution expansions to (3.22) - (3.28) must be written as

$$
\begin{aligned}
u &= tu_1(x,y) + O(t^2), \qquad v = tv_1(x,y) + O(t^2)\,, &(4.20)\\
\eta &= t^2\eta_2(x) + O(t^3), \qquad \xi = t^2\xi_2(y) + O(t^3)\,, &(4.21)\\
p &= p_0(x,y) + tp_1(x,y) + O(t^2) &(4.22)
\end{aligned}
$$

as $t \to 0$ with $\mathbf{x} = O(1)$.
And the condition $v(x,-1,t) = 0$ can be written as $tv_1(x,-1) + t^2v_2(x,-1) + \ldots = 0$ which gives the following,

$$
\begin{aligned}
v_1(x,-1) &= 0 \qquad \text{(of order t)}\,, &(4.23)\\
v_2(x,-1) &= 0 \qquad \text{(of order } t^2\text{)}\,. &(4.24)
\end{aligned}
$$

4.2. The Leading Order Boundary Value Problem

At the leading order, we find the following boundary value problem

$$
\left.
\begin{array}{c}
u_{1,x} + v_{1,y} = 0,\\
u_1 = -p_{0,x} \qquad , \qquad v_1 = -p_{0,y} - 1,\\
v_1(x,-1) = 0\,, \quad \eta_2 = \frac{1}{2}v_1(x,0)\,, \quad \xi_2 = \frac{1}{2}u_1(0,y)\,,\\
p_0(x,0) = 0\,, \quad p_0(0,y) = 0
\end{array}
\right\} \qquad (4.25)
$$

with $u_1, v_1 \to 0$ and $p_0 \to -y$ as $x \to \infty$.

Then the problem (4.25) is equivalent to the boundary value problem

$$\left.\begin{array}{c}\Delta p_0 = 0, \\ p_{0,y}(x,-1) = -1\,, \quad p_0(x,0) = 0\,, \quad p_0(0,y) = 0\end{array}\right\} \tag{4.26}$$

with $u_1, v_1 \to 0$ and $p_0 \to -y$ as $x \to \infty$.

4.2.1. The Leading Order Solution & Singularity

Solution to the problem (4.26) may be found by the standard "separation of variables" method (see Appendix A) in the form

$$p_0(x,y) = -y + \sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)^2\pi^2} \sin\left((2n+1)\frac{\pi}{2}y\right) e^{-(2n+1)\frac{\pi}{2}x}\,. \tag{4.27}$$

Then since $v_1 = -p_{0,y} - 1$ and $u_1 = -p_{0,x}$, we get

$$\begin{aligned} v_1(x,y) &= -\frac{4}{\pi} Re\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-(2n+1)\frac{\pi}{2}x} e^{i(2n+1)\frac{\pi}{2}y}\right], \\ u_1(x,y) &= \frac{4}{\pi} Im\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-(2n+1)\frac{\pi}{2}x} e^{i(2n+1)\frac{\pi}{2}y}\right] \end{aligned}$$

where $Re(z)$ denotes the real part of z and $Im(z)$ the imaginary part. On the other hand since $2\eta_2 = v_1(x,0) = -p_{0,y}(x,0) - 1$, we get

$$\eta_2 = -\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (e^{-\frac{\pi}{2}x})^{2n+1}}{2n+1}\,, \tag{4.28}$$

and similarly since $2\xi_2 = u_1(0,y) = -p_{0,x}(0,y)$, we get

$$\xi_2 = \frac{2}{\pi} Im \sum_{n=0}^{\infty} \frac{(-1)^n (e^{i\frac{\pi}{2}y})^{2n+1}}{2n+1}\,. \tag{4.29}$$

The series in (4.28) and (4.29) can be summed exactly (see Appendix B) to give

$$\eta_2 = -\frac{2}{\pi}\arctan(e^{-\frac{\pi}{2}x}) \quad , \quad \xi_2 = \frac{1}{\pi}\log(\tan\frac{\pi}{4}(1+y)) \tag{4.30}$$

and we see that ξ_2 reveals a singularity in the free-surface elevation as $y \to -1$ (see Figure 4.1).

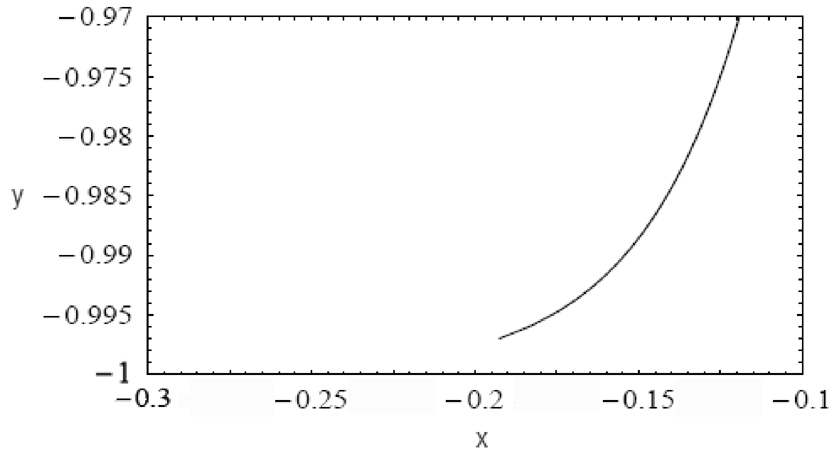

Figure 4.1. First order outer solution of ξ for $t = 0.001$

4.3. Outer Matching Conditions

This non-uniformity in the expansion about the point $(0, -1)$ suggest that the expansions (4.1)-(4.3) are outer expansions to this problem. To correctly capture the behavior in the neighborhood of the point $(0, -1)$ we require an inner region in which $x, y = o(1)$ as $t \to 0$. To motivate the form of the inner expansion we require the local behavior of the pressure p_0 as $(x^2 + y^2)^{\frac{1}{2}} \to 0$. First of all we change the variables, $\xi = x$ and $\eta = y + 1$, to translate the origin to the singularity and then use the polar coordinates $\xi = \rho\cos\theta$ and $\eta = \rho\sin\theta$ where (ρ, θ) are the standard polar coordinates based at the cartesian origin (Figure 4.2). Thus we require the local behavior of the pressure p_0 as $(\xi^2 + \eta^2)^{\frac{1}{2}} \to 0$ (*i.e.* as $\rho \to 0$) in the quarter plane $0 \leq \rho \leq \infty$, $0 \leq \theta \leq \frac{\pi}{2}$.

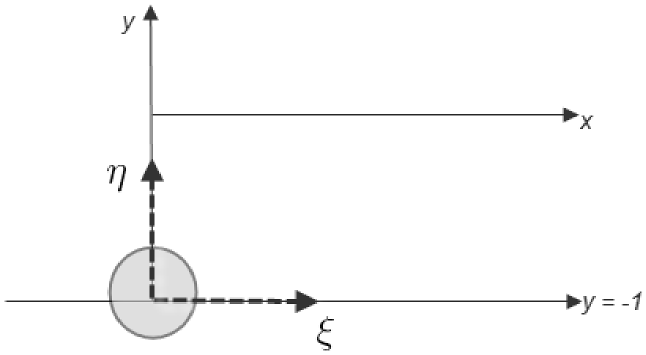

Figure 4.2. New coordinate axes ξ, η and the inner region

4.3.1. Behaviour of the Leading Order Solution Near the Singular Point

For the new coordinate axes ξ and η, the pressure p_0 becomes

$$p_0(\xi,\eta) = 1-\eta+\sum_{n=0}^{\infty}\frac{8(-1)^n}{(2n+1)^2\pi^2}\sin\left((2n+1)\frac{\pi}{2}(\eta-1)\right)e^{-(2n+1)\frac{\pi}{2}\xi}\ .$$

Then by using the standard polar coordinates and letting $p_0 = P$, the pressure P and the derivative of P with respect to θ becomes

$$P(\rho,\theta) = 1-\rho\sin\theta+\sum_{n=0}^{\infty}\frac{8(-1)^n}{(2n+1)^2\pi^2}\sin\left((2n+1)\frac{\pi}{2}(\rho\sin\theta-1)\right)e^{-(2n+1)\frac{\pi}{2}\rho\cos\theta}$$

and

$$P_\theta(\rho,\theta) = -\rho\cos\theta+\sum_{n=0}^{\infty}\frac{4(-1)^n\rho}{(2n+1)\pi}\cos\left((2n+1)\frac{\pi}{2}(\rho\sin\theta-1)-\theta\right)e^{-(2n+1)\frac{\pi}{2}\rho\cos\theta}\ .$$

