The Ablowitz–Ladik lattice system by means of the extended 
\((G'/G)\)-expansion method

İsmail Aslan
Department of Mathematics, Izmir Institute of Technology, Urla, Izmir 35430, Turkey

**Abstract**
We analyzed the Ablowitz–Ladik lattice system by using the extended \((G'/G)\)-expansion method. Further discrete soliton and periodic wave solutions with more arbitrary parameters are obtained. We observed that some previously known results can be recovered by assigning special values to the arbitrary parameters.

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1. Introduction
Nonlinear differential–difference equations (NDDEs) is a subject of immense breadth. Its spectrum ranges from the abstract through numerical techniques to many interesting applications. These applications range from such theoretical applications as the use of NDDEs in mechanical engineering, biophysics, condensed matter physics, and to many practical applications in such fields as atomic chains, molecular crystals, currents in electrical networks [1–4], etc. Many integrable NDDEs [5–8] are proposed in the literature. Contrary to difference equations which are being fully discretized, NDDEs are semi-discretized with some (or all) of their space variables discretized while time is usually kept continuous.

On the other hand, a considerable number of well-known analytic methods are successfully extended to NDDEs by the researchers [9–17]. However, no method obeys the strength and the flexibility for finding all solutions to all types of NDDEs. Recently, the \((G'/G)\)-expansion method [18] has become popular in the research community, and there has been a number of studies refining the initial idea [19–30]. The method is powerful in the sense that it takes full advantage of linear theory. It is reliable, efficient, and entirely algorithmic. Moreover, the \((G'/G)\)-expansion method delivers three types of exact solutions; hyperbolic, trigonometric, and rational functions. Not long ago, Zhang et al. [31] and Aslan [32] have successfully generalized the \((G'/G)\)-expansion method to some physically important NDDEs.

The objective of this study is to illustrate the applicability of the \((G'/G)\)-expansion method to the famous Ablowitz–Ladik lattice system. In order to get as widest results as possible, we add extra terms with negative powers to the ansatz considered in the standard \((G'/G)\)-expansion method. The rest of this paper is organized as follows. In Section 2, we describe our methodology for NDDEs, and state the main steps. In Section 3, we analyze the Ablowitz–Ladik lattice system. Finally, we give some concluding remarks in Section 4.

2. The extended \((G'/G)\)-expansion method for NDDEs
Consider a system of \(M\) polynomial NDDEs in the form
\[
P(u_{n+p_1}(x), \ldots, u_{n+p_k}(x), \ldots, u'_{n+p_1}(x), \ldots, u'_{n+p_k}(x), \ldots, u''_{n+p_1}(x), \ldots, u''_{n+p_k}(x)) = 0. \tag{1}
\]
where the dependent variable \( u_n \) have \( M \) components \( u_{\ell n} \) and so do its shifts, the continuous variable \( x \) has \( N \) components \( x_{\ell} \), the discrete variable \( n \) has \( Q \) components \( n_j \), the \( k \) shift vectors \( \mathbf{p}_j \in \mathbb{Z}^Q \), and \( \mathbf{u}^{(l)}(\mathbf{x}) \) denotes the collection of mixed derivative terms of order \( r \).

Step 1. For constructing traveling wave solutions to Eq. (1), we consider the wave transformation

\[
\mathbf{u}_{n+\mathbf{p}}(\mathbf{x}) = \mathbf{U}_n(\mathbf{p}, \xi_n), \quad \xi_n = \sum_{l=1}^Q d_l n_l + \sum_{j=1}^N C_j x_j + \zeta, \quad (s = 1, 2, \ldots, k),
\]

where the coefficients \( c_1, c_2, \ldots, c_N \) are constants to be determined and the phase \( \zeta \) are all constants. Then, Eq. (1) becomes

\[
P(\mathbf{U}_{n+\mathbf{p}}(\xi_n), \ldots, \mathbf{U}_{n+p_k}(\xi_n), \mathbf{U}_{n+p_1}(\xi_n), \mathbf{U}_{n+\mathbf{p}_k}(\xi_n), \mathbf{U}_{n+\mathbf{p}_2}(\xi_n), \ldots, \mathbf{U}_{n+\mathbf{p}_Q}(\xi_n)) = 0.
\]