On the boundaries of the quarter plane,

$$P_\theta = -\rho \qquad \text{on } \theta = 0 \qquad \text{clearly.}$$

On the other hand on $\theta = \frac{\pi}{2}$,

$$
\begin{aligned}
P_\theta &= \sum_{n=0}^{\infty} \frac{4(-1)^n \rho}{(2n+1)\pi} \cos\left[(2n+1)\frac{\pi}{2}(\rho-1) - \frac{\pi}{2}\right] , \\
&= \sum_{n=0}^{\infty} \frac{4(-1)^n \rho}{(2n+1)\pi} \sin\left[(2n+1)\frac{\pi}{2}(\rho-1)\right] , \\
&= \frac{4\rho}{\pi} Im \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (e^{i\frac{\pi}{2}(\rho-1)})^{2n+1} , \\
&= \frac{4\rho}{\pi} Im \arctan(e^{i\frac{\pi}{2}(\rho-1)}) , \\
&= \frac{2\rho}{\pi} \log(\tan\frac{\pi}{4}\rho) .
\end{aligned}
$$

Since we consider the local behavior of the pressure as $\rho \to 0$, expand $\tan\frac{\pi}{4}\rho$ into Taylor series at $\rho = 0$,

$$
P_\theta = \frac{2\rho}{\pi} \log\left(\frac{\pi}{4}\rho + \frac{(\frac{\pi}{4}\rho)^3}{3} + \dots\right) ,
$$

$$
P_\theta = \frac{2\rho}{\pi}\left[\log(\frac{\pi}{4}\rho) + O(\rho^2)\right] .
$$

Thus

$$
P_\theta = \frac{2\rho}{\pi}\log\rho + \frac{2\rho}{\pi}\log(\frac{\pi}{4}) + O(\rho^3) \qquad \text{on } \theta = \frac{\pi}{2}
$$

and

$$
P = 0 \qquad \text{on } \theta = \frac{\pi}{2}.
$$

Hence we are led to consider the following boundary problem in the quarter plane $0 \leq \rho \leq \infty, 0 \leq \theta \leq \frac{\pi}{2}$:

$$
\left.
\begin{array}{c}
\boldsymbol{\Delta} P = 0 , \\
P_\theta = -\rho \quad \text{on} \quad \theta = 0, \qquad P = 0 \quad \text{on} \quad \theta = \frac{\pi}{2} , \\
\text{and} \quad P_\theta = \frac{2}{\pi}\rho\log\rho + \frac{2}{\pi}\rho\log\frac{\pi}{4} + O(\rho^3) \quad \text{on} \quad \theta = \frac{\pi}{2} .
\end{array}
\right\} \tag{4.31}
$$

As we are interested in the solution for small ρ, we pose a coordinate expansion of the form

$$
P = \rho(\log\rho)g(\theta) + \rho h(\theta) + O(\rho) \qquad \text{as } \rho \to 0 ,
$$

and by using the polar form of the Laplace equation, we have

$$\mathbf{\Delta} P = \frac{1}{\rho} g(\theta) + \frac{1}{\rho}\Big(\log\rho g(\theta) + g(\theta) + h(\theta) \Big) + \frac{1}{\rho^2}\Big(\rho \log\rho g''(\theta) + \rho h''(\theta) \Big) = 0 \ ,$$

which gives

$$g(\theta) + g''(\theta) = 0 \qquad (\text{order } \frac{\log\rho}{\rho}) \ , \tag{4.32}$$

$$2g(\theta) + h(\theta) + h''(\theta) = 0 \qquad (\text{order } \frac{1}{\rho}) \ . \tag{4.33}$$

Then by using (4.32), (4.33) and the boundary conditions in (4.31), general solutions of $g(\theta)$ and $h(\theta)$ are

$$\begin{aligned} g(\theta) &= \frac{-2}{\pi} \cos\theta \ , \\ h(\theta) &= -\sin\theta + \frac{2}{\pi}\Big(1 - \log(\frac{\pi}{4}) \cos\theta + \frac{2}{\pi}\theta \sin\theta \Big) \ . \end{aligned}$$

Hence as $\rho \to 0$,

$$P = \rho \log\rho \Big(-\frac{2}{\pi}\cos\theta \Big) + \rho \Big(-\sin\theta + \frac{2}{\pi}(1 - \log(\frac{\pi}{4})) + \frac{2}{\pi}\theta \sin\theta \Big) + o(\rho) \ . \tag{4.34}$$

Next, we shall find the behavior of ξ_2 near the singular point. The closed-form expression of ξ_2 (4.30) in the new coordinate axes ξ and η becomes

$$\xi_2 = \frac{1}{\pi} \log(\tan \frac{\pi}{4}\eta) \ .$$

For the behavior of ξ_2 as $\rho \to 0$, we expand $\tan\frac{\pi}{4}\eta$ into Taylor Series at $\eta = 0$,

$$\begin{aligned} \xi_2 &= \frac{1}{\pi} \log\left(\frac{\pi}{4}\eta + \frac{(\frac{\pi}{4}\eta)^3}{3} + \dots \right), \\ \xi_2 &= \frac{1}{\pi}\log\frac{\pi}{4} + \frac{1}{\pi}\log\eta + O(\eta^2) \ . \end{aligned}$$

Thus we have the asymptotic matching conditions (limiting values of the outer solution as $\rho \to 0$) that will be needed later on such that

$$
\begin{aligned}
p_0 &= \rho \log \rho \Big(-\frac{2}{\pi}\cos\theta \Big) + \rho \Big(-\sin\theta + \frac{2}{\pi}(1-\log(\frac{\pi}{4})) + \frac{2}{\pi}\theta\sin\theta \Big) + o(\rho), \\
\xi_2 &= \frac{1}{\pi}\log\frac{\pi}{4} + \frac{1}{\pi}\log\eta + O(\eta^2) \qquad \text{as } \eta = \rho \to 0.
\end{aligned}
\tag{4.35}
$$

This indicates that $p_0 = O(\rho\log\rho)$ and $\xi_2 = O(\log\rho)$ as $\rho \to 0$.

The fluid velocities in the corner region are calculated from (4.25) by the equations $u_1 = -p_{0,\xi}$ and $v_1 = -p_{0,\eta} - 1$. Then by chain rule, we get derivatives of P with respect to ξ and η such that

$$
\begin{aligned}
p_{0,\xi} &= \Big[-\log\rho\frac{2}{\pi}\cos\theta - \frac{2}{\pi}\cos\theta - \sin\theta + \frac{2}{\pi}(1-\log\frac{\pi}{4})\cos\theta + \frac{2}{\pi}\theta\sin\theta \Big]\cos\theta \\
&- \Big[\rho\log\rho\frac{2}{\pi}\sin\theta - \rho\cos\theta - \frac{2}{\pi}\rho(1-\log\frac{\pi}{4})\sin\theta + \frac{2}{\pi}\rho\sin\theta + \frac{2}{\pi}\rho\theta\cos\theta \Big]\frac{\sin\theta}{\rho} ,
\end{aligned}
$$

$$
p_{0,\xi} = -\frac{2}{\pi}\log\rho - \frac{2}{\pi}\log\frac{\pi}{4} , \tag{4.36}
$$

and

$$
\begin{aligned}
p_{0,\eta} &= \Big[-\log\rho\frac{2}{\pi}\cos\theta - \frac{2}{\pi}\cos\theta - \sin\theta + \frac{2}{\pi}(1-\log\frac{\pi}{4})\cos\theta + \frac{2}{\pi}\theta\sin\theta \Big]\sin\theta \\
&- \Big[\rho\log\rho\frac{2}{\pi}\sin\theta - \rho\cos\theta - \frac{2}{\pi}\rho(1-\log\frac{\pi}{4})\sin\theta + \frac{2}{\pi}\rho\cos\theta + \frac{2}{\pi}\rho\theta\cos\theta \Big]\frac{\cos\theta}{\rho} ,
\end{aligned}
$$

$$
p_{0,\eta} = -1 + \frac{2}{\pi}\theta . \tag{4.37}
$$

Thus by using (4.36) and (4.37) the velocity components u_1, v_1 becomes

$$
u_1 = \frac{2}{\pi}\log\rho + \frac{2}{\pi}\log\frac{\pi}{4} , \tag{4.38}
$$

$$
v_1 = -\frac{2}{\pi}\theta , \tag{4.39}
$$

as $\rho \to 0$. This indicates that $u_1 = O(\log \rho)$ and $v_1 = O(1)$ as $\rho \to 0$.