Step 2. We initially predict the structure of the solutions \( \mathbf{U}_n = \mathbf{U}_n(\xi_n) \) to Eq. (3) in the finite series form

\[
\mathbf{U}_n = \sum_{l=-m}^m a_l \left( \frac{G}{C} \right)^l,
\]

where \( m \) (a positive integer) and \( a_l \)'s are constants to be determined and the function \( G = G(\xi_n) \) satisfies the auxiliary equation

\[
G'' + \lambda G' + \mu G = 0,
\]

where \( \lambda \) and \( \mu \) are constants and the primes denote derivatives with respect to \( \xi_n \). We note that the general solution of Eq. (5) is well known for us.

Step 3. A straightforward calculation leads to the identity

\[
\xi_{n+\mathbf{p}_j} = \xi_n + \phi_j, \quad \phi_j = p_{j1} n_1 + p_{j2} n_2 + \cdots + p_{jQ} n_Q,
\]

where \( p_{ji} \) is the \( j \)th component of the shift vector \( \mathbf{p}_j \). Thus, we can derive the expressions for the shift functions as follows:

\[
\mathbf{U}_{n+\mathbf{p}_j} = \sum_{l=-m}^m a_l \left( \frac{\lambda + C}{2} - \frac{\sqrt{\lambda^2 - 4\mu}}{2} \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \phi_j \right) \right)^l, \quad \lambda^2 - 4\mu > 0,
\]

\[
\mathbf{U}_{n+\mathbf{p}_j} = \sum_{l=-m}^m a_l \left( \frac{\lambda + C}{2} - \frac{\sqrt{\lambda^2 - 4\mu}}{2} \tan \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \phi_j \right) \right)^l, \quad \lambda^2 - 4\mu < 0,
\]

\[
\mathbf{U}_{n+\mathbf{p}_j} = \sum_{l=-m}^m a_l \left( \frac{\lambda + C}{2} - \frac{\sqrt{\lambda^2}}{2} \phi_j \right)^l, \quad \lambda^2 - 4\mu = 0.
\]

Step 4. From Eq. (3), we can easily determine the degree \( m \) of the ansatz (4) and the expressions (7a-c) by balancing the highest order nonlinear term \( s \) and the highest-order derivative term in \( \mathbf{U}_n \). Since \( \mathbf{U}_{n+\mathbf{p}_j} \) can be interpreted as being of degree zero in \( (G/C) \), the leading terms of \( \mathbf{U}_{n+\mathbf{p}_j} (\mathbf{p}_j \neq 0) \) will not affect the balance.

Step 5. Finally, substituting the ansatz (4) and the expressions (7a-c) along with (5) into Eq. (3), setting the coefficients of \( (G/C)^l \) \((l = 0, 1, 2, \ldots)\) to zero, we obtain a system of nonlinear algebraic equations from which the undetermined constants \( a_0, a_1, c_0, \lambda \) and \( \mu \) can be found explicitly. Then, by substituting these values into the ansatz (4), we derive traveling wave solutions to Eq. (1).

3. Solutions to the Ablowitz–Ladik lattice system

The Ablowitz–Ladik lattice system [8,10,33,34] is known as

\[
\begin{align*}
\frac{\partial u_n}{\partial t} &= (x + u_n v_n)(u_{n+1} + u_{n-1}) - 2x u_n, \\
\frac{\partial v_n}{\partial t} &= -(x + u_n v_n)(v_{n+1} + v_{n-1}) + 2x v_n,
\end{align*}
\]

where \( x \) is a constant. Eq. (8) was proposed by using inverse scattering method and has a rich mathematical structure. To look for traveling wave solutions of Eq. (8), we first let

\[
\begin{align*}
u_n &= U_n(\xi_n), \quad v_n = V_n(\xi_n), \quad \xi_n = dn + ct + \zeta,
\end{align*}
\]

where \( c, d \) are constants to be determined and \( \zeta \) denotes the phase shift. Then, Eq. (8) can be reduced to

\[
\begin{align*}
u_n &= U_n(\xi_n), \quad v_n = V_n(\xi_n), \quad \xi_n = dn + ct + \zeta,
\end{align*}
\]
where $U_n = U_n(\xi_n)$, $V_n = V_n(\xi_n)$ and the prime denotes derivative with respect to $\xi_n$. The balancing procedure in (10) leads to $m = 1$. Thus, we assume that the solution of Eq. (10) is in the form

$$
\begin{align*}
U_n &= a_0 + a_1 (\xi_n) + a_{-1} (\xi_n)^{-1}, \\
V_n &= b_0 + b_1 (\xi_n) + b_{-1} (\xi_n)^{-1},
\end{align*}
$$

(11)

where $a_0, a_1, a_{-1}, b_0, b_1,$ and $b_{-1}$ are constants to be specified.