CHAPTER 5

THE INNER VARIABLES

5.1. Inner Region Problem

In order to construct an inner solution to this problem when $\xi, \eta = o(1)$ as $t \to 0$, it is useful to examine where the magnitude of terms retained in deriving the velocity components (3.22)-(3.24) is equal to the terms neglected. The local analysis of the Chapter 4 shows that as $\rho = (\xi^2 + \eta^2)^{\frac{1}{2}} \to 0$;

$$\begin{aligned}
u &= O(t \log \rho) && \text{since } u = tu_1 + O(t^2)\ , && (5.1)\\
v &= O(t) && \text{since } v = tv_1 + O(t^2)\ , && (5.2)\\
P &= O(\rho \log \rho) && \text{by (4.35)}\ , && (5.3)\\
\xi &= O(t^2 \log \rho) && \text{since } \xi = t^2\xi_2 + O(t^3)\ . && (5.4)
\end{aligned}$$

Thus a typical retained term in (3.24) is $v_t = O(1)$ whereas a typical neglected term, which represents fluid inertia, is $uv_x = O((t^2 \log \rho)/\rho)$. These two terms are of equal magnitude when $t^2 \log \rho = O(\rho)$. If this is solved iteratively (by Newton-Raphson method), it is clear that when $\rho = O(-t^2 \log t)$, inertial terms are important and that in this region $v = O(t)$, $u = O(t \log t)$, $p = O(t^2 \log^2 t)$ and $\xi = O(t^2 \log t)$. These estimates motivate the introduction of the following inner variables:

$$X = -\frac{\xi}{t^2 \log t}, \quad Y = -\frac{\eta}{t^2 \log t}, \quad U = u, \quad V = v, \quad P = p. \tag{5.5}$$

5.1.1. Equations of the Problem in Inner Variables

By using (3.22), which is equivalent to $u_\xi + v_\eta = 0$, (5.5) leads to

$$
\begin{aligned}
\frac{\partial U}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial \eta} &= 0\,,\\
U_X\left(-\frac{1}{t^2\log t}\right) + V_Y\left(-\frac{1}{t^2\log t}\right) &= 0\,,\\
U_X + V_Y &= 0\,.
\end{aligned}
$$

Similarly by using (3.23), which is equivalent to $u_t + uu_\xi + vu_\eta = -p_\xi$, (5.5) leads to

$$
\begin{aligned}
&\frac{\partial U}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial U}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial U}{\partial Y}\frac{\partial Y}{\partial t} + U\left(\frac{\partial U}{\partial X}\frac{\partial X}{\partial \xi}\right) + V\left(\frac{\partial U}{\partial Y}\frac{\partial Y}{\partial \eta}\right) = -\frac{\partial P}{\partial X}\frac{\partial X}{\partial \xi}\,,\\
&U_t + U_X\left(\xi\frac{2t\log t + t}{(t^2\log t)^2}\right) + U_Y\left(\eta\frac{2t\log t + t}{(t^2\log t)^2}\right) + UU_X\left(-\frac{1}{t^2\log t}\right)\\
&\qquad + VU_Y\left(-\frac{1}{t^2\log t}\right) = -P_X\left(-\frac{1}{t^2\log t}\right)\,,\\
&U_t - X\left(\frac{2}{t} + \frac{1}{t\log t}\right)U_X - Y\left(\frac{2}{t} + \frac{1}{t\log t}\right)U_Y - \frac{1}{t^2\log t}UU_X\\
&\qquad - \frac{1}{t^2\log t}VU_Y = \frac{1}{t^2\log t}P_X.
\end{aligned}
$$

Similarly by using (3.24), which is equivalent to $v_t + uv_\xi + vv_\eta = -p_{\eta-1}$, (5.5) leads to

$$
\begin{aligned}
&\frac{\partial V}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial V}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial t} + U\left(\frac{\partial V}{\partial X}\frac{\partial X}{\partial \xi}\right) + V\left(\frac{\partial V}{\partial Y}\frac{\partial Y}{\partial \eta}\right) = -\frac{\partial P}{\partial Y}\frac{\partial Y}{\partial \eta} - 1\,,\\
&V_t + V_X\left(\xi\frac{2t\log t + t}{(t^2\log t)^2}\right) + V_Y\left(\eta\frac{2t\log t + t}{(t^2\log t)^2}\right) + UV_X\left(-\frac{1}{t^2\log t}\right)\\
&\qquad + VV_Y\left(-\frac{1}{t^2\log t}\right) = -P_Y\left(-\frac{1}{t^2\log t}\right) - 1\,,
\end{aligned}
$$

$$V_t - X\left(\frac{2}{t} + \frac{1}{t\log t}\right)V_X - Y\left(\frac{2}{t} + \frac{1}{t\log t}\right)V_Y - \frac{1}{t^2\log t}UV_X$$
$$-\frac{1}{t^2\log t}VV_Y = \frac{1}{t^2\log t}P_Y - 1.$$

And the free-surface conditions (3.18 and 3.19), which are equivalent to $V = \eta_t + u\eta_\xi$ and $p = 0$ on $y = \eta(x,t)$, become

$$\begin{aligned}
V &= \frac{\partial\eta}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial\eta}{\partial X}\frac{\partial X}{\partial t} + U\frac{\partial\eta}{\partial X}\frac{\partial X}{\partial\xi}, \\
V &= \eta_t - X\left(\frac{2}{t} + \frac{1}{t\log t}\right)\eta_X - \left(-\frac{1}{t^2\log t}\right)U\eta_X \quad \text{and} \quad P = 0.
\end{aligned}$$

Similarly (3.20) and (3.21), which are equivalent to $u = \xi_t + v\xi_\eta$ and $p = 0$ on $x = \xi(y,t)$, become

$$\begin{aligned}
U &= \frac{\partial\xi}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial\xi}{\partial Y}\frac{\partial Y}{\partial t} + V\frac{\partial\xi}{\partial Y}\frac{\partial Y}{\partial\eta}, \\
U &= \xi_t - Y\left(\frac{2}{t} + \frac{1}{t\log t}\right)\xi_Y - \left(-\frac{1}{t^2\log t}\right)V\xi_Y \quad \text{and} \quad P = 0.
\end{aligned}$$

Hence we have the inner region problem;

$$\left.\begin{array}{c}
U_X + V_Y = 0, \\[2ex]
U_t - X\left(\frac{2}{t} + \frac{1}{t\log t}\right)U_X - Y\left(\frac{2}{t} + \frac{1}{t\log t}\right)U_Y - \frac{1}{t^2\log t}UU_X \\
-\frac{1}{t^2\log t}VU_Y = \frac{1}{t^2\log t}P_X, \\[2ex]
V_t - X\left(\frac{2}{t} + \frac{1}{t\log t}\right)V_X - Y\left(\frac{2}{t} + \frac{1}{t\log t}\right)V_Y - \frac{1}{t^2\log t}UV_X \\
-\frac{1}{t^2\log t}VV_Y = \frac{1}{t^2\log t}P_Y - 1,
\end{array}\right\} \qquad (5.6)$$

in $0 \leq Y < \infty$, $X \geq \xi(Y,t)/(-t^2 \log t)$ and subject to free-surface conditions in the form

$$U = \xi_t - Y\left(\frac{2}{t} + \frac{1}{t\log t}\right)\xi_Y - \left(-\frac{1}{t^2\log t}\right)V\xi_Y \quad \text{and} \quad P = 0. \tag{5.7}$$

5.2. Expansion of the Inner Region Problem in Time

Matching conditions (4.35) and the equations $u_1 = -P_\xi$ and $v_1 = -P_\eta - 1$ should be written in terms of the inner variables and be applied as $R = (X^2+Y^2)^{\frac{1}{2}} \to \infty$. Here by using the relation $\rho = -Rt^2\log t$, we get

$$\left.\begin{aligned} P = \frac{2}{\pi}R\log R\cos\theta t^2\log t + \frac{4}{\pi}R\cos\theta t^2(\log t)^2 \\ +\frac{2}{\pi}R\cos\theta\log(-\log t)t^2\log t + R\sin\theta t^2\log t \\ -\frac{2}{\pi}R\cos\theta t^2\log t + \frac{2}{\pi}R\cos\theta\log(\frac{\pi}{4})t^2\log t - \frac{2}{\pi}R\theta\sin\theta t^2\log t, \\ \xi = \frac{\log Y}{\pi}t^2 + \frac{2}{\pi}t^2\log t + \frac{\log(-\log t)}{\pi}t^2 + \frac{\log(\frac{\pi}{4})}{\pi}t^2, \\ U = P_{1,X}t\log t + P_{2,X}t, \\ V = P_{1,Y}t\log t + (P_{2,Y}-1)t. \end{aligned}\right\} \tag{5.8}$$

Thus we proceed with the solution to the inner problem by posing expansions of the form

$$P = t^2(\log t)^2P_1 + t^2\log tP_2 + o(t^2\log t) \quad , \quad \xi = t^2\log t\xi_1 + t^2\xi_2 + o(t^2), \tag{5.9}$$

$$U = t\log tU_1 + tU_2 + o(t) \qquad , \qquad V = t\log tV_1 + tV_2 + o(t) \tag{5.10}$$

as $t \to 0$ with $X, Y = O(1)$. If we substitute the above expansions into the inner region problem (5.6)-(5.7), at the leading order we obtain

$$\left.\begin{aligned} U_{1,X} + V_{1,Y} = 0, \\ U_1 - 2XU_{1,X} - 2YU_{1,Y} - U_1U_{1,X} - V_1U_{1,Y} = P_{1,X}, \\ V_1 - 2XV_{1,X} - 2YV_{1,Y} - U_1V_{1,X} - V_1V_{1,Y} = P_{1,Y}, \end{aligned}\right\} \tag{5.11}$$

in the domain $0 \leq Y < \infty$, $X > -\xi_1$, subject to the free-surface conditions on $X = -\xi_1$,

$$U_1 = 2\xi_1 - 2Y\xi_{1,Y} - V_1\xi_{1,Y} \quad \text{and} \quad P_1 = 0. \tag{5.12}$$