Case 1: When $\lambda^2 - 4\mu > 0$, from (7a), we obtain

$$
\begin{align*}
U_{n\pm 1} &= \frac{1}{1 - \lambda^2 - 4\mu} \left( \frac{1 + \sqrt{\lambda^2 - 4\mu}}{2} \tan \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} d \right) \right)^l, \\
V_{n\pm 1} &= \frac{1}{1 - \lambda^2 - 4\mu} \left( \frac{1 + \sqrt{\lambda^2 - 4\mu}}{2} \tan \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} d \right) \right)^l.
\end{align*}
$$

(12a, 12b)

Substituting the ansatz (11) and the expressions (12a–b) along with Eq. (5) into Eq. (10), clearing the denominator and equating the coefficients of $(G'/\tilde{G})^{l}$ ($0 \leq l \leq 8$) to zero, we obtain a system of algebraic equations for $a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, c, d, \lambda,$ and $\mu$. Solving the resulted algebraic system (we will omit to display them for simplicity) simultaneously, we get the following solution sets and the corresponding hyperbolic function solutions to Eq. (8):

Case 1.1.

$$
\begin{align*}
c &= \mp 2 \sinh^2 (d \sqrt{-\mu}) / \sqrt{-\mu}, & a_0 &= \mp \sinh^2 (d \sqrt{-\mu}) / b_0, & b_{-1} &= \pm b_0 \sqrt{-\mu}, \\
(a_{-1} &= \mp \sqrt{2} \sinh^2 (d \sqrt{-\mu}) / b_0, & a_1 &= 0, & b_1 &= 0, & b_0 &= b_0, & d &= d, & \lambda &= 0, & \mu &= \mu),
\end{align*}
$$

(13)

$$
\begin{align*}
\begin{aligned}
u^r_{n,1}(t) &= \frac{\sinh^2 (d \sqrt{-\mu})}{b_0} \left( 1 \mp \frac{c}{c} \sinh \left( \sqrt{-\mu} a_0 \right) + \frac{c}{d} \cosh \left( \sqrt{-\mu} a_0 \right) \right), \\
\nu^r_{n,1}(t) &= b_0 \left( 1 \mp \frac{c}{c} \sinh \left( \sqrt{-\mu} a_0 \right) + \frac{c}{d} \cosh \left( \sqrt{-\mu} a_0 \right) \right),
\end{aligned}
\end{align*}
$$

(14)

where $\tilde{\xi}_n = dn \mp \left( 2 \sinh^2 (d \sqrt{-\mu}) / \sqrt{-\mu} \right) t + \zeta$ and $\mu < 0$, $\zeta$, $b_0$, $d$, $C_1$, $C_2$ are arbitrary constants.

Case 1.2.

$$
\begin{align*}
c &= \pm \sqrt{\mu} \sinh^2 (2d \sqrt{-\mu}) / \mu, & a_0 &= \pm \sqrt{\mu} \sinh^2 (2d \sqrt{-\mu}) / 2b_1, & a_{-1} &= -\sqrt{2} \sinh^2 (2d \sqrt{-\mu}) / 4b_1, \\
(b_0 &= \pm 2 \sqrt{-\mu} b_1, & b_{-1} &= \mp b_1, & a_1 &= \sqrt{\mu} \sinh^2 (2d \sqrt{-\mu}) / 4b_1, & b_1 &= b_1, & d &= d, & \lambda &= 0, & \mu &= \mu),
\end{align*}
$$

(15)

$$
\begin{align*}
\begin{aligned}
u^r_{n,2}(t) &= \frac{\sinh^2 (d \sqrt{-\mu})}{b_0} \left( c_1 \sinh \left( \sqrt{-\mu} a_0 \right) + c_2 \cosh \left( \sqrt{-\mu} a_0 \right) + c_1 \cosh \left( \sqrt{-\mu} a_0 \right) + c_2 \sinh \left( \sqrt{-\mu} a_0 \right) \right), \\
\nu^r_{n,2}(t) &= \sqrt{-\mu} b_1 \left( c_1 \sinh \left( \sqrt{-\mu} a_0 \right) + c_2 \cosh \left( \sqrt{-\mu} a_0 \right) + c_1 \cosh \left( \sqrt{-\mu} a_0 \right) + c_2 \sinh \left( \sqrt{-\mu} a_0 \right) \right),
\end{aligned}
\end{align*}
$$

(16)

where $\tilde{\xi}_n = dn \pm \left( \sqrt{\mu} \sinh^2 (2d \sqrt{-\mu}) / \mu \right) t + \zeta$ and $\mu < 0$, $\zeta$, $b_1$, $d$, $C_1$, $C_2$ are arbitrary constants.

Remark 1. The expressions (14) and (16) represent abundant traveling wave solutions to Eq. (8). If we let “$C_1 \neq 0$ and $C_2 = 0$” or “$C_1 = 0$ and $C_2 \neq 0$” in (14) respectively, then we get formal solitary wave solutions to Eq. (8) as follows

$$
\begin{align*}
\begin{aligned}
u^r_{n,3}(t) &= \frac{\sinh^2 (d \sqrt{-\mu})}{b_0} \left( 1 \mp \tanh \left( \sqrt{-\mu} \tilde{\xi}_n \right) \right), \\
\nu^r_{n,3}(t) &= b_0 \left( 1 \mp \tanh \left( \sqrt{-\mu} \tilde{\xi}_n \right) \right), \\
\nu^r_{n,4}(t) &= \frac{\sinh^2 (d \sqrt{-\mu})}{b_0} \left( 1 \mp \coth \left( \sqrt{-\mu} \tilde{\xi}_n \right) \right), \\
\nu^r_{n,4}(t) &= b_0 \left( 1 \mp \coth \left( \sqrt{-\mu} \tilde{\xi}_n \right) \right),
\end{aligned}
\end{align*}
$$

(17, 18)

where $\tilde{\xi}_n$ is as in (14). Moreover, we observe that our results (17) and (18) include the results of Baldwin et al. [10] and Wang [35] respectively. Thus, the solutions (14) and (16) are wider in the sense that they contain more arbitrary parameters.

Case 2: When $\lambda^2 - 4\mu < 0$, from (7b), we obtain
Substituting the ansatz (11) and the expressions (19a–b) along with Eq. (5) into Eq. (10), clearing the denominator and equating the coefficients of ($G/G$)'s corresponding trigonometric function solutions to Eq. (8):

where $n$ is as in (21). In addition, we note that our results (24) and (25) are not presented in [10,35].

\[ c = \mp 2i \pi \sin^2 \left( d \sqrt{d^2} \right) / \sqrt{d}. \ a_0 = -\alpha \sin^2 \left( d \sqrt{d^2} / b_0 \right), \ b_0 = b_0, \ a_{-1} = \pm i \sqrt{d} \pi \sin^2 \left( d \sqrt{d^2} / b_0 \right), \]

\[ a_1 = 0, \ b_1 = 0, \ a_{-1} = \pm i \sqrt{d} \pi \sin^2 \left( d \sqrt{d^2} / b_0 \right), \]

\[ u_{n,5}(t) = \frac{\sin \left( d \sqrt{d^2} \right)}{2} \left( -1 + i \cot \left( \sqrt{d} \xi_n \right) \right), \]

\[ v_{n,5}(t) = b_0 \left( 1 + i \cot \left( \sqrt{d} \xi_n \right) \right). \]

where $\xi_n = dn \mp \left( 2i \pi \sin^2 \left( d \sqrt{d^2} / \sqrt{d} \right) t + \zeta \right)$ and $\mu > 0, \xi, b_0, d, C_1, C_2$ are arbitrary constants.