Appropriate matching conditions are by the equations (5.8) that

$$P_1 \sim \frac{4}{\pi}X \quad , \quad \xi_1 \sim \frac{2}{\pi} \quad , \quad U_1 \sim \frac{4}{\pi} \quad , \quad V_1 \sim 0 \qquad \text{as } (X^2 + Y^2)^{\frac{1}{2}} \to \infty. \tag{5.13}$$

These matching conditions must be the solution or a part of the solution of the leading order problem. By this fact, the exact solution to this problem is

$$U_1 \equiv \frac{4}{\pi} \quad , \quad V_1 \equiv 0 \quad , \quad P_1 \equiv \frac{4}{\pi}\left(X + \frac{2}{\pi}\right) \quad , \quad \xi_1 \equiv \frac{2}{\pi}. \tag{5.14}$$

Similarly, the next equations in the hierarchy generated by our perturbation process gives the second order problem,

$$\left.\begin{aligned} U_{2,X} + V_{2,Y} = 0, \\ \tfrac{4}{\pi} + U_2 - 2XU_{2,X} - 2YU_{2,Y} - \tfrac{4}{\pi}U_{2,X} = P_{2,X}, \\ V_2 - 2XV_{2,X} - 2YV_{2,Y} - \tfrac{4}{\pi}V_{2,X} = P_{2,Y} - 1, \end{aligned}\right\} \tag{5.15}$$

to be solved in the fixed domain $0 \leq Y < \infty$, $X > -\frac{2}{\pi}$. The free-surface conditions on $X = -\frac{2}{\pi}$ are

$$P_2 = \frac{4}{\pi}\xi_2 \quad , \quad U_2 = \frac{2}{\pi} + 2\xi_2 - 2Y\xi_{2,Y} \tag{5.16}$$

and matching conditions are by the equations (5.8) that

$$P_2 \sim \frac{2}{\pi}\left\{(\lambda - 1)X + \log RX + (\frac{\pi}{2} - \theta)Y\right\} \qquad , \qquad \xi_2 \sim \frac{1}{\pi}(\lambda + \log Y), \tag{5.17}$$

$$U_2 \sim \frac{2}{\pi}\lambda - 1 + \log R \qquad , \qquad V_2 \sim -\frac{2}{\pi}\theta, \tag{5.18}$$

as $(X^2 + Y^2)^{\frac{1}{2}} \to \infty$. Where $\lambda = \log(-\log t) + \log(\frac{\pi}{4})$ is regarded as a constant in the light of the Van Dyke matching principle as applied to series containing logarithms. Here (R, θ) are the usual polar coordinates with respect to the Cartesian coordinates (X,Y).

An exact solution to the above quarter-plane problem (5.15)-(5.18) can be found using an *integral transform method*.

5.3. Solution of the Second Order Inner Region Problem

5.3.1. Reformulation and the Mellin Transform of the Boundary Value Problem

The boundary value problem in the quarter-plane defined by (5.15)-(5.18) is not straightforward to solve. To proceed we define new dependent and independent variables by

$$u = U_2 + \frac{4}{\pi}, \quad v = V_2, \quad p = P_2 - Y, \quad \xi = \xi_2, \quad X = x - \frac{2}{\pi}, \quad y = Y.$$

This reformulation gives the more symmetric system of equations for (5.15)-(5.16),

$$\left.\begin{aligned} u_x + v_y &= 0, \\ u - 2xu_x - 2yu_y &= p_x, \\ v - 2xv_x - 2yv_y &= p_y \end{aligned}\right\} \tag{5.19}$$

in the domain $0 \leq y < \infty$, $0 < x < \infty$ and subject to the free-surface conditions

$$p(0, y) = \frac{4}{\pi}\xi - y\,, \qquad u(0, y) = \frac{6}{\pi} + 2\xi - 2y\xi_y\,. \tag{5.20}$$

Eliminating the pressures in (5.19) gives a vorticity (ζ) equation in the form

$$\zeta + 2x\zeta_x + 2y\zeta_y = 0, \tag{5.21}$$

where $\zeta = u_y - v_x$. Equation (5.21) is readily solved by the method of characteristics, which shows that $d\zeta/dx = -\zeta/2x$ on the curves $dy/dx = y/x$. As the vorticity is bounded and the outer flow is irrotational the only solution of (5.21) is $\zeta \equiv 0$.

This enables us to introduce a velocity potential ϕ such that $u = \phi_x$, $v = \phi_y$. Upon integration of the equations in (5.19) we find that $p = p_0 + 3\phi - 2x\phi_x - 2y\phi_y$, where p_0 is a constant pressure which could be found by higher-order matching. By this velocity potential, our quarter-plane problem (5.15)-(5.18) becomes

$$\left.\begin{gathered} \boldsymbol{\Delta}\phi = 0 \\ \text{with the free-surface conditions on } x = 0 \text{ in the form} \\ p_0 + 3\phi - 2y\phi_y = \frac{4}{\pi}\xi - y\,, \\ \phi_x = \frac{6}{\pi} + 2\xi - 2y\xi_y \\ \text{the free-surface condition on } y = 0 \\ \phi_y = 0 \\ \text{and a matching condition as } (x^2+y^2)^{\frac{1}{2}} \to \infty, \\ \phi \sim \frac{2}{\pi}\left\{x\log(x^2+y^2)^{\frac{1}{2}} - y\tan^{-1}(\frac{y}{x}) + (\lambda+1)x\right\} + o((x^2+y^2)^{\frac{1}{2}}), \\ \xi \sim \frac{1}{\pi}(\log y + \lambda) + o(1). \end{gathered}\right\} \tag{5.22}$$

Since ϕ is unbounded as $r \to \infty$, some further manipulations are necessary before using integral transforms to solve this linear boundary value problem. We use standard polar coordinates (r, θ) based at the Cartesian origin and redefine the potential and free surface by

$$\left.\begin{aligned} \phi = \overline{\phi} + \frac{2}{\pi}\Big\{ r\cos\theta\log r - \theta r\sin\theta + (\lambda+1)r\cos\theta \Big\}, \\ \xi = \overline{\xi} + \frac{1}{\pi}(\log r + \lambda), \end{aligned}\right\} \tag{5.23}$$

so that $\overline{\phi}$ is harmonic with $\overline{\phi}_\theta(r,0) = 0$ and the free-surface conditions on $\theta = \frac{\pi}{2}$ can be written from (5.22) by using (5.23) as

$$p_0 + 3\overline{\phi} - 2r\overline{\phi}_r = \frac{4}{\pi}\Big\{ \overline{\xi} + \frac{1}{\pi}(\log r + \lambda) \Big\}, \tag{5.24}$$

$$\frac{1}{r}\overline{\phi}_\theta = -2\overline{\xi} + 2r\overline{\xi}_r. \tag{5.25}$$

Here we can easily see the conditions $\overline{\xi} = o(1)$ and $\overline{\phi} = o(r)$ apply as $r \to \infty$. The condition $\overline{\phi} = o(r)$ is still not good enough for a transform technique to be applied. Thus, if we make an estimation by taking into account the problem, a coordinate expansion must be in the form $\overline{\phi} = A\log r + B + O(1/r)$ for $r \gg 1$. By substituting this estimated expansion into the problem, we get the coordinate expansion as $\overline{\phi} = (4/3\pi^2)\log r + B + O(1/r)$ where B is a constant related to p_0 and λ, and $\overline{\xi} = O(1/r^2)$. $(B = (4/3\pi^2)\lambda + (8/9\pi^2) - (p_0/3))$. We now make a further redefinition of the potential (to obtain both the potential and free-surface vanishing as $r \to \infty$) using

$$\phi^* = \overline{\phi} - \frac{4}{6\pi^2}\log(1+r^2) - B, \qquad \xi^* = \overline{\xi}. \tag{5.26}$$

Here we write $\log(1+r^2)$ instead of $\log r$ as $r \to \infty$ to avoid introducing a singularity for $r \to 0$. Then we obtain $\phi^* = O(1/r)$ as $r \to \infty$ and the following boundary value problem:

$$\left.\begin{array}{c} \boldsymbol{\Delta}\phi^* = -\dfrac{4}{6\pi^2}\boldsymbol{\Delta}\log(1+r^2) \\ \\ \text{subject to } \phi^*_\theta = 0 \text{ and free-surface conditions on } \theta = \dfrac{\pi}{2} \\ 3\phi^* - 2r\phi^*_r = \dfrac{4}{\pi}\xi^* + F^*(r), \\ \dfrac{1}{r}\phi^*_\theta = -2\xi^* + 2r\xi^*_r, \\ \\ \text{where} \qquad F^*(r) = \dfrac{4}{3\pi^2}\left\{3\log r - \dfrac{3}{2}\log(1+r^2) + \dfrac{2r^2}{1+r^2} - 2\right\}. \end{array}\right\} \tag{5.27}$$

We now construct a solution to this problem with $\phi^* = O(1)$, $\xi^* = O(\log r)$ as $r \to 0$ and $\phi^* = O(1/r)$, $\xi^* = O(1/r^2)$ as $r \to \infty$ by using Mellin transforms.

The *Mellin transform* is a type of Fourier transform which can be derived by some substitutions from the Fourier transform (see Appendix C). The Mellin Transform of ϕ^* and ξ^* are defined in the usual way as

$$\mu[\phi^*(r,\theta)] \quad = \quad \hat{\phi}(p,\theta) = \int_0^\infty r^{p-1}\phi^*(r,\theta)dr, \tag{5.28}$$

$$\mu[\xi^*(r)] \quad = \quad \hat{\xi}(p) = \int_0^\infty r^{p-1}\xi^*(r)dr. \tag{5.29}$$

Given $\phi^* = O(1)$, $\xi^* = O(\log r)$ as $r \to 0$ and $\phi^* = O(1/r)$, $\xi^* = O(1/r^2)$ as $r \to \infty$ we expect $\hat{\phi}$ to exist and be analytic in the strip $0 < Re(p) < 1$ of the complex p-plane; $\hat{\xi}$ will similarly be analytic in $0 < Re(p) < 2$.

Taking transforms of both sides of the equation $\boldsymbol{\Delta}\phi^* = -\frac{4}{6\pi^2}\boldsymbol{\Delta}\log(1+r^2)$ and by using 'integration by parts' (see Appendix C), we have

$$\left\{\frac{\partial^2}{\partial\theta^2} + (p-2)^2\right\}\hat{\phi}(p-2,\theta) = \frac{2(p-2)}{3\pi\sin(\frac{\pi}{2}p)}$$

which can be equivalently written as

$$\left\{\frac{\partial^2}{\partial\theta^2} + p^2\right\}\hat{\phi}(p,\theta) = -\frac{2p}{3\pi\sin(\frac{\pi}{2}p)}\,. \tag{5.30}$$

The general solution of this ordinary differential equation can be obtained by the method of 'variation of parameters' as

$$\hat{\phi}(p,\theta) = A(p)\sin(p\theta) + B(p)\cos(p\theta) - \frac{2}{3\pi p\sin(\frac{\pi}{2}p)}\,. \tag{5.31}$$

Similarly taking the Mellin transform of the boundary condition $\phi^*_\theta = 0$ on $\theta = 0$ gives the transformed boundary condition $\hat{\phi}_\theta = 0$ on $\theta = 0$. Transforming the free-surface conditions on $\theta = \frac{\pi}{2}$ results in

$$(3+2p)\hat{\phi}(p,\frac{\pi}{2}) = \frac{4}{\pi}\hat{\xi}(p) - \frac{4(3+2p)}{6\pi p\sin(\frac{\pi}{2}p)}, \tag{5.32}$$

$$\hat{\phi}_\theta(p-1,\frac{\pi}{2}) = -2(1+p)\hat{\xi}(p). \tag{5.33}$$

Application of the boundary condition $\hat{\phi}_\theta = 0$ on $\theta = 0$ to the general solution (5.31) gives $A(p) = 0$. Then by the boundary conditions (5.32) and (5.33) we get two equations respectively

$$B(p) = \frac{4\hat{\xi}(p)}{\pi(3+2p)\cos(\frac{\pi}{2}p)}\,, \qquad B(p-1) = -\frac{2(1+p)\hat{\xi}(p)}{(p-1)\cos(\frac{\pi}{2}p)}$$

which yields the following difference equation

$$\frac{B(p)}{B(p-1)} = -\frac{2(p-1)}{\pi(3+2p)(1+p)}. \tag{5.34}$$

The solution to this difference equation is readily obtained by the standard methods (see Appendix D) as

$$B(p) = \frac{b(p)(-1)^p\Gamma(p)}{\pi^p\Gamma(p+\frac{5}{2})\Gamma(p+2)}\,, \tag{5.35}$$

where $b(p)$ is a solution of $b(p)/b(p-1) = 1$ and is as yet undetermined. This solution gives the transform of the free-surface elevation,

$$\hat{\xi}(p) = -\frac{b(p)(-1)^{p-1}\cos(\frac{\pi}{2}p)}{2p(p+1)\pi^{p-1}\Gamma(p+\frac{3}{2})}\,. \tag{5.36}$$

To determine $b(p)$ and hence complete the transform solution we firstly examine the behaviour of $\hat{\xi}(p)$ as $|p| \to \infty$ and choose $b(p)$ so as to ensure convergence of the Mellin inversion integral. Using Stirling's approximation to the Γ-function for large $|p| = |\mu+i\tau|$ (see Appendix E) we have

$$\hat{\xi}(\mu+i\tau) = \begin{cases} O\left(\dfrac{b(\mu+i\tau)}{\tau^{\mu+3}e^{i\tau\log\tau}}\right), & \tau \to +\infty \\ \\ O\left(\dfrac{b(\mu+i\tau)e^{-2\pi\tau}}{\tau^{\mu+3}e^{i\tau\log\tau}}\right), & \tau \to -\infty. \end{cases} \tag{5.37}$$

It is clear from this behaviour that to ensure convergence of the inversion integral we require $\mu > -3$ and

$$b(\mu+i\tau) = \begin{cases} O(1), & \tau \to +\infty \\ O(e^{2\pi\tau}), & \tau \to -\infty. \end{cases} \tag{5.38}$$

A function of period 1 which has this property is

$$b(p) = \frac{C}{(-1)^p \sin\pi p} \tag{5.39}$$

where C is a constant. With this justification for the choice of $b(p)$ and by using the following Mellin inversion formulas of $\hat{\xi}(p)$ and $\hat{\phi}(p,\theta)$

$$\begin{aligned} \mu^{-1}[\hat{\xi}(p)] &= \xi^*(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-p}\hat{\xi}(p)dp\,, \\ \mu^{-1}[\hat{\phi}(p,\theta)] &= \phi^*(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-p}\hat{\phi}(p,\theta)dp \end{aligned}$$

we have,

$$\xi^*(r) = \frac{C\pi}{4}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(1/\pi r)^p}{p(1+p)sin(\frac{\pi}{2}p)\Gamma(p+\frac{3}{2})}dp \tag{5.40}$$

for $\quad (0 < c < 2)\,,$

$$\phi^*(r,\theta) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\left\{\frac{C\cos p\theta}{\pi^p\sin(\pi p)p(1+p)\Gamma(p+\frac{5}{2})} - \frac{2}{3\pi\sin(\frac{\pi}{2}p)}\right\}r^{-p}dp \tag{5.41}$$

for $\quad (0 < d < 1)\,.$

Of particular interest now is the form of free-surface that this integral solution represents. The line integral (5.41) may be turned into a counter integral in $Re(p) > 0$ by noting that the integrand decays on the semicircle $p = Re^{i\theta}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and as a simple application of residue theorem (see Appendix F), gives

$$\xi^*(r) = -\frac{C}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(1/\pi r)^{2n}}{2n(2n+1)\Gamma(2n+\frac{3}{2})}\,. \tag{5.42}$$

The ratio test reveals that this series is convergent for all $r \neq 0$. In fact for large r the series is asymptotic and $\xi^* = O(1/r^2)$ clearly from (5.42) as expected. At $r = 0$ the above series diverges and a rather different approach to the evaluation of ξ^* is needed for small r as we can see from the tables of ξ^* (Table 5.1, 5.2 and Figure 5.1)

Table 5.1. Table of ξ^* as $r \to \infty$ for small r

ξ^*	$r = 10^{-5}$	$r = 10^{-3}$	$r = 10^{-2}$
$n = 1$	$1,43340153 \times 10^7$	$1433,40153$	$14,3340153$
$n = 2$	$-2,766360613 \times 10^{14}$	$-2,764927354 \times 10^6$	$-262,3020603$
$n = 3$	$3,73347886 \times 10^{21}$	$3,730714209 \times 10^9$	$3471,177076$
$n = 4$	$-3,461390054 \times 10^{28}$	$-3,457659713 \times 10^{12}$	$-31142,72719$

Table 5.2. Table of ξ^* as $r \to \infty$ for large r

ξ^*	$r = 10^{-1}$	$r = 0,5$	$r = 1$	$r = 2$
$n = 1$	$0,143340153$	$0,005733606121$	$0,00143340153$	$0,0003583503826$
$n = 2$	$0,1156765455$	$0,005689344349$	$0,00143063517$	$0,000358177485$
$n = 3$	$0,1194100246$	$0,005689583292$	$0,001430638903$	$0,0003581775434$
$n = 4$	$0,1190638856$	$0,005689582406$	$0,0014306389$	$0,0003581775434$
$n = 5$	$0,1190868988$	$0,005689582408$	$0,0014306389$	
$n = 6$	$0,119085755$	$0,005689582408$		
$n = 7$	$0,119085799$			
$n = 8$	$0,1190857977$			
$n = 9$	$0,1190857977$			