**Case 2.2.**

\[ c = \pm i \pi \sin^2 \left( 2d \sqrt{d^2} / \sqrt{d} \right) \ a_0 = \mp i \pi \sin \left( 2d \sqrt{d^2} / \sqrt{d} \right) / 2 \sqrt{d} \ b_1, \ a_{-1} = \alpha \sin^2 \left( 2d \sqrt{d^2} / 4b_1 \right), \ b_0 = \pm 2i \sqrt{d} \]

\[ d = d, \ b_1 = -\mu b_1, \ a_{-1} = -\pi \sin^2 \left( 2d \sqrt{d^2} / 4b_1 \right), \]

\[ u_{n,6}(t) = -\frac{\sin \left( 2d \sqrt{d^2} \right)}{4b_1} \left( -C_1 \cos \left( \sqrt{d} \xi_n \right) + C_2 \sin \left( \sqrt{d} \xi_n \right) \right) / C_2 \cos \left( \sqrt{d} \xi_n \right) - C_1 \sin \left( \sqrt{d} \xi_n \right) \right) 

\[ + \left( C_1 \cos \left( \sqrt{d} \xi_n \right) - C_2 \sin \left( \sqrt{d} \xi_n \right) \right) 

\[ v_{n,6}(t) = \sqrt{d} b_1 \left( -C_1 \cos \left( \sqrt{d} \xi_n \right) + C_2 \sin \left( \sqrt{d} \xi_n \right) \right) / C_2 \cos \left( \sqrt{d} \xi_n \right) - C_1 \sin \left( \sqrt{d} \xi_n \right) \right) 

\[ + \left( C_1 \cos \left( \sqrt{d} \xi_n \right) - C_2 \sin \left( \sqrt{d} \xi_n \right) \right) \]

where $\xi_n$ is as in (21). In addition, we note that our results (24) and (25) are not presented in [10,35].

\[ U_{n+1} = \sum_{l=-1}^{1} a_l \left( \frac{\lambda}{2} + \frac{G^l + \lambda}{2} \right) \left( 1 \pm \frac{\lambda}{2} \right) d \right)^l, \]

\[ V_{n+1} = \sum_{l=-1}^{1} b_l \left( \frac{\lambda}{2} + \frac{G^l + \lambda}{2} \right) \left( 1 \pm \frac{\lambda}{2} \right) d \right)^l. \]

Substituting the ansatz (11) and the expressions (26a–b) along with Eq. (5) into Eq. (10), clearing the denominator and equating the coefficients of ($G/G$)'s corresponding trigonometric function solutions to Eq. (8):

From (7c), we obtain

\[ U_{n+1} = \sum_{l=-1}^{1} a_l \left( \frac{\lambda}{2} + \frac{G^l + \lambda}{2} \right) \left( 1 \pm \frac{\lambda}{2} \right) d \right)^l, \]

\[ V_{n+1} = \sum_{l=-1}^{1} b_l \left( \frac{\lambda}{2} + \frac{G^l + \lambda}{2} \right) \left( 1 \pm \frac{\lambda}{2} \right) d \right)^l. \]
\{c = 0, \ a_1 = -d^2 A / b_1, \ a_{-1} = 0, \ a_0 = 0, \ b_{-1} = 0, \ b_0 = 0, \ b_1 = b_1, \ d = d, \ \dot{r} = 0, \ \mu = 0\}, \quad (27)
\begin{align*}
\{ & u_{t, A} (t) = \frac{A_1}{\pi A_2}, \\
& v_{t, A} (t) = \frac{A_1}{\pi A_3},
\end{align*} \quad (28)

where $A_1$, $A_2$, and $A_3$ are arbitrary constants.

**Remark 3.** The expression (28) is a non-constant steady-state (time independent) solution to Eq. (8) and not derived in [10,35].

4. Conclusion

By means of the extended $(G'/G)$-expansion method, we obtained various kinds of exact discrete solutions to the Ablowitz–Ladik lattice system. Our approach provides traveling wave solutions from which one can easily construct solitary and periodic waves by setting special values to the parameters. We checked the correctness of the solutions by putting them back into the original equation with the aid of MATHEMATICA, this provides an extra measure of confidence in the results. We predict that the extended $(G'/G)$-expansion method will be a promising method for exactly solving NDDEs.

References