Since the line integral can not be made into a contour integral by addition of a semicircle in the left-p-plane (due to the growth in the gamma function) we consider a rectangular contour as shown in Figure F.2. The contribution from the line segments L_1 and L_2 can be made arbitrarily small owing to the estimate (5.37) and the use of ML lemma. Accordingly we obtain

$$\xi^*(r) = \frac{C\pi}{4}\left\{\frac{1}{2\pi i}\int_{-c'-i\infty}^{-c'+i\infty}\frac{(1/\pi r)^p}{p(p+1)\sin(\frac{\pi}{2}p)\Gamma(p+\frac{3}{2})}dp + Res(p=0,-1)\right\} \quad (5.43)$$

where $1 < c' < 2$. This integral is bounded above by

$$\left|\frac{1}{2\pi i}\int_{-c'-i\infty}^{-c'+i\infty}\frac{(1/\pi r)^p}{p(p+1)\sin(\frac{\pi}{2}p)\Gamma(p+\frac{3}{2})}dp\right| \leq D(\pi r)^{c'} \quad (5.44)$$

where D is an O(1) constant. Evaluating the residues at the double pole and single pole at $p = 0, -1$ leads to the following asymptotic expansion, valid as $r \to 0$,

$$\xi^*(r) = \frac{C\pi}{4}\left\{\frac{2}{\pi\Gamma(\frac{3}{2})}\left(\log(\frac{1}{\pi r}) - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})}\right)\right\} + o(r). \quad (5.45)$$

We can see this expression of ξ^* is valid for small r from the Table (5.3).

Table 5.3. Table of ξ^* as $r \to 0$

	$r = 10^{-2}$	$r = 10^{-3}$	$r = 10^{-4}$	$r = 10^{-10}$	$r = 10^{-100}$
ξ^*	$0,7794213341$	$1,505288349$	$2,23751709$	$6,635052144$	$72,59925604$

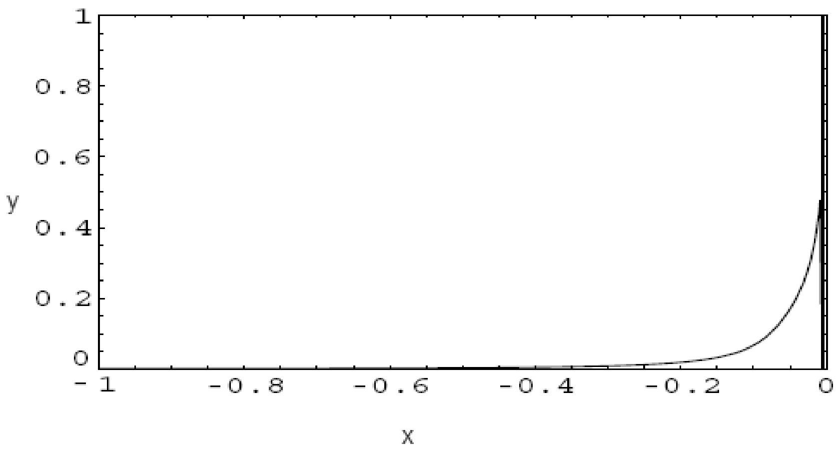

Figure 5.1. The function ξ^* as $r \to \infty$

Recalling that the physical free-surface elevation was $\xi_2 = \xi^* + (1/\pi)(\log r + \lambda)$, we can eliminate the logarithmic term by the choice $C = 1/\sqrt{\pi}$. Thus we get the form of free surface $\xi^*(r)$ for small r

$$\xi_2 = \frac{1}{\pi}\left\{ \log(-\log t) + \log(\frac{1}{4}) - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} + \frac{\pi^2 r}{4} \right\} + o(r), \tag{5.46}$$

and the form of free surface $\xi^*(r)$ for large r

$$\xi_2 = \frac{1}{\pi}\left\{ \log r + \log(-\log t) + \log(\frac{\pi}{4}) \right\} - \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n (1/\pi r)^{2n}}{2n(2n+1)\Gamma(2n+\frac{3}{2})}. \tag{5.47}$$

The solution is found to contain no singularities, so that only two asymptotic regions are necessary and confirms our choice of solution of the leading order problem. The correction to the leading order free-surface elevation, $\xi_2(Y)$ can be written as

$$\xi_2(Y) = \frac{1}{\pi}(\log Y + \lambda) + \frac{\sqrt{\pi}}{4}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(1/\pi Y)^p}{p(1+p)sin(\frac{\pi}{2}p)\Gamma(p+\frac{3}{2})}dp \qquad (5.48)$$

with $0 < c < 2$. The integral appearing in (5.48) is of $O(1/Y^2)$ as $Y \to \infty$ so that $\xi_2(Y)$ matches the outer solution as $Y \to \infty$ satisfactorily as we see from the Table (5.4).

Table 5.4. Comparison of the solutions of $\xi(y,t)$ near intersection point

	Outer Solution	Inner Solution
$y = -0.4$	-0.000343406	-0.000383189
$y = -0.6$	-0.000572539	-0.00058969
$y = -0.8$	-0.000938495	-0.000942706
$y = -0.9$	-0.00129468	-0.00129572
$y = -0.99$	-0.00246841	-0.00246781
$y = -0.999$	-0.00364112	-0.00358329
$y = -0.9999$	-0.00481382	-0.003893
$y = -0.99991$	$-0,00486748$	-0.0038951
$y = -0.99992$	-0.00492746	-0.0038974
$y = -0.99993$	-0.00499547	-0.0039002
$y = -0.99994$	-0.00507398	-0.00390249
$y = -0.99995$	-0.00516683	-0.00390493
$y = -0.99996$	-0.00528048	-0.00390739
$y = -0.99997$	-0.00542699	$+0.8004156$
$y = -0.99998$	-0.0056335	1.06512×10^9

And as $Y \to 0$ we can show that

$$\xi_2(Y) = \frac{1}{\pi}\left\{ \log(-\log t) + \log(\frac{1}{4}) - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} + \frac{\pi^2 Y}{4} \right\} + o(Y), \qquad (5.49)$$

Equation (5.49) gives the free-surface in a linear manner.

By using the inner solutions of ξ, we have the following graph. Figure 5.2 is drawn by using inner solution of ξ as $Y \to \infty$ between $(-0.995, -0.99997)$ and inner solution of ξ as $Y \to 0$ between $(-0.99997, -1)$ for $t = 0.05366531459995$, $t = 0.0491934955049954$, $t = 0.0447213595499958$ respectively.

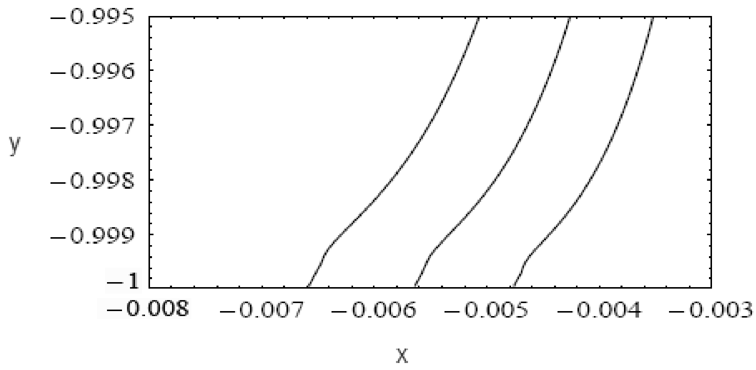

Figure 5.2. The function ξ in inner region

CHAPTER 6

CONCLUSION

In this work asymptotic solution of the two-dimensional dam-break problem was studied. We derived the leading-order asymptotic solution of the two-dimensional dam-break problem by the same methodology of the paper King & Needham (1994) and get the same solution as Korobkin & Yilmaz (2009).

In the first part, starting from mathematical formulation of the dam-break problem, the leading-order dam-break problem is solved and the singularity near the intersection (corner) point is analyzed. This introduced the necessity of the inner solution. The behavior of the solution (outer solution) near the intersection point analyzed in order to construct an inner region and thereby an inner solution.

In the second part, the dimensions of the inner region are specified appropriately which reveals a leading-order and correction to the leading order in the inner region. The exact solution to the leading-order inner region problem was derived by using the corresponding asymptotic matching conditions of the outer solution. Using an integral transform method, the correction to the leading order is obtained in two form which valid for small r and large r.

Finally, we compare the outer solution and the inner solution for large r of the free-surface ξ , such that they matche on a small part of the neighbourhood of the intersection point as we see from table (5.4). We sketched the graph of the free-surface near the intersection point for $t = 0.05366531459995$ (see Figure 6.1).

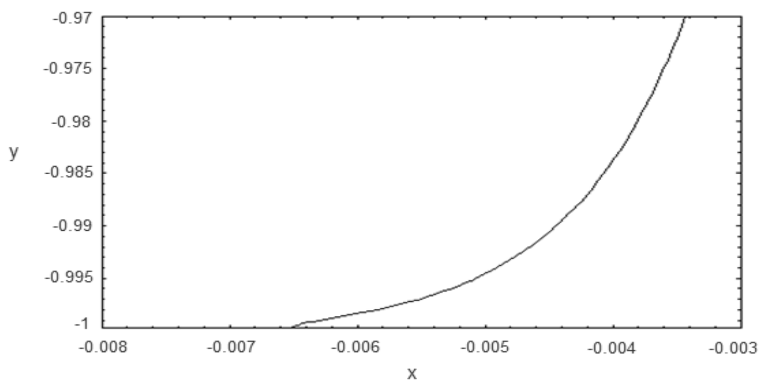

Figure 6.1. The function ξ with inner and outer solutions

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APPENDIX A

SEPARATION OF VARIABLES

A.1. Solution of the Leading Order Outer Problem Using Separation of Variables

Problem:

$$\Delta p_0 = 0,$$

$$p_{0,y}(x,-1) = -1\,, \quad p_0(x,0) = 0\,, \quad p_0(0,y) = 0$$

with $u_1, v_1 \to 0$ and $p_0 \to -y$ as $x \to \infty$.

Solution:

If we replace $p_0(x,y) = X(x)Y(y)$ into equation $\triangle p_0 = 0$ and dividing $X(x)Y(y)$ we get,

$$\frac{X_{xx}}{X} + \frac{Y_{yy}}{Y} = 0$$

or equivalently

$$\frac{X_{xx}}{X} = -\frac{Y_{yy}}{Y} = \lambda^2 \qquad \text{for some constant } \lambda \geq 0$$

which means

$$X_{xx}(x) - \lambda^2 X_x = 0, \tag{A.1}$$

$$Y_{yy}(y) + \lambda^2 Y_y = 0. \tag{A.2}$$

From the solution of equations (A.1) and (A.2) we get,

$$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}, \tag{A.3}$$

$$Y(y) = c_3 y + c_4 \sin(\lambda y) \tag{A.4}$$

or equivalently

$$p_0(x,y) = c_5 \sin(\lambda y)e^{\lambda x} + c_6 \sin(\lambda y)e^{-\lambda x} \qquad \text{for } \lambda > 0. \tag{A.5}$$

Applying the boundary condition at infinity, we get $c_5 = 0$, then

$$p_0(x,y) = c_3 y + c_6 \sin(\lambda y)e^{-\lambda x} \qquad \text{for } \lambda \geq 0. \tag{A.6}$$

Applying the boundary condition at $y = -1$, we get $c_3 = -1$ and $\lambda = (2n+1)\frac{\pi}{2}$ for $n = 0, 1, \ldots$ Similarly if the boundary condition at $x = 0$ is applied to the equation (A.6), we get $c_6 = \dfrac{8(-1)^n}{(2n+1)^2}\pi^2$ for $n = 0, 1, \ldots$ which gives the series solution of the above boundary value problem.

$$p_0(x,y) = -y + \sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)^2\pi^2} \sin\left((2n+1)\frac{\pi}{2}y\right)e^{-(2n+1)\frac{\pi}{2}x}\,. \tag{A.7}$$

APPENDIX B

SUMMATION OF SERIES

B.1. Useful Identities for the Summation

The below identities are useful on the steps of the exact summation of the series:

- $\tan^{-1}(ix) = i\tanh^{-1}(x)$,
- $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$,
- $\left|\frac{1-e^{ix}}{1+e^{ix}}\right| = \tan(\frac{x}{2})$.

B.2. Summation of the Leading Order Solutions of Free Surfaces in Outer Region

Since $\eta_2(x) = \frac{1}{2}(-\frac{\partial P_0}{\partial y}(x,0) - 1)$;

$$\begin{aligned}\eta_2(x) &= -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(e^{-\frac{\pi}{2}x})^{2n+1}}{2n+1} ,\\ \eta_2(x) &= -\frac{2}{\pi}\tan^{-1}(e^{-\frac{\pi}{2}x}) .\end{aligned}$$

Since $\xi_2(y) = -\frac{1}{2}\frac{\partial P_0}{\partial x}(0,y)$;

$$
\begin{aligned}
\xi_2(y) &= \frac{2}{\pi} Im \sum_{n=0}^{\infty} \frac{(-1)^n (e^{i\frac{\pi}{2}y})^{2n+1}}{2n+1} \,, \\
&= \frac{2}{\pi} Im \tan^{-1}(e^{i\frac{\pi}{2}y}) \,, \\
&= \frac{2}{\pi} Im[-i \tanh^{-1}(e^{i\frac{\pi}{2}(1+y)})] \,, \\
&= \frac{2}{\pi} Im[-\frac{i}{2} \log \left(\frac{1+e^{i\frac{\pi}{2}(1+y)}}{1-e^{i\frac{\pi}{2}(1+y)}} \right)] \,, \\
&= \frac{2}{\pi} Im[\frac{i}{2} \log \left(\frac{1-e^{i\frac{\pi}{2}(1+y)}}{1+e^{i\frac{\pi}{2}(1+y)}} \right)] \,, \\
&= \frac{2}{\pi}\frac{1}{2} \log \left| \frac{1-e^{i\frac{\pi}{2}(1+y)}}{1+e^{i\frac{\pi}{2}(1+y)}}) \right| \,, \\
&= \frac{1}{\pi} \log \big(\tan \frac{\pi}{4}(1+y) \big) \,.
\end{aligned}
$$

APPENDIX C

MELLIN TRANSFORM

C.1. Derivation of the Mellin Transform from Standard Fourier Transform

The standard Fourier transform of a function $g(\xi)$ is defined by

$$\mathcal{F}[g(\xi)] = G(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\xi=-\infty}^{\infty} e^{-i\alpha\xi} g(\xi) d\xi \tag{C.1}$$

and the inverse Fourier transform of $G(\alpha)$ is defined by

$$\mathcal{F}^{-1}[G(\alpha)] = g(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\alpha=-\infty}^{\infty} e^{i\alpha\xi} G(\alpha) d\alpha. \tag{C.2}$$

If we do the substitutions $\xi = \log x$ and $\alpha = ip - ic$ (p is a complex number and c is a real number) to the equations (C.1) and (C.2) we get

$$G(\alpha) = G(ip - ic) = \frac{1}{\sqrt{2\pi}} \int_{x=0}^{\infty} x^{p-c-1} g(\log x) dx, \tag{C.3}$$

$$g(\xi) = g(\log x) = \frac{1}{\sqrt{2\pi} i} \int_{p=c-i\alpha}^{c+i\alpha} x^{c-p} G(ip - ic) dp. \tag{C.4}$$

Moreover if we do the substitutions $\frac{1}{\sqrt{2\pi}} x^{-c} g(\log x) = f(x)$ and $G(ip - ic) = \hat{f}(p)$ to the equations (C.3) and (C.4) we get the definition of the Mellin transform and the inverse Mellin transform

$$\hat{f}(p) = \int_{x=0}^{\infty} x^{p-1} f(x) dx, \tag{C.5}$$

$$f(x) = \frac{1}{2\pi i} \int_{p=c-i\alpha}^{c+i\alpha} x^{-p} \hat{f}(p) dp. \tag{C.6}$$

We can also denote the Mellin transform of the function f(x) as $\mu[f(x)]$ and the inverse Mellin transform of the function $\hat{f}(p)$ as $\mu^{-1}[\hat{f}(p)]$.

C.2. Mellin Transform of the Equation $\boldsymbol{\Delta}\phi^* = -\frac{4}{6\pi^2}\boldsymbol{\Delta}\log(1+r^2)$

By using the definition of Laplace operator in polar form

$$\boldsymbol{\Delta}\phi^* = \frac{\partial^2}{\partial r^2}\phi^* + \frac{1}{r}\frac{\partial}{\partial r}\phi^* + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^*.$$

Applying Mellin transform to this three terms, we have

$$\underbrace{\int_0^\infty r^{p-1}\frac{\partial^2}{\partial r^2}\phi^* dr}_{I_1} + \underbrace{\int_0^\infty r^{p-1}\frac{1}{r}\frac{\partial}{\partial r}\phi^* dr}_{I_2} + \underbrace{\int_0^\infty r^{p-1}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^* dr}_{I_3}$$

Using 'integration by parts',

$$\begin{aligned}
I_1 &= lim_{a\to\infty}[r^{p-1}\frac{\partial}{\partial r}\phi^*]\Big|_{r=0}^{a} - \int_0^\infty (p-1)r^{p-2}\frac{\partial}{\partial r}\phi^* dr \\
&= -(p-1)\left(lim_{a\to\infty}[r^{p-2}\frac{\partial}{\partial r}\phi^*]\Big|_{r=0}^{a} - \int_0^\infty (p-2)r^{p-3}\phi^* dr\right) \\
&= (p-1)(p-2)\int_0^\infty r^{p-3}\phi^* dr \\
&= (p-1)(p-2)\hat{\phi}(p-2,0).
\end{aligned}$$

Similarly

$$\begin{aligned}
I_2 &= lim_{a\to\infty}[r^{p-2}\phi^*]\Big|_{r=0}^{a} - \int_0^\infty (p-2)r^{p-3}\phi^* dr \\
&= -(p-2)\int_0^\infty r^{p-3}\phi^* dr \\
&= -(p-2)\hat{\phi}(p-2,0)
\end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_0^\infty r^{p-3} \frac{\partial^2}{\partial\theta^2}\phi^* dr \\ &= \frac{\partial^2}{\partial\theta^2}\int_0^\infty r^{p-3}\phi^* dr \\ &= \frac{\partial^2}{\partial\theta^2}\hat{\phi}(p-2,0) \end{aligned}$$

Thus the left-hand side of the equation $\mathbf{\Delta}\phi^* = -\frac{4}{6\pi^2}\mathbf{\Delta}\log(1+r^2)$ becomes

$$\left\{\frac{\partial^2}{\partial\theta^2} + (p-2)^2\right\}\hat{\phi}(p-2,0)$$

Apply Mellin transform to the right-hand side,

$$-\frac{2}{3\pi^2}\int_0^\infty r^{p-1}\frac{4}{(1+r^2)^2}dr = -(p-2)\pi\csc(\frac{\pi}{2}p)$$

for $0 < Re(p) < 4$. Hence, Mellin transform of the equation $\mathbf{\Delta}\phi^* = -\frac{4}{6\pi^2}\mathbf{\Delta}\log(1+r^2)$ gives

$$\left\{\frac{\partial^2}{\partial\theta^2} + (p-2)^2\right\}\hat{\phi}(p-2,0) = \frac{2(p-2)}{3\pi\sin(\frac{\pi}{2}p)}.$$

APPENDIX D

DIFFERENCE EQUATIONS

Definition D.1 *The general form of the linear difference equation of first order is*

$$y_{k+1} - p_k y_k = q_k \ , \tag{D.1}$$

where p_k and q_k are given functions. If q_k is identically zero, then the difference equation becomes homogeneous, otherwise inhomogeneous.
The general solution of the equation (D.1) consists of the sum of the solution to the homogeneous equation and any particular solution of the inhomogeneous equation.

D.1. Solution of the $\mathbf{y_{k+1} = R(k) y_k}$ Type Difference Equations

Let R_k be a rational function of k. It can be represented as follows:

$$R_k = \frac{C(k-\alpha_1)(k-\alpha_2)\cdots(k-\alpha_n)}{(k-\beta_1)(k-\beta_2)\cdots(k-\beta_m)} \ ,$$

where C and the α's and β's are constants. Since

$$\Gamma(k+1-\alpha_i) = (k-\alpha_i)\Gamma(k-\alpha_i) \ ,$$

it is clear that the equation

$$y_{k+1} = R(k) y_k$$

has the solution

$$y_k = AC^k \frac{\Gamma(k-\alpha_1)\Gamma(k-\alpha_2)\cdots\Gamma(k-\alpha_n)}{\Gamma(k-\beta_1)\Gamma(k-\beta_2)\cdots\Gamma(k-\beta_m)},$$

where A is an arbitrary constant.

APPENDIX E

GAMMA FUNCTION AND ITS PROPERTIES

E.1. Definition of the Gamma Function

If the real part of the complex number z is positive $(Re(z) > 0)$, the Gamma function can be defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

E.2. Stirling's Approximation to the Gamma Function for Large z

$$\Gamma(z) \sim e^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \cdots \right)$$

as $z \to \infty$ in $|arg z| < \pi$.

APPENDIX F

RESIDUE THEOREM

F.1. Residue Theorem

If a function f has only a finite number of singular points interior to a given simple closed contour $\mathcal{C}$, then they must be isolated. The following theorem is a precise statement of the fact that if, moreover, f is analytic on $\mathcal{C}$ and $\mathcal{C}$ is described in the positive sense, then the value of the integral of f around $\mathcal{C}$ is $2\pi i$ times the sum of the residues at those singular points.

Theorem F.1 *If $\mathcal{C}$ is a positively oriented simple closed contour within and on which a function is analytic except for a finite number of singular points z_k ($k = 1, 2, \ldots, n$) interior to $\mathcal{C}$, then*

$$\int_{\mathcal{C}} f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z). \tag{F.1}$$

F.2. Evaluation of the Complex Integral (5.41) on the Semi-Circle

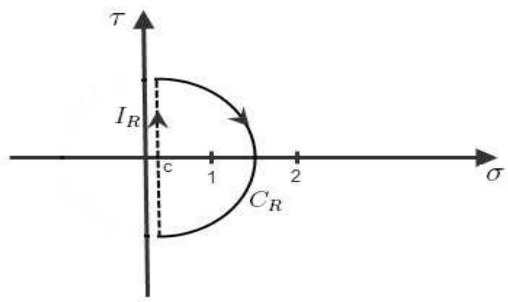

Figure F.1. The semi-circle contour.

By using the *Residue theorem*; integral over the semi-circle in the negative orientation is equals to $-2\pi i$ times sum of the residues at poles inside the semi-circle. We can easily see that the integral over $C_R \to 0$ as $R \to \infty$ such that

$$\Big| \int_{C_R} \frac{(1/\pi r)^p}{p(p+1)\sin\frac{\pi}{2}p\Gamma(p+\frac{3}{2})} dp \Big| \tag{F.2}$$

Let $p = Re^{i\theta}$, $-\pi/2 < \theta < \pi/2$, (F.2) becomes;

$$\Big| \int_{C_R} \frac{(1/\pi r)^{Re^{i\theta}}}{Re^{i\theta}(Re^{i\theta}+1)\sin\frac{\pi}{2}Re^{i\theta}\Gamma(Re^{i\theta}+\frac{3}{2})} d(Re^{i\theta}) \Big|$$

(by using Stirling's approximation of Gamma function for large p)

$$\begin{aligned}
&\leq \frac{\pi R(1/\pi r)^{R\cos\theta}}{R(1+Re^{i\theta})e^{-(R\cos\theta+\frac{3}{2})}(Re^{i\theta}+\frac{3}{2})} \\
&\leq \frac{\pi e^{\frac{3}{2}}(e/\pi r)^{R\cos\theta}}{(1+Re^{i\theta})e^{(R\cos\theta+1)}\log(Re^{i\theta+\frac{3}{2}})}
\end{aligned}$$

which tends to zero as $R \to \infty$.

For the integration over I_R, there is infinitely many poles at $p = 2n$, $n = 0, 1, 2, \ldots$ from sine function and poles at $n = 0, -1$. We calculate the residues at poles $p = 2n$, $n = 1, 2, \ldots$ which are in our semi-circle. Hence we get

$$\begin{aligned}
&\int_{c-i\infty}^{c+i\infty} \frac{(1/\pi r)^p}{p(p+1)\sin\frac{\pi}{2}p\Gamma(p+\frac{3}{2})} dp \\
&= \int_{I_R} \frac{(1/\pi r)^p}{p(p+1)\sin\frac{\pi}{2}p\Gamma(p+\frac{3}{2})} dp \\
&= -2\pi i \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n(1/\pi r)^{2n}}{2n(2n+1)\Gamma(2n+\frac{3}{2})}.
\end{aligned}$$

F.3. Evaluation of the Complex Integral (5.41) on the Rectangular Region

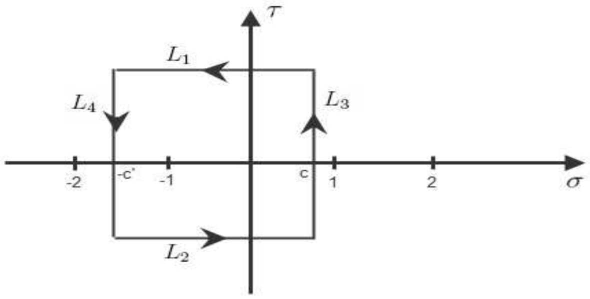

Figure F.2. The rectangular contour.

Integral over L_1 and L_2 tends to zero. Thus our integral (over L_3) equals to $\{$integral over $L_4\}+2\pi i\{$Res$(p = 0, -1)\}$. Since integral over L_4 is bounded by $D(\pi r)^{c'}$, we have

$$\int_{c-i\infty}^{c+i\infty} \frac{(1/\pi r)^p}{p(p+1)\sin\frac{\pi}{2}p\Gamma(p+\frac{3}{2})}dp = 2\pi i Res(p=0,-1) + o(r).$$

Applying two times *'l'hospital rule'* we get the residue at the double pole $p = 0$,

$$Res(p=0) = \frac{2}{\pi\Gamma(\frac{3}{2})}\left\{ \log(\frac{1}{\pi r}) - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} \right\}$$

and the residue at the single pole $p = -1$,

$$Res(p=-1) = \frac{\pi r}{\Gamma(\frac{1}{2})}$$

Hence

$$\int_{c-i\infty}^{c+i\infty} \frac{(1/\pi r)^p}{p(p+1)\sin\frac{\pi}{2}p\Gamma(p+\frac{3}{2})}dp = 2\pi i \frac{2}{\pi\Gamma(\frac{3}{2})}\left\{ \log(\frac{1}{\pi r}) - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} + \frac{\pi^2 r}{4} \right\} + o(r).$$